Aubry-Mather Theory and Periodic Solutions of the Forced Burgers Equation

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Dedicated with admiration to Professor Hong-ci Huang on the occasion of his retirement

Abstract

Consider a Hamiltonian system with Hamiltonian of the form $H(x, t, p)$ where H is convex in p and periodic in x, and t and $x \in \mathbb{R}^1$. It is well-known that its smooth invariant curves correspond to smooth Z^2-periodic solutions of the PDE

$$u_t + H(x, t, u)_x = 0\,.$$

In this paper, we establish a connection between the Aubry-Mather theory of invariant sets of the Hamiltonian system and Z^2-periodic weak solutions of this PDE by realizing the Aubry-Mather sets as closed subsets of the graphs of these weak solutions. We show that the complement of the Aubry-Mather set on the graph can be viewed as a subset of the *generalized* unstable manifold of the Aubry-Mather set, defined in (2.24). The graph itself is a backward-invariant set of the Hamiltonian system. The basic idea is to embed the globally minimizing orbits used in the Aubry-Mather theory into the characteristic fields of the above PDE. This is done by making use of one- and two-sided minimizers, a notion introduced in [12] and inspired by the work of Morse on geodesics of type A [26]. The asymptotic slope of the minimizers, also known as the rotation number, is given by the derivative of the homogenized Hamiltonian, defined in [21]. As an application, we prove that the Z^2-periodic weak solution of the above PDE with given irrational asymptotic slope is unique. A similar connection also exists in multidimensional problems with the convex Hamiltonian, except that in higher dimensions, two-sided minimizers with a specified asymptotic slope may not exist. © 1999 John Wiley & Sons, Inc.

1 Introduction

Consider the following Hamiltonian system:

$$\frac{dx_i}{dt} = \frac{\partial H}{\partial p_i}\,, \quad \frac{dp_i}{dt} = -\frac{\partial H}{\partial x_i}\,, \tag{1.1}$$

$i = 1, \ldots, n$. We assume that the Hamiltonian $H = H(\boldsymbol{x}, \boldsymbol{p})$ is convex in $\boldsymbol{p}$ and periodic in $\boldsymbol{x}$ with period $[0, 1]^n$. A central issue in the theory of dynamical systems is to look for invariant tori of (1.1). As in [17], we will restrict ourselves to the case of Lagrangian tori, i.e., tori that can be parametrized in the form $G_S = \{(\boldsymbol{x}, \nabla S) : \boldsymbol{x} \in \mathbb{R}^n\}$ for some function $S = S(\boldsymbol{x})$. It is well-known [1] that a necessary and

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sufficient condition for G_S to be invariant is that S satisfies

$$H(\boldsymbol{x}, \nabla S) = \text{const}\,, \tag{1.2}$$

where the constant depends on S. In general, S itself is not periodic, but ∇S is. By rewriting (1.2) as

$$H(\boldsymbol{x}, \boldsymbol{c} + \nabla S) = \text{const}\,, \tag{1.3}$$

we can require S to be periodic.

The task is then reduced to the PDE problem (1.3). Classical KAM theory establishes the existence of smooth solutions of (1.3) for some values of $\boldsymbol{c}$, which give rise to the KAM tori. For $H(\boldsymbol{x}, \boldsymbol{p}) = \frac{1}{2}|\boldsymbol{p}|^2 + V(\boldsymbol{x})$, under suitable conditions on V, such smooth solutions exist for most large values of $\boldsymbol{c}$.

On the other hand, it is relatively straightforward, by using the theory of viscosity solutions [9, 10, 20], to prove that (1.3) always has a periodic weak solution (the viscosity solution) for any $\boldsymbol{c} \in \mathbb{R}^n$ [21]. We are naturally led to the question of what the implications are of these weak solutions from the point of view of the Hamiltonian system (1.1). The main purpose of this paper is to establish a connection between these weak solutions and the Aubry-Mather theory [3, 23, 28].

Consider a smooth, area-preserving map F of an annulus $A = \{(r, \theta) : a \le r \le b\}$:

$$r_1 = f(r, \theta)\,, \quad \theta_1 = g(r, \theta)\,, \tag{1.4}$$

where

$$f(r, \theta + 2\pi) = f(r, \theta), \quad g(r, \theta + 2\pi) = g(r, \theta) + 2\pi\,. \tag{1.5}$$

F is called a monotone twist map if $\partial g/\partial r$ never vanishes.

Let $\Gamma = \{(r, \theta) : r = r(\theta), 0 \le \theta \le 2\pi\}$ be a smooth, invariant curve of F. $F|_\Gamma$ can be viewed as a smooth circle map. By the classical result of Poincaré [2], we can associate with F a rotation number

$$\lim_{j \to \pm\infty} \frac{\theta_j}{j} = \alpha(\Gamma) \tag{1.6}$$

where $(r_j, \theta_j) = F^j(r_0, \theta_0), (r_0, \theta_0) \in \Gamma$. $\alpha(\Gamma)$ is independent of the initial point on Γ. It is clear that the notion of rotation number can be extended to any orbit for which the limit in (1.6) exists.

Denote by Γ_a and Γ_b the inner and outer boundaries of A, respectively, and let $\alpha_a = \alpha(\Gamma_a)$ and $\alpha_b = \alpha(\Gamma_b)$. Without loss of generality, let us assume $\alpha_a < \alpha_b$. The basic result of the Aubry-Mather theory states the following:

THEOREM 1.1 [3, 4, 23, 28]. *For any $\alpha \in [\alpha_a, \alpha_b]$, there exists an invariant Aubry-Mather set Γ_α with rotation number α; i.e., every orbit in Γ_α has a rotation number α. Moreover, Γ_α is a subset of a Lipschitz-continuous curve.*

When α is rational, Γ_α consists of periodic orbits. When α is irrational, it follows from a classical result of Denjoy [2] that Γ_α is either a smooth curve or a

Cantor subset of a Lipschitz curve with countably many gaps. In the latter case, we sometimes speak of Cantori. In either case, every orbit in Γ_α is dense in Γ_α.

