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Auslander's formula and correspondence for exact categories

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Abstract

The Auslander correspondence is a fundamental result in Auslander-Reiten theory. In this paper we introduce the category $\mathrm{mod}_{\mathsf{adm}}(\mathcal{E})$ of admissibly finitely presented functors and use it to give a version of Auslander correspondence for any exact category $\mathcal{E}$. An important ingredient in the proof is the localization theory of exact categories. We also investigate how properties of $\mathcal{E}$ are reflected in $\mathrm{mod}_{\mathsf{adm}}(\mathcal{E})$, for example being (weakly) idempotent complete or having enough projectives or injectives. Furthermore, we describe $\mathrm{mod}_{\mathsf{adm}}(\mathcal{E})$ as a subcategory of $\mathrm{mod}(\mathcal{E})$ when $\mathcal{E}$ is a resolving subcategory of an abelian category. This includes the category of Gorenstein projective modules and the category of maximal Cohen-Macaulay modules as special cases. Finally, we use $\mathrm{mod}_{\mathsf{adm}}(\mathcal{E})$ to give a bijection between exact structures on an idempotent complete additive category $\mathcal{C}$ and certain resolving subcategories of $\mathrm{mod}(\mathcal{C})$.

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1 Introduction

To study representations of finite-dimensional algebras, Auslander advocated a functorial point of view, which he developed in [6, 7, 10, 11, 12]. This means that to study the category $\operatorname{mod}(\Lambda)$ of finite-dimensional modules over a finite-dimensional algebra Λ, one should study the category $\operatorname{mod}(\operatorname{mod}(\Lambda))$ of finitely presented functors $F\colon \operatorname{mod}(\Lambda)^{\mathrm{op}} \to \mathrm{Ab}$ to abelian groups, and translate the results back to $\operatorname{mod}(\Lambda)$. For example, almost split sequences in $\operatorname{mod}(\Lambda)$ naturally arise when taking minimal projective presentations of simple objects of projective dimension 2 in $\operatorname{mod}(\operatorname{mod}(\Lambda))$, and representation finiteness of Λ can be detected by investigating when $\operatorname{mod}(\operatorname{mod}(\Lambda))$ is equivalent to $\operatorname{mod}(\Gamma)$ for some finite-dimensional algebra Γ. In the latter case, Γ inherits homological properties from the categorical properties of $\operatorname{mod}(\operatorname{mod}(\Lambda))$. More precisely, the class of algebras Γ is equal to the class of Auslander algebras, i.e. the algebras satisfying

$$\operatorname{dom.dim}\Gamma \geq 2 \geq \operatorname{gl.dim}\Gamma$$

where $\operatorname{dom.dim}\Gamma$ is the dominant dimension and $\operatorname{gl.dim}\Gamma$ is the global dimension of Γ. This recovers the famous Auslander correspondence.

On the other hand, in representation theory one often considers categories which are not abelian. If the category $\operatorname{mod}(\Lambda)$ is too complicated, one might need to restrict to studying subcategories of $\operatorname{mod}(\Lambda)$, for example the Gorenstein projective modules [16, 19, 28, 29, 59], or the modules of finite projective dimension [15, 41, 52, 55]. Furthermore, Cohen-Macaulay representations [7, 8, 23, 27, 46, 49, 51, 58, 65], their graded counterparts [13, 14], and their connection to McKay quivers [1, 9] have also been actively studied from a representation theoretic point of view. In all these cases, the categories are exact in the sense of Quillen, but not necessarily abelian. Unfortunately, if $\mathcal{E}$ is an exact category, then $\operatorname{mod}(\mathcal{E})$ does not have nice categorical properties in general, see [34, 45]. This makes it harder to deduce results about $\mathcal{E}$ by studying $\operatorname{mod}(\mathcal{E})$, as in Auslander's philosophy. In this paper we introduce a different candidate to study instead of $\operatorname{mod}(\mathcal{E})$, which *takes into account the exact structure of* $\mathcal{E}$.

Definition 1.1 (Definition 3.1)**.** Let $\mathcal{E}$ be an exact category. We define $\operatorname{mod}_{\mathsf{adm}}(\mathcal{E})$ to be the full subcategory of $\operatorname{mod}(\mathcal{E})$ consisting of those functors F that admit a projective representation

$$\operatorname{Hom}_{\mathcal{E}}(-,X) \xrightarrow{\operatorname{Hom}_{\mathcal{E}}(-,f)} \operatorname{Hom}_{\mathcal{E}}(-,Y) \to F \to 0$$

where $f\colon X \to Y$ is an admissible morphism in $\mathcal{E}$.

We first characterize the categories of the form $\operatorname{mod}_{\mathsf{adm}}(\mathcal{E})$, analogously to the characterization of Auslander algebras. This is done by introducing the notion of an Auslander exact category. In the following we write

$${}^{\perp}\mathcal{P} := \{E \in \mathcal{E} \mid \operatorname{Hom}_{\mathcal{E}}(E,\mathcal{P}) = 0\} \quad \text{and} \quad \operatorname{cogen}(\mathcal{P}) := \{E \in \mathcal{E} \mid \exists \text{ inflation } E \rightarrowtail P \text{ with } P \in \mathcal{P}\}.$$

Definition 1.2 (Definition 4.1)**.** Let $\mathcal{E}$ be an exact category with enough projectives $\mathcal{P}$. We say that $\mathcal{E}$ is an *Auslander exact category* if it satisfies the following:

(i) $({}^{\perp}\mathcal{P}, \operatorname{cogen}(\mathcal{P}))$ is a torsion pair in $\mathcal{E}$;

(ii) If $E' \in {}^{\perp}\mathcal{P}$, then any morphism $f\colon E \to E'$ is admissible with image in ${}^{\perp}\mathcal{P}$;

(iii) $\operatorname{Ext}^1_{\mathcal{E}}({}^{\perp}\mathcal{P}, \mathcal{P}) = 0$;

(iv) $\operatorname{gl.dim}(\mathcal{E}) \leq 2$.

Theorem 1.3 (Theorem 4.7)**.** *The following hold:*

1. *If $\mathcal{E}$ is an exact category, then $\operatorname{mod}_{\mathsf{adm}}(\mathcal{E})$ is an Auslander exact category.*
2. *If $\mathcal{E}'$ is an Auslander exact category, then there exists an exact equivalence $\mathcal{E}' \cong \operatorname{mod}_{\mathsf{adm}}(\mathcal{E})$ where $\mathcal{E}$ is an exact category. Furthermore, $\mathcal{E}$ is uniquely determined up to exact equivalence.*

In fact, we show that the association $\mathcal{E} \mapsto \mathrm{mod}_{\mathsf{adm}}(\mathcal{E})$ is an equivalence from the 2-category of exact categories and left exact functors to the 2-category of Auslander exact categories and exact functors preserving projective objects. It can also be interpreted as a left adjoint to an inclusion of 2-categories, see Corollary 3.12.

Note that if $\mathcal{Q}$ is the subcategory of projective objects in $\mathrm{mod}_{\mathsf{adm}}(\mathcal{E})$, then we have the following identifications:

1. $F \in \mathcal{Q}$ if and only if F is isomorphic to a representable functor.
2. $F \in {}^{\perp}\mathcal{Q}$ if and only if F is an *effaceable functor*, i.e. there exist a projective representation
$$\mathrm{Hom}_{\mathcal{E}}(-, X) \xrightarrow{\mathrm{Hom}_{\mathcal{E}}(-,f)} \mathrm{Hom}_{\mathcal{E}}(-, Y) \to F \to 0$$
where $f\colon X \to Y$ is a deflation. These functors have already been studied by several authors under the name *defects*, see [22, 34, 36].
3. $F \in \mathrm{cogen}(\mathcal{Q})$ if and only if there exist a projective representation
$$\mathrm{Hom}_{\mathcal{E}}(-, X) \xrightarrow{\mathrm{Hom}_{\mathcal{E}}(-,f)} \mathrm{Hom}_{\mathcal{E}}(-, Y) \to F \to 0$$
where $f\colon X \to Y$ is an inflation.

It follows that the subcategory of effaceable functors is the torsion part of a torsion pair in $\mathrm{mod}_{\mathsf{adm}}(\mathcal{E})$. In fact, it is a two-sided admissibly percolating subcategory of $\mathrm{mod}_{\mathsf{adm}}(\mathcal{E})$, which enables us to use the localisation theory developed in [26] and [43] to study it. In particular, we extend Auslander's formula to exact categories [6, 50].

Theorem 1.4 (Proposition 3.5 and Theorem 3.9)**.** *The following holds:*

1. $\mathrm{eff}(\mathcal{E})$ *is a two-sided admissibly percolating subcategory of* $\mathrm{mod}_{\mathsf{adm}}(\mathcal{E})$*. In particular, the quotient* $\mathrm{mod}_{\mathsf{adm}}(\mathcal{E})/\mathrm{eff}(\mathcal{E})$ *exists (in the category of exact categories);*
2. $\mathrm{mod}_{\mathsf{adm}}(\mathcal{E})/\mathrm{eff}(\mathcal{E})$ *is equivalent to* $\mathcal{E}$ *as an exact category.*

If $\mathcal{E}$ is idempotent complete, then we also show that $\mathrm{mod}_{\mathsf{adm}}(\mathcal{E})$ is the smallest resolving subcategory containing $\mathrm{eff}(\mathcal{E})$.

If $\mathcal{E}$ has enough injectives, then we obtain a characterization of $\mathrm{mod}_{\mathsf{adm}}(\mathcal{E})$ which is similar to module categories of Auslander algebras.

Theorem 1.5 (Theorem 4.21)**.** *Let* $\mathcal{E}'$ *be an exact category with enough projectives* $\mathcal{P}'$*. Then there exists an exact equivalence* $\mathcal{E}' \cong \mathrm{mod}_{\mathsf{adm}}(\mathcal{E})$ *where* $\mathcal{E}$ *is an exact category with enough injectives if and only if the following hold:*

1. $\mathrm{dom.\,dim}(\mathcal{E}') \geq 2 \geq \mathrm{gl.\,dim}(\mathcal{E}')$*;*
2. *Any morphism* $X \to E$ *with* $E \in {}^{\perp}\mathcal{P}'$ *is admissible;*
3. *For any* $E \in \mathcal{E}'$ *there exists an admissible left* $\mathcal{P}'$*-approximation* $E \to P$*.*

Furthermore, in this case $\mathcal{E}'$ *has enough injectives.*

We also analyse the properties of $\mathrm{mod}_{\mathsf{adm}}(\mathcal{E})$ when $\mathcal{E}$ has enough projectives, see Section 4.4.

One of our main examples of an exact category $\mathcal{E}$ is the category of Cohen-Macaulay modules, and it turns out that $\mathrm{mod}_{\mathsf{adm}}(\mathcal{E})$ has a simple description in this case. Let R be a commutative Cohen-Macaulay local ring and let Λ be a Cohen-Macaulay-finite R-order, i.e. a noetherian R-algebra which is maximal Cohen-Macaulay as an R-module. Let M be an additive generator of the subcategory $\mathrm{CM}(\Lambda)$ of maximal Cohen-Macaulay modules containing Λ as a summand. Let $\Gamma = \mathrm{End}_{\Lambda}(M)$ and let e be the idempotent corresponding to the direct summand Λ. Then we have
$$\mathrm{mod}_{\mathsf{adm}}(\Gamma) = \{M \in \mathrm{mod}(\Gamma) \mid Me \in \mathrm{CM}(\Lambda)\}.$$

Similar descriptions hold for any resolving subcategory of a module category.

The study of exact structures on a fixed idempotent complete additive category $\mathcal{C}$ has recently attracted a lot of interest [17, 20, 21, 34, 36, 61, 62, 63]. In particular, they form a lattice [21, 61] and the association $\mathcal{E} \mapsto \operatorname{eff}(\mathcal{E})$ gives a bijection to certain Serre subcategories of $\operatorname{mod}(\mathcal{E})$ [22, 34, 36]. We obtain a similar characterization by replacing $\operatorname{eff}(\mathcal{E})$ with $\operatorname{mod}_{\mathsf{adm}}(\mathcal{E})$ and Serre subcategories by resolving subcategories. In the following, $\mathcal{P}^2(\mathcal{C})$ denotes the subcategory of $\operatorname{mod}(\mathcal{C})$ consisting of functors F admitting a projective resolution

$$0 \to \operatorname{Hom}_{\mathcal{E}}(-, X) \to \operatorname{Hom}_{\mathcal{E}}(-, Y) \to \operatorname{Hom}_{\mathcal{E}}(-, Z) \to F \to 0.$$

Also, $\underline{\operatorname{mod}}(\mathcal{C})$ denote the stable category of $\operatorname{mod}(\mathcal{C})$ modulo projectives, and $\operatorname{Tr}\colon \underline{\operatorname{mod}}(\mathcal{C}) \to \underline{\operatorname{mod}}(\mathcal{C}^{\mathrm{op}})^{\mathrm{op}}$ denotes the Auslander-Bridger transpose [10]. For a subcategory $\mathcal{X}$ of $\operatorname{mod}(\mathcal{C})$ we let $\operatorname{Tr}(\mathcal{X})$ denote the subcategory of $\operatorname{mod}(\mathcal{C}^{\mathrm{op}})$ consisting of all functors F which are isomorphic in $\underline{\operatorname{mod}}(\mathcal{C}^{\mathrm{op}})$ to $\operatorname{Tr}(F)$ for some $F \in \mathcal{X}$. For the notion of grade see Definition 5.7.

Theorem 1.6 (Theorem 5.8)**.** *Let $\mathcal{C}$ be an idempotent complete additive category. The association $\mathcal{E} \mapsto \operatorname{mod}_{\mathsf{adm}}(\mathcal{E})$ gives a bijection between the following:*

1. *Exact structures $\mathcal{S}$ on $\mathcal{C}$, where $\mathcal{E} = (\mathcal{C}, \mathcal{S})$ is the corresponding exact category;*
2. *Subcategories $\mathcal{X}$ of $\operatorname{mod}(\mathcal{C})$ satisfying the following:*
   - *(a) $\mathcal{X}$ is a resolving subcategory of $\mathcal{P}^2(\mathcal{C})$ and $\operatorname{Tr}(\mathcal{X})$ is a resolving subcategory of $\mathcal{P}^2(\mathcal{C}^{\mathrm{op}})$;*
   - *(b) $\mathcal{X}$ and $\operatorname{Tr}(\mathcal{X})$ have no objects of grade 1.*

It was shown in [25, 30] that exact structures on an abelian category $\mathcal{A}$ correspond to closed subfunctors of $\operatorname{Ext}^1_{\mathcal{A}}(-,-)$. Such structures are one of the main objects of study in relative homological algebra [31, 32], see also [2, 3, 4, 5] for the case of Artin algebras. We obtain the following corollary for abelian categories.

Corollary 1.7 (Corollary 5.15)**.** *Let $\mathcal{A}$ be an abelian category. Then there exists a bijection between the following:*

1. *Exact structures on $\mathcal{A}$;*
2. *Resolving subcategories $\mathcal{X}$ of $\operatorname{mod}(\mathcal{A})$ for which $\operatorname{Tr}(\mathcal{X})$ is a resolving subcategory of $\operatorname{mod}(\mathcal{A}^{\mathrm{op}})$.*

Structure of the paper

In Section 2 we recall the quotients of exact categories that occur in the formulation of Auslander's formula. In particular, we show that localization by a two-sided admissible percolating subcategory reflects admissible morphisms, see Theorem 2.10. We end by investigating torsion theory on exact categories where the torsion part is a two-sided admissible percolating subcategory.

In Section 3 we study the category $\operatorname{mod}_{\mathsf{adm}}(\mathcal{E})$. In particular, we show in Theorem 3.7 that it satisfies a universal property, in Theorem 3.9 that Auslander's formula holds, and in Corollary 3.12 that the association $\mathcal{E} \mapsto \operatorname{mod}_{\mathsf{adm}}(\mathcal{E})$ can be made into a left adjoint 2-functor. We finish the section by showing that $\operatorname{mod}_{\mathsf{adm}}(\mathcal{E})$ has the properties of an Auslander exact category.

In Section 4 we introduce Auslander exact categories, and we show that $\operatorname{mod}_{\mathsf{adm}}(-)$ gives an equivalence onto the 2-category of Auslander exact categories, see Theorem 4.7. We then investigate how properties of $\mathcal{E}$ are reflected in $\operatorname{mod}_{\mathsf{adm}}(\mathcal{E})$, for example having enough injectives, see Theorem 4.21, and having enough projectives, see Proposition 4.25. We end by characterizing $\operatorname{mod}_{\mathsf{adm}}(\mathcal{E})$ for Gorenstein projective modules, see Corollary 4.28, and for maximal Cohen-Macaulay modules, see Corollary 4.29.

In Section 5 we investigate different exact structures on an idempotent complete additive category by studying $\operatorname{mod}_{\mathsf{adm}}(\mathcal{E})$. In particular, we show that $\operatorname{mod}_{\mathsf{adm}}(\mathcal{E})$ is the smallest resolving subcategory of $\mathcal{P}^2(\mathcal{E})$ containing $\operatorname{eff}(\mathcal{E})$, see Proposition 5.4. In the last part we show Theorem 1.6, which is the main result of the section.

Conventions

All categories $\mathcal{C}$ are assumed to be essentially small. We also assume that they are additive, i.e. that $\mathrm{Hom}_{\mathcal{C}}(-,-)$ is enriched over abelian groups and that $\mathcal{C}$ admit finite direct sums. All subcategories are assumed to be full and closed under isomorphisms.

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2 Localizations/Quotients of exact categories

This section is preliminary in nature, we establish basic conventions and terminology. We recall the notion of an exact category as introduced by Quillen in [56]. We recall the notion of a *two-sided admissibly percolating subcategory* as in [43] (these types of subcategories are called *subcategories that localize* in [26]). Given an exact category $\mathcal{E}$ and a two-sided admissibly percolating subcategory $\mathcal{A} \subseteq \mathcal{E}$, the quotient $\mathcal{E}/\mathcal{A}$ can be realized as a localization $\Sigma_{\mathcal{A}}^{-1}\mathcal{E}$.

2.1 Exact categories

Definition 2.1.

1. A *conflation category* is an additive category $\mathcal{C}$ together with a chosen class of kernel-cokernel pairs, called *conflations*, closed under isomorphisms. The kernel part of a conflation is called an *inflation* and the cokernel part of a conflation is called a *deflation*. We depict inflations by the symbol $\rightarrowtail$ and deflations by $\twoheadrightarrow$.
2. An additive functor $F\colon \mathcal{C} \to \mathcal{D}$ of conflation categories is called *exact* or *conflation-exact* if conflations are mapped to conflations.
3. A map $f\colon X \to Y$ in a conflation category $\mathcal{C}$ is called *admissible* if f admits a factorization $X \twoheadrightarrow I \rightarrowtail Y$. Admissible morphisms are depicted by $X \xrightarrow{f} Y$.
4. A cochain $\cdots \to X^{n-1} \xrightarrow{d^{n-1}} X^n \xrightarrow{d^n} X^{n+1} \to \dots$ in a conflation category $\mathcal{C}$ is called *acyclic* or *exact* if each d^i is admissible and $\ker(d^{i+1}) = \mathrm{im}(d^i)$.

Remark 2.2. Note that if $f\colon X \to Y$ is admissible, the factorization $X \twoheadrightarrow I \rightarrowtail Y$ is unique up to isomorphism. Moroever, f admits a kernel and cokernel and $\mathrm{im}(f) := \ker(\mathrm{coker}(f)) \cong I \cong \mathrm{coker}(\ker(f)) =: \mathrm{coim}(f)$.

Definition 2.3. A conflation category $\mathcal{E}$ is called an *exact category* if $\mathcal{E}$ satisfies the following two dual lists of axioms:

R0 For any $X \in \mathcal{E}$, $X \twoheadrightarrow 0$ is a deflation.

R1 Deflations are closed under composition.

R2 Pullbacks along deflations exist, moreover, deflations are stable under pullbacks.

The dual list is given by:

L0 For any $X \in \mathcal{E}$, $0 \rightarrowtail X$ is an inflation.

L1 Inflations are closed under composition.

L2 Pushouts along inflations exist, moreover, inflations are stable under pushouts.

Given an exact category $\mathcal{E}$, we let $\mathsf{K}^{\mathsf{b}}(\mathcal{E})$ denote the homotopy category of bounded complexes in $\mathcal{E}$, and we let $\mathbf{Ac}^{\mathsf{b}}(\mathcal{E})$ denote the subcategory of $\mathsf{K}^{\mathsf{b}}(\mathcal{E})$ consisting of the acyclic complexes. Note that $\mathbf{Ac}^{\mathsf{b}}(\mathcal{E})$ is not closed under isomorphism in $\mathsf{K}^{\mathsf{b}}(\mathcal{E})$ unless $\mathcal{E}$ is weakly idempotent complete, see [24, Proposition 10.14]. However, $\mathbf{Ac}^{\mathsf{b}}(\mathcal{E})$ is still a triangulated subcategory of $\mathsf{K}^{\mathsf{b}}(\mathcal{E})$, see [24, Corollary 10.5]. The Verdier localization $\mathsf{K}^{\mathsf{b}}(\mathcal{E})/\mathbf{Ac}^{\mathsf{b}}(\mathcal{E})$ yields the triangulated category $\mathsf{D}^{\mathsf{b}}(\mathcal{E})$, called the bounded derived category of $\mathcal{E}$. The natural embedding $\mathcal{E} \hookrightarrow \mathsf{D}^{\mathsf{b}}(\mathcal{E})$, mapping objects to stalk complexes in degree zero, is fully faithful. Moreover, conflations correspond to triangles under this embedding.

