Augmenting the Connectivity of Geometric Graphs

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Abstract

Let G be a connected plane geometric graph with n vertices. In this paper, we study bounds on the number of edges required to be added to G to obtain 2-vertex or 2-edge connected plane geometric graphs. In particular, we show that for G to become 2-edge connected, $\frac{2n}{3}$ additional edges are required in some cases and that $\frac{6n}{7}$ additional edges are always sufficient. For the special case of plane geometric trees, these bounds decrease to $\frac{n}{2}$ and $\frac{2n}{3}$, respectively.

1 Introduction

A classical problem in graph theory is that of augmenting the connectivity of a graph G by adding to it as few edges as possible. The problem of increasing the connectivity of a connected graph to make it 2-vertex or 2-edge connected using the smallest possible number of edges can be solved in linear time [8]. For $k = 3, 4$ polynomial time algorithms for augmenting a $k-1$-vertex connected graph to a k-vertex connected graph have been known for some time (see [24, 15]); only recently a polynomial time algorithm for this problem has been found for any fixed k [17]. A survey in which these problems are described within a more generic framework is given in [20].

The problem of increasing the connectivity of planar graphs was studied by Kant [18, 19]. He proved that it is NP-hard to determine the minimum number of edges required to be added to augment a given planar graph into a 2-vertex connected planar graph. The corresponding problem for 2-edge connectivity, i.e, determining the minimum number of edges we have to add to augment a given planar graph into a 2-edge connected planar graph, is open.

In this paper we study the following problem: let G be a connected plane geometric graph. How many edges must be added to G in such a way that the plane geometric graph we obtain is 2-edge or 2-vertex connected? We show that for G to become 2-edge connected, $\frac{2n}{3}$ additional edges are required in some cases and that $\frac{6n}{7}$ additional edges are always sufficient. If G is a plane geometric tree (a plane connected geometric graph with n vertices and $n-1$ edges), the addition of $\frac{2n}{3}$ edges is always sufficient to make it 2-edge

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connected and $\frac{n}{2}$ edges are sometimes required. Moreover, if G has b blocks, then it can be completed to a 2-vertex connected plane geometric graph by adding at most $b-1$ edges.

A closely related problem for geometric graphs was studied by Rappaport [22]. He proved that the problem of deciding whether a plane geometric graph G which is a set of polygonal chains admits a simple circuit (a geometric graph which is a cycle) is NP-complete. This paper was perhaps the first of a series of papers by several authors in which the objective is to find planar geometric graphs that have some specific structure [2, 12, 13, 14, 16].

Let us recall some standard notations and definitions. Let $G = (V, E)$ be a graph. A graph is k-vertex connected (resp. k-edge connected) if the deletion of any set of at most $k-1$ vertices (resp. $k-1$ edges) of G results in a connected graph. If G is a graph and e one of its edges, $G-e$ will denote the graph obtained by removing e from G. Similarly $G+e$ will denote the graph obtained by adding to G an edge e not in G. An edge e with vertices u and v will be denoted by uv.

A vertex v of a graph G is called a *cut vertex* if $G-v$ is not connected. A graph with no cut vertices is called a *block*. Given a graph G, a maximal subgraph that has no cut vertices is called a block of G. Observe that if G is a tree, the blocks of G are its edges. A block of a graph with at least 3 vertices is 2-vertex connected. An edge e of a graph G is called a *bridge* if $G-e$ is not connected.

According to [11], a *geometric graph* is a graph G such that its vertex set is a set of points on the plane in general position (no three points being collinear), and its edge set is a set of line segments joining pairs of vertices of G. A geometric graph G is *plane* if no two edges of G intersect except at a common vertex. Plane geometric graphs are also known in the literature as plane straight line graphs.

All geometric graphs considered here will be plane. We will also assume that all our graphs have n vertices, $n \geq 3$, and that they are *connected*. A plane geometric graph G is called a *triangulation* if all of its faces, except perhaps for the unbounded face of G, are triangles.

The paper is organized as follows. In Section 2 we study the problem of obtaining 2-vertex connected graphs, in Section 3 we study the problem of adding edges to a plane geometric tree to obtain a 2-edge connected plane geometric graph, and in Section 4 we study the problem of obtaining 2-edge connected plane geometric graphs from generic plane geometric graphs. We make some concluding remarks in Section 5.

2 Two-Vertex Connected Plane Geometric Graphs

A set of points in the plane is in *convex position* if the elements of the set are the vertices of a convex polygon. Observe that if G is a plane geometric graph with at least two vertices and whose vertices are in convex position, then G is an outerplanar graph, and thus it has at least two vertices of degree two or less. Therefore the edge and vertex connectivity of G are at most two. Hence we only study problems regarding the completion of plane geometric graphs to 2-vertex and 2-edge connected graphs.

In this section, we solve the problem of finding the maximum number of edges that must be added to a plane geometric graph to obtain a 2-vertex connected plane geometric graph.