Moser [27] made the important observation that the results of Aubry and Mather can be naturally extended to Lagrangian systems of the form $\int L(x,t,\dot{x})dt$ satisfying the Legendre condition $\partial^2 L/\partial \dot{x}^2 > 0$. Here L is assumed to be periodic in (x,t) with period $[0,1]^2$. The corresponding Euler-Lagrange equation can be written in the Hamiltonian form

$$\frac{dx}{dt} = \frac{\partial H}{\partial p}, \quad \frac{dp}{dt} = -\frac{\partial H}{\partial x}, \tag{1.7}$$

where $H = H(x,t,p)$ is the Legendre transform of L with respect to $\dot{x}$. (1.7) is the equation for the characteristics of the first-order Hamilton-Jacobi equation

$$u_t + H(x,t,u)_x = 0\,. \tag{1.8}$$

It is easy to see that if u is a smooth Z^2-periodic solution of (1.8), then the graph $\{(x,u(x,0)) : 0 \le x \le 1\}$ is an invariant curve of the time-1 map

$$F:\ (x(0),p(0)) \to (x(1),p(1)) \tag{1.9}$$

associated with (1.7). The main purpose of Section 2 is to study the situation when we have a weak Z^2-periodic solution of (1.8).

The basic idea behind making a connection between (1.3) and (1.8) with the Aubry-Mather theory is to imbed the globally minimizing orbits of Aubry and Mather into the characteristic field of (1.3) as a special class of characteristics called two-sided minimizers. To construct the entire solution, it is necessary to consider also one-sided minimizers. In the geometric context, such one-sided minimizers are the geodesic rays introduced in [7]. These one-sided minimizers give additional information to the standard Aubry-Mather approach. While the Aubry-Mather theory recognizes two-sided minimizers that account for a subset of the graph of u and the configuration space, the PDE approach gives a much better picture of what constitutes the rest of the configuration space. In particular, it shows that the remaining part on the graph of the solutions of (1.8) lies on the generalized unstable manifold of the Aubry-Mather set. Stated in another way, the full graph of $u(\cdot,0)$, which may contain vertical discontinuities, is a backward-invariant set of the time-1 map: If (x_0,u_0) is on the graph, then $F^{-1}(x_0,u_0)$ is also on the graph.

Crucial to the Aubry-Mather theory is the notion of global minimizers (two-sided minimizers in the present paper). This concept goes back to Morse and Hedlund in their work on geodesics of type A [16, 26]. Aubry constructed the Aubry-Mather set by gathering all globally minimizing orbits of the twist map with a given rotation number. Moser [28] recognized the connection between the work of Morse and Hedlund and the work of Aubry and Mather (see also [5]). In [12], the notion of minimizers was introduced in order to construct time-dependent statistically stationary solutions of (1.8) when the Hamiltonian is random. This was possible since in the random case, minimizers have an intrinsic meaning.

This connection can also be exploited to study the structure of Z^2-periodic solutions of (1.8). For example, it proves the existence of periodic solutions with infinitely many shocks at any fixed time. We can also show, using the minimal properties of the Aubry-Mather sets with irrational rotation numbers, that Z^2-periodic weak solutions of (1.8) with given irrational asymptotic slope are unique.

The generalization of the Aubry-Mather theory to higher dimensions has been less successful in the continuous case. The work of Bangert [5] and Mather [25] suggests that the variety of globally minimizing orbits depends crucially on the convexity of a function, called the minimum average action [24, 25]. We explain in this paper that the minimum average action is nothing but the Legendre transform of the homogenized Hamiltonian, the study of which motivated the work in [11, 13, 21]. In this way, we make a natural connection with homogenization theory.

Although this PDE approach can be exploited to give independent proofs of the main results in the Aubry-Mather theory, we will not insist on doing this here. Instead, we will restrict our interest to exploring the relationship between periodic solutions of the PDE and the Aubry-Mather sets.

Some aspects of ideas explored here can be found in [14, 18].

2 One-Dimensional Problem

Let us start with a one-dimensional problem

$$u_t + H(x,t,u)_x = 0 \tag{2.1}$$

where $H(x,t,u) = f(u) + V(x,t)$ is periodic in (x,t) with period $[0,1]^2$ and convex in u. We will sometimes use p in place of u. For the special case when $H(x,t,u) = \frac{1}{2}u^2 + V(x,t)$, we get

$$u_t + \left(\frac{1}{2}u^2\right)_x = -V_x(x,t)\,. \tag{2.2}$$

This is the forced Burgers equation. The characteristics associated with (2.1) satisfy

$$\frac{dx}{dt} = \frac{\partial H}{\partial p}\,, \quad \frac{dp}{dt} = -\frac{\partial H}{\partial x}\,. \tag{2.3}$$

This is the simplest nonintegrable Hamiltonian system.

Without mentioning any further, we will always be interested in weak solutions of (2.1) satisfying the entropy condition [19]. For a comprehensive treatment, including results on regularity of solutions used in this paper, see [22]. Under the convexity assumption, this means that $u(x+,t) \le u(x-,t)$ for all $(x,t) \in \mathbb{R}^1 \times \mathbb{R}^1$.

(2.1) admits a conservation law

$$\frac{d}{dt}\int_0^1 u\,dx = 0\,. \tag{2.4}$$

Therefore $\langle u \rangle = \int_0^1 u\,dx$ is conserved by the dynamics of (2.1). It is natural to parametrize Z^2-periodic solutions by the value of $\langle u \rangle$.

LEMMA 2.1 *For any $c \in \mathbb{R}^1$, there exists a space-time periodic solution* (*with period* $[0,1]^2$) *such that $\langle u \rangle = c$.*

PROOF: This is proved in [17] using the viscosity method. Consider the following:

$$u_t^\varepsilon + H(x,t,u^\varepsilon)_x = \varepsilon u_{xx}^\varepsilon \,. \tag{2.5}$$

It is shown in [17] that there exists Z^2-periodic solutions of (2.5) u^ε such that $\langle u^\varepsilon \rangle = c$ (see also [30]). Moreover, u^ε satisfies the following a priori estimates:

$$\|u^\varepsilon\|_{L^\infty([0,1]^2)} \le C_1 \,, \quad \|u^\varepsilon\|_{\mathrm{BV}([0,1]^2)} \le C_2 \,,$$

where C_1 and C_2 are constants independent of ε. It follows that there exists a subsequence, still denoted by $\{u^\varepsilon\}$, and a function $u \in L^\infty([0,1]^2) \cap \mathrm{BV}([0,1]^2)$ such that

$$u^\varepsilon \to u$$

a.e., as $\varepsilon \to 0$. It is straightforward to see that u is a weak Z^2-periodic solution of (2.1). □

Remark. In general, such solutions are not unique [17].