2.2 Localizations via two-sided admissibly percolating subcategories

Throughout this section, let $\mathcal{E}$ be an exact category.

Definition 2.4. A full nonempty subcategory $\mathcal{A} \subseteq \mathcal{E}$ is called an *admissibly deflation-percolating subcategory* if the following axioms are satisfied:

A1 The category $\mathcal{A}$ is a *Serre subcategory* of $\mathcal{E}$, i.e. given a conflation $X \rightarrowtail Y \twoheadrightarrow Z$ in $\mathcal{E}$, we have that $Y \in \mathcal{A} \Leftrightarrow X, Z \in \mathcal{A}$.

A2 All morphisms $f\colon X \to A$ with $A \in \mathcal{A}$ are admissible with image in $\mathcal{A}$, i.e. factors as

$$X \twoheadrightarrow \operatorname{im}(f) \rightarrowtail A$$

and $\operatorname{im}(f) \in \mathcal{A}$.

Dually, $\mathcal{A}$ is called an *admissibly inflation-percolating subcategory* if $\mathcal{A}^{\mathsf{op}}$ is an admissibly deflation-percolating subcategory in $\mathcal{E}^{\mathsf{op}}$. Finally, $\mathcal{A}$ is a *two-sided admissibly percolating subcategory* if it is both an admissibly deflation and admissibly inflation-percolating subcategory.

Remark 2.5.

1. Since an admissibly deflation-percolating subcategory $\mathcal{A} \subseteq \mathcal{E}$ is closed under extensions, it inherits an exact structure from $\mathcal{E}$. It follows from axiom **A2** that under this exact structure $\mathcal{A}$ becomes an abelian category.
2. Two-sided admissibly percolating subcategories are called *localizing subcategories* in [26].

The following definition is based on [64, Definition 1.12] and [26, Definition 4.0.36].

Definition 2.6. Let $\mathcal{A} \subseteq \mathcal{E}$ be an admissibly deflation-percolating subcategory. A morphism $f\colon X \to Y$ in $\mathcal{E}$ is called a *weak $\mathcal{A}$-isomorphism* or simply a *weak isomorphism* if $\mathcal{A}$ is understood, if f is admissible and $\ker(f), \operatorname{coker}(f) \in \mathcal{A}$. Morphisms endowed with $\sim$ are weak isomorphisms. The set of weak isomorphisms is denoted by $\Sigma_{\mathcal{A}}$.

Proposition 2.7. *Let $\mathcal{A} \subseteq \mathcal{E}$ be an admissibly deflation-percolating subcategory.*

1. *The set $\Sigma_{\mathcal{A}}$ is a right multiplicative system.*
2. *The set $\Sigma_{\mathcal{A}}$ is saturated, i.e. $Q(f)$ is an isomorphism in $\Sigma_{\mathcal{A}}^{-1}\mathcal{E}$ if and only if $f \in \Sigma_{\mathcal{A}}$.*
3. *If $Q(M) \cong 0$ for an object $M \in \mathcal{E}$, then $M \in \mathcal{A}$.*
4. *Pullbacks along weak isomorphisms exist, moreover, weak isomorphisms are stable under pullbacks.*
5. *Admissible morphisms are stable under pullbacks along weak isomorphisms.*

Proof. We refer the reader to [43] for the first four statements. The last statement follows from [43, Corollary 6.10] and the fact that deflations are preserved under pullbacks. □

The following theorem yields that the localization $\Sigma_{\mathcal{A}}^{-1}\mathcal{E}$ satisfies the universal property of the quotient $\mathcal{E}/\mathcal{A}$.

Theorem 2.8. *Let $\mathcal{A} \subseteq \mathcal{E}$ be a two-sided admissibly percolating subcategory and write $Q\colon \mathcal{E} \to \Sigma_{\mathcal{A}}^{-1}\mathcal{E}$ for the localization functor.*

1. *Q sends conflations in $\mathcal{E}$ to kernel-cokernel pairs in $\Sigma_{\mathcal{A}}^{-1}\mathcal{E}$. Furthermore, closing this class of kernel-cokernel pairs under isomorphism makes $\Sigma_{\mathcal{A}}^{-1}\mathcal{E}$ into an exact category.*

2. *The functor Q satisfies the 2-universal property of a quotient $\mathcal{E}/\mathcal{A}$ in the category of exact categories, i.e. for any exact category $\mathcal{F}$ and exact functor $F\colon \mathcal{E} \to \mathcal{F}$ such that $F(\mathcal{A}) \cong 0$, there exists an exact functor $F'\colon \Sigma_{\mathcal{A}}^{-1}\mathcal{E} \to \mathcal{F}$ such that $F' \circ Q = F$.*

3. *The localization sequence $\mathcal{A} \to \mathcal{E} \xrightarrow{Q} \Sigma_{\mathcal{A}}^{-1}\mathcal{E}$ induces the Verdier localization sequence*

$$\mathsf{D}^{\mathsf{b}}_{\mathcal{A}}(\mathcal{E}) \to \mathsf{D}^{\mathsf{b}}(\mathcal{E}) \to \mathsf{D}^{\mathsf{b}}(\mathcal{E}/\mathcal{A})$$

where $\mathsf{D}^{\mathsf{b}}_{\mathcal{A}}(\mathcal{E})$ is the thick full triangulated subcategory of $\mathsf{D}^{\mathsf{b}}(\mathcal{E})$ generated by $\mathcal{A}$ under the canonical embedding $\mathcal{E} \hookrightarrow \mathsf{D}^{\mathsf{b}}(\mathcal{E})$.

Explicitly, a sequence $X \xrightarrow{f} Y \xrightarrow{g} Z$ in $\Sigma_{\mathcal{A}}^{-1}\mathcal{E}$ is a conflation if and only if there exists a commutative diagram

$$\begin{tikzcd}
Q(X') \arrow[r, "Q(f')"] \arrow[d, "\cong"] & Q(Y') \arrow[r, "Q(g')"] \arrow[d, "\cong"] & Q(Z') \arrow[d, "\cong"] \\
X \arrow[r, "f"] & Y \arrow[r, "g"] & Z
\end{tikzcd}$$

where the vertical maps are isomorphisms in $\Sigma_{\mathcal{A}}^{-1}\mathcal{E}$, and where $X' \xrightarrow{f'} Y' \xrightarrow{g'} Z'$ is a conflation in $\mathcal{E}$.

2.3 Admissible morphisms in $\mathcal{E}/\mathcal{A}$

In this section we show that the localization functor $Q\colon \mathcal{E} \to \mathcal{E}/\mathcal{A}$ reflects admissible morphisms when $\mathcal{A}$ is a two-sided admissibly percolating subcategory.

Proposition 2.9. *Let $\mathcal{E}$ be an exact category, and let $\mathcal{A}$ be an admissibly deflation-percolating subcategory of $\mathcal{E}$. Let $f\colon X \to Y$ and $g\colon Y \to Z$ be morphisms in $\mathcal{E}$. The following holds:*

1. *If f is an inflation with cokernel in $\mathcal{A}$ and g is a deflation, then $g \circ f = f' \circ g'$ where g' is a deflation and f' is an inflation with cokernel in $\mathcal{A}$;*

2. *If f is a weak isomorphism and g is admissible, then $g \circ f$ is admissible;*

3. *If g is a weak isomorphism and $g \circ f$ is admissible, then f is admissible.*

Proof. 1. This follows from [43, Corollary 6.10].

2. Write $f = f_2 \circ f_1$ and $g = g_2 \circ g_1$ where f_1, g_1 are deflations and f_2, g_2 are inflations. By (1) we have that $g_1 \circ f_2$ can be written as a composite $f_2' \circ g_1'$ where g_1' is a deflation and f_2' is an inflation. In particular, $g_1 \circ f_2$ is admissible. Hence the composite $g \circ f = g_2 \circ (g_1 \circ f_2) \circ f_1$ is admissible, which proves the claim.

3. Taking the pullback of g along $g \circ f$, we get a commutative diagram

$$\begin{tikzcd}
X' \arrow[r, "h"] \arrow[d, "g'"] & Y \arrow[d, "g"] \\
X \arrow[r, "g \circ f"] & Z.
\end{tikzcd}$$

Since $g \circ f\colon X \to Z$ factors through $g\colon Y \to Z$ via f, there exists a map $i\colon X \to X'$ satisfying $g' \circ i = 1_X$ and $h \circ i = f$. Since weak isomorphisms are preserved by pullbacks, g' must be a weak isomorphism, and hence a deflation since it has a right inverse. Therefore, i is an inflation with cokernel in $\mathcal{A}$. Also, since admissible morphisms are preserved under pullbacks by weak isomorphisms, h must be admissible. Since $f = h \circ i$, we get that f is admissible by the first part of the proof $\square$

Theorem 2.10. *A map $f\colon X \to Y$ is admissible in $\mathcal{E}$ if and only if $Q(f)$ is admissible in $\mathcal{E}/\mathcal{A}$. In other words, the localization functor Q reflects admissible morphisms.*

Proof. As $Q(f)$ is admissible in $\Sigma_{\mathcal{A}}^{-1}\mathcal{E}$, there is a deflation $A \twoheadrightarrow B$ and an inflation $C \rightarrowtail D$ in $\mathcal{E}$ and isomorphisms $B \cong C$, $A \cong X$ and $D \cong Y$ in $\Sigma_{\mathcal{A}}^{-1}\mathcal{E}$ making the diagram

$$\begin{tikzcd} A \arrow[r, two heads] \arrow[d, "\cong"] & B \arrow[r, "\cong"] & C \arrow[r, tail] & D \arrow[d, "\cong"] \\ X \arrow[rrr, "Q(f)"] & & & Y \end{tikzcd}$$

commutative in $\Sigma_{\mathcal{A}}^{-1}\mathcal{E}$. Let $B \xrightarrow{\sim} B' \xleftarrow{\sim} C$ be a roof in $\mathcal{E}$ representing the isomorphism $B \cong C$, (by Proposition 2.7.(2), $\Sigma_{\mathcal{A}}$ is saturated and thus both arrows in this roof belong to $\Sigma_{\mathcal{A}}$). Taking the pushout D' of $C \rightarrowtail D$ along the weak isomorphism $C \xrightarrow{\sim} B'$ yields an inflation $B' \rightarrowtail D'$ in $\mathcal{E}$ and an isomorphism $D' \cong D$ in $\Sigma_{\mathcal{A}}^{-1}\mathcal{E}$ by the dual of Proposition 2.7.(5). The composition $A \twoheadrightarrow B \xrightarrow{\sim} B' \rightarrowtail D'$ is admissible as $B \xrightarrow{\sim} B'$ is admissible. Thus, we have a commutative diagram

$$\begin{tikzcd} A \arrow[r, two heads] \arrow[d, "\cong"] & B'' \arrow[r, tail] & D' \arrow[d, "\cong"] \\ X \arrow[rr, "Q(f)"] & & Y \end{tikzcd}$$

in $\Sigma_{\mathcal{A}}^{-1}\mathcal{E}$.

Since $A \cong X$, there is a roof $X \xrightarrow{\sim} \overline{X} \xleftarrow{\sim} A$ in $\mathcal{E}$ representing this map. By the dual of Proposition 2.7.(4) and Proposition 2.7.(5), taking the pushout of the admissible map $A \twoheadrightarrow B'' \rightarrowtail D'$ along the weak isomorphism $A \xrightarrow{\sim} \overline{X}$, we obtain an admissible map $\overline{X} \twoheadrightarrow \overline{B} \rightarrowtail \overline{D}$ with $\overline{D} \cong D'$ in $\Sigma_{\mathcal{A}}^{-1}\mathcal{E}$. By Proposition 2.9. (2), the composition $X \xrightarrow{\sim} \overline{X} \twoheadrightarrow \overline{B}$ is admissible and thus factors as $X \twoheadrightarrow U \rightarrowtail \overline{B}$. By axiom **L1**, the composition $U \rightarrowtail \overline{B} \rightarrowtail \overline{D}$ is an inflation. Hence, we have the commutative diagram

$$\begin{tikzcd} X \arrow[r, two heads] \arrow[d, "1"] & U \arrow[r, tail] & \overline{D} \arrow[d, "\cong"] \\ X \arrow[rr, "Q(f)"] & & Y \end{tikzcd}$$

in $\Sigma_{\mathcal{A}}^{-1}\mathcal{E}$.

As $\overline{D} \cong Y$ in $\Sigma_{\mathcal{A}}^{-1}\mathcal{E}$, there is a roof $\overline{D} \xrightarrow{\sim} \overline{Y} \xleftarrow{\sim} Y$ in $\mathcal{E}$ representing this map. Clearly, the compositions $X \xrightarrow{f} Y \xrightarrow{\sim} \overline{Y}$ and $X \twoheadrightarrow U \rightarrowtail \overline{D} \xrightarrow{\sim} \overline{Y}$ are equal in $\Sigma_{\mathcal{A}}^{-1}\mathcal{E}$. As $\Sigma_{\mathcal{A}}$ is a left multiplicative system, there exists a weak isomorphism $\overline{Y} \to Z$ such that the compositions $X \xrightarrow{f} Y \xrightarrow{\sim} \overline{Y} \xrightarrow{\sim} Z$ and $X \twoheadrightarrow U \rightarrowtail \overline{D} \xrightarrow{\sim} \overline{Y} \xrightarrow{\sim} Z$ are equal in $\mathcal{E}$. Since the latter composition is admissible by the dual of Proposition 2.9. (2), it follows that $X \xrightarrow{f} Y \xrightarrow{\sim} \overline{Y} \xrightarrow{\sim} Z$ is admissible. Finally, by Proposition 2.9. (3) we get that f is admissible. □

2.4 Torsion theory with two-sided admissibly percolating subcategories

In this section we show that for any torsion pair $(\mathcal{T}, \mathcal{F})$ in an exact category $\mathcal{E}$ such that $\mathcal{T}$ satisfies axiom **A2**, $\mathcal{T} \subseteq \mathcal{E}$ is a two-sided admissibly percolating subcategory. Moreover, the torsion and torsionfree functor enjoy further properties.

Recall the notion of a torsion theory in an exact category.

Definition 2.11. Let $\mathcal{E}$ be an exact category. A *torsion theory* or *torsion pair* is a pair $(\mathcal{T}, \mathcal{F})$ of full subcategories closed under isomorphism such that

1. $\mathrm{Hom}_{\mathcal{E}}(T, F) = 0$ for all $T \in \mathcal{T}$ and $F \in \mathcal{F}$;
2. for any object $M \in \mathcal{E}$, there exists a conflation $T \rightarrowtail M \twoheadrightarrow F$ with $T \in \mathcal{T}$ and $F \in \mathcal{F}$.

For a subcategory $\mathcal{X}$ of $\mathcal{E}$ we let ${}^{\perp}\mathcal{X}$ and $\mathcal{X}^{\perp}$ denote the subcategories consisting of all objects $E \in \mathcal{E}$ satisfying $\operatorname{Hom}_{\mathcal{E}}(E, X) = 0$ or $\operatorname{Hom}_{\mathcal{E}}(X, E) = 0$ for all $X \in \mathcal{X}$, respectively. The following proposition is standard (see for example [54]).

Proposition 2.12. *Let $(\mathcal{T}, \mathcal{F})$ be a torsion pair in an exact category $\mathcal{E}$.*

1. *$\mathcal{T} = {}^{\perp}\mathcal{F}$ and $\mathcal{F} = \mathcal{T}^{\perp}$;*
2. *The subcategory $\mathcal{T} \subseteq \mathcal{E}$ is closed under extensions and epimorphic quotients. Dually, the subcategory $\mathcal{F} \subseteq \mathcal{E}$ is closed under extensions and subobjects. In particular, $\mathcal{T}$ and $\mathcal{F}$ inherit exact structures from $\mathcal{E}$.*
3. *For each $M \in \mathcal{E}$, there is a unique (up to isomorphism) conflation $\mathfrak{t}M \rightarrowtail M \twoheadrightarrow \mathfrak{f}M$ with $\mathfrak{t}M \in \mathcal{T}$ and $\mathfrak{f}M \in \mathcal{F}$.*
4. *The inclusion functor $\mathcal{T} \to \mathcal{E}$ has right adjoint $\mathfrak{t}$ such that $\mathfrak{t}(M) = \mathfrak{t}M$ as above. Dually, the inclusion functor $\mathcal{F} \to \mathcal{E}$ has a left adjoint $\mathfrak{f}$ such that $\mathfrak{f}(M) = \mathfrak{f}M$.*

The following definition is adapted from [54].

Definition 2.13. An additive functor $F\colon \mathcal{E} \to \mathcal{F}$ between exact categories is called *left exact* if a conflation $X \overset{f}{\rightarrowtail} Y \overset{g}{\twoheadrightarrow} Z$ is mapped to an exact sequence $F(X) \overset{F(f)}{\rightarrowtail} F(Y) \xrightarrow{F(g)} F(Z) \twoheadrightarrow \operatorname{coker}(F(g))$. The notion of a *right exact functor* is defined dually.

Remark 2.14. If $F\colon \mathcal{E} \to \mathcal{F}$ is left exact and $h\colon Y \to Z$ is an admissible morphism in $\mathcal{E}$ with kernel $f\colon X \rightarrowtail Y$, then $F(h)\colon F(Y) \to F(Z)$ is an admissible morphism in $\mathcal{F}$ with kernel $F(f)\colon F(X) \rightarrowtail F(Y)$. Indeed, choosing a factorization $Y \overset{g'}{\twoheadrightarrow} \operatorname{im}(f) \overset{g''}{\rightarrowtail} Z$ of g and applying F to the conflation $X \overset{f}{\rightarrowtail} Y \overset{g'}{\twoheadrightarrow} \operatorname{im}(f)$ we get an admissible morphism $F(g')\colon F(Y) \to F(\operatorname{im}(f))$ with kernel $F(f)\colon F(X) \rightarrowtail F(Y)$. The claim now follows since $F(g) = F(g'') \circ F(g')$ and $F(g'')$ is an inflation.

Remark 2.15. Note that a left exact (resp right exact) functor between exact categories does not necessarily preserve all kernels (resp cokernels), but only kernels of admissible morphisms.

Proposition 2.16. *Let $\mathcal{E}$ be an exact category with a torsion pair $(\mathcal{T}, \mathcal{F})$ such that $\mathcal{T} \subseteq \mathcal{E}$ satisfies axiom **A2**.*

1. *The subcategory $\mathcal{T} \subseteq \mathcal{E}$ is two-sided admissibly percolating.*
2. *The functor $\mathfrak{t}$ is left exact and the functor $\mathfrak{f}$ preserves inflations and deflations.*
3. *The functor f reflects admissible morphisms.*

Proof. 1. We first show axiom **A2**^{op}. Let $f\colon T \to X$ be a map with $X \in \mathcal{T}$. As $(\mathcal{T}, \mathcal{F})$ is a torsion pair, there is a conflation $\mathfrak{t}X \rightarrowtail X \twoheadrightarrow \mathfrak{f}X$. Note that f factors through $\mathfrak{t}X \rightarrowtail X$ as $\operatorname{Hom}(\mathcal{T}, \mathcal{F}) = 0$. By axiom **A2**, the map $T \to \mathfrak{t}X$ is admissible with image in $\mathcal{T}$. It follows that the composition $T \to \mathfrak{t}X \rightarrowtail X$ is admissible with image in $\mathcal{T}$. This shows axiom **A2**^{op}.

We now show axiom **A1**. Let $X \rightarrowtail Y \twoheadrightarrow Z$ be a conflation in $\mathcal{E}$. Assume first that $Y \in \mathcal{T}$. By axiom **A2**, the map $X \rightarrowtail Y$ is admissible with image in $\mathcal{T}$ and thus $X \in \mathcal{T}$. Similarly, by axiom **A2**^{op} yields that $Y \twoheadrightarrow Z$ is admissible with image in $\mathcal{T}$ and thus $Z \in \mathcal{T}$. Conversely, if $X, Z \in \mathcal{T}$, then $Y \in \mathcal{T}$ as $\mathcal{T} \subseteq \mathcal{E}$ is extension-closed.