Theorem 1. *Let G be a connected plane geometric graph with b blocks. Then G can be completed to a 2-vertex connected plane geometric graph by adding at most $b-1$ edges to G. This bound is tight.*

Proof. The proof proceeds by induction on the number of blocks of G. Recall that G contains at least three vertices. If G has exactly one block, then it is already 2-connected and no edges need to be added.

Suppose, then, that G has at least two blocks, and let v be a cut vertex of G. We now prove that we can add one edge to G so that we obtain a plane geometric graph with fewer blocks than G. This will prove our result. Let us divide the vertices of $G - v$ into two disjoint sets: a set V_1 formed by the vertices of one of the components of $G - v$, and a set V_2 containing the remaining vertices of $G - v$. By construction, the vertices of V_1 are in a block different from those containing the elements of V_2. A folklore result for plane geometric graphs asserts that any plane geometric graph G can be completed to a triangulation $\mathcal{T}$. Observe next that no triangulation contains a cut vertex. It follows now that there is an edge e in $\mathcal{T}$ that joins two vertices, one in V_1 and the other in V_2, for otherwise v would be a cut vertex in $\mathcal{T}$. By adding $e = (v_1, v_2)$ to G we obtain a plane geometric graph in which the edges contained in any simple path from v_1 to v_2 are in a new common block.

To prove that the bound is tight, we observe that if G is a *zig-zag* path whose vertices are in convex position, it has exactly $n - 1$ blocks (its edges), and to make it 2-vertex connected, we need to add to G exactly $n - 2$ edges. ∎

3 Two-Edge Connected Plane Geometric Graphs from Trees

We observe next that a similar idea to that used in the proof of Theorem 1 can be used to increase the edge connectivity of a plane geometric graph.

Let e be a bridge of a plane geometric graph G, and let H_1 and H_2 be the components of $G - e$. As in the proof of Theorem 1, if we add edges to G until we get a triangulation, there must be an edge $f \neq e$ joining a vertex in H_1 to a vertex in H_2. Clearly e is no longer a bridge in $G + f$. Thus we have:

Lemma 1. *Let G be a plane geometric graph with k bridges. Then G can be completed to a 2-edge connected plane geometric graph by adding at most k edges to G.*

It is straightforward to see that in some cases, k edges are necessary.

3.1 Method 1 for Trees

The next lemma will be useful for improving the previous bound for the case when G is a tree.

Lemma 2. *Let G be a plane geometric graph and $e = uv$ a bridge of G. Let H_1 and H_2 be the components of $G - e$ such that $u \in H_1$. Then if H_1 has more than one vertex, there is an edge $f = v'w$ such that $G + f$ is planar, $v' \in H_2$, $w \in H_1$, $w \neq u$, and thus e is no longer a bridge of $G + f$.*

Proof. Let $\mathcal{T}$ be a triangulation that contains G as a subgraph. In $\mathcal{T}$ we must necessarily have at least one edge $f = v'w$ connecting a vertex $w \neq u$ of H_1 to v, for otherwise u would be a cut vertex of $\mathcal{T}$. ∎

We can now prove:

Lemma 3. *Let G be a plane geometric tree with h leaves. Then G can be completed to a 2-edge connected plane geometric graph by adding to it at most $\lfloor \frac{n+h-2}{2} \rfloor$ edges.*

Proof. Let S be the set of edges of G such that none of its vertices is a leaf; let $|S| = m$, $m + h = n - 1$. Observe that for any edge $e \in S$, the components of $G - e$ have at least two vertices.

The proof is constructive, and in each step we add a new edge creating a cycle that contains at least two bridges of G. We start with a leaf, v, of G. Let $e = uv$ be the edge of G incident to v. Since the component of $G - e$ containing u has at least two vertices, then by Lemma 2 there is an edge $f = vw$ such that $G_1 = G + f$ is a plane geometric graph, $u \neq w$. Moreover G_1 contains a cycle C_1 with at least two edges of G. Let $H_1 = C_1$, the only 2-edge connected component of G_1.

If G_1 is not 2-edge connected, let $e = uv$ (if it exists) be a bridge of G_1 such that v is a vertex of H_1 and the second component F of $G_1 - e$ has at least two vertices. Note that F is a tree. By Lemma 2, there is an edge $f = v'w$, $u \neq w$, such that w is a vertex of F and $G_1 + f$ is plane. Observe that when we add f to G_1 we create a cycle C_2 containing at least two bridges of G_1, e and a bridge in the path from u to w in G. Let $G_2 = G_1 + f$, and H_2 be the subgraph of G_2 obtained by adding to H_1 the edges of C_2 not in H_1. Thus, the only 2-edge connected component of G_2 is H_2.