Define v by $v(x,t) = \int_0^x u(y,t)dy$. Obviously v satisfies a Hamilton-Jacobi equation of the form

$$v_t + H(x,t,v_x) = f(t)$$

where f is a periodic function of t. Let $v = \tilde{v} + g(t)$ where g satisfies $g'(t) = f(t) - \int_0^1 f(t)dt = f(t) - \langle f \rangle$. We have

$$\tilde{v}_t + H(x,t,\tilde{v}_x) = \langle f \rangle \,.$$

Therefore, without loss of generality, we can write the above equation as

$$v_t + H(x,t,v_x) = \lambda \tag{2.6}$$

where λ is a constant and $v - cx$ is periodic. We will show below that λ is the homogenized Hamiltonian at c (see Section 3).

Let $L = L(x,t,\alpha)$ be the Lagrangian defined as the Legendre transform of $H(x,t,u)$ with respect to u. For any path γ on a finite interval (a,b), we define the action of γ as

$$A_{a,b}(\gamma) = \int_a^b L(\gamma(s), s, \dot{\gamma}(s))ds \,. \tag{2.7}$$

Solutions of (2.6) admit a variational representation [20]:

$$v(x,t) = \inf_{\xi(t)=x} \left\{ v(\xi(s),s) + \int_s^t L(\xi(\tau),\tau,\dot{\xi}(\tau))d\tau \right\} + \lambda(t-s) \tag{2.8}$$

for $s < t$. This will be a basic tool for our discussion.

DEFINITION 2.2 (Minimizers [12]) Let γ be a characteristic satisfying (2.3) on an interval I. γ is called a *minimizer* if it is action-minimizing under compact perturbation; i.e., for any $\delta\gamma$ defined on a finite subinterval I' of I, $I' = [a, b]$, with $\delta\gamma(a) = \delta\gamma(b) = 0$, we have

$$A_{a,b}(\gamma + \delta\gamma) \geq A_{a,b}(\gamma)\,. \tag{2.9}$$

We will be interested in one-sided minimizers defined on $(-\infty, t]$ and two-sided minimizers defined on $(-\infty, +\infty)$.

LEMMA 2.3 *Let* $\gamma : [0, t] \to \mathbb{R}^1$ *be a minimizing path in* (2.8) *and* $\gamma(t) = x$*. Then* $L_\alpha(x, t, \dot\gamma(t))$ *belongs to the set of supergradients of* $v(\cdot, t)$ *at* x*.*

Remark. One can show (see [17, 20]) that $u_x \leq K$ for some constant K depending only on H. Therefore v is semiconcave.

PROOF: First we observe that

$$v(x,t) = v(\gamma(s), s) + \int_s^t L(\gamma(\tau), \tau, \dot\gamma(\tau))d\tau + \lambda(t - s) \tag{2.10}$$

holds for all $s \in [0, t]$. We want to show that

$$\overline{\lim}_{\lambda\to 0+}\frac{v(x+\lambda y, t) - v(x,t)}{\lambda} \leq L_\alpha(x, t, \dot\gamma(t))y\,. \tag{2.11}$$

For $\lambda > 0$, define $\gamma_\lambda : [t - \varepsilon, t] \to \mathbb{R}^1$ by $\gamma_\lambda(s) = \gamma(s) + \frac{s-t+\varepsilon}{\varepsilon}\lambda y$. Then $\gamma_\lambda(t) = x + \lambda y$, $\gamma_\lambda(t - \varepsilon) = \gamma(t - \varepsilon)$, and

$$v(x+\lambda y, t) - v(\gamma(t-\varepsilon), t-\varepsilon) \leq \int_{t-\varepsilon}^t L(\gamma_\lambda(s), s, \dot\gamma_\lambda(s))ds + \lambda\varepsilon\,.$$

Therefore we have

$$\frac{v(x+\lambda y, t) - v(x,t)}{\lambda} \leq \frac{1}{\lambda}\int_{t-\varepsilon}^t \{L(\gamma_\lambda(s), s, \dot\gamma_\lambda(s)) - L(\gamma(s), s, \dot\gamma(s))\}ds\,.$$

This implies

$$\overline{\lim}_{\lambda\to 0+}\frac{v(x+\lambda y, t) - v(x,t)}{\lambda} \leq$$
$$\int_{t-\varepsilon}^t \left\{\frac{s-t+\varepsilon}{\varepsilon}L_x(\gamma(s), s, \dot\gamma(s))y + \frac{1}{\varepsilon}L_\alpha(\gamma(s), s, \dot\gamma(s))y\right\}ds\,.$$

Taking the limit as $\varepsilon \to 0$, we get (2.11). □

LEMMA 2.4 *If* $v(\cdot, t)$ *is differentiable at* x*, then there exists a unique minimizer* $\gamma : [0, t] \to \mathbb{R}^1$ *such that* $\gamma(t) = x$ *and* (2.10) *holds. Furthermore,*

$$\dot\gamma(t) = H_p(x, t, \nabla v(x,t))\,.$$

This is because $\dot\gamma(t)$ has to satisfy

$$\nabla v(x,t) = L_\alpha(x, t, \dot\gamma(t))\,.$$

Similarly, we have the following:

LEMMA 2.5 *Let $\gamma : [0,t] \to \mathbb{R}^1$ be a minimizing path such that* (2.10) *holds. Then for $s < t$, $v(\cdot, s)$ is differentiable at $x = \gamma(s)$.*

The proof goes along the same lines as in the proof of Lemma 2.5 by constructing $\gamma_\lambda : [s, s+\varepsilon] \to \mathbb{R}^1$: $\gamma_\lambda(\tau) = \gamma(\tau) + \frac{s+\varepsilon-\tau}{\varepsilon}\lambda y$. We then have $\gamma_\lambda(s) = \gamma(s) + \lambda y, \gamma_\lambda(s+\varepsilon) = \gamma(s+\varepsilon)$. Furthermore, we have

$$v(\gamma(s)+\lambda y, s) - v(\gamma(s), s) \geq \int_s^{s+\varepsilon} \{L(\gamma(\tau), \tau, \dot{\gamma}(\tau)) - L(\gamma_\lambda(\tau), \tau, \dot{\gamma}_\lambda(\tau))\} d\tau \, .$$

Taking the limit as $\lambda \to 0$ and $\varepsilon \to 0$, we get

$$\lim_{\lambda \to 0+} \frac{v(\gamma(s)+\lambda y, s) - v(\gamma(s), s)}{\lambda} \geq L_\alpha(\gamma(s), s, \dot{\gamma}(s)) y \, .$$

Similarly, by considering the interval $[s-\varepsilon, s]$, we get the reversed inequality.