2. Let $X \overset{a}{\rightarrowtail} Y \overset{b}{\twoheadrightarrow} Z$ be a conflation in $\mathcal{E}$. By axiom **A2** the maps $\mathfrak{t}(a)$ and $\mathfrak{t}(b)$ are admissible. As $\mathfrak{t}(a)$ is monic, it is an inflation. Since $\mathfrak{t}$ is a right adjoint, $\mathfrak{t}$ commutes with kernels, and thus the kernel of $\mathfrak{t}(b)$ in $\mathcal{T}$ is given by $\mathfrak{t}(a)$. Since $\mathcal{T}$ is abelian and the inclusion $\mathcal{T} \to \mathcal{E}$ is exact, it follows that $\mathfrak{t}(a)$ is also the kernel of $\mathfrak{t}(b)$ in $\mathcal{E}$. This shows that $\mathfrak{t}$ is left exact. In particular, it follows that

$\mathrm{im}(\mathfrak{t}(b)) \cong \mathrm{coker}(\mathfrak{t}(a))$. By axiom **L1**, the composition $\mathrm{im}(\mathfrak{t}(b)) \rightarrowtail \mathfrak{t}Z \rightarrowtail Z$ is an inflation and we write L for its cokernel. Consider the following two commutative diagrams:

$$\begin{tikzcd}
\mathfrak{t}X \arrow[r, tail] \arrow[d, tail] & \mathfrak{t}Y \arrow[r, two heads] \arrow[d, tail] & \mathrm{im}(\mathfrak{t}(b)) \arrow[d, tail] \\
X \arrow[r, tail] \arrow[d, two heads] & Y \arrow[r, two heads] \arrow[d, two heads] & Z \arrow[d, two heads] \\
\mathfrak{f}X \arrow[r] & \mathfrak{f}Y \arrow[r] & L
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{im}(\mathfrak{t}(b)) \arrow[r, tail] \arrow[d, equal] & \mathfrak{t}Z \arrow[r, two heads] \arrow[d, tail] & \mathrm{coker}(\mathfrak{t}(b)) \arrow[d] \\
\mathrm{im}(\mathfrak{t}(b)) \arrow[r, tail] \arrow[d, two heads] & Z \arrow[r, two heads] \arrow[d, two heads] & L \arrow[d] \\
0 \arrow[r, tail] & \mathfrak{f}Z \arrow[r, equal] & \mathfrak{f}Z
\end{tikzcd}$$

Applying the Nine Lemma to both diagrams, we obtain conflations

$$\mathfrak{f}X \rightarrowtail \mathfrak{f}Y \twoheadrightarrow L \quad \text{and} \quad \mathrm{coker}(\mathfrak{t}(b)) \rightarrowtail L \twoheadrightarrow \mathfrak{f}Z.$$

By axiom **R1**, the composition $\mathfrak{f}Y \twoheadrightarrow L \twoheadrightarrow fZ$ is a deflation. The result follows.

3. Let $f\colon X \to Y$ be a morphism in $\mathcal{E}$. Consider the following commutative diagram:

$$\begin{tikzcd}
\mathfrak{t}X \arrow[r, tail] \arrow[d, "\mathfrak{t}(f)"] & X \arrow[r, two heads, "\sim"] \arrow[d, "f"] & \mathfrak{f}X \arrow[d, "\mathfrak{f}(f)"] \\
\mathfrak{t}Y \arrow[r, tail] & Y \arrow[r, two heads, "\sim"] & \mathfrak{f}Y
\end{tikzcd}$$

If f is admissible, then $\mathfrak{f}(f)$ is admissible as $\mathfrak{f}$ preserves inflations and deflations. Conversely, if $\mathfrak{f}(f)$ is admissible, then the composition $X \twoheadrightarrow \mathfrak{f}X \xrightarrow{\mathfrak{f}(f)} \mathfrak{f}Y$ is admissible. Therefore the composition $X \xrightarrow{f} Y \twoheadrightarrow \mathfrak{f}Y$ is admissible, and hence f is admissible by Proposition 2.9.(3). □

Remark 2.17. Despite the functor $\mathfrak{f}$ preserving inflations and deflations, it need not be conflation-exact. Indeed, consider the standard torsion theory on the category abelian groups. The short exact sequence $\mathbb{Z} \overset{\cdot n}{\rightarrowtail} \mathbb{Z} \twoheadrightarrow \mathbb{Z}/n\mathbb{Z}$ is mapped to $\mathbb{Z} \overset{\cdot n}{\to} \mathbb{Z} \to 0$.

3 Auslander's formula for exact categories

Given an exact category $\mathcal{E}$, we consider the category $\mathrm{mod}_{\mathsf{adm}}(\mathcal{E})$ of admissibly presented functors and the subcategory $\mathrm{eff}(\mathcal{E}) \subseteq \mathrm{mod}_{\mathsf{adm}}(\mathcal{E})$ of effaceable functors (see Definition 3.1 below). We show that Auslander's formula extends to exact categories, i.e. the categories $\mathcal{E}$ and $\mathrm{mod}_{\mathsf{adm}}(\mathcal{E})/\,\mathrm{eff}(\mathcal{E})$ are equivalent as exact categories. Moreover, $\mathrm{eff}(\mathcal{E}) \subseteq \mathrm{mod}_{\mathsf{adm}}(\mathcal{E})$ is a two-sided admissibly percolating subcategory that arises as the torsion part of a torsion theory on $\mathrm{mod}_{\mathsf{adm}}(\mathcal{E})$.

3.1 Basic definitions and results

Write $\Upsilon\colon \mathcal{E} \to \mathrm{Mod}(\mathcal{E}) : A \mapsto \mathrm{Hom}_{\mathcal{E}}(-, A)$ for the Yoneda embedding . We start with the following definition.

Definition 3.1. Let $\mathcal{E}$ be a conflation category.

1. Let $\mathrm{mod}(\mathcal{E})$ be the full subcategory of $\mathrm{Mod}(\mathcal{E})$ consisting of those contravariant functors $F\colon \mathcal{E} \to \mathsf{Ab}$ that admit a projective representation

$$\Upsilon(X) \xrightarrow{\Upsilon(f)} \Upsilon(Y) \to F \to 0.$$

2. Let $\mathrm{mod}_{\mathsf{adm}}(\mathcal{E})$ be the full subcategory of $\mathrm{Mod}(\mathcal{E})$ consisting of those functors F that admit a projective representation

$$\Upsilon(X) \xrightarrow{\Upsilon(f)} \Upsilon(Y) \to F \to 0$$

such that $f\colon X \to Y$ is an admissible morphism in $\mathcal{E}$. We refer to f as a underlying or a representing morphism of F.

3. Let $\operatorname{eff}(\mathcal{E})$ be the full subcategory of $\operatorname{Mod}(\mathcal{E})$ consisting of those functors F that admit a projective representation
$$\Upsilon(X) \xrightarrow{\Upsilon(f)} \Upsilon(Y) \to F \to 0$$
such that $f\colon X \twoheadrightarrow Y$ is a deflation in $\mathcal{E}$.

4. Let $\mathcal{F}$ be the full subcategory of $\operatorname{Mod}(\mathcal{E})$ consisting of those functors F that admit a projective representation
$$\Upsilon(X) \xrightarrow{\Upsilon(f)} \Upsilon(Y) \to F \to 0$$
such that $f\colon X \rightarrowtail Y$ is an inflation in $\mathcal{E}$.

Lemma 3.2. *Every $F \in \operatorname{mod}_{\mathsf{adm}}(\mathcal{E})$ fits into an exact sequence*
$$0 \to \Upsilon(\ker(f)) \to \Upsilon(X) \xrightarrow{\Upsilon(f)} \Upsilon(Y) \to F \to 0.$$
in $\operatorname{Mod}(\mathcal{E})$.

Proof. The Yoneda embedding $\Upsilon\colon \mathcal{E} \to \operatorname{Mod}(\mathcal{E})$ is a left exact covariant functor. The result follows as f admits a kernel. □

The following proposition will be used multiple times throughout the text.

Proposition 3.3. *Let $\mathcal{C}$ be an additive category and write $\Upsilon\colon \mathcal{C} \to \operatorname{Mod}(\mathcal{C})$ for the Yoneda embedding. Consider a commutative diagram*
$$\begin{tikzcd} A \arrow[r, "\beta"] \arrow[d, "f"] & C \arrow[d, "g"] \\ B \arrow[r, "\alpha"] & D \end{tikzcd}$$
in $\mathcal{E}$ such that g admits a kernel $k\colon \ker(g) \to C$ and such that the cospan $B \xrightarrow{\alpha} D \xleftarrow{g} C$ admits a pullback E. Write $F = \operatorname{coker}(\Upsilon(f))$, $G = \operatorname{coker}(\Upsilon(g))$ and $\eta\colon F \to G$ for the induced map. Consider the commutative diagram
$$\begin{tikzcd} & & \ker(g') \arrow[r, equal] \arrow[d, "k'"] & \ker(g) \arrow[d, "k"] & & \\ \ker(g) \oplus A \arrow[r, "(0\ 1)"] \arrow[d, "(k'\ \beta'')"] & A \arrow[r, "\beta''"] \arrow[d, "f"] & E \arrow[r, "\beta'"] \arrow[d, "g'"] & C \arrow[r, "\binom{1}{0}"] \arrow[d, "g"] & C \oplus B \arrow[d, "(g\ \alpha)"] \\ E \arrow[r, "g'"] & B \arrow[r, equal] & B \arrow[r, "\alpha"] & D \arrow[r, equal] & D \end{tikzcd} \tag{1}$$
where $ECBD$ is a pullback square and $\beta = \beta'\beta''$. Applying Υ and taking the cokernel of the vertical maps induces the epi-mono factorization
$$\ker(\eta) \rightarrowtail F \twoheadrightarrow \operatorname{im}(\eta) \rightarrowtail G \twoheadrightarrow \operatorname{coker}(\eta)$$
of η in $\operatorname{Mod}(\mathcal{C})$.

Proof. This is a standard diagram chase argument. □

3.2 The quotient $\operatorname{mod}_{\mathsf{adm}}(\mathcal{E})/\operatorname{eff}(\mathcal{E})$

Throughout this section, $\mathcal{E}$ denotes an exact category.

Proposition 3.4. *The category $\operatorname{mod}_{\mathsf{adm}}(\mathcal{E})$ is an extension-closed subcategory of $\operatorname{Mod}(\mathcal{E})$. In particular, $\operatorname{mod}_{\mathsf{adm}}(\mathcal{E})$ inherits an exact structure from $\operatorname{Mod}(\mathcal{E})$.*

Proof. Let $0 \to F \to G \to H \to 0$ be a short exact sequence in the abelian category $\mathrm{Mod}(\mathcal{E})$ and assume that $F, H \in \mathrm{mod}_{\mathsf{adm}}(\mathcal{E})$. Let $\Upsilon(\ker(f)) \xrightarrow{\Upsilon(k_f)} \Upsilon(A) \xrightarrow{\Upsilon(f)} \Upsilon(B) \to F$ and $\Upsilon(\ker(h)) \xrightarrow{\Upsilon(k_h)} \Upsilon(C) \xrightarrow{\Upsilon(h)} \Upsilon(D) \to H$ be projective resolutions of F and H respectively with f, h admissible morphisms. By the Horseshoe Lemma and the fact that the Yoneda embedding is faithful, one finds a projective resolution $\Upsilon(\ker(h) \oplus \ker(f)) \xrightarrow{\Upsilon\left(\begin{smallmatrix} k_h & 0 \\ l' & k_f \end{smallmatrix}\right)} \Upsilon(C \oplus A) \xrightarrow{\Upsilon\left(\begin{smallmatrix} h & 0 \\ l & f \end{smallmatrix}\right)} \Upsilon(D \oplus B) \to G$ for some morphisms $l\colon C \to B$ and $l'\colon \ker(h) \to A$. Let $C \xrightarrow{h'} \mathrm{im}(h) \xrightarrow{h''}$ and be a factorization of h into a deflation and an inflation. We obtain the following morphism of acyclic complexes in $\mathcal{E}$:

$$\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] \arrow[d] & 0 \arrow[r, tail] \arrow[d] & \ker(h) \arrow[r, tail, "-k_h"] \arrow[d, "l'"] & C \arrow[r, two heads, "-h'"] \arrow[d, "l"] & \mathrm{im}(h) \arrow[r] \arrow[d, dotted, "l''"] & 0 \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r, tail] & \ker(f) \arrow[r, tail, "k_f"] & A \arrow[r, "f"] & B \arrow[r, two heads] & \mathrm{coker}(f) \arrow[r, two heads] & 0 \arrow[r] & \cdots
\end{tikzcd}$$

By [53, Lemma 1.1], the cone is an acyclic complex as well. It follows that the map $\left(\begin{smallmatrix} h' & 0 \\ l & f \end{smallmatrix}\right)\colon C \oplus A \to \mathrm{im}(h) \oplus B$ is admissible. Also, the map $\left(\begin{smallmatrix} h'' & 0 \\ 0 & 1_B \end{smallmatrix}\right)\colon \mathrm{im}(h) \oplus B \to D \oplus B$ is an inflation as the direct sum of inflations are inflations. It follows that $\left(\begin{smallmatrix} h & 0 \\ l & f \end{smallmatrix}\right) = \left(\begin{smallmatrix} h'' & 0 \\ 0 & 1_B \end{smallmatrix}\right)\left(\begin{smallmatrix} h' & 0 \\ l & f \end{smallmatrix}\right)$ is admissible. This shows that $G \in \mathrm{mod}_{\mathsf{adm}}(\mathcal{E})$. $\square$

Proposition 3.5. *The pair* $(\mathrm{eff}(\mathcal{E}), \mathcal{F})$ *is a torsion pair in* $\mathrm{mod}_{\mathsf{adm}}(\mathcal{E})$ *such that* $\mathrm{eff}(\mathcal{E}) \subseteq \mathrm{mod}_{\mathsf{adm}}(\mathcal{E})$ *satisfies axiom* ***A2****. In particular,* $\mathrm{eff}(\mathcal{E}) \subseteq \mathrm{mod}_{\mathsf{adm}}(\mathcal{E})$ *is a two-sided admissibly percolating subcategory.*

Proof. We first show that $(\mathrm{eff}(\mathcal{E}), \mathcal{F})$ yields a torsion pair in $\mathrm{mod}_{\mathsf{adm}}(\mathcal{E})$. To that end, let $G \in \mathrm{mod}_{\mathsf{adm}}(\mathcal{E})$ and choose a projective presentation $\Upsilon(X) \xrightarrow{\Upsilon(f)} \Upsilon(Y) \to G \to 0$ where f is admissible. Let $X \overset{f'}{\twoheadrightarrow} X' \overset{f''}{\rightarrowtail} Y$ be the deflation-inflation factorization of f. The commutative diagram

$$\begin{tikzcd}
X \arrow[r, equal] \arrow[d, two heads, "f'"] & X \arrow[r, two heads, "f'"] \arrow[d, "f"] & X' \arrow[d, tail, "f''"] \\
X' \arrow[r, tail, "f''"] & Y \arrow[r, equal] & Y
\end{tikzcd}$$

in $\mathcal{E}$ induces a sequence $F \xrightarrow{\phi} G \xrightarrow{\psi} H$ in $\mathrm{mod}_{\mathsf{adm}}(\mathcal{E})$ where $F = \mathrm{coker}(\Upsilon(f')) \in \mathrm{eff}(\mathcal{E})$ and $H = \mathrm{coker}(\Upsilon(f'')) \in \mathcal{F}$. A straightforward diagram chase shows that $F \xrightarrow{\phi} G \xrightarrow{\psi} H$ is a short exact sequence.

We now show that $\mathrm{Hom}_{\mathrm{mod}_{\mathsf{adm}}(\mathcal{E})}(\mathrm{eff}(\mathcal{E}), \mathcal{F}) = 0$. Let $\eta\colon F \to H$ be a morphism in $\mathrm{mod}_{\mathsf{adm}}(\mathcal{E})$ with $F \in \mathrm{eff}(\mathcal{E})$ and $H \in \mathcal{F}$. Choose projective presentations $\Upsilon(X) \xrightarrow{\Upsilon(f)} \Upsilon(Y) \to F \to 0$ and $\Upsilon(X') \xrightarrow{\Upsilon(h)} \Upsilon(Y') \to H \to 0$ where f is a deflation and h is an inflation in $\mathcal{E}$. Lifting the morphism η to a map between the projective presentations and using that the Yoneda embedding is fully faithful, we find morphisms $\alpha\colon X \to X'$ and $\beta\colon Y \to Y'$ such that $h \circ \alpha = \beta \circ f$ and making the diagram

$$\begin{tikzcd}
\Upsilon(X) \arrow[r, "\Upsilon(f)"] \arrow[d, "\Upsilon(\alpha)"] & \Upsilon(Y) \arrow[r] \arrow[d, "\Upsilon(\beta)"] & F \arrow[r] \arrow[d, "\eta"] & 0 \\
\Upsilon(X') \arrow[r, "\Upsilon(h)"] & \Upsilon(Y') \arrow[r] & F' \arrow[r] & 0
\end{tikzcd}$$

commutative. Moreover, using Lemma 3.2, we find that $\alpha \circ k_f = 0$ (where $k_f\colon \ker(f) \rightarrowtail X$ is the kernel of f). Since $h \circ \alpha \circ k_f = \beta \circ f \circ k_f = 0$ and h is an inflation, it follows that $\alpha \circ k_f = 0$. Hence α factors through f via a morphism $\gamma\colon Y \to X'$. Since $h \circ \gamma \circ f = h \circ \alpha = \beta \circ f$ and f is a deflation, we get that $h \circ \gamma = \beta$. Hence $\Upsilon(\beta)$ factors through $\Upsilon(h)$, so η must be 0.

To prove that $\mathrm{eff}(\mathcal{E}) \subseteq \mathrm{mod}_{\mathsf{adm}}(\mathcal{E})$ satisfies axiom **A2**, assume we are given a morphism $\eta\colon F \to G$ with $G \in \mathrm{mod}_{\mathsf{adm}}(\mathcal{E})$ and $F \in \mathrm{eff}(\mathcal{E})$. Choose projective presentations $\Upsilon(A) \xrightarrow{\Upsilon(f)} \Upsilon(B) \to F \to 0$ and

$\Upsilon(C) \xrightarrow{\Upsilon(g)} \Upsilon(D) \to G \to 0$ where f is admissible and g is a deflation. Lifting η to these projective presentation gives maps $\alpha\colon B \to D$ and $\beta\colon A \to C$ making the square

$$\begin{tikzcd} A \arrow[r, "\beta"] \arrow[d, "f"] & C \arrow[d, "g"] \\ B \arrow[r, "\alpha"] & D \end{tikzcd}$$

commutative. By taking the pullback of α along g we get the diagram (1) in Proposition 3.3, which induces the factorization

$$\ker(\eta) \rightarrowtail F \twoheadrightarrow \operatorname{im}(\eta) \rightarrowtail G \twoheadrightarrow \operatorname{coker}(\eta)$$

of η in $\operatorname{Mod}(\mathcal{E})$. Since g' and $(\begin{smallmatrix} g & \alpha \end{smallmatrix})$ are deflations, it follows that $\operatorname{im}(\eta)$ and $\operatorname{coker}(\eta)$ are in $\operatorname{eff}(\mathcal{E})$. Now let $A \overset{f'}{\twoheadrightarrow} B' \overset{f''}{\rightarrowtail} B$ be a deflation-inflation factorization of f. Taking the pullback E' of f'' along g', we get a commutative diagram

$$\begin{tikzcd} \ker(f) \arrow[r, tail] \arrow[d] & A \arrow[r, two heads, "f'"] \arrow[d] & B' \arrow[d, equal] \\ \ker(g) \arrow[r, tail] \arrow[d, equal] & E' \arrow[r, two heads] \arrow[d, tail] & B' \arrow[d, tail, "f''"] \\ \ker(g) \arrow[r, tail, "k'"] & E \arrow[r, two heads, "g'"] & B \end{tikzcd}$$

where the rows are exact and where the composite $A \to E' \rightarrowtail E$ is equal to β''. Since the pullback of an inflation along a deflation is an inflation [24, Proposition 2.15], the map $E' \rightarrowtail E$ is an inflation. Also, by [24, Proposition 2.12] the top left square is pushout square and the induced map $\ker(g) \oplus A \twoheadrightarrow E'$ is a deflation. Since $(\begin{smallmatrix} k' & \beta'' \end{smallmatrix})\colon \ker(g) \oplus A \to E$ is equal to the composite $\ker(g) \oplus A \twoheadrightarrow E' \rightarrowtail E$, it must be admissible. Hence $\ker(\eta) \in \operatorname{mod}_{\mathsf{adm}}(\mathcal{E})$, which proves axiom **A2**. Finally, by Proposition 2.16, $\operatorname{eff}(\mathcal{E}) \subseteq \operatorname{mod}_{\mathsf{adm}}(\mathcal{E})$ is a two-sided admissibly percolating subcategory. □

Corollary 3.6. *The quotient* $\operatorname{mod}_{\mathsf{adm}}(\mathcal{E})/\operatorname{eff}(\mathcal{E})$ *is an exact category.*

3.3 Auslander's formula for exact categories

In this section $\mathcal{E}$ denotes an exact category.