We iterate this process by adding in each step a new cycle to the 2-edge connected component. In a generic step i, we search for a bridge $e = uv$ of G_i such that v is a vertex of H_i and the second component F of $G_i - e$ has at least two vertices, and then we define the graphs G_{i+1} and H_{i+1} as before. The process stops when the graph obtained, G_i, is 2-edge connected or all the edges of G_i not in H_i are leaves, that is, until the bridge e we were seeking above does not exist. In this last case, we use an extra edge to eliminate the bridges on the remaining leaves. Clearly during the process we added at most $\lfloor \frac{m+1}{2} \rfloor + h - 1 = \lfloor \frac{n+h-2}{2} \rfloor$ edges. ∎

As a corollary we have:

Corollary 1. *Let G be a plane geometric graph that is a path. Then it can always be completed to a 2-edge connected plane geometric graph by adding at most $\lfloor n/2 \rfloor$ edges; the bound is tight.*

The bound is again achieved when G is a zig-zag path and its vertices are the vertices of a convex polygon.

The bound given in Lemma 3 is, however, poor if G has many leaves. We present below a different method which gives better results for this case.

3.2 Two Lemmas

Our objective now is to show that any plane geometric tree can be completed to a 2-edge connected plane geometric graph with the addition of at most $\frac{2n}{3}$ edges.

Before we prove this result, some remarks are in order. In general, we want to increase the edge connectivity of a plane geometric graph G. To achieve this we will take a second plane geometric graph H (not necessarily connected) and consider the union of G and H. The main requirement is that $G \cup H$ also be a plane geometric graph. We will also use the following technical trick that will allow us to simplify our proofs. In what follows it could happen that H has some edges in common with G. If an edge uv is an edge in G and H, we will consider u and v to be joined by two edges, i.e. we will admit multiple edges. We will color the edges of G black, and the edges of H red. Thus if u and v are joined by two edges, one will be black, and the other red. The black edges will always remain, while the red edges may be deleted and inserted throughout the procedures.

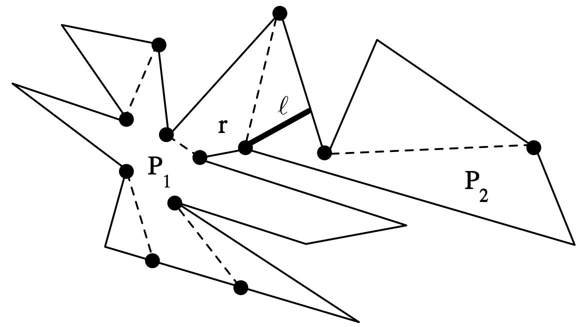

Figure 1: Illustration of Lemma 5.

In all of the figures here, red edges will be represented by dashed curves. We stress that the edge represented by the dashed curve is in fact a straight line segment, and that for our purposes we will consider double edges as non-intersecting.

For a trivial example, to obtain a 2-edge connected plane geometric graph from a plane geometric tree G, we can proceed as follows. Let H be isomorphic to G, and let $G' = G \cup H$. Then G' is 2-edge connected. Indeed any two vertices in G' are joined by two edge disjoint paths, a black path in G, and a red path in H.

The next lemma will prove useful and will allow duplicated edges to be eliminated.

Lemma 4. *Let $G' = G \cup H$ be a planar geometric graph such that G' is 2-edge connected. Edges of G are coloured black and edges of H are coloured red. Let u and v be two vertices of G' that are joined by a black and a red edge, e and e' respectively. Then we can either eliminate e' or substitute it by another red edge f such that $G' - e'$ or $G' - e' + f$ is 2-edge connected. In the second case, f can be chosen such that it does not create a new double edge.*

Proof. If there is a cycle which uses e and bypasses e', we can eliminate e', and $G' - e'$ remains 2-edge connected. Suppose there is no such cycle. Then e is a bridge of $G' - e'$. By Lemma 2, there is an edge $f \neq e$ such that $G' - e' + f$ is a plane geometric graph. It is easy to see that $G' - e' + f$ is 2-edge connected. Clearly f is not part of a double edge. ∎

A *plane geometric perfect matching* of a point set P with $2m$ elements is a set of m disjoint segments joining pairs of elements of P. We will use the two following results which are interesting in themselves.

Lemma 5. *Let P be a simple polygon and let $R = \{r_1, \ldots, r_l\}$ be the set of reflex vertices of P. Let A be a subset of the vertex set of P with an even number of elements such that $R \subset A$. Then there is a plane geometric perfect matching $\mathcal{M}$ of A such that the line segments determined by $\mathcal{M}$ are contained in the interior or lie on the boundary of P.*

Proof. The proof is by induction on the number of reflex vertices of P. The result is clearly true if P is convex. Suppose then that P has at least one reflex vertex r. Let ℓ be an open line segment contained in the interior of P such that ℓ splits P into two polygons P_1 and P_2 such that one endpoint of ℓ is r, and r is a convex vertex in both of P_1 and P_2; see Figure 1.

Let A_1 and A_2 be the subsets of $A - \{r\}$ that are vertices of P_1 and P_2 respectively. One of them, say A_1, has an even number of elements, while A_2 has an odd number of elements.