Now we are ready to prove the following:

THEOREM 2.6 *Let u be a Z^2-periodic solution of* (2.1)*. For each point $(x,t) \in [0,1] \times \mathbb{R}^1$, there exists a one-sided minimizer $\gamma : (-\infty, t] \to \mathbb{R}^1$ such that $\gamma(t) = x$ and for any $s < t$,*

$$u(\gamma(s)+, s) = u(\gamma(s)-, s) \, , \quad \dot{\gamma}(s) = H_u(\gamma(s), s, u(\gamma(s), s)) \, , \tag{2.12}$$

$$v(\gamma(s_2), s_2) - v(\gamma(s_1), s_1) = \int_{s_1}^{s_2} L(\gamma(s), s, \dot{\gamma}(s)) ds + \lambda(s_2 - s_1) \, , \tag{2.13}$$

for $s_1 < s_2 \leq t$.

PROOF: First, let us consider the case of a finite interval $[-k, t]$ where k is an integer, $k > -t$. Denote by $\gamma^k : [-k, t] \to \mathbb{R}^1$ the minimizing path for the variational problem in (2.8). From Lemma 2.5, we know that (2.12) and (2.13) holds for γ^k. Now Theorem 2.6 follows from a simple limiting procedure on k as $k \to +\infty$. □

To make a connection with the Aubry-Mather theory, we need the analogue of the rotation number, which will be the asymptotic slope of one-sided minimizers. To this end, we define ($|I|$ is the length of the interval I):

$$\bar{L}(\alpha) = \lim_{|I| \to \infty} \inf \frac{1}{|I|} \int_I L(\alpha t + \xi(t), t, \alpha + \dot{\xi}(t)) dt \tag{2.14}$$

where the infimum is taken over paths ξ that are piecewise C^1 and vanish at the endpoints of I. This is usually called the minimal average action. We denote by $\bar{H}$ the Legendre transform of $\bar{L}$. It can be shown that the limit in (2.14) exists, independent of how it is taken [11, 25].

LEMMA 2.7 *$\bar{L}$ is strictly convex and differentiable at irrational points.*

This result is proved in [5, 23].

Consequently, $\bar{H}$ is C^1, and all flat pieces on the graph of $\bar{H}$ have rational slopes. Let ξ be a path defined on $[-t, 0]$ that vanishes at the endpoints. Using (2.8), we have

$$\frac{v(\xi(0),0) - v(-\beta t + \xi(-t), -t)}{t} \leq \frac{1}{t}\int_{-t}^{0} L(\beta s + \xi(s), s, \beta + \dot{\xi}(s))ds + \lambda \tag{2.15}$$

for any $\beta \in \mathbb{R}^1$. Taking the limit as $t \to \infty$, we get

$$c\beta \leq \bar{L}(\beta) + \lambda\,. \tag{2.16}$$

On the other hand, let β^* be a limit point of $\gamma(s)/s$ as $s \to -\infty$ for a one-sided minimizer γ; then (2.14) becomes an equality,

$$c\beta^* = \bar{L}(\beta^*) + \lambda\,. \tag{2.17}$$

Hence we have

$$\lambda = \sup_{\beta}\{c\beta - \bar{L}(\beta)\} = \bar{H}(c)\,. \tag{2.18}$$

From Lemma 2.7, we have

$$\beta^* = \bar{H}'(c)\,. \tag{2.19}$$

We have arrived at the following:

LEMMA 2.8 *Let γ be a one-sided minimizer associated with a Z^2-periodic solution u; then*

$$\lim_{t \to -\infty} \frac{\gamma(t)}{t} = \bar{H}'(c) \tag{2.20}$$

where $c = \langle u \rangle$.

In particular, this lemma asserts that all one-sided minimizers associated with a periodic solution of (2.1) have the *same* asymptotic slope.

LEMMA 2.9 *Associated with any Z^2-periodic solution u, there exist genuine two-sided minimizers γ satisfying* (2.10) *for all $t, s \in \mathbb{R}^1$.*

PROOF: Take a sequence of one-sided minimizers γ^k defined on $(-\infty, k]$. Without loss of generality, we can assume that $\gamma^k(0) \in [0,1]$. This can always be arranged by translating γ^k in the x-direction by a suitable amount. We also know that $u(\gamma^k(0), 0)$ is bounded in k since u is bounded. Hence $\dot{\gamma}^k(0)$ is bounded. Therefore there exists a subsequence, still denoted by $\{\gamma^k\}$, that converges uniformly on compact sets to some path γ defined on $\mathbb{R}^1$. It is easy to see that the limit γ is a genuine two-sided minimizer. □

Unlike in the random case [12], two-sided minimizers are usually not unique. They are the globally minimizing orbits defined by Aubry and Mather. We can now define the Aubry-Mather set associated with a Z^2-periodic solution u as

$$\Gamma_{\mathrm{AM}}(u) = \{(\gamma(0), u(\gamma(0), 0)), \gamma \text{ is a genuine two-sided minimizer}\}\,.$$

This is well-defined since u is continuous along two-sided minimizers.

Recall that $F : \mathbb{R}^2 \to \mathbb{R}^2$, is the time-1 map associated with (2.3). We have the following:

THEOREM 2.10 *$\Gamma_{\mathrm{AM}}(u)$ is a closed, invariant set of F.*

$\Gamma_{\mathrm{AM}}(u)$ is obviously invariant. It is closed since the limit of two-sided minimizers is also a two-sided minimizer.