Theorem 3.7 (Universal property)**.** *1. The Yoneda embedding* $\Upsilon\colon \mathcal{E} \to \operatorname{mod}_{\mathsf{adm}}(\mathcal{E})$ *is left exact;*

2. If $\mathcal{F}$ *is an exact category and* $\Phi\colon \mathcal{E} \to \mathcal{F}$ *is a left exact functor, then there exists a functor* $\overline{\Phi}\colon \operatorname{mod}_{\mathsf{adm}}(\mathcal{E}) \to \mathcal{F}'$*, unique up to isomorphism, which is exact and satisfies* $\overline{\Phi} \circ \Upsilon = \Phi$*.*

Proof. 1. This follows from Lemma 3.2

2. Uniqueness is clear since any exact functor $\operatorname{mod}_{\mathsf{adm}}(\mathcal{E}) \to \mathcal{F}$ is uniquely determined (up to isomorphism) by its restriction to the representable functors. We prove existence. For each $F \in \operatorname{mod}_{\mathsf{adm}}(\mathcal{E})$ choose a projective presentation $\Upsilon(A) \xrightarrow{\Upsilon(f)} \Upsilon(B) \to F \to 0$ with f admissible (where if $F = \Upsilon(E)$ we simply choose $0 \to \Upsilon(E) \xrightarrow{1} \Upsilon(E) \to 0$). Since Φ is left exact, it preserves admissible morphisms, and hence $\Phi(f)$ is admissible in $\mathcal{F}$. We define $\overline{\Phi}(F) = \operatorname{coker}(\Phi(f))$. Let $\eta\colon F \to F'$ be a morphism in $\operatorname{mod}_{\mathsf{adm}}(\mathcal{E})$ between objects with fixed projective presentations $\Upsilon(A) \xrightarrow{\Upsilon(f)} \Upsilon(B) \to F \to 0$ and $\Upsilon(A') \xrightarrow{\Upsilon(f')} \Upsilon(B') \to F' \to 0$. Choose maps $\alpha\colon A \to A'$ and $\beta\colon B \to B'$ in $\mathcal{E}$ making the diagram

$$\begin{tikzcd} \Upsilon(A) \arrow[r, "\Upsilon(f)"] \arrow[d, "\Upsilon(\alpha)"] & \Upsilon(B) \arrow[r] \arrow[d, "\Upsilon(\beta)"] & F \arrow[r] \arrow[d, "\eta"] & 0 \\ \Upsilon(A') \arrow[r, "\Upsilon(f')"] & \Upsilon(B') \arrow[r] & F' \arrow[r] & 0 \end{tikzcd}$$

commutative. We define $\overline{\Phi}(\eta)\colon \overline{\Phi}(F) \to \overline{\Phi}(F')$ to be the unique morphism in $\mathcal{F}$ making the diagram

$$\begin{tikzcd}
\Phi(A) \arrow[r, "\Phi(f)"] \arrow[d, "\Phi(\alpha)"] & \Phi(B) \arrow[r] \arrow[d, "\Phi(\beta)"] & \overline{\Phi}(F) \arrow[r] \arrow[d, "\overline{\Phi}(\eta)"] & 0 \\
\Phi(A') \arrow[r, "\Phi(f')"] & \Phi(B') \arrow[r] & \overline{\Phi}(F') \arrow[r] & 0
\end{tikzcd}$$

commutative. It is straightforward to check that $\overline{\Phi}(\eta)$ is independent of choice of α and β, and that $\overline{\Phi}\colon \operatorname{mod}_{\mathsf{adm}}(\mathcal{E}) \to \mathcal{F}$ is a well-defined functor. It only remains to show that $\overline{\Phi}$ is exact. Let $F \rightarrowtail G \twoheadrightarrow H$ be a conflation in $\operatorname{mod}_{\mathsf{adm}}(\mathcal{E})$. Choose projective presentations $\Upsilon(A) \xrightarrow{\Upsilon(f)} \Upsilon(B) \to F \to 0$ and $\Upsilon(C) \xrightarrow{\Upsilon(h)} \Upsilon(D) \to H \to 0$ of F and H, and use the Horseshoe Lemma as in Proposition 3.4 to obtain the following commutative diagram in $\mathcal{E}$:

$$\begin{tikzcd}
\ker(f) \arrow[r, tail] \arrow[d, tail] & \ker(f) \oplus \ker(h) \arrow[r, two heads] \arrow[d, tail] & \ker(h) \arrow[d, tail] \\
A \arrow[r, tail] \arrow[d, "f"] & A \oplus C \arrow[r, two heads] \arrow[d] & C \arrow[d, "h"] \\
B \arrow[r, tail] & B \oplus D \arrow[r, two heads] & D
\end{tikzcd}$$

where the rows are are split kernel-cokernels pairs and the columns are left exact sequences. Since Φ is left exact and preserves split kernel-cokernel pairs, by applying the 3×3 lemma twice [24, Corollary 3.6] we get that $\overline{\Phi}(F) \rightarrowtail \overline{\Phi}(G) \twoheadrightarrow \overline{\Phi}(H)$ is a conflation. This shows that $\overline{\Phi}$ is exact. □

Corollary 3.8. *The Yoneda embedding $\Upsilon\colon \mathcal{E} \to \operatorname{mod}_{\mathsf{adm}}(\mathcal{E})$ has an exact left adjoint $L\colon \operatorname{mod}_{\mathsf{adm}}(\mathcal{E}) \to \mathcal{E}$ satisfying $L \circ \Upsilon = 1_{\mathcal{E}}$ and $\ker(L) = \operatorname{eff}(\mathcal{E})$.*

Proof. Applying Theorem 3.7 to the identity functor on $\mathcal{E}$ gives an exact functor $L\colon \operatorname{mod}_{\mathsf{adm}}(\mathcal{E}) \to \mathcal{E}$ satisfying $L \circ \Upsilon = 1_{\mathcal{E}}$. Explicitly, given $F \in \operatorname{mod}_{\mathsf{adm}}(\mathcal{E})$ with projective presentation $\Upsilon(A) \xrightarrow{\Upsilon(f)} \Upsilon(B) \to F$ where f admissible, we have $L(F) \cong \operatorname{coker}(f)$. Note that $\Upsilon(B) \to \Upsilon(\operatorname{coker}(f))$ factors as $\Upsilon(Y) \to F \xrightarrow{\alpha} \Upsilon(\operatorname{coker}(f))$ for a unique map α. One readily verifies that any map $F \to \Upsilon(Z)$ must factor through α in a unique way, and hence we have natural isomorphisms

$$\operatorname{Hom}_{\operatorname{mod}_{\mathsf{adm}}(\mathcal{E})}(F, \Upsilon(Z)) \cong \operatorname{Hom}_{\mathcal{E}}(\operatorname{coker}(f), Z) \cong \operatorname{Hom}_{\mathcal{E}}(L(F), Z)$$

which shows that L is left adjoint to Υ. Finally, note that $L(F) \cong 0$ if and only if $\operatorname{coker}(f) \cong 0$. As f is admissible, $\operatorname{coker}(f) \cong 0$ implies that f is a deflation. This shows that $\ker(L) = \operatorname{eff}(\mathcal{E})$. □

We now prove Auslander's formula for exact categories.

Theorem 3.9 (Auslander's formula for exact categories)**.** *The functor $L\colon \operatorname{mod}_{\mathsf{adm}}(\mathcal{E}) \to \mathcal{E}$ induces an equivalence*

$$\operatorname{mod}_{\mathsf{adm}}(\mathcal{E}) / \operatorname{eff}(\mathcal{E}) \cong \mathcal{E}$$

of exact categories.

Proof. Since $\ker(L) = \operatorname{eff}(\mathcal{E})$, it follows that L induces an exact functor $\operatorname{mod}_{\mathsf{adm}}(\mathcal{E}) / \operatorname{eff}(\mathcal{E}) \to \mathcal{E}$, which is an equivalence by [38, Proposition 1.3]. □

Corollary 3.10. *1. The Yoneda embedding $\Upsilon\colon \mathcal{E} \to \operatorname{mod}_{\mathsf{adm}}(\mathcal{E})$ lifts to a triangle equivalence $\mathsf{K}^{\mathsf{b}}(\mathcal{E}) \to \mathsf{D}^{\mathsf{b}}(\operatorname{mod}_{\mathsf{adm}}(\mathcal{E}))$;*

2. there is a natural commutative diagram

$$\begin{tikzcd}
\mathsf{D}^{\mathsf{b}}{}_{\mathrm{eff}(\mathcal{E})}(\mathrm{mod}_{\mathsf{adm}}(\mathcal{E})) \arrow[r] \arrow[d, "\simeq"] & \mathsf{D}^{\mathsf{b}}(\mathrm{mod}_{\mathsf{adm}}(\mathcal{E})) \arrow[r] \arrow[d, "\simeq"] & \mathsf{D}^{\mathsf{b}}(\mathrm{mod}_{\mathsf{adm}}(\mathcal{E})/\,\mathrm{eff}(\mathcal{E})) \arrow[d, "\simeq"] \\
\left\langle \boldsymbol{Ac}^{\mathsf{b}}(\mathcal{E}) \right\rangle_{thick} \arrow[r] & \mathsf{K}^{\mathsf{b}}(\mathcal{E}) \arrow[r] & \mathsf{D}^{\mathsf{b}}(\mathcal{E})
\end{tikzcd}$$

whose rows are Verdier localization sequences.

Proof. 1. This follows immediately since any object in $\mathrm{mod}_{\mathsf{adm}}(\mathcal{E})$ has finite projective dimension by Lemma 3.2.

2. The upper Verdier localization sequence is induced by the localization sequence

$$\mathrm{eff}(\mathcal{E}) \to \mathrm{mod}_{\mathsf{adm}}(\mathcal{E}) \to \mathrm{mod}_{\mathsf{adm}}(\mathcal{E})/\,\mathrm{eff}(\mathcal{E}).$$

The lower sequence is simply the definition of $\mathsf{D}^{\mathsf{b}}(\mathcal{E})$ for an exact category. The functor L' induces the equivalence $\mathsf{D}^{\mathsf{b}}(\mathrm{mod}_{\mathsf{adm}}(\mathcal{E})/\,\mathrm{eff}(\mathcal{E})) \xrightarrow{\simeq} \mathsf{D}^{\mathsf{b}}(\mathcal{E})$ and the Yoneda embedding induces the equivalence $\mathsf{K}^{\mathsf{b}}(\mathcal{E}) \to \mathsf{D}^{\mathsf{b}}(\mathrm{mod}_{\mathsf{adm}}(\mathcal{E}))$. The result follows. □

In the last part of this section we show that $\mathcal{E} \mapsto \mathrm{mod}_{\mathsf{adm}}(\mathcal{E})$ can be considered as a 2-functor.

Definition 3.11. 1. Let $\mathbf{Ex}$ denote the 2-category with objects exact categories, with 1-morphisms exact functors, and with 2-morphisms natural transformations;

2. Let $\mathbf{Ex}_L$ denote the 2-category with objects exact categories, with 1-morphisms left exact functors, and with 2-morphisms natural transformations.

Corollary 3.12. *The association $\mathcal{E} \mapsto \mathrm{mod}_{\mathsf{adm}}(\mathcal{E})$ induces a 2-functor*

$$\mathrm{mod}_{\mathsf{adm}}(-)\colon\ \mathbf{Ex}_L \to \mathbf{Ex}$$

which is left adjoint to the inclusion $\mathbf{Ex} \subseteq \mathbf{Ex}_L$, i.e. for which the restriction

$$\mathrm{Hom}_{\mathbf{Ex}}(\mathrm{mod}_{\mathsf{adm}}(\mathcal{E}), \mathcal{E}') \to \mathrm{Hom}_{\mathbf{Ex}_L}(\mathcal{E}, \mathcal{E}')$$

via $\Upsilon\colon \mathcal{E} \to \mathrm{mod}_{\mathsf{adm}}(\mathcal{E})$ is an equivalence of categories.

Proof. Using Theorem 3.7, for each left exact functor $\Phi\colon \mathcal{E} \to \mathcal{E}'$ we choose an exact functor

$$\mathrm{mod}_{\mathsf{adm}}(\Phi)\colon\ \mathrm{mod}_{\mathsf{adm}}(\mathcal{E}) \to \mathrm{mod}_{\mathsf{adm}}(\mathcal{E}')$$

making the square

$$\begin{tikzcd}
\mathcal{E} \arrow[r, "\Phi"] \arrow[d, "\Upsilon"] & \mathcal{E}' \arrow[d, "\Upsilon"] \\
\mathrm{mod}_{\mathsf{adm}}(\mathcal{E}) \arrow[r, "\mathrm{mod}_{\mathsf{adm}}(\Phi)"] & \mathrm{mod}_{\mathsf{adm}}(\mathcal{E}')
\end{tikzcd}$$

commutative. Since each natural transformation $\Phi \to \Phi'$ can be extended uniquely to a natural transformation $\mathrm{mod}_{\mathsf{adm}}(\Phi) \to \mathrm{mod}_{\mathsf{adm}}(\Phi')$, we get a 2-functor $\mathrm{mod}_{\mathsf{adm}}(-)\colon\ \mathbf{Ex}_L \to \mathbf{Ex}$. Finally, the restriction $\mathrm{Hom}_{\mathbf{Ex}}(\mathrm{mod}_{\mathsf{adm}}(\mathcal{E}), \mathcal{E}') \to \mathrm{Hom}_{\mathbf{Ex}_L}(\mathcal{E}, \mathcal{E}')$ is an equivalence by Theorem 3.7 and the fact that any natural transformation $\Psi \circ \Upsilon \to \Psi' \circ \Upsilon$ where $\Psi, \Psi' \in \mathrm{Hom}_{\mathbf{Ex}}(\mathrm{mod}_{\mathsf{adm}}(\mathcal{E}), \mathcal{F})$ can be extended uniquely to a natural transformation $\Psi \to \Psi'$. □

3.4 The torsion pair $(\mathrm{eff}(\mathcal{E}), \mathrm{cogen}(\mathcal{Q}))$ in $\mathrm{mod}_{\mathsf{adm}}(\mathcal{E})$

Fix an exact category $\mathcal{E}$ and set $\mathcal{Q} = \mathrm{Proj}(\mathrm{mod}_{\mathsf{adm}}(\mathcal{E}))$. In this section, we show that $(\mathrm{eff}(\mathcal{E}), \mathcal{F})$ as in Proposition 3.5 satisfies several useful properties. In the next section, we show that these properties can be used to characterize the image of the 2-functor $\mathrm{mod}_{\mathsf{adm}}(-)\colon \mathsf{Ex}_L \to \mathsf{Ex}$, which leads to a generalization of Auslander correspondence.

Definition 3.13. Let $\mathcal{X} \subseteq \mathcal{E}$ be a subcategory and $n \geq 0$ an integer.

1. We write $\mathrm{gen}_k(\mathcal{X})$ for the full subcategory of $\mathcal{E}$ consisting of those objects E such that there is an exact sequence $X_k \to \cdots \to X_2 \to X_1 \to E \to 0$ with $X_i \in \mathcal{X}$ for $1 \leq i \leq k$. If $k = 1$ we write $\mathrm{gen}_1(\mathcal{X}) = \mathrm{gen}(\mathcal{X})$
2. We write $\mathrm{cogen}_k(\mathcal{X})$ for the full subcategory of $\mathcal{E}$ consisting of those objects E such that there is an exact sequence $0 \to E \to X_1 \to X_2 \to \cdots \to X_k$ with $X_i \in \mathcal{X}$ for $1 \leq i \leq k$. If $k = 1$ we write $\mathrm{cogen}_1(\mathcal{X}) = \mathrm{cogen}(\mathcal{X})$

In the following ${}^{\perp}\mathcal{Q}$ and $\mathrm{cogen}(\mathcal{Q})$ are defined inside the exact category $\mathrm{mod}_{\mathsf{adm}}(\mathcal{E})$.

Proposition 3.14. *Consider the torsion pair $(\mathrm{eff}(\mathcal{E}), \mathcal{F})$ in $\mathrm{mod}_{\mathsf{adm}}(\mathcal{E})$ as in Proposition 3.5. The following hold:*

1. $\mathrm{eff}(\mathcal{E}) = {}^{\perp}\mathcal{Q}$;
2. $\mathcal{F} = \mathrm{cogen}(\mathcal{Q})$;
3. $\mathrm{Ext}^1_{\mathrm{mod}_{\mathsf{adm}}(\mathcal{E})}(\mathrm{eff}(\mathcal{E}), \mathcal{Q}) = 0$;
4. *The Yoneda functor $\Upsilon\colon \mathcal{E} \to \mathrm{mod}_{\mathsf{adm}}(\mathcal{E})$ gives an equivalence $\mathcal{E} \cong \mathcal{Q}$.*

Proof. 1. Clearly $\mathcal{Q} \subseteq \mathcal{F}$, hence the inclusion $\mathrm{eff}(\mathcal{E}) \subseteq {}^{\perp}\mathcal{Q}$ follows from $\mathrm{Hom}_{\mathrm{mod}_{\mathsf{adm}}(\mathcal{E})}(\mathrm{eff}(\mathcal{E}), \mathcal{F}) = 0$. Conversely, let $G \in {}^{\perp}\mathcal{Q}$. Choose a projective presentation $\Upsilon(A) \xrightarrow{\Upsilon(f)} \Upsilon(B) \to G \to 0$ with f admissible in $\mathcal{E}$. Since the composite $\Upsilon(A) \xrightarrow{\Upsilon(f)} \Upsilon(B) \to \Upsilon(\mathrm{coker}(f))$ is 0, the map $\Upsilon(B) \to \Upsilon(\mathrm{coker}(f))$ factors through G and must therefore be 0 since $G \in {}^{\perp}\mathcal{Q}$. Using that Υ is faithful, it follows that the map $B \to \mathrm{coker}(f)$ is 0, so f must be a deflation. This shows that $G \in \mathrm{eff}(\mathcal{E})$.

2. We first show $\mathcal{F} \subseteq \mathrm{cogen}(\mathcal{Q})$. Choose $F \in \mathcal{F}$ and let $\Upsilon(X) \xrightarrow{\Upsilon(f)} \Upsilon(Y) \to F \to 0$ be a projective presentation with f an inflation. Let $g\colon Y \twoheadrightarrow Z$ denote the cokernel of f. By considering the two rightmost squares in diagram (1) in Proposition 3.3, we get that the commutative diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d, tail, "f"] & 0 \arrow[r] \arrow[d] & Y \arrow[d, two heads, "g"] \\ Y \arrow[r, two heads, "g"] & Z \arrow[r, equal] & Z \end{tikzcd}$$

yields a short exact sequence $F \rightarrowtail \Upsilon(Z) \twoheadrightarrow H$ where $H = \mathrm{coker}(\Upsilon(g)) \in \mathrm{eff}(\mathcal{E})$. Thus $F \in \mathrm{cogen}(\mathcal{Q})$. Conversely, let $G \in \mathrm{cogen}(\mathcal{Q})$, then there is an inflation $G \rightarrowtail \Upsilon(Z)$ by definition. As $(\mathrm{eff}(\mathcal{E}), \mathcal{F})$ is a torsion pair, G fits into a conflation $F \rightarrowtail G \twoheadrightarrow H$ with $F \in \mathrm{eff}(\mathcal{E})$ and $H \in \mathcal{F}$. As $\Upsilon(Z) \in \mathcal{F}$, the composition $F \rightarrowtail G \rightarrowtail \Upsilon(Z)$ is zero. It follows that $F \cong 0$ and thus $G \cong H \in \mathcal{F}$. This shows the equality $\mathcal{F} = \mathrm{cogen}(\mathcal{Q})$.

3. Let $E \in \mathrm{eff}(\mathcal{E})$. Then there exists a conflation $\ker(e) \overset{k}{\rightarrowtail} A \overset{e}{\twoheadrightarrow} B$ in $\mathcal{E}$ which gives a projective resolution

$$0 \to \Upsilon(\ker(e)) \xrightarrow{\Upsilon(k)} \Upsilon(A) \xrightarrow{\Upsilon(e)} \Upsilon(B) \to E \to 0$$

of E. Let $\Upsilon(Z) \in \mathcal{Q}$ be arbitrary. Applying the functor $\mathrm{Hom}(-, \Upsilon(Z))$ to the projective resolution and using that the Υ is fully faithful yields the left exact sequence

$$0 \to \mathrm{Hom}(B, Z) \xrightarrow{-\circ e} \mathrm{Hom}(A, Z) \xrightarrow{-\circ k} \mathrm{Hom}(\ker(e), Z)$$

of abelian groups. Since the sequence is exact at $\mathrm{Hom}(A, Z)$, it follows that $\mathrm{Ext}^1(E, \Upsilon(Z)) = 0$.

4. Let $P \in \mathcal{Q}$. Note that P is torsion-free and thus there exists a conflation $A \overset{f}{\rightarrowtail} B \twoheadrightarrow C$ such that $\Upsilon(A) \xrightarrow{\Upsilon(f)} \Upsilon(B) \to P$ is a short exact sequence. Since P is projective, the latter sequence is split and thus f is a split inflation as Υ is fully faithful. It follows that $P \cong \Upsilon(C)$, which proves the claim. □

Corollary 3.15. *Let $\mathcal{E}$ be an exact category. The following hold:*

1. *The category $\mathcal{E}$ is weakly idempotent complete if and only if $\operatorname{mod}_{\mathsf{adm}}(\mathcal{E})$ is weakly idempotent complete.*

2. *The category $\mathcal{E}$ is idempotent complete if and only if $\operatorname{mod}_{\mathsf{adm}}(\mathcal{E})$ is idempotent complete.*

Proof. The if direction follows as $\mathcal{E}$ is equivalent to $\operatorname{Proj}(\operatorname{mod}_{\mathsf{adm}}(\mathcal{E}))$. Conversely, assume that $\mathcal{E}$ is (weakly) idempotent complete, and let f be an idempotent endomorphism (respectively a map which admits a right inverse) in $\operatorname{mod}_{\mathsf{adm}}(\mathcal{E})$. Then $Q(f)$ is idempotent (respectively admits a right inverse), and hence $Q(f)$ is admissible by assumption. Therefore, by Theorem 2.10 we have that f is admissible in $\operatorname{mod}_{\mathsf{adm}}(\mathcal{E})$. Hence $\operatorname{mod}_{\mathsf{adm}}(\mathcal{E})$ is (weakly) idempotent complete, which proves the claim. □

3.5 Admissibly presented functors and the category of sheaves

Throughout this section, $\mathcal{E}$ denotes an exact category. The category $\operatorname{mod}_{\mathsf{adm}}(\mathcal{E})$ is defined as a subcategory of $\operatorname{mod}(\mathcal{E})$, depending on the exact structure. In this section, we show how the category $\operatorname{mod}_{\mathsf{adm}}(\mathcal{E})$ occurs naturally by comparing the localization sequence $\operatorname{eff}(\mathcal{E}) \to \operatorname{mod}_{\mathsf{adm}}(\mathcal{E}) \to \operatorname{mod}_{\mathsf{adm}}(\mathcal{E})/\operatorname{eff}(\mathcal{E})$ from Theorem 3.9 to similar localization sequences appearing in the literature (see Proposition 3.23). In order to do so, we first recall the notion of a weakly effaceable functor as introduced by Grothendieck in [40] (following the terminology of [47]).