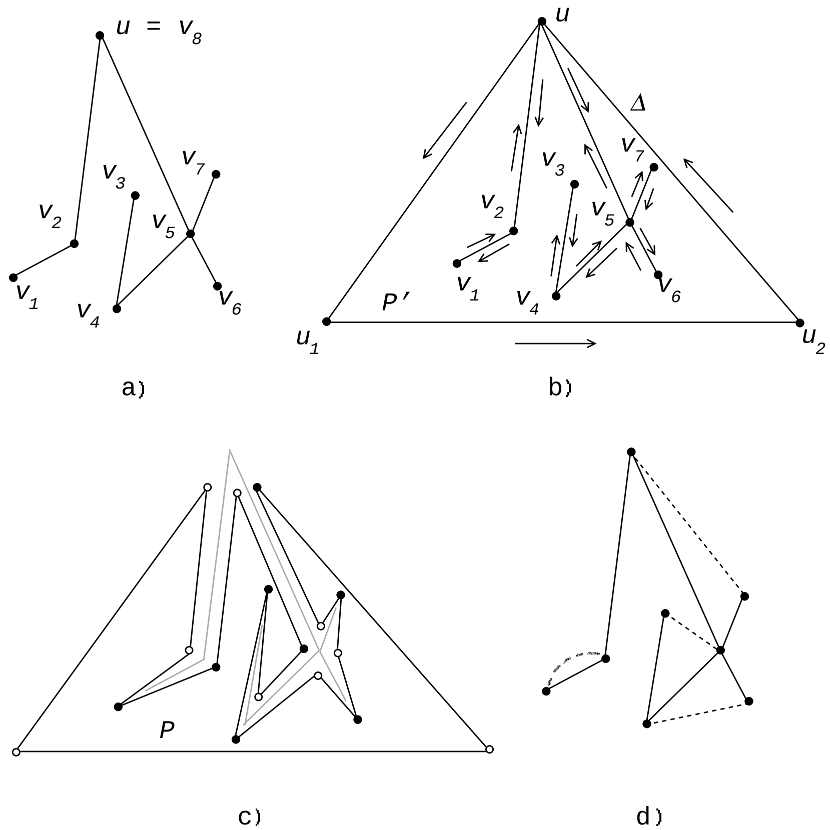

Figure 2: Part a) shows the given tree. Part b) shows the weakly simple polygon $P' = (u, u_1, u_2, u, v_5, v_7, v_5, v_6, v_5, v_4, v_3, v_4, v_5, u, v_2, v_1, v_2, u)$. Part c) shows the simple polygon P obtained from P'. The copies chosen to form the matching $\mathcal{M}$ are in black. Part d) shows the tree augmented with $\mathcal{M}$. Edges of $\mathcal{M}$ are represented by dashed curves.

Since r is no longer a reflex vertex, both A_1 and A_2 have fewer reflex elements than A. By adding r to A_2, and applying induction on P_1 and A_1, and P_2 and A_2, respectively, the result follows. ∎

Lemma 6. *Let G be a plane geometric tree with an even number of vertices. Then there is a perfect matching $\mathcal{M}$ on the set of vertices of G such that the graph G' obtained by adding to G the edges of $\mathcal{M}$ is a plane geometric graph, possibly with multiple edges. If two vertices of G' are joined by two edges, one of them belongs to G and the other to $\mathcal{M}$.*

Proof. Let G be the given plane geometric tree and $\triangle$ a triangle that encloses all the vertices of G, such that one vertex of G, say $u = v_n$ is also a vertex of $\triangle$. By *duplicating* the edges of G and traversing externally the edges, starting at u, we obtain a weakly simple polygon P' with $2(n-1)+3$ edges (a weakly simple polygon is a closed polygonal chain without self-crossings). Figure 2b shows how this weakly simple polygon P' is built.

It is well-known that any weakly simple polygon P' can be transformed into a simple polygon P very close to it (see for example [3, 6, 14]). In our case, we proceed as follows. Each time that a vertex v of the tree appears in P', if it appears in the sequence v_i, v, v_k, then substitute vertex v by a vertex v' on the bisector of the clockwise angle $v_i v v_k$ and placed it at a distance arbitrarily small ϵ from v. Linking these new vertices in the same order as they are created we obtain the simple polygon P being sought (see Figure 2c). Note that G lies on the complement of P.

Let us call these added vertices v' copies of v. If v has degree k, the set S_v of copies of v has k elements, except for S_u that contains as many copies of u as the degree of u plus one. Notice that, by construction, at most one copy of each vertex v can be reflex in P.

Therefore, by choosing in each set S_v the reflex copy of v, if it exists, or an arbitrary copy of v otherwise, we can form a perfect matching of these n copies in the interior of P. Besides, if ϵ is small enough we can substitute each segment $v'w'$ between copies by the segment vw between vertices of G, without producing crosses (but perhaps producing duplicated edges), thus obtaining the matching sought (see Figure 2d).

∎

3.3 Method 2 for Trees

We now outline how to complete a plane geometric tree G to a 2-edge connected plane geometric graph using few edges. Remember that the edges of G are coloured black and the other edges are coloured red. In a nutshell, the algorithm used to accomplish this task is as follows:

Algorithm 1

Phase 1. Matching
Given G, construct $\mathcal{M}$ and G' as in Lemma 6. Let $C_1, \ldots, C_s$ be the 2-edge connected components of G'.