In principle, Γ_{AM} may depend on u, not just its average. The standard Aubry-Mather set associated with a particular asymptotic slope α is the union of all $\Gamma_{\mathrm{AM}}(u)$ such that $\langle u\rangle \subset \nabla \bar{L}(\alpha)$, the set of subgradients of $\bar{L}$ at α. In other words,

$$\tilde{\Gamma}_{\mathrm{AM}}(\alpha) = \bigcup\{\Gamma_{\mathrm{AM}}(u), \langle u\rangle = c \text{ where } c \text{ satisfies } \bar{H}'(c) = \alpha\}\,.$$

Since $\bar{H}'$ is a continuous increasing function, such c always exists for any given α. Therefore the Aubry-Mather set exists for any given rotation number.

It is more informative to visualize the Aubry-Mather set in the xt-plane as the union of all two-sided minimizers. We will denote this set by $\tilde{\Gamma}(u)$. The complement of this set are all one-sided minimizers that are eventually absorbed by the shocks. This set is denoted by $D(u)$. For $D(u)$ and $\tilde{\Gamma}(u)$, we have the following elementary facts: $\tilde{\Gamma}(u)$ is a closed set. Hence $D(u)$ is open. Each connected component of $D(u)$ corresponds to a gap in $\tilde{\Gamma}(u)$. Each gap contains a nonempty connected set that constitutes the shock. The gap can be identified as the domain of attraction [17] for the shock in the sense that any one-sided minimizer inside the gap is eventually absorbed by the shock. Since shocks can neither disappear nor bifurcate, each gap is an infinite strip, and the boundary consists of two two-sided minimizers. When the asymptotic slope is irrational, the existence of one gap implies the existence of infinitely many gaps inside the period and hence infinitely many shocks. When the asymptotic slope is a rational number p/q, we know from the work of Aubry and Mather [5] that the gaps are periodic with period (p, q) in the sense that the boundary curves γ_1 and γ_2 satisfy

$$\gamma_i(\cdot - p) = \gamma_i(\cdot) - q \tag{2.21}$$

for $i = 1, 2$. Curves satisfying (2.21) will be called (p, q)-periodic.

LEMMA 2.11 *Assume that $\bar{H}'(c) = p/q$ where p and q are relatively prime integers. There exists a unique, single-valued, (p, q)-periodic shock curve $S : \mathbb{R}^1 \to \mathbb{R}^1$ inside each gap.*

PROOF: Let $\eta : \mathbb{R}^1 \to \mathbb{R}^1$ be a shock curve inside a gap G. It is obvious that such an object exists. Define $T^k(\eta(\cdot)) = \eta(\cdot - kp) + kq$. $T^k(\eta)$ is also a shock curve inside G. The set $\{(x,t)d : (x,t)$ belongs to the graph of $T^k(\eta)$ for infinitely many k's$\}$ is the graph of the shock curve stated in the lemma. □

We will call this shock the main shock inside G. Its main feature is that it extends from $t = -\infty$ to $t = +\infty$.

Returning now to the set $\Gamma_{\mathrm{AM}}(u)$ defined earlier, let us denote its projection to the x-axis (restricted to $[0,1]$) by $\Gamma_0(u)$. The complement of $\Gamma_0(u)$ is a union of intervals that are the gaps discussed earlier projected to the x-axis.

We turn now to the Lipschitz property of the Aubry-Mather set. This is a simple consequence of the following known result:

LEMMA 2.12 *Let $t_0 > 0$. Assume that $(\gamma_1, p_1), (\gamma_2, p_2) : [-t_0, t_0] \to \mathbb{R}^1$ satisfy the ODE* (2.3) *and $\gamma_1(s) < \gamma_2(s)$ for $s \in [-t_0, t_0]$. Then there exists a constant C, depending only on t_0, such that*

$$\left|\frac{\dot\gamma_1(0) - \dot\gamma_2(0)}{\gamma_1(0) - \gamma_2(0)}\right| \le C\,. \tag{2.22}$$

The proof is elementary and will be given in the appendix. As a consequence, since two-sided minimizers never intersect each other, we have the following:

LEMMA 2.13 *$u(\cdot, 0)$ restricted to $\Gamma_0(u)$ is Lipschitz-continuous.*

LEMMA 2.14 *Assume that $\bar H'(c)$ is irrational. Let G be a gap for $\tilde\Gamma(u)$ and $\gamma_1, \gamma_2 : (-\infty, +\infty) \to \mathbb{R}^1$ be its boundary curves. Then*

$$\gamma_2(t) - \gamma_1(t) \to 0 \tag{2.23}$$

as $t \to \pm\infty$.

PROOF: Let $(a_0, b_0) = (\gamma_1(0), \gamma_2(0))$. Its images under F^i, $i \in Z$, are denoted by $(a_i, b_i) = (\gamma_1(i), \gamma_2(i))$. Since these intervals are disjoint and their total length is finite, we must have $b_i - a_i \to 0$ as $i \to \pm\infty$. Applying the Lipschitz property, we get $\dot\gamma_2(i) - \dot\gamma_1(i) \to 0$. Since γ_1 and γ_2 satisfy (2.3), we get (2.23) from a simple estimate on the ODE. □

As a consequence, any one-sided minimizer inside the gap must be asymptotic to the boundaries of the gap.

When the asymptotic slope is rational, we also have a similar result.

LEMMA 2.15 *Assume that $\bar H'(c)$ is rational. Let G be a gap for $\tilde\Gamma(u)$ and $\gamma_1, \gamma_2 : (-\infty, +\infty) \to \mathbb{R}^1$ be its boundary curves. Let $\gamma : (-\infty, t] \to \mathbb{R}^1$ be a one-sided minimizer inside G. Then either*

$$\gamma(s) - \gamma_1(s) \to 0 \quad \textit{or} \quad \gamma_2(s) - \gamma(s) \to 0$$

as $s \to -\infty$.