Definition 3.16. A functor $F \in \operatorname{Mod}(\mathcal{E})$ is called *weakly effaceable* if for every $C \in \mathcal{E}$ and every $x \in F(C)$, there exists a deflation $\lambda\colon C' \twoheadrightarrow C$ such that $F(\lambda)(x) = 0$. The full subcategory of $\operatorname{Mod}(\mathcal{E})$ of weakly effaceable functors is denoted by $\operatorname{Eff}(\mathcal{E})$.

The following proposition states that effaceable functors are weakly effaceable functors which are finitely presented.

Proposition 3.17. *Let $F \in \operatorname{Mod}(\mathcal{E})$, then $F \in \operatorname{eff}(\mathcal{E})$ if and only if $F \in \operatorname{Eff}(\mathcal{E}) \cap \operatorname{mod}(\mathcal{E})$.*

Proof. Assume that $F \in \operatorname{eff}(\mathcal{E})$ and let $f\colon A \twoheadrightarrow B$ be a deflation in $\mathcal{E}$ such that $F \cong \operatorname{coker}(\Upsilon(f))$. For any $C \in \mathcal{E}$ and $x' \in F(C)$, there exists a morphism $x\colon C \to B$ representing x'. By axiom **R2**, we obtain the following pullback square:

$$\begin{tikzcd} C' \arrow[r, "y"] \arrow[d, two heads, "\lambda"] & A \arrow[d, two heads, "f"] \\ C \arrow[r, "x"] & B \end{tikzcd}$$

Commutativity of the square yields $F(\lambda)(x) = 0$. This shows that $F \in \operatorname{Eff}(\mathcal{E}) \cap \operatorname{mod}(\mathcal{E})$.

Conversely, let $F \in \operatorname{Eff}(\mathcal{E}) \cap \operatorname{mod}(\mathcal{E})$. As F is finitely presented, there is a map $f\colon A \to B$ such that $F \cong \operatorname{coker}(\Upsilon(f))$. The identity 1_B yields an element $1 \in F(B)$. As $F \in \operatorname{Eff}(\mathcal{E})$, there exists a deflation $\lambda\colon C \twoheadrightarrow B$ such that $F(\lambda)(1) = 0$. Hence there exists a map $u\colon C \to A$ such that $fu = \lambda 1_B = \lambda$. By [44, Proposition 3.4], the map $\begin{pmatrix} 0 & f \end{pmatrix}\colon C \oplus A \twoheadrightarrow B$ is a deflation in $\mathcal{E}$. Clearly, $\operatorname{coker}(\Upsilon(\begin{pmatrix} 0 & f \end{pmatrix})) \cong F$ and thus $F \in \operatorname{eff}(\mathcal{E})$. □

Remark 3.18. It is shown in [35, Lemma 2.13] that $\operatorname{eff}(\mathcal{E}) \subseteq \operatorname{coh}(\mathcal{E})$ where $\operatorname{coh}(\mathcal{E})$ are the coherent functors, and thus $\operatorname{eff}(\mathcal{E}) = \operatorname{Eff}(\mathcal{E}) \cap \operatorname{coh}(\mathcal{E})$. In particular, $\operatorname{eff}(\mathcal{E}) \subseteq \operatorname{coh}(\mathcal{E})$ is a Serre subcategory and thus $\operatorname{eff}(\mathcal{E})$ is abelian.

Corollary 3.19. 1. *The category $\operatorname{Eff}(\mathcal{E})$ is a localizing Serre subcategory of $\operatorname{Mod}(\mathcal{E})$.*

2. *The category $\operatorname{eff}(\mathcal{E})$ is an admissibly-deflation percolating subcategory of $\operatorname{mod}(\mathcal{E})$.*

Proof. That $\mathrm{Eff}(\mathcal{E})$ is a localizing Serre subcategory of $\mathrm{Mod}(\mathcal{E})$ is shown in [48, Appendix A.2]. Hence, by Proposition 3.17, $\mathrm{eff}(\mathcal{E}) \subseteq \mathrm{mod}(\mathcal{E})$ is a Serre subcategory. That $\mathrm{eff}(\mathcal{E}) \subseteq \mathrm{mod}(\mathcal{E})$ satisfies axiom **A2** follows from Proposition 3.3. □

Remark 3.20. By proposition 2.7, $\Sigma_{\mathrm{eff}(\mathcal{E})} \subseteq \mathrm{Mor}(\mathrm{mod}(\mathcal{E}))$ is a right multiplicative system. Following [43], the localization functor $Q\colon \mathrm{mod}(\mathcal{E}) \to \mathrm{mod}(\mathcal{E})[\Sigma_{\mathrm{eff}(\mathcal{E})}^{-1}]$ endows $\mathrm{mod}(\mathcal{E})[\Sigma_{\mathrm{eff}(\mathcal{E})}]$ with a deflation-exact structure (that is, $\mathrm{mod}(\mathcal{E})[\Sigma_{\mathrm{eff}(\mathcal{E})}^{-1}]$ satisfies axioms **R0**,**R1** and **R2**), moreover, $\mathrm{mod}(\mathcal{E})[\Sigma_{\mathrm{eff}(\mathcal{E})}^{-1}]$ satisfies the universal property of a quotient of deflation-exact categories. As such, $\mathrm{mod}(\mathcal{E})[\Sigma_{\mathrm{eff}(\mathcal{E})}^{-1}] \simeq \mathrm{mod}(\mathcal{E})/\,\mathrm{eff}(\mathcal{E})$.

We are now in a position the consider the following commutative diagram for an exact category $\mathcal{E}$:

$$\begin{tikzcd}
\mathrm{eff}(\mathcal{E}) \arrow[r] \arrow[d, equal] & \mathrm{mod}_{\mathsf{adm}}(\mathcal{E}) \arrow[r] \arrow[d] & \mathrm{mod}_{\mathsf{adm}}(\mathcal{E})/\,\mathrm{eff}(\mathcal{E}) \simeq \mathcal{E} \arrow[d] \\
\mathrm{eff}(\mathcal{E}) \arrow[r] \arrow[d] & \mathrm{mod}(\mathcal{E}) \arrow[r] \arrow[d, "\iota"] & \mathrm{mod}(\mathcal{E})/\,\mathrm{eff}(\mathcal{E}) \arrow[d] \\
\mathrm{Eff}(\mathcal{E}) \arrow[r] & \mathrm{Mod}(\mathcal{E}) \arrow[r, "Q_w"'] & \mathrm{Sh}^{\mathrm{add}}(\mathcal{E}) \simeq \mathrm{Lex}(\mathcal{E})
\end{tikzcd}$$

Here the bottom row is a Serre localization sequence of abelian categories as in [48, Appendix A.2] (see also [39, III.5.b]). The middle row is obtained by Remark 3.20 and the top row is Auslander's formula as in Theorem 3.9.

Remark 3.21. In [63], Rump considers a localization theory of left abelian categories. By [60, Proposition 5], left abelian categories with enough projectives are precisely categories of the form $\mathrm{mod}(\mathcal{A})$ for an additive category $\mathcal{A}$. Any left abelian category has an exact structure given by all kernel-cokernel pairs [63, Proposition 2]. Rump defines a category $Q_l(\mathcal{E})$ which is obtained by localizing the left abelian category $\mathrm{mod}(\mathcal{E})$ at a multiplicative system of morphisms, moveover, $Q_l(\mathcal{E})$ is exact (see [63, Theorem 1]).

There is a natural functor $\theta\colon \mathrm{mod}(\mathcal{E})/\,\mathrm{eff}(\mathcal{E}) \to Q_l(\mathcal{E})$ which is a localization functor by [42, Lemma 5.2]. By [63, Example 3], $\mathrm{eff}(\mathcal{E})$ is a two-sided admissibly percolating subcategory if $\mathcal{E}$ is left quasi-abelian and then θ is an equivalence. Hence, the second row reduces to Rump's localization sequence in many cases. Furthermore, [63, corollary 2] shows that the embedding $\mathcal{E} \to \mathbf{Q}_l(\mathcal{E})$ is an equivalence if and only if $\mathcal{E}$ is abelian. In this case, the top and middle line of this diagram are equivalent.

We conclude this section by recovering $\mathrm{mod}_{\mathsf{adm}}(\mathcal{E})$ directly from the bottom two localization sequences. We need the following lemma.

Lemma 3.22. *Let $E \in \mathrm{eff}(\mathcal{E})$ and let $\Upsilon(\ker(g)) \rightarrowtail \Upsilon(X) \xrightarrow{\Upsilon(g)} \Upsilon(Y) \twoheadrightarrow E$ be a resolution of E.*

1. *There exists an $X' \in \mathcal{E}$ such that $\begin{pmatrix} 0 & g \end{pmatrix}\colon X' \oplus X \twoheadrightarrow Y$ is a deflation.*
2. *If $\mathcal{E}$ is weakly idempotent complete, g is a deflation.*

Proof. As $E \in \mathrm{eff}(\mathcal{E})$, there is a resolution $\Upsilon(\ker(f)) \to \Upsilon(A) \xrightarrow{\Upsilon(f)} \Upsilon(B) \twoheadrightarrow E$ with f a deflation. By the Comparison Theorem, there is a homotopy equivalence between the two resolutions. As the Yoneda embedding is faithful, we obtain a homotopy equivalence between $\ker(g) \to X \xrightarrow{g} Y \to 0$ and $\ker(f) \to A \xrightarrow{f} B \to 0$. By [24, Proposition 10.14], g is a deflation in the weak idempotent completion $\widehat{\mathcal{E}}$ of $\mathcal{E}$. This shows the first part, the second now follows from [44, Proposition 3.4]. □

Proposition 3.23. *The category $\mathrm{mod}_{\mathsf{adm}}(\mathcal{E})$ is obtained as the inverse image $(Q_w \circ \iota)^{-1}(\Upsilon(\mathcal{E}))$.*

Proof. Let $F \in (Q_w \circ \iota)^{-1}(\Upsilon(\mathcal{E}))$. As $F \in \mathrm{mod}(\mathcal{E})$, there is a map $f\colon A \to B$ such that $F \cong \mathrm{coker}(\Upsilon(f))$. As $Q_w \circ \iota(F) \in \Upsilon(\mathcal{E})$, there is a $C \in \mathcal{E}$ such that $Q_w \circ \iota(F) \cong \Upsilon(C)$. As $\mathrm{Eff}(\mathcal{E})$ is a localizing Serre subcategory of $\mathrm{Mod}(\mathcal{E})$ (see Corollary 3.19), the quotient functor Q_w has a right adjoint R. The unit of the adjunction $Q_w \dashv R$ yields a map $u_F\colon \iota(F) \to R\Upsilon(C)$ such that $Q_w(u_F)$ is an isomorphism. It follows that $\ker(u_F), \mathrm{coker}(u_F) \in \mathrm{Eff}(\mathcal{E})$.

Consider the following diagram

$$\begin{tikzcd}
 & \Upsilon(A) \arrow[d, "\Upsilon(f)"] & & \\
 & \Upsilon(B) \arrow[d, two heads] \arrow[dr, dotted, "\Upsilon(g)"] & & \\
\ker(u_F) \arrow[r, tail] & F \arrow[r, "\sim"] & \Upsilon(C) \arrow[r, two heads] & \operatorname{coker}(u_F)
\end{tikzcd}$$

As the Yoneda functor is fully faithful, there is a map $g\colon B \to C$ such that the above diagram commutes. Note that $\operatorname{coker}(u_F) \cong \operatorname{coker}(\Upsilon(g))$. By Lemma 3.22, there is an object $B' \in \mathcal{E}$ such that $\begin{pmatrix} 0 & g \end{pmatrix}\colon B' \oplus B \to C$ is a deflation. Write K for the kernel of $\begin{pmatrix} 0 & g \end{pmatrix}$. Consider the commutative diagram:

$$\begin{tikzcd}
B' \oplus A \arrow[r, equal] \arrow[d, "h"] & B' \oplus A \arrow[r] \arrow[d, "\begin{pmatrix} 1'_B & 0 \\ 0 & f \end{pmatrix}"] & K \arrow[r] \arrow[d] & 0 \arrow[d] \\
K \arrow[r, tail] & B' \oplus B \arrow[r, equal] & B' \oplus B \arrow[r, two heads, "\begin{pmatrix} 0 & g \end{pmatrix}"] & C
\end{tikzcd}$$

Using Proposition 3.3, the above diagram yields that $\ker(u_F) \cong \operatorname{coker}(\Upsilon(h))$. As $\ker(u_F) \in \operatorname{Eff}(\mathcal{E})$, Lemma 3.22 yields that $F \in \operatorname{mod}_{\mathsf{adm}}(\mathcal{E})$. □

4 Auslander correspondence for exact categories

In this section we show that $\operatorname{mod}_{\mathsf{adm}}(-)\colon \mathbf{Ex}_L \to \mathbf{Ex}$ gives an equivalence onto a subcategory $\mathbf{AEx}$ of $\mathbf{Ex}$, which we describe explicitly (see below). In particular, we obtain an extension of Auslander correspondence to exact categories.

4.1 The 2-category $\mathbf{AEx}$

For an exact category $\mathcal{E}$ with enough projective objects, we let $\operatorname{gl.dim}(\mathcal{E})$ denote the global dimension of $\mathcal{E}$.

Definition 4.1. Let $\mathcal{E}$ be an exact category with enough projectives $\mathcal{P} = \operatorname{Proj}(\mathcal{E})$. We say that $\mathcal{E}$ is an *Auslander exact category* if it satisfies the following conditions:

(i) $({}^{\perp}\mathcal{P}, \operatorname{cogen}(\mathcal{P}))$ is a torsion pair in $\mathcal{E}$;

(ii) ${}^{\perp}\mathcal{P}$ satisfies axiom **A2**, i.e. all morphisms $f\colon E \to E'$ with $E' \in {}^{\perp}\mathcal{P}$ are admissible with image in ${}^{\perp}\mathcal{P}$;

(iii) $\operatorname{Ext}^1_{\mathcal{E}}({}^{\perp}\mathcal{P}, \mathcal{P}) = 0$;

(iv) $\operatorname{gl.dim}(\mathcal{E}) \leq 2$.

Definition 4.2. Let $\mathbf{AEx}$ denote the 2-category with objects Auslander exact categories, with 1-morphisms exact functors preserving projective objects, and with 2-morphisms natural transformations.

Corollary 4.3. $\operatorname{mod}_{\mathsf{adm}}(-)\colon \mathbf{Ex}_L \to \mathbf{Ex}$ *factors through* $\mathbf{AEx}$.

Proof. By Lemma 3.2 we know that $\operatorname{gl.dim}(\operatorname{mod}_{\mathsf{adm}}(\mathcal{E})) \leq 2$. The claim now follows from Proposition 3.5 and Proposition 3.14 □

4.2 The equivalence $\mathsf{Ex}_L \simeq \mathsf{AEx}$

In this section we show that $\mathrm{mod}_{\mathsf{adm}}(-)\colon \mathsf{Ex}_L \to \mathsf{AEx}$ is an equivalence of 2-categories.

Proposition 4.4. *Let $\mathcal{E} \in \mathsf{AEx}$ and let $\mathcal{P} = \mathrm{Proj}(\mathcal{E})$. The following hold:*

1. *${}^{\perp}\mathcal{P} \subseteq \mathcal{E}$ is a two-sided admissibly percolating subcategory of $\mathcal{E}$;*
2. *Any weak ${}^{\perp}\mathcal{P}$-isomorphism $P \xrightarrow{\sim} M$ with $P \in \mathcal{P}$ is a split inflation.*
3. *For every $M \in \mathrm{cogen}(\mathcal{P})$ there exists a conflation $M \overset{\sim}{\rightarrowtail} P \twoheadrightarrow T$ with $P \in \mathcal{P}$ and $T \in {}^{\perp}\mathcal{P}$.*

Proof. 1. This follows from Proposition 2.16.

2. If $f\colon P \xrightarrow{\sim} M$ be a weak ${}^{\perp}\mathcal{P}$-isomorphism, then f is admissible and $\ker(f), \mathrm{coker}(f) \in {}^{\perp}\mathcal{P}$. Since $\mathrm{Hom}({}^{\perp}\mathcal{P}, P) = 0$, it follows that $\ker(f) = 0$, and hence f is an inflation. Since $\mathrm{Ext}^1_{\mathcal{E}}(\mathrm{coker}(f), P) = 0$, we get that f must be a split inflation.

3. Let $M \in \mathrm{cogen}(\mathcal{P})$. By definition there is a conflation $M \rightarrowtail Q \twoheadrightarrow N$ with $Q \in \mathcal{P}$. Choose a conflation $\mathfrak{t}N \rightarrowtail N \twoheadrightarrow \mathfrak{f}N$ with $\mathfrak{t}N \in {}^{\perp}\mathcal{P}$ and $\mathfrak{f}N \in \mathrm{cogen}(\mathcal{P})$. Taking the pullback of $Q \twoheadrightarrow N$ along $\mathfrak{t}N \rightarrowtail N$, we obtain the following commutative diagram:

$$\begin{tikzcd}
 & \mathfrak{f}N \arrow[r, equal] & \mathfrak{f}N \\
M \arrow[r, tail] \arrow[d, equal] & Q \arrow[r, two heads] \arrow[u, two heads] & N \arrow[u, two heads] \\
M \arrow[r, tail] & P \arrow[r, two heads] \arrow[u, tail] & \mathfrak{t}N \arrow[u, tail]
\end{tikzcd}$$

Since $\mathfrak{f}N \in \mathrm{cogen}(\mathcal{P})$ and $\mathrm{gl.dim}(\mathcal{E}) \leq 2$, it follows that $\mathfrak{f}N$ have projective dimension ≤ 1. Thus P must be projective. The lower row is therefore the desired conflation. □

Proposition 4.5. *Let $\mathcal{E} \in \mathsf{AEx}$ and let $\mathcal{P} = \mathrm{Proj}(\mathcal{E})$. The composition $\mathcal{P} \overset{\iota}{\hookrightarrow} \mathcal{E} \xrightarrow{Q} \mathcal{E}/{}^{\perp}\mathcal{P}$ is an equivalence of categories.*

Proof. We first show that the composition $Q \circ \iota$ is essentially surjective. Let $M \in \mathrm{Ob}(\mathcal{E}/{}^{\perp}\mathcal{P}) = \mathrm{Ob}(\mathcal{E})$. Choose a conflation $\mathfrak{t}M \rightarrowtail M \overset{\sim}{\twoheadrightarrow} \mathfrak{f}M$ with $\mathfrak{t}M \in {}^{\perp}\mathcal{P}$ and $\mathfrak{f}M \in \mathrm{cogen}(\mathcal{P})$. By Proposition 4.4 . (3) there exists a conflation $\mathfrak{f}M \overset{\sim}{\rightarrowtail} P \twoheadrightarrow T$ with $P \in \mathcal{P}$ and $T \in {}^{\perp}\mathcal{P}$. The composition $M \overset{\sim}{\twoheadrightarrow} \mathfrak{f}M \overset{\sim}{\rightarrowtail} P$ yield that $M \cong \mathfrak{f}M \cong P$ in $\Sigma^{-1}_{{}^{\perp}\mathcal{P}}\mathcal{E}$. Hence $Q \circ \iota$ is essentially surjective.

We now show that $Q \circ \iota$ is faithful. Let $f\colon P_1 \to P_2$ be a map in $\mathcal{P}$ such that $Q(f) = 0$. As ${}^{\perp}\mathcal{P}$ is a two-sided admissibly percolating subcategory of $\mathcal{E}$, the set $\Sigma_{{}^{\perp}\mathcal{P}}$ is a left multiplicative system. It follows that there exists a weak isomorphism $P_2 \xrightarrow{\sim} Y$ such that the composition $P_1 \xrightarrow{f} P_2 \xrightarrow{\sim} Y$ is zero. By Proposition 4.4.(2), $P_2 \xrightarrow{\sim} Y$ is monic. It follows that f is zero.