Phase 2. Merging components with 2 and 4 vertices
Add and delete some suitable red edges of G' to obtain a geometric graph G'' such that G is still a subgraph of G'', G' and G'' have the same number of edges, and all the 2-edge connected components of G'' contain at least 6 nodes.

Phase 3. Merging components with at least 6 edges
Add some extra edges to G'' by using the techniques described in Lemma 2 to eliminate all bridges.

End Algorithm 1

Observe that at the end of Phase 1, the components $C_1, \ldots, C_s$ have the following properties:

1. They are vertex disjoint.
2. They are joined by exactly $s-1$ bridges. Moreover, the graph T' whose vertices are $C_1, \ldots, C_s$, two of which are adjacent if there is a bridge that joins them, is a tree.
3. If we add an edge joining a vertex of C_i to a vertex in C_j, the resulting graph has a new 2-edge connected component containing all the vertices of the components in the path joining C_i to C_j in T'.
4. If a component C_i of G' has k edges of $\mathcal{M}$, then C_i has exactly $2k$ nodes, since $\mathcal{M}$ is a perfect matching. Moreover, these nodes are connected by exactly $2k-1$ edges of G.

For the tree shown in Figure 3 (left), the graph G' obtained by adding $\mathcal{M}$ to it has two 2-edge connected components, one of which has two vertices, and the other six. They are joined by the bridge uw.

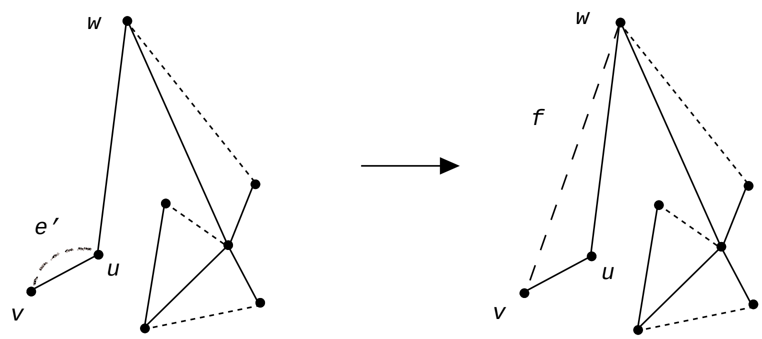

Figure 3: The 2-edge component uv can be joined to another component by deleting edge e' and adding edge f.

Now we show the details of Phase 2. Complete G' to a triangulation $\mathcal{T}$. Given any 2-edge component C_i of G', as $\mathcal{T}$ has to be 2-edge connected, then there must exist in $\mathcal{T}$ at least one empty triangle uvw such that $u, v \in C_i$ and $w \notin C_i$. These empty triangles uvw will be used in the process of merging components, where some of the red edges of G' will be changed for other edges of $\mathcal{T}$, using the method given in the following lemma.

Lemma 7. *Let G' be a plane geometric graph obtained by adding a set E of red edges to a plane geometric tree G (the black edges). Let C_i be a 2-edge connected component of G'. Suppose that there is an empty triangle uvw such that $u, v \in C_i$, $w \notin C_i$, and that in the path Π from u to v in G (a path formed by black edges contained in C_i) there is a double edge, i.e., a black edge $e = u'v'$ and a red copy e'. Then we can join C_i with the component containing the vertex w, using the same number of red edges as G'.*

Proof. Without loss of generality, suppose that, when edge $u'v'$ is deleted from the tree G, u, u' and w are in the same component of G, and v and v' are in the other component. Let Π_1 be the path joining u' and w in G and let Π_2 be the path joining v' and v in G. As Π_1, edge $u'v', \Pi_2$ and edge vw form a cycle, necessarily vw is neither an edge of G nor an edge of E. Then, by adding the new red edge vw to G' and deleting the red copy e' we keep the 2-edge connectivity of C_i and we join it to the 2-edge component containing w. ∎

As a component C_i of G' with two vertices consists of a double edge, a black edge uv and a red copy e', and there is an empty triangle uvw in $\mathcal{T}$, then we can apply the previous lemma to join C_i to the 2-edge component containing w, by adding $f = uw$ (or $f = vw$) and deleting e'. See Figure 3 for an example. Note that $G' + f - e'$ is a plane geometric graph. This process can be applied to the new graph obtained, to successively eliminate all the 2-edge connected components with two vertices. Thus, a new graph G'' is obtained with the same number of edges as G' and no 2-edge connected components with two vertices.

We now show how to join the 2-edge connected components of G'' with four vertices to other components of G'' without increasing the number of edges of G''. Let C_i be a

2-edge connected component of G'' with four vertices. There are five different ways (up to isomorphisms) in which the black and red edges are distributed in C_i. These are depicted in Figure 4. Cases a), b) and c) correspond to the different ways in which C_i can appear in G' or G'', knowing that the red edges are in $\mathcal{M}$. Cases d) and e) can occur when we join a 2-edge connected component with two vertices to another one using the previous method.