PROOF: Assume that γ is to the left of the main shock S. For any $\delta > 0$, let us fix a point $(x_0, t_0) \in G$ such that $x_0 - \gamma_1(t_0) < \delta$ and (x_0, t_0) is also to the left of S. Let ξ be the one-sided minimizer that passes through (x_0, t_0). Consider the set of translates of $(\xi(t_0), t_0) : \{(\xi(t_0) + lq, t_0 + lp) : l \in Z\}$. It is intuitively quite clear that there exists an l^* such that for $l < l^*$, $\xi(t_0) + lq) > \gamma(t_0 + lp)$. This is possible since all translates of ξ are distinct. Since the set of shocks in G is connected, we must have $T^l\xi(s) > \gamma(s)$ on the interval where they are both defined; i.e., γ is to the left of $T^l\xi$. Since

$$T^l\xi(t_0 + lp) - \gamma_1(t_0 + lp) = x_0 + lq - \gamma_1(t_0) + lq < \delta\,,$$

we have

$$0 < \gamma(t_0 + lp) - \gamma_1(t_0 + lp) < \delta$$

for all $l < l^*$. Using the Lipschitz property (2.22), we get:

$$|\dot\gamma(t_0 + lp) - \dot\gamma_1(t_0 + lp)| < C\delta\,.$$

Therefore we get

$$|\gamma(s) - \gamma_1(s)| < C\delta$$

for $s < t_0 + l^*p$. □

DEFINITION 2.16 Let $\xi : \mathbb{R}^1 \to \mathbb{R}^1$ be a trajectory (in the configuration space) satisfying (2.3), and $(\xi(\cdot), \beta(\cdot))$ be the corresponding trajectory in the phase space. Denote by $(x(\cdot), p(\cdot))$ a generic solution of (2.3). The generalized unstable manifold associated with ξ is the set

$$U(\xi) = \{(x(0), u(0)) : |x(s) - \xi(s)| + |p(s) - \beta(s)| \to 0 \text{ as } s \to -\infty\} \tag{2.24}$$

Usually unstable manifolds are defined for hyperbolic invariant sets. In that case, it can be shown that the unstable manifold is actually a manifold. In the current situation, it is not clear that the object defined in (2.24) is a manifold.

THEOREM 2.17 *The graph of $u(\cdot, 0)$ inside a gap lies on the unstable manifold associated with one of the boundary curves of the gap. More precisely, let G be a gap, and γ_1 and γ_2 be the boundaries of G; then $\{(x, u(x, 0)) : u(x+, 0) = u(x-, 0) = u(x, 0), (x, 0) \in G\} \subset U(\gamma_1) \cup U(\gamma_2)$.*

PROOF: Let x_0 be a point such that $(x_0, 0) \in G, u(x_0+, 0) = u(x_0-, 0)$. Let $(x(\cdot), p(\cdot))$ be the solution of (2.3) with initial data $x(0) = x_0, p(0) = u(x_0+, 0)$. We have already shown that either $|x(s) - \gamma_1(s)| \to 0$ or $|x(s) - \gamma_2(s)| \to 0$ as $s \to -\infty$. The convergence of p follows from the Lipschitz property. □

In the case where the boundary curve is hyperbolic, convergence in (2.22) is exponentially fast.

LEMMA 2.18 *The graph of $u(\cdot, 0)$ is invariant under backward iterations of F.*

This is interesting since the graph of $u(\cdot, 0)$ only has vertical gaps.

Lemma 2.18 can be proved using either the theory of backward characteristics developed by Dafermos [8] or directly from Lemmas 2.3 through 2.5. The latter approach extends to multidimensions.

Remark. Let B_c be the union of the graphs of all periodic solutions of (2.1) at $t = 0$ with average equal to c. Then B_c is also backward invariant. In general, this set is not forward invariant. The irreversibility is introduced since the notion of entropy solutions for (2.1) depends on the direction of the time variable.

To construct similar sets that are invariant under the forward iterations of F, we only have to reverse the sign of t in (2.1) and consider

$$\tilde{u}_\tau - H(x, -\tau, \tilde{u})_x = 0 \tag{2.25}$$

where $\tau = -t$. In this way, we find one-sided minimizers that are defined on $[t, +\infty)$, denoted by F_c. The Aubry-Mather set is the intersection of the forward and backward invariant sets constructed in this way.

As an application of the Aubry-Mather theory to the study of Z^2-periodic solutions of (2.1), we prove the following:

THEOREM 2.19 *If $\bar{H}'(c)$ is irrational, then the Z^2-periodic solution of* (2.1) *with $\langle u \rangle = c$ is unique.*

PROOF: Assume there are two such solutions u_1 and u_2. We define v_1 and v_2 as the primitive functions of u_1 and u_2, respectively. Since the asymptotic slope is irrational, we have $\tilde{\Gamma}(u_1) = \tilde{\Gamma}(u_2) = \tilde{\Gamma}$, $D(u_1) = D(u_2)$.

Let γ be a two-sided minimizer. Then

$$v_i(\gamma(t), t) - v_i(\gamma(s), s) = \int_s^t L(\gamma(\tau), \tau, \dot{\gamma}(\tau))d\tau + (t - s)\bar{H}'(c)\,.$$

It is convenient to view the Aubry-Mather set as a subset of the torus $\mathbb{T}^2 = [0, 1]^2$, still denoted by $\tilde{\Gamma}$. Since γ is dense in $\tilde{\Gamma}$, we have on $\tilde{\Gamma}$

$$v_1 = v_2 + C^* \tag{2.26}$$

where C^* is a constant.