Lastly, we show that $Q \circ \iota$ is full. As $\Sigma_{{}^{\perp}\mathcal{P}}$ is a left multiplicative system, we can represent a morphism $f \in \mathrm{Hom}_{\mathcal{E}/{}^{\perp}\mathcal{P}}(P_1, P_2)$ as a roof $P_1 \longrightarrow X \xleftarrow{\sim} P_2$. Then $P_2 \xrightarrow{\sim} X$ has a left inverse $h\colon X \xrightarrow{\sim} P_2$ by Proposition 4.4.(2). The composite $P_1 \to X \xrightarrow{h} P_2$ yields the desired morphism $g\colon P_1 \to P_2$ such that $Q \circ \iota(g) = f$. □

Proposition 4.6. *Let $\mathcal{E} \in \mathsf{AEx}$ and let $\mathcal{P} = \mathrm{Proj}(\mathcal{E})$ be endowed with the exact structure coming from the equivalence in Proposition 4.5. The following hold:*

1. *The functor $\mathcal{P} \to \mathcal{E}$ is left exact;*
2. *The induced exact functor $\mathrm{mod}_{\mathsf{adm}}(\mathcal{P}) \to \mathcal{E}$ is an equivalence.*

Proof. 1. Let $P_1 \xrightarrow{f} P_2 \xrightarrow{g} P_3$ be a sequence in $\mathcal{P}$, and assume that $Q(P_1) \xrightarrow{Q(f)} Q(P_2) \xrightarrow{Q(g)} Q(P_3)$ is a conflation in $\mathcal{E}/{}^{\perp}\mathcal{P}$. Since $Q(f)$ and $Q(g)$ are admissible in $\mathcal{E}/{}^{\perp}\mathcal{P}$, it follows that f and g are admissible in $\mathcal{E}$ by Theorem 2.10. Hence, it only remains to show that f is a kernel of g. Since $\mathcal{E}$ has enough projectives, it suffices to check that any morphism $h\colon P \to P_2$ with $P \in \mathcal{P}$ satisfying $g \circ h = 0$ must factor uniquely through f. In other words, it suffices to show that the sequence

$$\mathrm{Hom}_{\mathcal{E}}(P, P_1) \xrightarrow{f\circ -} \mathrm{Hom}_{\mathcal{E}}(P, P_2) \xrightarrow{g\circ -} \mathrm{Hom}_{\mathcal{E}}(P, P_3)$$

is left exact. Now by Proposition 4.5 we have isomorphisms $\mathrm{Hom}_{\mathcal{E}}(P, P_i) \cong \mathrm{Hom}_{\mathcal{E}/{}^{\perp}\mathcal{P}}(Q(P), Q(P_i))$ for $1 \leq i \leq 3$. Hence the sequence above is isomorphic to the sequence

$$\mathrm{Hom}_{\mathcal{E}/{}^{\perp}\mathcal{P}}(Q(P), Q(P_1)) \xrightarrow{Q(f)\circ -} \mathrm{Hom}_{\mathcal{E}/{}^{\perp}\mathcal{P}}(Q(P), Q(P_2)) \xrightarrow{Q(g)\circ -} \mathrm{Hom}_{\mathcal{E}/{}^{\perp}\mathcal{P}}(Q(P), Q(P_3))$$

which is left exact since $Q(P_1) \xrightarrow{Q(f)} Q(P_2) \xrightarrow{Q(g)} Q(P_3)$ is a conflation. This proves the claim.

2. By Theorem 3.7 there exists an exact functor $\mathrm{mod}_{\mathsf{adm}}(\mathcal{P}) \to \mathcal{E}$ extending the inclusion $\mathcal{P} \to \mathcal{E}$. Since both $\mathrm{mod}_{\mathsf{adm}}(\mathcal{P})$ and $\mathcal{E}$ are exact categories with enough projectives, and since the exact functor $\mathrm{mod}_{\mathsf{adm}}(\mathcal{P}) \to \mathcal{E}$ restricts to an equivalence on the projective objects, it must be an equivalence itself. □

Theorem 4.7 (First version of Auslander correspondence)**.** $\mathrm{mod}_{\mathsf{adm}}(-)\colon \mathbf{Ex}_L \to \mathbf{AEx}$ *is an equivalence of* 2*-categories.*

Proof. We define a 2-functor $\mathbf{AEx} \to \mathbf{Ex}_L$ as follows: An object $\mathcal{E} \in \mathbf{AEx}$ is sent to $\mathrm{Proj}(\mathcal{E})$ endowed with the exact structure coming from the equivalence in Proposition 4.5. A 1-morphism $\Psi\colon \mathcal{E} \to \mathcal{E}'$ is sent to the restriction $\Psi|_{\mathrm{Proj}(\mathcal{E})}\colon \mathrm{Proj}(\mathcal{E}) \to \mathrm{Proj}(\mathcal{E}')$ (which can be seen to be left exact by using that the inclusion $\mathrm{Proj}(\mathcal{E}) \to \mathcal{E}$ is left exact). Finally, a natural transformation $\Psi \to \Psi'$ is sent to the restriction $\Psi|_{\mathrm{Proj}(\mathcal{E})} \to \Psi'|_{\mathrm{Proj}(\mathcal{E})}$. It follows from Theorem 3.9 and Proposition 4.6 that this gives a quasi-inverse to $\mathrm{mod}_{\mathsf{adm}}(-)$, which proves the claim. □

Remark 4.8. Note that $\mathcal{E}$ is an abelian category if and only if $\mathrm{mod}_{\mathsf{adm}}(\mathcal{E})$ is an abelian category. Indeed, if $\mathcal{E}$ is abelian, then $\mathrm{mod}_{\mathsf{adm}}(\mathcal{E}) = \mathrm{mod}(\mathcal{E})$ is abelian since $\mathcal{E}$ has weak kernels [37]. Conversely, if $\mathrm{mod}_{\mathsf{adm}}(\mathcal{E})$ is abelian then $\mathrm{eff}(\mathcal{E})$ must be a Serre subcategory of $\mathrm{mod}_{\mathsf{adm}}(\mathcal{E})$, and hence the localization $\mathcal{E} \cong \mathrm{mod}_{\mathsf{adm}}(\mathcal{E})/\mathrm{eff}(\mathcal{E})$ is an abelian category. It follows from this that $\mathrm{mod}_{\mathsf{adm}}(-)\colon \mathbf{Ex}_L \to \mathbf{AEx}$ restricts to an equivalence of 2-categories

$$\mathrm{mod}(-)\colon \mathbf{Ab}_L \to \mathbf{AAb}$$

where $\mathbf{Ab}_L$ and $\mathbf{AAb}$ are the 2-subcategories of $\mathbf{Ex}_L$ and $\mathbf{AEx}$ consisting of the abelian categories. Note that axiom **A2** in the definition of Auslander exact categories reduces to requiring ${}^{\perp}\mathcal{P}$ to be closed under subobjects in this case. Note that Beligiannis obtained a similar characterization in [18, Theorem 6.6],

Using this equivalence, we deduce some further properties of Auslander exact categories.

Corollary 4.9. *Let* $\mathcal{E} \in \mathbf{AEx}$ *and* $\mathcal{P} = \mathrm{Proj}(\mathcal{E})$*. The following hold:*

1. *The subcategory* $\mathrm{cogen}(\mathcal{P})$ *consists precisely of the objects of projective dimension at most one;*
2. *The inclusion functor* $\mathcal{P} \to \mathcal{E}$ *has a left adjoint.*

Proof. This holds since the analogous properties are true for $\mathrm{mod}_{\mathsf{adm}}(\mathcal{E}')$ for an exact category $\mathcal{E}'$ by Corollary 3.8 and Proposition 3.14. (2). □

4.3 Injectives and dominant dimension

In this section we investigate how the property of an exact category $\mathcal{E}$ having enough injectives is reflected in $\mathrm{mod}_{\mathsf{adm}}(\mathcal{E})$. In particular, we obtain a more familiar version of Auslander correspondence.

Lemma 4.10. *Let* $\mathcal{E}$ *be an exact category. An object* I *is injective in* $\mathcal{E}$ *if and only if* $\Upsilon(I)$ *is injective in* $\mathrm{mod}_{\mathsf{adm}}\,\mathcal{E}$*.*

Proof. Assume I is injective in $\mathcal{E}$. Let $F \in \operatorname{mod}_{\mathsf{adm}}(\mathcal{E})$, and choose an admissible morphism $f\colon X \to Y$ such that

$$0 \to \Upsilon(\ker f) \to \Upsilon(X) \xrightarrow{\Upsilon(f)} \Upsilon(Y) \to F \to 0$$

is exact. Applying $\operatorname{Hom}_{\operatorname{mod}_{\mathsf{adm}}(\mathcal{E})}(-, \Upsilon(I))$ and using that the Yoneda embedding is fully faithful, we get the sequence

$$0 \to \operatorname{Hom}_{\operatorname{mod}_{\mathsf{adm}}(\mathcal{E})}(F, \Upsilon(I)) \to \operatorname{Hom}_{\mathcal{E}}(Y, I) \to \operatorname{Hom}_{\mathcal{E}}(X, I) \to \operatorname{Hom}_{\mathcal{E}}(\ker f, I) \to 0.$$

It is exact since I is injective in $\mathcal{E}$. Hence $\operatorname{Ext}^i_{\operatorname{mod}_{\mathsf{adm}}(\mathcal{E})}(F, \Upsilon(I)) = 0$ for all $i > 0$ and all $F \in \operatorname{mod}_{\mathsf{adm}}(\mathcal{E})$, so $\Upsilon(I)$ is injective in $\operatorname{mod}_{\mathsf{adm}}(\mathcal{E})$.

Conversely, assume $\Upsilon(I)$ is injective in $\operatorname{mod}_{\mathsf{adm}}(\mathcal{E})$, and let $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ be a conflation in $\mathcal{E}$. Then the sequence

$$0 \to \Upsilon(X) \xrightarrow{\Upsilon(f)} \Upsilon(Y) \xrightarrow{\Upsilon(g)} \Upsilon(Z) \to \operatorname{coker}(\Upsilon(g)) \to 0$$

is exact in $\operatorname{mod}_{\mathsf{adm}}(\mathcal{E})$. Since $\operatorname{coker}(\Upsilon(g)) \in \operatorname{eff}(\mathcal{E})$, we get that $\operatorname{Hom}_{\operatorname{mod}_{\mathsf{adm}}(\mathcal{E})}(\operatorname{coker}(\Upsilon(g)), \Upsilon(I)) = 0$. Therefore, applying $\operatorname{Hom}_{\operatorname{mod}_{\mathsf{adm}}(\mathcal{E})}(-, \Upsilon(I))$ to the sequence above and using that

$$\operatorname{Ext}^i_{\operatorname{mod}_{\mathsf{adm}}(\mathcal{E})}(\operatorname{coker}(\Upsilon(g)), \Upsilon(I)) = 0$$

for all $i > 0$ since $\Upsilon(I)$ is injective, we get an exact sequence

$$0 \to \operatorname{Hom}_{\mathcal{E}}(Z, I) \xrightarrow{-\circ g} \operatorname{Hom}_{\mathcal{E}}(Y, I) \xrightarrow{-\circ f} \operatorname{Hom}_{\mathcal{E}}(X, I) \to 0.$$

This shows that I is injective in $\mathcal{E}$. □

Definition 4.11. Let $\mathcal{E}$ be an exact category with enough projectives. The *dominant dimension* of $\mathcal{E}$, denoted $\operatorname{dom.dim}(\mathcal{E})$, is the largest integer n such that for any projective object $P \in \mathcal{E}$ there exists an exact sequence

$$0 \to P \to I_1 \to \cdots \to I_n \to C \to 0$$

with I_k being projective and injective for $1 \leq k \leq n$.

Lemma 4.12. *Let $\mathcal{E}$ be an exact category. The following are equivalent:*

1. *$\mathcal{E}$ has enough injectives;*
2. $\operatorname{dom.dim}(\operatorname{mod}_{\mathsf{adm}}(\mathcal{E})) \geq 2$*;*
3. $\operatorname{dom.dim}(\operatorname{mod}_{\mathsf{adm}}(\mathcal{E})) \geq 1$.

Proof. Assume $\mathcal{E}$ has enough injectives. By Proposition 3.14.(4), a projective object in $\operatorname{mod}_{\mathsf{adm}}(\mathcal{E})$ is of the form $\Upsilon(X)$. As $\mathcal{E}$ has enough injectives, there is an exact sequence

$$0 \to X \to I_1 \xrightarrow{f} I_2 \to \operatorname{coker}(f) \to 0$$

with I_1 and I_2 injective. Applying the Yoneda embedding we get a left exact sequence

$$0 \to \Upsilon(X) \to \Upsilon(I_1) \to \Upsilon(I_2)$$

in $\operatorname{mod}_{\mathsf{adm}}(\mathcal{E})$, and by Lemma 4.10 we know that $\Upsilon(I_1)$ and $\Upsilon(I_2)$ are projective and injective in $\operatorname{mod}_{\mathsf{adm}}(\mathcal{E})$. This shows that $\operatorname{dom.dim}(\operatorname{mod}_{\mathsf{adm}}(\mathcal{E})) \geq 2$. Since the implication (2) $\Rightarrow$ (3) is clear, it only remains to show (3) $\Rightarrow$ (1). Let X be an arbitrary object in $\mathcal{E}$. By assumption there exists an inflation $\Upsilon(X) \to \Upsilon(I)$ where $\Upsilon(I)$ is injective in $\operatorname{mod}_{\mathsf{adm}}(\mathcal{E})$, and thus I is injective in $\mathcal{E}$ by Lemma 4.10. Since the localization functor $Q\colon \operatorname{mod}_{\mathsf{adm}}(\mathcal{E}) \to \operatorname{mod}_{\mathsf{adm}}(\mathcal{E})/\operatorname{eff}(\mathcal{E}) \simeq \mathcal{E}$ is exact, it follows that the induced morphism $X \to I$ is an inflation in $\mathcal{E}$. This proves the claim. □

Our next goal is to show that $\operatorname{dom.dim}(\mathcal{E}) \geq 2$ implies some of the other criteria for categories in **PEx**.

Lemma 4.13. *Let $\mathcal{E}$ be an exact category with enough projectives $\mathcal{P} = \mathrm{Proj}(\mathcal{E})$. Assume* $\mathrm{dom.\,dim}(\mathcal{E}) \geq 2$. *Then* $\mathrm{Ext}^1_{\mathcal{E}}({}^{\perp}\mathcal{P}, \mathcal{P}) = 0$.

Proof. Let $X \in {}^{\perp}\mathcal{P}$ and $P \in \mathcal{P}$ be arbitrary. Choose a conflation $0 \to P \to I_0 \to C \to 0$ and $0 \to C \to I_1 \to C' \to 0$ with I_0 and I_1 projective and injective in $\mathcal{E}$. Applying $\mathrm{Hom}_{\mathcal{E}}(X, -)$ to the second sequence, we get a monomorphism $\mathrm{Hom}_{\mathcal{E}}(X, C) \to \mathrm{Hom}_{\mathcal{E}}(X, I_1)$ and since $\mathrm{Hom}_{\mathcal{E}}(X, I_1) = 0$ by assumption on X, it follows that $\mathrm{Hom}_{\mathcal{E}}(X, C) = 0$. Applying $\mathrm{Hom}_{\mathcal{E}}(X, -)$ to the first sequence we get an exact sequence $\mathrm{Hom}_{\mathcal{E}}(X, C) \to \mathrm{Ext}^1_{\mathcal{E}}(X, P) \to \mathrm{Ext}^1_{\mathcal{E}}(X, I)$. Now $\mathrm{Hom}_{\mathcal{E}}(X, C) = 0$ by the discussion above and $\mathrm{Ext}^1_{\mathcal{E}}(X, I) = 0$ since I is injective. Hence $\mathrm{Ext}^1_{\mathcal{E}}(X, P) = 0$, which proves the claim. □

Lemma 4.14. *Let $\mathcal{E}$ be an exact category with enough projectives $\mathcal{P} = \mathrm{Proj}(\mathcal{E})$. Assume* $\mathrm{dom.\,dim}(\mathcal{E}) \geq 1$. *Then ${}^{\perp}\mathcal{P}$ is closed under admissible subobjects.*

Proof. Let $E \rightarrowtail X$ be an inflation with $X \in {}^{\perp}\mathcal{P}$. We want to show that $E \in {}^{\perp}\mathcal{P}$. Since $\mathrm{dom.\,dim}(\mathcal{E}) \geq 1$, it suffices to show that $\mathrm{Hom}_{\mathcal{E}}(E, P) = 0$ for all P which are projective and injective. But since P is injective, the induced map $\mathrm{Hom}_{\mathcal{E}}(X, P) \to \mathrm{Hom}_{\mathcal{E}}(E, P)$ is an epimorphism. Since $\mathrm{Hom}_{\mathcal{E}}(X, P) = 0$ by assumption, it follows that $\mathrm{Hom}_{\mathcal{E}}(E, P) = 0$. □

Definition 4.15. Let $\mathcal{Y}$ be a subcategory of an exact category $\mathcal{E}$. A morphism $f\colon X \to Y$ in $\mathcal{E}$ with $Y \in \mathcal{Y}$ is called an *(admissible) left-$\mathcal{Y}$ approximation* if f is an admissible morphism and any morphism $X \to Y'$ with $Y' \in \mathcal{Y}$ factors through f. The subcategory $\mathcal{Y}$ is called *(admissibly) covariantly finite* if for all objects X in $\mathcal{E}$ there exists an admissible left $\mathcal{Y}$-approximation $X \to Y$.

Lemma 4.16. *Let $\mathcal{E}$ be an exact category with enough projectives $\mathcal{P} = \mathrm{Proj}(\mathcal{E})$. The following hold:*

1. *If $\mathcal{E}$ is an Auslander exact category, then $\mathcal{P}$ is admissibly covariantly finite in $\mathcal{E}$;*
2. *If $\mathcal{P}$ is admissibly covariantly finite in $\mathcal{E}$ and* $\mathrm{dom.\,dim}(\mathcal{E}) \geq 1$, *then* $({}^{\perp}\mathcal{P}, \mathrm{cogen}(\mathcal{P}))$ *is a torsion pair.*

Proof. 1. For $E \in \mathcal{E}$, choose a conflation $\mathfrak{t}E \rightarrowtail \mathcal{E} \twoheadrightarrow \mathfrak{f}E$ with $\mathfrak{t}E \in {}^{\perp}\mathcal{P}$ and $\mathfrak{f}E \in \mathrm{cogen}(\mathcal{P})$. By Proposition 4.4 . (3) we can choose a conflation $\mathfrak{f}E \rightarrowtail P \twoheadrightarrow T$ with $P \in \mathcal{P}$ and $T \in {}^{\perp}\mathcal{P}$. Since $\mathrm{Hom}_{\mathcal{E}}(\mathfrak{t}E, \mathcal{P}) = 0$, any morphism $E \to Q$ with $Q \in \mathcal{P}$ must factor through $\mathfrak{f}E$. Also, since $\mathrm{Ext}^1_{\mathcal{E}}(T, \mathcal{P}) = 0$, any morphism $\mathfrak{f}E \to Q$ with $Q \in \mathcal{P}$ must factor through P. This shows that $E \twoheadrightarrow \mathfrak{f}E \rightarrowtail P$ is the required left $\mathcal{P}$-approximation.

2. Let X be an arbitrary object in $\mathcal{E}$, and choose an admissible left $\mathcal{P}$-approximation $f\colon X \to P$. Consider the conflation $\ker(f) \rightarrowtail X \twoheadrightarrow \mathrm{im}(f)$. Let I be a projective and injective object in $\mathcal{E}$. Applying $\mathrm{Hom}_{\mathcal{E}}(-, I)$ to the conflation above, we get a short exact sequence of abelian groups

$$\mathrm{Hom}_{\mathcal{E}}(\mathrm{im}(f), I) \rightarrowtail \mathrm{Hom}_{\mathcal{E}}(X, I) \twoheadrightarrow \mathrm{Hom}_{\mathcal{E}}(\ker(f), I).$$

Since f is a left $\mathcal{P}$-approximation and I is projective, it follows that $\mathrm{Hom}_{\mathcal{E}}(\mathrm{im}(f), I) \rightarrowtail \mathrm{Hom}_{\mathcal{E}}(X, I)$ is surjective, so $\mathrm{Hom}_{\mathcal{E}}(\ker(f), I) = 0$. Since $\mathrm{dom.\,dim}(\mathcal{E}) \geq 1$, any projective object is a subobject of a projective and injective object. Hence $\ker(f) \in {}^{\perp}\mathcal{P}$. Since $\mathrm{im}(f) \in \mathrm{cogen}(\mathcal{P})$, this proves the claim. □

We finish this section by showing that if $\mathcal{E}$ has enough injectives, then $\mathrm{mod}_{\mathsf{adm}}(\mathcal{E})$ has enough injectives. In the following we let $\mathcal{I}$ denote the subcategory of injective objects in $\mathcal{E}$.

Remark 4.17. For all $n \geq 1$, $\mathrm{cogen}_n(\mathcal{I}) \subseteq \mathrm{cogen}_{n-1}(\mathcal{I}) \subseteq \cdots \subseteq \mathrm{cogen}_1(\mathcal{I})$. If $\mathcal{E}$ has enough injectives, then $\mathcal{E} = \mathrm{cogen}_1(\mathcal{I}) = \mathrm{cogen}_2(\mathcal{I}) = \cdots$.

Lemma 4.18. *Let $\mathcal{E}$ be an exact category, let $n \geq 1$ be an integer, and let $X \rightarrowtail Y \twoheadrightarrow Z$ be a conflation in $\mathcal{E}$. The following hold:*

1. *If $X \in \mathrm{cogen}_n(\mathcal{I})$ and $Z \in \mathrm{cogen}_n(\mathcal{I})$, then $Y \in \mathrm{cogen}_n(\mathcal{I})$;*
2. *If $X \in \mathrm{cogen}_n(\mathcal{I})$ and $Y \in \mathrm{cogen}_{n-1}(\mathcal{I})$, then $Z \in \mathrm{cogen}_{n-1}(\mathcal{I})$.*

Proof. 1. Choose exact sequences $0 \to X \to I_1^X \to \cdots \to I_n^X \to C^X \to 0$ and $0 \to Z \to I_1^Z \to \cdots \to I_n^Z \to C^Z \to 0$ with each I_i^X, I_j^Z injective. By the Horseshoe Lemma, there is an exact sequence

$$0 \to Y \to I_1^X \oplus I_Z^1 \to \cdots \to I_n^X \oplus I_n^Z \to C^X \oplus C_Z \to 0.$$

This shows that $Y \in \operatorname{cogen}_n(\mathcal{I})$.