In cases b) and d) we can eliminate the red edges $u'v$ and $u'v'$, respectively, retaining the 2-edge connectivity of C_i. We can then join C_i to another component of G'' by using an extra edge of $\mathcal{T}$. In case a) we can eliminate edge uu' and add a red edge connecting u to v, reducing the configuration to case e).

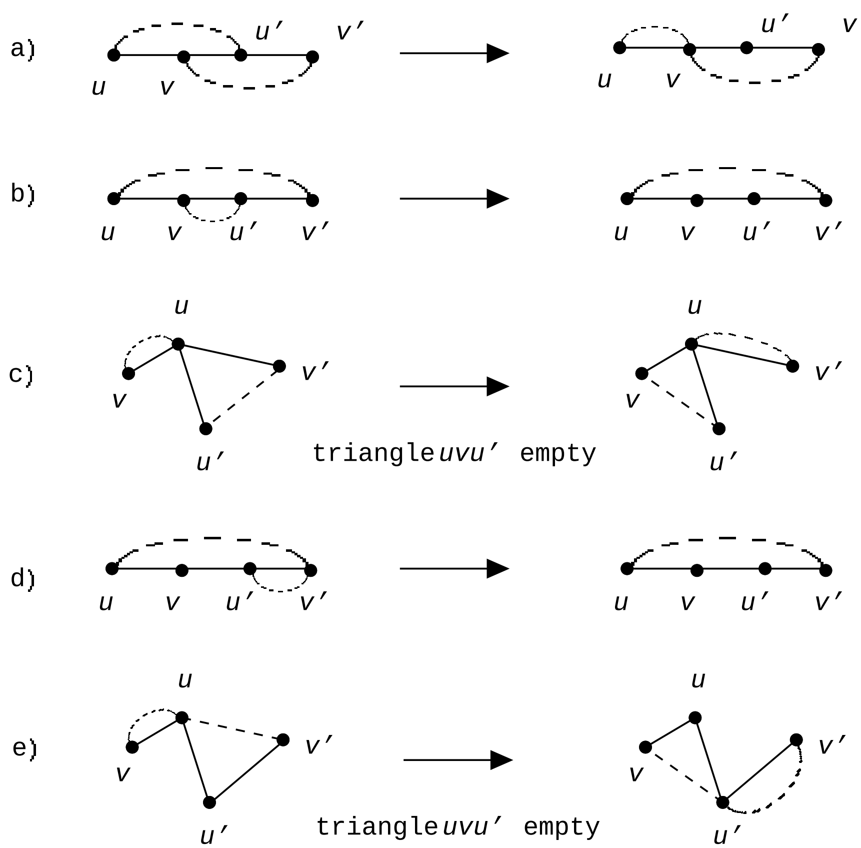

Figure 4: Components with four vertices.

We now show how to deal with cases c) and e). In both cases, C_i consists of a triangle together with a double edge. Let us assume that the vertices of C_i are labelled as in c) and e) in Figure 4, where the triangle is, counterclockwise, $uu'v'$, and the double edge is uv.

Case c) We know that there is at least one empty triangle Δ of $\mathcal{T}$ with two vertices in C_i and the other vertex $w \notin C_i$. Then, if the two vertices of Δ in C_i are vu, or vu' or vv', we can apply the method given in the previous lemma. Otherwise, since segment uv must belong to at least one empty triangle of $\mathcal{T}$ (or two of them if uv is not an edge on the external face of $\mathcal{T}$), then uvv', or uvu', or both must be empty triangles of $\mathcal{T}$. Hence, without loss of generality, suppose that uvu' is an empty triangle of $\mathcal{T}$. Then, we can modify the red edges of C_i by taking as new red edges vu' and uv' (this last edge becoming a double edge), instead of $u'v'$ and uv. This is the case shown in Figure 4c.

Again, if the two vertices of Δ in C_i are $v'u$, or $v'u'$, we can apply the method of the previous lemma. Otherwise, necessarily $uv'u'$, or $uv'v$, or both, are empty triangles of $\mathcal{T}$ and the only possibility left for Δ is to be a triangle of type $uu'w$. However, we have the two empty triangles uvu' and $uu'w$, and since it is impossible to have three empty triangles with the same side uu' in $\mathcal{T}$, then $uv'u'$ cannot be an empty triangle and necessarily $uv'v$ must be the empty triangle. Then taking as new red edges of C_i the edges vv' and uu', we can apply the method of Lemma 7.

Case e) This is solved in a similar way. Without loss of generality, suppose that the red edge in the triangle $uu'v'$ is the edge uv'. If the two vertices of Δ in C_i are vu, or vu' or vv', we apply the previous lemma. Otherwise, uvv', or uvu', or both, are empty triangles of $\mathcal{T}$. If the triangle uvv' is empty, then we can delete the two red edges of C_i, add the red edge vv' and use an extra red edge to join C_i with another 2-edge component. Suppose then, that uvu' is an empty triangle of $\mathcal{T}$. We can modify the red edges of C_i by taking as new red edges vu' and $u'v'$, this last edge becoming a double edge (see Figure 4e).