Let $(x, t) \in [0, 1] \times \mathbb{R}^1$, and ξ be a one-sided minimizer associated with u_1 such that $\xi(t) = x$. Then

$$v_1(\xi(t), t) - v_1(\xi(s), s) = \int_s^t L(\xi(\tau), \tau, \dot{\xi}(\tau))d\tau + (t - s)\bar{H}'(c)$$

for $s < t$. On the other hand, we also have

$$v_2(\xi(t), t) - v_2(\xi(s), s) \leq \int_s^t L(\xi(\tau), \tau, \dot{\xi}(\tau))d\tau + (t - s)\bar{H}'(c)\,.$$

Hence

$$v_1(\xi(t), t) - v_2(\xi(t), t) \leq v_1(\xi(s), s) - v_2(\xi(s), s)\,.$$

As $s \to -\infty$, ξ must have an α-limit point in $\tilde{\Gamma}$. This implies

$$v_1(\xi(t), t) - v_2(\xi(t), t) \leq C^*\,.$$

Similarly, we also have

$$v_1(\xi(t), t) - v_2(\xi(t), t) \geq C^*\,.$$

Hence

$$v_1(x,t) - v_2(x,t) \equiv C^* \,.$$

□

3 Connection with Homogenization Theory

LEMMA 3.1 *Let u be a Z^2-periodic solution of* (2.1) *with $\langle u \rangle = c$. Then*

$$\int_{[0,1]^2} H(x,t,u(x,t))dx\,dt = \bar{H}(c)\,. \tag{3.1}$$

This follows from (2.18).

The left-hand side of (3.1) can be interpreted as the average Hamiltonian. In particular, Lemma 3.1 asserts that the average Hamiltonian depends only on the average of u, even though there may be more than one Z^2-periodic solution associated with a particular c.

Consider the problem

$$v_t^\varepsilon + H(\tfrac{x}{\varepsilon}, \tfrac{t}{\varepsilon}, v_x^\varepsilon) = 0 \tag{3.2}$$

with initial data $v^\varepsilon(x,0) = v_0(x)$. We assume that v_0 is bounded and continuous.

THEOREM 3.2 *$v^\varepsilon \to \bar{v}$ uniformly on compact subsets of $\mathbb{R}^1 \times \mathbb{R}^1$, where $\bar{v}$ is the solution of*

$$\bar{v}_t + \bar{H}(\bar{v}_x) = 0 \tag{3.3}$$

with initial data $\bar{v}(x,0) = v_0(x)$.

For this reason, $\bar{H}$ is also referred to as the homogenized Hamiltonian.

An equivalent formulation of Theorem 3.2 can be obtained by considering the variational problem

$$\min \int_I L\Big(\tfrac{x(t)}{\varepsilon}, \tfrac{t}{\varepsilon}, \dot{x}(t)\Big)dt \tag{3.4}$$

where I is a finite interval. We then have the following:

THEOREM 3.3 *The variational problem* (3.4) *Γ-converges to*

$$\min \int_I \bar{L}(\dot{x}(t))dt\,.$$

Theorem 3.2 was first proved in [20, 21] for general, time-independent Hamiltonians; see also [13]. Theorem 3.3 was proved (for more general situations) in [11]. As was indicated at the end of [11], Theorem 3.2 is an easy consequence of Theorem 3.3.

4 Multidimensional Problem

Let us now return to the multidimensional case and consider (1.1) and (1.3). It is useful to think about the geometric example of a torus $\mathbb{T}^n$ equipped with a Riemannian metric g. In this case (1.1) is the equation of the geodesics, and the Lagrangian is a homogeneous function of degree 1:

$$L(x, \dot{x}) = (\sum_{i,j} g_{i,j}(x)\dot{x}_i\dot{x}_j)^{\frac{1}{2}} .$$

It was shown in [21] that for any given $\boldsymbol{c}$, periodic weak solutions of (1.3) always exist for a unique constant $\bar{H}(\boldsymbol{c})$. An outline of the proof there is as follows:

Consider the perturbed problem

$$H(x, \boldsymbol{c} + \nabla v^{\varepsilon}) + \varepsilon v^{\varepsilon} = 0 .$$

It is easy to show that $|\nabla v^{\varepsilon}|$ and $|\varepsilon v^{\varepsilon}|$ are uniformly bounded. One can then extract a subsequence of $(v^{\varepsilon} - \min_{[0,1]^n} v^{\varepsilon}, -\varepsilon v^{\varepsilon})$ that converges uniformly on $[0,1]^n$ to (v, λ) where v is a periodic Lipschitz-continuous function on $[0,1]^n$ and λ is a constant. v is a periodic viscosity solution of (1.3) with the constant equal to $\lambda = \bar{H}(\boldsymbol{c})$.

As in Section 2, we can still define one- and two-sided minimizers associated with v. Lemmas 2.3 through 2.5 still hold; only notational changes are required for their proofs. The analogue of (2.12) holds in multidimensions and can be derived simply from (2.8) and (2.10). We can also define the minimal averaged Lagrangian and homogenized Hamiltonian, $\bar{L}$ and $\bar{H}$, in the same way as in Section 2. These are still convex functions. There is a very significant difference, however—namely, that $\bar{L}$ is no longer strictly convex for higher-dimensional problems. This is most easily seen in the geometric example for which the minimal averaged Lagrangian is given by the stable norm [6, 15]:

$$\bar{L}(\alpha) = ||\alpha||_s = \lim_{m\to\infty} \frac{1}{m}\tilde{d}(0, m\alpha)$$

where $\tilde{d}$ is the distance in the metric g lifted to $\mathbb{R}^n$. We will denote the unit ball of the stable norm by B. Its Wulff shape, the convex dual of B, will be denoted by W_B [29]. The average Hamiltonian $\bar{H}$ is given by the indicator function of the Wulff shape associated with the unit ball of the stable norm. Bangert [6] has shown that the stable norm of the so-called Hedlund example of Z^3-periodic Riemannian metrics is given by

$$||\alpha||_s = C_1|\alpha_1| + C_2|\alpha_2| + C_3|\alpha_3|$$

where $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ and C_1, C_2, and C_3 are constants associated with the Riemannian metric. In this case B is an octahedron. This is the most extreme case where B is only strictly convex at its six vertices.

The loss of strict convexity of $\bar{L}$ means that $\bar{H}$ is not continuously differentiable everywhere. As a result, Lemma 2.5 is changed to the following:

LEMMA 2.4′ *Let γ be a one-sided minimizer associated with a periodic solution of* (1.3). *Then as $t \to -\infty$, all limiting points of $\gamma(t)/t$ belong to $\nabla \bar{H}(\boldsymbol{c})$, the set of subgradients of $\bar{H}$ at $\boldsymbol{c}$.*

The proof is the same as the proof of Lemma 2.5. This gives a very simple proof of the main results in [6] in a more general form. Contrary to the one-dimensional situation considered in Section 2, the one-sided A-characteristics may not have the same asymptotic slope.