2. As $X \in \operatorname{cogen}_n(\mathcal{I})$ and $n \geq 1$, there is a conflation $X \overset{i}{\rightarrowtail} I \twoheadrightarrow C$ with $I \in \operatorname{Inj}(\mathcal{E})$ and $C \in \operatorname{cogen}_{n-1}(\mathcal{I})$.. Taking the pushout of i along $X \rightarrowtail Y$ we obtain the commutative diagram

$$\begin{tikzcd}
X \arrow[r, tail] \arrow[d, tail, "i"] & Y \arrow[r, two heads] \arrow[d, tail] & Z \arrow[d, equal] \\
I \arrow[r, tail] \arrow[d, two heads] & U \arrow[r, two heads] \arrow[d, two heads] & Z \\
C \arrow[r, equal] & C &
\end{tikzcd}$$

where the rows and columns are conflations. As I is injective, the middle row is a split conflation and thus $U \cong I \oplus Z$. Applying the first part of this lemma to the conflation $Y \rightarrowtail I \oplus Z \twoheadrightarrow C$ yields that $I \oplus Z \in \operatorname{cogen}_{n-1}(\mathcal{I})$. Hence there is an exact sequence

$$0 \to I \oplus Z \to I_1 \to \cdots \to I_{n-1} \to D \to 0$$

with each $I_j \in \operatorname{Inj}(\mathcal{E})$. The composition $I \overset{\binom{1}{0}}{\rightarrowtail} I \oplus Z \rightarrowtail I_1$ is a split inflation as I is injective. It follows that $I_1 \cong I \oplus I_1'$ and $I_1' \in \operatorname{Inj}(\mathcal{E})$ as injectives are closed under direct summands. The exact sequence above becomes

$$0 \to I \oplus Z \xrightarrow{\left(\begin{smallmatrix} 1 & 0 \\ 0 & f \end{smallmatrix}\right)} I \oplus I_1' \to \cdots \to I_{n-1} \to D \to 0$$

for some map $f \colon Z \to I_1'$. Taking the pushout of the canonical projection $I \oplus Z \twoheadrightarrow Z$ along $I \oplus Z \overset{\left(\begin{smallmatrix} 1 & 0 \\ 0 & f \end{smallmatrix}\right)}{\rightarrowtail} I \oplus I_1'$, one sees that $f \colon Z \rightarrowtail I_1'$ is an inflation. Hence the sequence

$$0 \to Z \xrightarrow{f} I_1' \to I_2 \to \cdots \to I_{n-1} \to D \to 0$$

is exact, which shows that $Z \in \operatorname{cogen}_{n-1}(\mathcal{I})$. □

Proposition 4.19. *Let $\mathcal{E}$ be an exact category with enough projectives and assume*

$$\operatorname{dom.dim}(\mathcal{E}) \geq 2 \geq \operatorname{gl.dim}(\mathcal{E}).$$

Then $\mathcal{E}$ has enough injectives.

Proof. For any projective $P \in \mathcal{E}$, there exists an exact sequence

$$0 \to P \to I_1 \to I_2 \to C \to 0$$

where I_1 and I_2 are projective and injective. Since $\operatorname{gl.dim}(\mathcal{E}) \leq 2$ and C is a cokernel of an admissible morphism between injective objects, it follows that C is injective. Hence $P \in \operatorname{cogen}_3(\mathcal{I})$.

Now let $X \in \mathcal{E}$ be an arbitrary object, and choose an exact sequence $0 \to P_2 \xrightarrow{f} P_1 \to P_0 \to X \to 0$ with P_0, P_1, P_2 projective in $\mathcal{E}$. Applying Lemma 4.18 to the conflation $P_2 \xrightarrow{f} P_1 \to \operatorname{coker}(f)$ and using that $P_2, P_1 \in \operatorname{cogen}_3(\mathcal{I})$, it follows that $\operatorname{coker}(f) \in \operatorname{cogen}_2(\mathcal{I})$. Similarly, applying Lemma 4.18 to the conflation $\operatorname{coker}(f) \to P_0 \to X$ we get that $X \in \operatorname{cogen}_1(\mathcal{I})$. This proves the claim. □

Definition 4.20. Let IEx_L be the 2-subcategory of Ex_L consisting of the exact categories with enough injectives.

Combining the results of this section, we obtain the following theorem.

Theorem 4.21 (Second version of Auslander correspondence)**.** *Let $\mathcal{E}$ be an exact category with enough projectives $\mathcal{P} = \mathrm{Proj}(\mathcal{E})$ and satisfying the following conditions:*

(a) $\mathrm{dom.\,dim}(\mathcal{E}) \geq 2 \geq \mathrm{gl.\,dim}(\mathcal{E})$*;*

(b) Any morphism $X \to E$ with $E \in {}^{\perp}\mathcal{P}$ is admissible;

(c) $\mathcal{P}$ is admissibly covariantly finite.

The following hold:

1. *$\mathcal{E}$ is an Auslander exact category with enough injectives;*
2. *The equivalence of 2-categories $\mathrm{mod}_{\mathsf{adm}}(-)\colon \mathbf{Ex}_L \to \mathbf{AEx}$ restricts to an equivalence from $\mathbf{IEx}_L$ onto the subcategory of $\mathbf{AEx}$ consisting of the exact categories satisfying (a), (b) and (c) above.*

Proof. 1. The category $\mathcal{E}$ has enough injectives by Proposition 4.19, and is an Auslander exact category by Lemma 4.13, Lemma 4.14 and Lemma 4.16. (2).

2. We only need to verify that $\mathcal{E}$ has enough injectives if and only if $\mathrm{mod}_{\mathsf{adm}}(\mathcal{E})$ satisfies (a), (b) and (c) above. The "only if" direction follows from Lemma 4.12, and the "if" direction follows from Lemma 4.12 and Lemma 4.16. (1). □

Remark 4.22. Note that the abelian categories satisfying (a) in Theorem 4.21 are precisely the free abelian categories, see [18, Theorem 6.6]

Remark 4.23. We explain how to recover the classical Auslander correspondence. Let k be a field. Assume Λ is a representation-finite finite-dimensional algebra, and choose an additive generator M of $\mathrm{mod}\,\Lambda$. We then have an equivalences of categories

$$\mathrm{mod}\,\Gamma \cong \mathrm{mod}(\mathrm{mod}(\Lambda))$$

where $\Gamma = \mathrm{End}_{\Lambda}(M)$. Since $\mathrm{mod}(\Lambda)$ is an abelian category with enough injectives, we get that

$$\mathrm{dom.\,dim}\,\Gamma \geq 2 \geq \mathrm{gl.\,dim}\,\Gamma$$

by Theorem 4.21. This gives one direction of the classical Auslander correspondence.

Conversely, assume Γ is a finite-dimensional algebra satisfying $\mathrm{dom.\,dim}\,\Gamma \geq 2 \geq \mathrm{gl.\,dim}\,\Gamma$. Since Γ is finite-dimensional, the subcategory $\mathrm{proj}\,\Gamma$ of projective objects in $\mathrm{mod}(\Gamma)$ is a covariantly finite subcategory. Hence, $\mathrm{mod}(\Gamma)$ satisfies (a), (b) and (c) in Theorem 4.21, and combining this with Remark 4.8 we therefore get that $\mathrm{proj}\,\Gamma$ is an abelian category with enough injectives. Hence, $(\mathrm{proj}\,\Gamma)^{\mathrm{op}} \cong \mathrm{mod}(\Lambda')$ for some finite-dimensional algebra Λ'. Composing with the duality

$$\mathrm{Hom}_k(-,k)\colon\ \mathrm{mod}({\Lambda'}^{\mathrm{op}}) \xrightarrow{\cong} (\mathrm{mod}(\Lambda'))^{\mathrm{op}}$$

we get that $\mathrm{proj}\,\Gamma \cong \mathrm{mod}({\Lambda'}^{\mathrm{op}})$. This recovers the other direction of the classical Auslander correspondence.

4.4 Projective objects

In this section we investigate how the property of an exact category $\mathcal{E}$ having enough projectives is reflected in $\mathrm{mod}_{\mathsf{adm}}(\mathcal{E})$.

Lemma 4.24. *An object P of an exact category $\mathcal{E}$ is projective if and only if $\Upsilon(P) \in {}^{\perp}\,\mathrm{eff}(\mathcal{E})$.*

Proof. Let $F \in \mathrm{mod}_{\mathsf{adm}}(\mathcal{E})$, and consider the projective presentation

$$\Upsilon(X) \xrightarrow{\Upsilon(f)} \Upsilon(Y) \to F \to 0$$

where $f\colon X \to Y$ is admissible. Applying $\mathrm{Hom}_{\mathrm{mod}_{\mathsf{adm}}(\mathcal{E})}(\Upsilon(P), -)$ gives an exact sequence

$$\mathrm{Hom}_{\mathcal{E}}(P, X) \xrightarrow{f\circ -} \mathrm{Hom}_{\mathcal{E}}(P, Y) \to \mathrm{Hom}_{\mathrm{mod}_{\mathsf{adm}}(\mathcal{E})}(\Upsilon(P), F) \to 0$$

of abelian groups. Now by definition P is projective if and only if for any deflation $f\colon X \twoheadrightarrow Y$ the induced map

$$\mathrm{Hom}_{\mathcal{E}}(P, X) \xrightarrow{f\circ -} \mathrm{Hom}_{\mathcal{E}}(P, Y)$$

is an epimorphism. Hence, by the exact sequence above we get that P is projective if and only $\mathrm{Hom}_{\mathrm{mod}_{\mathsf{adm}}(\mathcal{E})}(\Upsilon(P), F) = 0$ for any $F \in \mathrm{eff}(\mathcal{E})$. This proves the claim. □

Proposition 4.25. *Let $\mathcal{E}$ be an exact category and write $\mathcal{Q}$ for* $\mathrm{Proj}(\mathrm{mod}_{\mathsf{adm}}(\mathcal{E}))$. *The following are equivalent:*

1. *The category $\mathcal{E}$ has enough projectives;*
2. $(\mathrm{gen}(\mathcal{Q} \cap {}^{\perp}\mathrm{eff}(\mathcal{E})), \mathrm{eff}(\mathcal{E}))$ *is a torsion pair in* $\mathrm{mod}_{\mathsf{adm}}(\mathcal{E})$*;*
3. *For all $P \in \mathcal{Q}$, there exists an exact sequence*

$$0 \to F' \to P \to F'' \to 0$$

with $F' \in \mathrm{gen}(\mathcal{Q} \cap {}^{\perp}\mathrm{eff}(\mathcal{E}))$ and $F'' \in \mathrm{eff}(\mathcal{E})$.

Proof. Assume (1). We first show that $\mathrm{Hom}(\mathrm{gen}(\mathcal{Q} \cap {}^{\perp}\mathrm{eff}(\mathcal{E})), \mathrm{eff}(\mathcal{E})) = 0$. Let $g\colon E \to F$ be a map with $E \in \mathrm{gen}(\mathcal{Q} \cap {}^{\perp}\mathrm{eff}(\mathcal{E}))$ and $F \in \mathrm{eff}(\mathcal{E})$. By definition, there is an object $E' \in \mathcal{Q} \cap {}^{\perp}\mathrm{eff}(\mathcal{E})$ and a deflation $E' \twoheadrightarrow E$. By Proposition 3.14. (4) the functor $\Upsilon\colon \mathcal{E} \to \mathcal{Q}$ is an equivalence and thus there is an object $P \in \mathcal{E}$ such that $\Upsilon(P) \cong E'$. By Lemma 4.24, $P \in \mathrm{Proj}(\mathcal{E})$. Let $f\colon X \twoheadrightarrow Y$ be a deflation in $\mathcal{E}$ such that $\mathrm{coker}(\Upsilon(f)) \cong F$. Since $\Upsilon(P)$ is projective and Υ is fully faithful, there exists a map $h\colon P \to Y$ such that the composite $\Upsilon(P) \twoheadrightarrow E \xrightarrow{g} F$ is equal to $\Upsilon(P) \xrightarrow{\Upsilon(h)} \Upsilon(Y) \twoheadrightarrow F$. Now since f is a deflation and P is projective, h must factor through f, and hence the composite above must be 0. As $\Upsilon(P) \twoheadrightarrow E$ is surjective, g must be zero. We conclude that $\mathrm{Hom}(\mathrm{gen}(\mathcal{Q} \cap {}^{\perp}\mathrm{eff}(\mathcal{E})), \mathrm{eff}(\mathcal{E})) = 0$.

Let $F \in \mathrm{mod}_{\mathsf{adm}}(\mathcal{E})$ and let $F \cong \mathrm{coker}(\Upsilon(f))$ for some admissible morphism $f\colon X \to Y$ in $\mathcal{E}$. As $\mathcal{E}$ has enough projectives, there is a deflation $P \overset{p}{\twoheadrightarrow} Y$ with $P \in \mathrm{Proj}(\mathcal{E})$. The pullback Z of f along p yields an admissible morphism $g\colon Z \to P$. Note that this pullback square induces a conflation $Z \rightarrowtail X \oplus P \overset{(f\ p)}{\twoheadrightarrow} Y$ by (the dual of) [24, Proposition 2.12]. Let $F' = \mathrm{coker}(\Upsilon(g))$ and let $F'' = \mathrm{coker}(\Upsilon(f\ p)) \in \mathrm{eff}(\mathcal{E})$. Considering the two rightmost squares in diagram (1) in Proposition 3.3, we see that we have a conflation $F' \rightarrowtail F \twoheadrightarrow F''$. It remains to show that $F' \in \mathrm{gen}(\mathcal{Q} \cap {}^{\perp}\mathrm{eff}(\mathcal{E}))$. By construction there is a deflation $\Upsilon(P) \twoheadrightarrow F'$ and as $P \in \mathrm{Proj}(\mathcal{E})$, Lemma 4.24 yields that $\Upsilon(P) \in {}^{\perp}\mathrm{eff}(\mathcal{E})$, thus $F' \in \mathrm{gen}(\mathcal{Q} \cap {}^{\perp}\mathrm{eff}(\mathcal{E}))$. This shows the implication (1) ⇒ (2).

The implication (2) ⇒ (3) is trivial. Now assume (3). Let $A \in \mathcal{E}$. By assumption there is a short exact sequence

$$F' \rightarrowtail \Upsilon(A) \twoheadrightarrow F''$$

with $F' \in \mathrm{gen}(\mathcal{Q} \cap {}^{\perp}\mathrm{eff}(\mathcal{E}))$ and $F'' \in \mathrm{eff}(\mathcal{E})$. By definition there is a deflation $\Upsilon(P) \twoheadrightarrow F'$ with $P \in \mathrm{Proj}(\mathcal{E})$. Letting K be the kernel of the map $\Upsilon(P) \twoheadrightarrow F' \rightarrowtail \Upsilon(A)$, we get an exact sequence

$$0 \to K \to \Upsilon(P) \to \Upsilon(A) \to F'' \to 0$$

in $\mathrm{mod}_{\mathsf{adm}}(\mathcal{E})$. Applying the functor $L\colon \mathrm{mod}_{\mathsf{adm}}(\mathcal{E}) \to \mathcal{E}$ in Corollary 3.8 and using that $L(F'') = 0$ since $F'' \in \mathrm{eff}(\mathcal{E})$, we get the conflation $L(K) \rightarrowtail P \twoheadrightarrow A$ in $\mathcal{E}$. Hence $\mathcal{E}$ has enough projectives. □

4.5 Gorenstein projectives and Cohen-Macaulay modules

If $\mathcal{E}'$ is an extension closed subcategory of an exact category $\mathcal{E}$, then $\mathcal{E}'$ inherits an exact structure from $\mathcal{E}$. In this case, since the inclusion $\mathcal{E}' \to \mathcal{E}$ is fully faithful and exact, applying the 2-functor $\mathrm{mod}_{\mathsf{adm}}(-)$ we get a fully faithful exact functor

$$\mathrm{mod}_{\mathsf{adm}}(\mathcal{E}') \to \mathrm{mod}_{\mathsf{adm}}(\mathcal{E}).$$

Hence $\mathrm{mod}_{\mathsf{adm}}(\mathcal{E}')$ is equivalent to a full subcategory of $\mathrm{mod}_{\mathsf{adm}}(\mathcal{E})$. Our goal in this section is to characterize this subcategory in certain examples. In the following $Q\colon \mathrm{mod}_{\mathsf{adm}}(\mathcal{E}) \to \mathcal{E}$ denotes the localization functor, $\Upsilon(\mathcal{E}') \subseteq \mathrm{mod}_{\mathsf{adm}}(\mathcal{E})$ denotes the subcategory of objects of the form $\Upsilon(E)$ with $E \in \mathcal{E}'$, and $Q^{-1}(\mathcal{E}') \subseteq \mathrm{mod}_{\mathsf{adm}}(\mathcal{E})$ denotes the subcategory of objects F satisfying $Q(F) \in \mathcal{E}'$.

Lemma 4.26. *Let $\mathcal{E}$ be an exact category, and let $\mathcal{E}'$ be a subcategory of $\mathcal{E}$ closed under extensions and kernels of deflations. Then $\mathrm{mod}_{\mathsf{adm}}(\mathcal{E}')$ is equivalent to the subcategory $Q^{-1}(\mathcal{E}') \cap \mathrm{gen}_2(\Upsilon(\mathcal{E}'))$ of $\mathrm{mod}_{\mathsf{adm}}(\mathcal{E})$.*

Proof. An object $F \in \mathrm{mod}_{\mathsf{adm}}(\mathcal{E})$ is in the essential image of the functor $\mathrm{mod}_{\mathsf{adm}}(\mathcal{E}') \to \mathrm{mod}_{\mathsf{adm}}(\mathcal{E})$ if and only if there exists a projective presentation

$$\Upsilon(X) \xrightarrow{\Upsilon(f)} \Upsilon(Y) \to F \to 0 \tag{2}$$

where $X, Y \in \mathcal{E}'$ and f is admissible in $\mathcal{E}'$. Since $Q(F) = \mathrm{coker}(f)$, it is clear that the essential image is contained in $Q^{-1}(\mathcal{E}') \cap \mathrm{gen}_2(\Upsilon(\mathcal{E}'))$. Conversely, if F is in $Q^{-1}(\mathcal{E}') \cap \mathrm{gen}_2(\Upsilon(\mathcal{E}'))$, then there exists a projective presentation (2) where $X, Y \in \mathcal{E}'$. Since Q preserves admissible morphisms, it follows that f is admissible and $\mathrm{coker}(f) \in \mathcal{E}'$. But then f must be admissible in $\mathcal{E}'$, since $\mathcal{E}'$ is closed under kernels of deflations. This proves the claim. □

We get the following corollary for subcategories of abelian categories with enough projectives.

Proposition 4.27. *Let $\mathcal{P}$ be an additive category, and assume $\mathrm{mod}(\mathcal{P})$ is an abelian category. Let $\mathcal{X}$ be a subcategory of $\mathrm{mod}(\mathcal{P})$ closed under extensions, kernels of deflations, and which contain $\Upsilon(\mathcal{P})$. Then*

$$\mathrm{mod}_{\mathsf{adm}}(\mathcal{X}) = \{F \in \mathrm{mod}(\mathcal{X}) \mid F|_{\mathcal{P}} \in \mathcal{X}\}.$$

Proof. Note that the localization $Q\colon \mathrm{mod}(\mathrm{mod}(\mathcal{P})) \to \mathrm{mod}(\mathcal{P})$ is given by the restriction functor $Q(F) = F|_{\mathcal{P}}$. The claim follows therefore from Lemma 4.26 and the fact that $\mathrm{mod}(\mathcal{X})$ can be identified with the subcategory $\mathrm{gen}_2(\Upsilon(\mathcal{X}))$ of $\mathrm{mod}(\mathcal{A})$. □

In particular, Proposition 4.27 holds in the following cases:

- $\mathcal{X}$ is the subcategory $\mathrm{GP}(\mathcal{P})$ of Gorenstein-projective objects in $\mathrm{mod}(\mathcal{P})$.
- $\mathcal{X}$ is the subcategory of objects of projective dimension $\leq i$ for some integer i.
- $\mathcal{X}$ is the subcategory of objects of finite projective dimension.
- $\mathcal{X} = {}^{>0}\!\perp U = \{M \in \mathrm{mod}(\mathcal{P}) \mid \mathrm{Ext}^i_{\mathrm{mod}(\mathcal{P})}(M, U) = 0 \text{ for all } i > 0\}$ for some object $U \in \mathrm{mod}(\mathcal{P})$. In particular, this covers orthogonal categories of cotilting modules as studied in [45] and [34].
- R is a commutative Cohen-Macaulay local ring, Λ is an R-order, i.e. a noetherian R-algebra which is maximal Cohen-Macaulay as an R-module, and $\mathcal{X}$ is the subcategory

  $$\mathrm{CM}(\Lambda) = \{M \in \mathrm{mod}(\Lambda) \mid M \text{ is maximal Cohen-Macaulay as an } R\text{-module}\}$$

  of maximal Cohen-Macaulay modules of Λ.