Again, if the two vertices of Δ in C_i are $v'u$, or $v'u'$, we can apply the previous lemma. Otherwise, necessarily $v'u'u$, or $v'u'v$, or both, are empty triangles, and Δ must be a triangle of type $uu'w$. As before, triangles $uu'w$, $uu'v$ and $u'uv'$ cannot be empty at the same time, so the empty triangle is $v'u'v$. But then, we can delete the red edges vu' and $u'v'$, add the red edge vv' and use an extra red edge to join C_i with another 2-edge component.

Summarizing, we have just proved:

Lemma 8. *Let G be a plane geometric tree with n vertices. If n is even, then by adding at most $\frac{n}{2}$ edges to G, we can obtain a plane geometric graph all of whose 2-edge connected components have at least six vertices.*

We can now prove:

Theorem 2. *Any plane geometric tree G with $n \geq 6$ vertices can be completed to a 2-edge connected plane geometric graph by adding at most $\lfloor 2n/3 \rfloor - 1$ edges if n is even, and at most $\lfloor 2(n+1)/3 \rfloor - 1$ edges if n is odd.*

Proof. For n even, and using Lemma 8, by adding at most $\frac{n}{2}$ edges to G we can obtain a plane geometric graph such that all its 2-edge connected components have at least 6 vertices. We then have at most $s = \lfloor n/6 \rfloor$ components joined by exactly $s-1$ bridges. Each of the bridges can be eliminated by adding an extra edge to G'' as in Lemma 2.

For n odd, we choose an arbitrary vertex v and we *add* a new vertex v' (as if v had been duplicated) connected to v by an edge of length ϵ, arbitrarily small. We apply the lemma for this new tree and then we delete v' and we connect to v the edges adjacent to v'. ∎

Observe that at this point we may have some double edges, which can now be eliminated as in Lemma 4 without increasing the total number of edges.

With respect to the computational complexity of the proposed method, notice that phase 1 of Algorithm 1, building a matching using vertices of a simple polygon P, as described in Lemma 5, can be theoretically done in linear time. First, triangulate P, and delete some diagonals to obtain a convex partition of P. This process can be done in linear time. The other two steps, the assignment of an even number of vertices to each convex region and the matching among vertices in each region, can be done again with the same complexity.

Similarly, phases 2 and 3 of the algorithm are linear. This is clear for phase 3 because we can obtain a compatible triangulation in linear time, and also the 2-edge-connected components and the bridges can be calculated with the same complexity. In phase 2, merging components with 2 or 4 vertices, some of the edges of the triangulation are chosen as new red edges, but the number of changes and candidates for changes are again $O(n)$. In order to choose one of these new edges, we have to decide, given an empty triangle uvw, whether the duplicated edge $u'v'$ is in the black path from w to v or not. However, by doing a linear time preprocessing step in the tree, this decision can be made in constant time.

4 Connectivity for Arbitrary Plane Geometric Graphs

In the previous section, we provided a bound on the maximum number of edges that need to be added to a plane geometric tree to obtain a 2-edge connected plane geometric graph. In this section we study the same problem for arbitrary plane connected geometric graphs.

We now construct plane geometric graphs that need at least $\frac{2n-2}{3}$ added edges to make them 2-edge connected.

Let G_1 be a triangulation with n_1 vertices, k_1 of which belong to the external face of G. Then G_1 has $f_1 = 2n_1 - k_1 - 1$ faces and $e_1 = 3n_1 - k_1 - 3$ edges. In each internal face of G_1, place an extra vertex adjacent to a vertex of the face. We also add k_1 vertices in the external face, close enough to the edges of the external face of G_1. Each added vertex is adjacent to one vertex of G_1 as shown in Figure 5. Let G_2 be the graph thus obtained. G_2 has $n = 3n_1 - 2$ vertices, $2n_1 - 2$ of which have degree one. Clearly, to make G_2 2-edge connected, we need to add an edge for each of the $2n_1 - 2$ vertices of G_2 of degree one. We have proved:

Lemma 9. *There are plane geometric graphs with n vertices that need at least $\frac{2n-2}{3}$ added edges to make them 2-edge connected.*

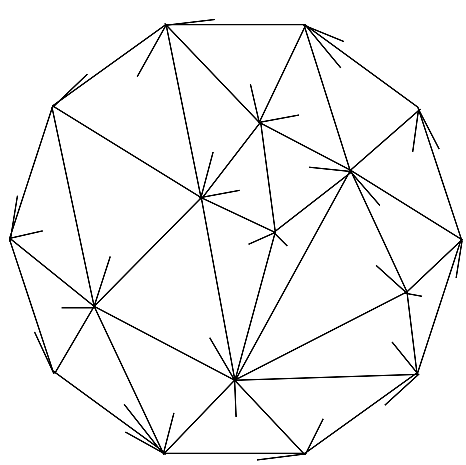

Figure 5: A graph that requires $\frac{2n-2}{3}$ extra edges.