We can define the Aubry-Mather set as

$$\Gamma_{\mathrm{AM}}(v) = \{(x, \nabla v(x)) : \text{ there exists a two-sided minimizer passing through } x\,.\}$$

Lemmas 2.7 and 2.8 still hold. Therefore this is a nonempty set, and it is invariant under the dynamics of (1.1). The Lipschitz property also holds and can be proved in a similar way.

The loss of strict convexity of $\bar{L}$ also implies that the set of possible asymptotic slopes of two-sided minimizers can be a very small set, in contrast to the one-dimensional situation when there exist two-sided minimizers of arbitrary asymptotic slope. In the multidimensional case, we know that this set contains the set $\{\nabla \bar{H}(\boldsymbol{c}) : \bar{H}$ is differentiable at $\boldsymbol{c}\}$. Bangert [5] showed that for the Hedlund example, it contains no other elements. Therefore $\tilde{\Gamma}_{\mathrm{AM}}(\alpha)$ is not defined for all α.

Consider now the variational problem (2.12) in multidimensions. For some values of α, this variational problem may not have a solution. A natural idea is to relax the original problem and admit as solutions probability measures on the space of trajectories. This is Mather's notion of minimal measure [24, 25]. For the Lagrangian systems, Mather showed that this measure can be realized on the tangent bundle instead of on the space of trajectories. However, in the more general situation with more than one independent variable as considered in [11], the minimal measure will be defined on the space of trajectories, or more appropriately, maps.

5 Concluding Remarks

For one-dimensional problems, our understanding of the basic issues is quite complete. However, there are several refined questions that are outstanding. One is the rate of convergence of the viscous limit for the periodic solutions (problem 3 in [17]). Another interesting question is the minimum regularity of Lipschitz-continuous solutions. Clearly the result has to depend on the asymptotic slope of the minimizers. It is easy to see that there are periodic solutions that are Lipschitz-continuous but not better.

For higher-dimensional problems, the amount of information obtained from this theory depends heavily on the convexity of $\bar{L}$. In the case of Hedlund's example, the Mather set is so small that it gives little information about the global properties of the geodesic flow. Any general information on the convexity and properties of $\bar{L}$ and $\bar{H}$ can be helpful.

Finally, to understand whether there is any limitation on this PDE approach to the minimizing orbits, it would be interesting to prove or disprove that any globally minimizing solutions of (1.1) can be imbedded into a periodic solution of (1.3).

Appendix: Proof of Lemma 2.12

We will work with the assumption that $0 < t_0 \ll 1$ and use C's to denote generic constants that depend only on H. We will prove the lower bound. The proof of the upper bound goes in the same way.

For $f : [0, t_0] \to \mathbb{R}^1$, define $\|f\| = \sup_{0\le t\le t_0} |f(t)|$. Let

$$g(x,p,t) = H_{px}(x,p,t)H_p(x,p,t) - H_{pp}(x,p,t)H_x(x,p,t) + H_{pt}(x,p,t)\,.$$

Then

$$\dot\gamma_i(t) = \dot\gamma_i(0) + \int_0^t g(\gamma_i(s), p_i(s), s)ds$$

for $i = 1, 2$. Subtracting the two identities, we get

$$\begin{aligned}\|\dot\gamma_2 - \dot\gamma_1\| &\le |\dot\gamma_2(0) - \dot\gamma_1(0)| + t_0(\|g_x\|\,\|\gamma_2 - \gamma_1\| + \|g_p\|\,\|p_2 - p_1\|)\\ &\le |\dot\gamma_2(0) - \dot\gamma_1(0)| + C_1 t_0(\|g_x\|\,\|\gamma_2 - \gamma_1\| + \|g_p\|\,\|\dot\gamma_2 - \dot\gamma_1\|)\,.\end{aligned}$$

Hence we have

$$\|\dot\gamma_2 - \dot\gamma_1\| \le (1 + C_2t_0)|\dot\gamma_2(0) - \dot\gamma_1(0)| + C_3t_0\|\gamma_2 - \gamma_1\|\,. \tag{A.1}$$

Similarly, from

$$\gamma_2(t) - \gamma_1(t) = \gamma_2(0) - \gamma_1(0) + \int_0^t [H_p(\gamma_2(s), p_2(s), s) - H_p(\gamma_1(s), p_1(s), s)]ds$$

we get

$$\|\gamma_2 - \gamma_1\| \le (1 + C_4t_0)|\gamma_2(0) - \gamma_1(0)| + C_5t_0|\dot\gamma_2(0) - \dot\gamma_1(0)|\,.$$

Next we have

$$\begin{aligned}\gamma_2(t) - \gamma_1(t) &= \gamma_2(0) - \gamma_1(0) + t(\dot\gamma_2(0) - \dot\gamma_1(0))\\ &\qquad + \int_0^t\int_0^s [g(\gamma_2(\tau), p_2(\tau), \tau) - g(\gamma_1(\tau), p_1(\tau), \tau)]d\tau ds > 0\,.\end{aligned}$$

Therefore,

$$\gamma_2(0) - \gamma_1(0) + t_0(\dot\gamma_2(0) - \dot\gamma_1(0)) + Ct_0^2(\|\gamma_2 - \gamma_1\| + \|p_2 - p_1\|) > 0\,.$$

The last inequality can be rewritten as

$$\gamma_2(0) - \gamma_1(0) + t_0(\dot\gamma_2(0) - \dot\gamma_1(0)) + Ct_0^2(\|\gamma_2 - \gamma_1\| + \|\dot\gamma_2 - \dot\gamma_1\|) > 0\,. \tag{A.2}$$

Assume that $\dot\gamma_2(0) - \dot\gamma_1(0) < 0$. Otherwise, we already have

$$\frac{\dot\gamma_2(0) - \dot\gamma_1(0)}{\gamma_2(0) - \gamma_1(0)} > 0\,.$$

Combining (A.1) and (A.2), we get

$$\frac{\dot\gamma_2(0)-\dot\gamma_1(0)}{\gamma_2(0)-\gamma_1(0)} > -\frac{1+Ct_0}{t_0}\,. \tag{A.3}$$

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