We can also prove:

Lemma 10. *Any plane geometric graph G can be completed to a 2-edge connected plane geometric graph by adding at most $\frac{6n}{7}$ edges.*

Proof. Observe first that any plane geometric graph with n vertices, $c \geq 2$ faces, a edges, and b bridges satisfies: $3c + 2b \leq 2a$. Using the Euler's formula, $n + c = a + 2$, we obtain $2b \leq 3n - a - 6$. Then any geometric graph with at least $a \geq \frac{9n}{7}$ edges has at most $\frac{6n}{7}$ bridges, and thus can be completed to a 2-edge connected graph by adding at most one edge per bridge.

Suppose then that a is less than $\frac{9n}{7}$. Choose any spanning tree T of G. For any edge $e = uv$ of G not in T, choose a new vertex v_e close enough to v, in the edge uv, remove e from G and add the new edge uv_e (essentially edge uv_e is the same as edge uv). The graph thus obtained is a tree with exactly $a + 1$ vertices. Then by Theorem 2 the graph can be completed by adding to it at most

$$\frac{2(\frac{9n}{7} + 1)}{3} - 1 \leq \frac{6n}{7}$$

edges. Lastly, by deleting each added vertex v_e and reconnecting the adjacent edges to v_e to v, where v is the endpoint of the edge $e = uv$ closest to v_e, we obtain the desired result. ∎

Let $k(n)$ be the smallest integer such that any connected plane geometric graph with n vertices can be augmented to a 2-edge connected plane geometric graph by adding at most $k(n)$ edges. Combining Lemmas 9 and 10 we have:

Theorem 3. *For $n \geq 6$,*

$$\frac{2n-2}{3} \leq k(n) \leq \frac{6n}{7}.$$

We conclude by presenting a result that can in some instances help to reduce the number of edges that need to be added to some plane geometric graphs to make them 2-edge connected. We say that a vertex of a graph G is *odd* if it has odd degree. A simple polygon such that some of its edges belong to G and the others are line segments joining pairs of visible vertices of G will be called a *compatible cycle*.

Lemma 11. *Let G be a plane geometric graph. If there is a compatible cycle that contains all the odd vertices of G, then G can be completed to a 2-edge connected plane geometric graph by adding at most $\lfloor n/2 \rfloor$ edges.*

Proof. Recall first that any graph has an even number of odd vertices. Let $e = uv$ be a bridge of G, and let G_1 and G_2 be the components of $G - e$, $u \in G_1$. Since G_1 and G_2 must have an even number of odd vertices, and u and v are the only vertices that change their degrees in G_1 and G_2 with respect to the degrees in G of the vertices, then necessarily G_1 and G_2 contain an odd number of odd vertices of G. In particular, each of them contains at least one odd vertex of G.

Suppose that G contains $2k$ odd vertices, $k \geq 1$, and let P be a compatible cycle of G containing all the odd vertices of G. Suppose that the odd vertices of G appear in the order $i_0, \ldots, i_{2k-1}$ along P. For $j = 0, \ldots, 2k-1$, let P_j be the path contained in P joining i_j to i_{j+1}, addition taken *mod* $2k$.

Let S_0 be the set containing all P_j for j even, and S_1 the set of paths P_j, j odd. We claim that if we add to G a red edge for each edge of a path in S_0, the resulting graph

G' contains no bridges, and is therefore 2-edge connected. Observe that double edges are allowed as before.

Suppose otherwise that G' contains a bridge e. As we proved above, each of the components of $G - e$, G_1 and G_2, has an odd number of odd vertices of G. It follows that at least one of the paths in S_0 must join an odd vertex of G_1 to one in G_2. This path generates a red path in G' from G_1 to G_2 that bypasses e, contradicting the assumption that e is a bridge.

By symmetry, the graph obtained from G by adding a red edge for each edge in a path of S_1 contains no bridges, and thus is 2-edge connected. Observe that by Lemma 4, double edges can be eliminated without increasing the total number of edges, keeping the graphs 2-edge connected.

Since either S_0 or S_1 contains at most $\lfloor \frac{n}{2} \rfloor$ edges, the result follows. ∎

In particular, this result implies that any connected plane geometric graph whose vertices are in convex position can be completed to a 2-edge connected plane geometric graph with at most $\lfloor \frac{n}{2} \rfloor$ edges.

5 Conclusions

In this paper we have considered the problem of calculating the minimum number of edges that can make every connected plane geometric graph with n vertices 2-edge or 2-vertex connected. Upper and lower bounds were obtained for arbitrary connected plane geometric graphs, and for plane geometric graphs which are trees.

Finally, we close with two conjectures:

Conjecture 1. *Any plane geometric tree with n vertices can be completed to a 2-edge connected plane geometric graph by adding at most $\frac{n}{2} \pm c$ edges, c being a constant.*

Conjecture 2. *Any connected plane geometric graph with n vertices can be completed to a 2-edge connected plane geometric graph by adding at most $\frac{2n}{3} \pm c$ edges, c being a constant.*

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