We end this section by restating Proposition 4.27 when the subcategories of Gorenstein projective modules or Cohen-Macaulay modules are of finite type. Here, for an exact category $\mathcal{E}'$ with an additive generator M and endomorphism ring $\Gamma = \mathrm{End}_{\mathcal{E}'}(M)$, we let $\mathrm{mod}_{\mathsf{adm}}(\Gamma)$ be the subcategory of $\mathrm{mod}(\Gamma)$ which corresponds to the subcategory $\mathrm{mod}_{\mathsf{adm}}(\mathcal{E}')$ under the equivalence $\mathrm{mod}(\Gamma) \cong \mathrm{mod}(\mathcal{E}')$.

Corollary 4.28. *Let Λ be a noetherian ring, and assume the subcategory $\mathrm{GP}(\Lambda)$ of Gorenstein projective modules in $\mathrm{mod}(\Lambda)$ has an additive generator $M \in \mathrm{GP}(\Lambda)$. Assume also for simplicity that Λ is a summand of M. Set $\Gamma = \mathrm{End}_\Lambda(M)$ and let $e \in \Gamma$ be the idempotent corresponding to Λ, so that $e\Gamma e \cong \Lambda$. Then*

$$\mathrm{mod}_{\mathsf{adm}}(\Gamma) = \{M \in \mathrm{mod}(\Gamma) \mid Me \in \mathrm{GP}(\Lambda)\}.$$

Corollary 4.29. *Let Λ be an R-order over a local commutative Cohen-Macaulay ring R, and assume the subcategory $\mathrm{CM}(\Lambda)$ of maximal Cohen-Macaulay has an additive generator $M \in \mathrm{CM}(\Lambda)$. Assume also for simplicity that Λ is a summand of M. Set $\Gamma = \mathrm{End}_\Lambda(M)$ and let $e \in \Gamma$ be the idempotent corresponding to Λ. Then*

$$\mathrm{mod}_{\mathsf{adm}}(\Gamma) = \{M \in \mathrm{mod}(\Gamma) \mid Me \in \mathrm{CM}(\Lambda)\}.$$

5 Characterizing exact structures via resolving subcategories

Let $\mathcal{C}$ be an idempotent complete additive category. In this section we characterize the subcategories of $\operatorname{mod}(\mathcal{C})$ of the form $\operatorname{mod}_{\mathsf{adm}}(\mathcal{E})$ where $\mathcal{E}$ is an exact category obtained by choosing an exact structure on $\mathcal{C}$.

5.1 Resolving subcategories

Fix an idempotent complete exact category $\mathcal{E}$. Our goal in this section is to investigate the relationship between $\operatorname{eff}(\mathcal{E})$ and $\operatorname{mod}_{\mathsf{adm}}(\mathcal{E})$ as subcategories of $\operatorname{mod}(\mathcal{E})$.

Definition 5.1. A subcategory $\mathcal{X}$ of $\mathcal{E}$ is called *resolving* if it is closed under extensions, direct summands, and kernel of deflations, and if it is *generating*, i.e. $\operatorname{gen}(\mathcal{X}) = \mathcal{E}$.

Definition 5.2. Let $\mathcal{P}^2(\mathcal{E})$ denote the subcategory of $\operatorname{mod}(\mathcal{E})$ consisting of all F for which there exists an exact sequence

$$0 \to \operatorname{Hom}_{\mathcal{E}}(-, X) \to \operatorname{Hom}_{\mathcal{E}}(-, Y) \to \operatorname{Hom}_{\mathcal{E}}(-, Z) \to F \to 0.$$

Lemma 5.3. *$\mathcal{P}^2(\mathcal{E})$ is closed under extensions, direct summands and kernels of epimorphisms.*

Proof. $\mathcal{P}^2(\mathcal{E})$ is the subcategory of objects of projective dimension ≤ 2 in the category $\operatorname{mod}_\infty(\mathcal{E})$ consisting of all F which admits an exact sequence

$$\cdots \to \operatorname{Hom}_{\mathcal{E}}(-, X_2) \to \operatorname{Hom}_{\mathcal{E}}(-, X_1) \to F \to 0.$$

Since $\operatorname{mod}_\infty(\mathcal{E})$ is closed under extensions, direct summands and kernels of epimorphisms by [33, Proposition 2.6], the same must hold for $\mathcal{P}^2(\mathcal{E})$. □

Proposition 5.4. $\operatorname{mod}_{\mathsf{adm}}(\mathcal{E})$ *is the smallest resolving subcategory of* $\mathcal{P}^2(\mathcal{E})$ *containing* $\operatorname{eff}(\mathcal{E})$.

Proof. We first show that $\operatorname{mod}_{\mathsf{adm}}(\mathcal{E})$ is a resolving subcategory of $\mathcal{P}^2(\mathcal{E})$. Note that $\operatorname{mod}_{\mathsf{adm}}(\mathcal{E})$ is closed under extensions by Proposition 3.4 and is idempotent complete by Corollary 3.15, and therefore closed under direct summands. Also, since $\operatorname{mod}_{\mathsf{adm}}(\mathcal{E})$ contains the representable functors, it is generating in $\mathcal{P}^2(\mathcal{E})$. To show closure under kernels of deflations, it suffices by [33, Lemma 2.5] to show that if $0 \to K \to \operatorname{Hom}_{\mathcal{E}}(-, E) \xrightarrow{p} G \to 0$ is an exact sequence with $G \in \operatorname{mod}_{\mathsf{adm}}(\mathcal{E})$, then $K \in \operatorname{mod}_{\mathsf{adm}}(\mathcal{E})$. To this end, let $0 \to F \to G \xrightarrow{q} H \to 0$ be a conflation with $F \in \operatorname{eff}(\mathcal{E})$ and $H \in \mathcal{F}$. Then we get a commutative diagram

$$\begin{tikzcd}
K' \arrow[r, tail] \arrow[d, "r"] & \operatorname{Hom}_{\mathcal{E}}(-, E) \arrow[r, two heads] \arrow[d, "p"] & H \arrow[d, equal] \\
F \arrow[r, tail] & G \arrow[r, two heads, "q"] & H
\end{tikzcd}$$

where the rows are conflations. It follows that the left hand square is a pullback square, so r is an epimorphism and $\ker(r) \cong \ker(p) \cong K$. Since H has projective dimension ≤ 1, it follows that K' is projective, and hence $K' \cong \operatorname{Hom}_{\mathcal{E}}(-, E')$ for some $E' \in \mathcal{E}$ since $\mathcal{E}$ is idempotent complete. Hence $K' \in \operatorname{mod}_{\mathsf{adm}}(\mathcal{E})$, and therefore r is admissible in $\operatorname{mod}_{\mathsf{adm}}(\mathcal{E})$ by Proposition 3.5. Hence $K \cong \ker(r) \in \operatorname{mod}_{\mathsf{adm}}(\mathcal{E})$, so $\operatorname{mod}_{\mathsf{adm}}(\mathcal{E})$ is closed under kernels of deflations and therefore a resolving subcategory of $\mathcal{P}^2(\mathcal{E})$.

Now let $\mathcal{X} \in \mathcal{P}^2(\mathcal{E})$ be a resolving subcategory containing $\operatorname{eff}(\mathcal{E})$. Then by Proposition 4.4.(3) we have that $\mathcal{X}$ must contain $\mathcal{F}$. Since $(\operatorname{eff}(\mathcal{E}), \mathcal{F})$ is a torsion pair in $\operatorname{mod}_{\mathsf{adm}}(\mathcal{E})$ and $\mathcal{X}$ is closed under extensions, $\mathcal{X}$ must contain $\operatorname{mod}_{\mathsf{adm}}(\mathcal{E})$. This proves the claim. □

5.2 The Auslander-Bridger transpose

Fix an idempotent complete additive category $\mathcal{C}$. We let $\underline{\operatorname{mod}}(\mathcal{C})$ denote the projectively stable category of $\operatorname{mod}(\mathcal{C})$. Explicitly, $\underline{\operatorname{mod}}(\mathcal{C})$ has the same objects as $\operatorname{mod}(\mathcal{C})$, and

$$\operatorname{Hom}_{\underline{\operatorname{mod}}(\mathcal{C})}(F, G) = \operatorname{Hom}_{\operatorname{mod}(\mathcal{C})}(F, G)/\mathcal{P}(F, G)$$

where $\mathcal{P}(F,G)$ denotes the set of morphisms $F \to G$ factoring through a representable functor.

We now recall the Auslander-Bridger transpose, which was first introduced and studied in [6] and [10].

Definition 5.5. The *Auslander-Bridger transpose* is a functor

$$\mathrm{Tr}\colon \underline{\mathrm{mod}}(\mathcal{C}) \to \underline{\mathrm{mod}}(\mathcal{C}^{\mathrm{op}})^{\mathrm{op}}$$

defined as follows:

1. For each object $F \in \underline{\mathrm{mod}}(\mathcal{C})$ choose a projective presentation

$$\mathrm{Hom}_{\mathcal{C}}(-,X) \xrightarrow{\mathrm{Hom}_{\mathcal{C}}(-,f)} \mathrm{Hom}_{\mathcal{C}}(-,Y) \to F \to 0$$

in $\mathrm{mod}(\mathcal{C})$ and set

$$\mathrm{Tr}(F) = \mathrm{coker}(\mathrm{Hom}_{\mathcal{C}}(Y,-) \xrightarrow{\mathrm{Hom}_{\mathcal{C}}(f,-)} \mathrm{Hom}_{\mathcal{C}}(X,-)).$$

2. For each morphism morphism $\underline{\phi}\colon F \to G$ in $\underline{\mathrm{mod}}(\mathcal{C})$ choose a representative $\phi\colon F \to G$ in $\mathrm{mod}(\mathcal{C})$ and a lift

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{C}}(-,X) \arrow[r, "\mathrm{Hom}_{\mathcal{C}}(-,f)"] \arrow[d, "\mathrm{Hom}_{\mathcal{C}}(-,k)"] & \mathrm{Hom}_{\mathcal{C}}(-,Y) \arrow[r] \arrow[d, "\mathrm{Hom}_{\mathcal{C}}(-,h)"] & F \arrow[r] \arrow[d, "\phi"] & 0 \\
\mathrm{Hom}_{\mathcal{C}}(-,X') \arrow[r, "\mathrm{Hom}_{\mathcal{C}}(-,f')"] & \mathrm{Hom}_{\mathcal{C}}(-,Y') \arrow[r] & G \arrow[r] & 0
\end{tikzcd}$$

and define $\mathrm{Tr}(\underline{\phi})\colon \mathrm{Tr}(F) \to \mathrm{Tr}(G)$ to be the image in $\underline{\mathrm{mod}}(\mathcal{C})$ of the unique morphism in $\mathrm{mod}(\mathcal{C})$ making the diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{C}}(Y',-) \arrow[r, "\mathrm{Hom}_{\mathcal{C}}(f',-)"] \arrow[d, "\mathrm{Hom}_{\mathcal{C}}(h,-)"] & \mathrm{Hom}_{\mathcal{C}}(X',-) \arrow[r] \arrow[d, "\mathrm{Hom}_{\mathcal{C}}(k,-)"] & \mathrm{Tr}(G) \arrow[r] \arrow[d] & 0 \\
\mathrm{Hom}_{\mathcal{C}}(Y,-) \arrow[r, "\mathrm{Hom}_{\mathcal{C}}(f,-)"] & \mathrm{Hom}_{\mathcal{C}}(X,-) \arrow[r] & \mathrm{Tr}(F) \arrow[r] & 0
\end{tikzcd}$$

commutative.

By definition we have that $\mathrm{Tr}^2 \cong 1_{\underline{\mathrm{mod}}(\mathcal{C})}$, and Tr therefore induces an equivalence between $\underline{\mathrm{mod}}(\mathcal{C})$ and $\underline{\mathrm{mod}}(\mathcal{C}^{\mathrm{op}})^{\mathrm{op}}$. Since Tr is a functor to the projectively stable category, the object $\mathrm{Tr}(F)$ is well-defined in $\mathrm{mod}(\mathcal{C})$ up to projective summands. By abuse of notation, we sometimes write $\mathrm{Tr}(F)$ for a choice of an object in $\mathrm{mod}(\mathcal{C}^{\mathrm{op}})$ which is isomorphic in $\underline{\mathrm{mod}}(\mathcal{C}^{\mathrm{op}})$ to the image of $\mathrm{Tr}\colon \underline{\mathrm{mod}}(\mathcal{C}) \to \underline{\mathrm{mod}}(\mathcal{C}^{\mathrm{op}})^{\mathrm{op}}$ at $F \in \underline{\mathrm{mod}}(\mathcal{C})$.

In the following, for a functor $F \in \mathrm{Mod}(\mathcal{C})$ we let $F^* \in \mathrm{Mod}(\mathcal{C}^{\mathrm{op}})$ denote the functor given by

$$F^*(X) = \mathrm{Hom}_{\mathrm{Mod}(\mathcal{C})}(F, \mathrm{Hom}_{\mathcal{C}}(-,X))$$

and we let $\mathrm{ev}_F\colon F \to F^{**}$ denote the natural transformation sending $x \in F(X)$ to

$$\mathrm{ev}_F(x)\colon F^* \to \mathrm{Hom}_{\mathcal{C}}(X,-) \quad \mathrm{ev}_F(x)(\psi) = \psi(x).$$

Finally, for $i > 0$ we let $\mathrm{Ext}^i_{\mathcal{C}}(F,\mathcal{C}) \in \mathrm{Mod}(\mathcal{C}^{\mathrm{op}})$ denote the functor given by

$$\mathrm{Ext}^i_{\mathcal{C}}(F,\mathcal{C})(X) = \mathrm{Ext}^i_{\mathrm{Mod}(\mathcal{C})}(F, \mathrm{Hom}_{\mathcal{C}}(-,X)).$$

Proposition 5.6 (Proposition 6.3 in [6])**.** *For any $F \in \mathrm{mod}(\mathcal{C})$ we have an exact sequence*

$$0 \to \mathrm{Ext}^1_{\mathcal{C}^{\mathrm{op}}}(\mathrm{Tr}(F),\mathcal{C}^{\mathrm{op}}) \to F \xrightarrow{\mathrm{ev}_F} F^{**} \to \mathrm{Ext}^2_{\mathcal{C}^{\mathrm{op}}}(\mathrm{Tr}(F),\mathcal{C}^{\mathrm{op}}) \to 0.$$

We end this subsection by recalling the definition of the grade, which was first introduced in [57].

Definition 5.7. The *grade* of $F \in \mathrm{Mod}(\mathcal{C})$, denoted $\mathrm{grade}\, F$, is the biggest integer $i \geq 0$ such that $\mathrm{Ext}^j_{\mathrm{Mod}(\mathcal{C})}(F, \mathrm{Hom}_{\mathcal{C}}(-,X)) = 0$ for all $j < i$ and $X \in \mathcal{C}$. Here we let $\mathrm{Ext}^0_{\mathrm{Mod}(\mathcal{C})}(F, \mathrm{Hom}_{\mathcal{C}}(-,X)) := \mathrm{Hom}_{\mathrm{Mod}(\mathcal{C})}(F, \mathrm{Hom}_{\mathcal{C}}(-,X)))$.

5.3 Main theorem

Fix an idempotent complete additive category $\mathcal{C}$. We are now ready to state our main theorem relating subcategories of $\mathrm{mod}(\mathcal{C})$ with exact structures on $\mathcal{C}$. Here, for a subcategory $\mathcal{X}$ of $\mathrm{mod}(\mathcal{C})$ we let $\mathrm{Tr}(\mathcal{X})$ denote the subcategory of $\mathrm{mod}(\mathcal{C}^{\mathrm{op}})$ consisting of all objects G which are isomorphism in $\underline{\mathrm{mod}}(\mathcal{C}^{\mathrm{op}})$ to an object $\mathrm{Tr}(F)$ with $F \in \mathcal{X}$. Since we encounter multiple exact structures on $\mathcal{C}$, we use the letters $\mathbb{C}$, $\mathbb{D}$ and $\mathbb{E}$ for classes of conflations in $\mathcal{C}$. Thus a conflation category or an exact category is written as a pair $(\mathcal{C}, \mathbb{C})$, and the associated Auslander exact category is denoted by $\mathrm{mod}_{\mathsf{adm}}(\mathbb{C})$.

Theorem 5.8. *Let $\mathcal{C}$ be an idempotent complete additive category. The association $\mathbb{C} \mapsto \mathrm{mod}_{\mathsf{adm}}(\mathbb{C})$ gives a bijection between:*

1. *Classes of conflations $\mathbb{C}$ in $\mathcal{C}$ such that $(\mathcal{C}, \mathbb{C})$ is an exact category;*
2. *Subcategories $\mathcal{X}$ of $\mathrm{mod}(\mathcal{C})$ satisfying the following:*
   (a) *$\mathcal{X}$ is a resolving subcategory of $\mathcal{P}^2(\mathcal{C})$ and $\mathrm{Tr}(\mathcal{X})$ is a resolving subcategory of $\mathcal{P}^2(\mathcal{C}^{\mathrm{op}})$;*
   (b) *$\mathcal{X}$ and $\mathrm{Tr}(\mathcal{X})$ have no objects of grade 1.*

Our goal in this section is to prove Theorem 5.8.

Lemma 5.9. *Assume $\mathcal{X}$ is a subcategory of $\mathrm{mod}(\mathcal{C})$ satisfying condition (2a) in Theorem 5.8, and let $F \in \mathcal{X}$. The following hold:*

1. *$F^* \cong \mathrm{Hom}_{\mathcal{C}}(Z, -)$ for some $Z \in \mathcal{C}$;*
2. *The morphism $\mathrm{ev}_F \colon F \to F^{**}$ is admissible in $\mathcal{X}$;*
3. *$\mathrm{Ext}^1_{\mathcal{C}}(F, \mathcal{C}) \in \mathrm{Tr}(\mathcal{X})$ and $\mathrm{Ext}^2_{\mathcal{C}}(F, \mathcal{C}) \in \mathrm{Tr}(\mathcal{X})$.*

Proof. 1. Recall that there exists an exact sequence

$$0 \to F^* \to \mathrm{Hom}_{\mathcal{C}}(X, -) \xrightarrow{\mathrm{Hom}_{\mathcal{C}}(f,-)} \mathrm{Hom}_{\mathcal{C}}(Y, -) \to \mathrm{Tr}(F) \to 0.$$

Since $\mathrm{Tr}(F) \in \mathrm{Tr}(\mathcal{X}) \in \mathcal{P}^2(\mathcal{C}^{\mathrm{op}})$, it follows that F^* is projective in $\mathrm{mod}(\mathcal{C}^{\mathrm{op}})$ and hence $F^* \cong \mathrm{Hom}_{\mathcal{C}}(Z, -)$ for some $Z \in \mathcal{C}$ since $\mathcal{C}$ is idempotent complete.

2. Consider the exact sequence

$$0 \to \mathrm{Ext}^1_{\mathcal{C}^{\mathrm{op}}}(\mathrm{Tr}(F), \mathcal{C}^{\mathrm{op}}) \to F \xrightarrow{\mathrm{ev}_F} F^{**} \to \mathrm{Ext}^2_{\mathcal{C}^{\mathrm{op}}}(\mathrm{Tr}(F), \mathcal{C}^{\mathrm{op}}) \to 0.$$

from Proposition 5.6. Since F^{**} is representable by part (1), it is contained in $\mathcal{X}$. Therefore, since $\mathcal{X}$ is closed under kernels of epimorphisms, it suffices to show that $\mathrm{Ext}^2_{\mathcal{C}^{\mathrm{op}}}(\mathrm{Tr}(F), \mathcal{C}^{\mathrm{op}})$ is in $\mathcal{X}$. Now by choosing an exact sequence

$$0 \to \mathrm{Hom}_{\mathcal{C}}(Z, -) \to \mathrm{Hom}_{\mathcal{C}}(Y, -) \xrightarrow{\mathrm{Hom}_{\mathcal{C}}(f,-)} \mathrm{Hom}_{\mathcal{C}}(X, -) \to \mathrm{Tr}(F) \to 0$$

we immediately see that $\mathrm{Ext}^2_{\mathcal{C}^{\mathrm{op}}}(\mathrm{Tr}(F), \mathcal{C}^{\mathrm{op}}) \cong \mathrm{Tr}(\mathrm{im}(\mathrm{Hom}_{\mathcal{C}}(f, -)))$. Since $\mathrm{Tr}(\mathcal{X})$ is closed under kernels of epimorphisms, we get that $\mathrm{im}(\mathrm{Hom}_{\mathcal{C}}(f, -)) \in \mathrm{Tr}(\mathcal{X})$, and hence $\mathrm{Ext}^2_{\mathcal{C}^{\mathrm{op}}}(\mathrm{Tr}(F), \mathcal{C}^{\mathrm{op}}) \in \mathcal{X}$.

3. Proposition 5.6 applied to $\mathrm{Tr}(F)$ gives an exact sequence

$$0 \to \mathrm{Ext}^1_{\mathcal{C}}(F, \mathcal{C}) \to \mathrm{Tr}(F) \xrightarrow{\mathrm{ev}_{\mathrm{Tr}(F)}} \mathrm{Tr}(F)^{**} \to \mathrm{Ext}^2_{\mathcal{C}}(F, \mathcal{C}) \to 0.$$

Hence, the claim follows from part (2) applied to $\mathrm{Tr}(F)$ and $\mathrm{Tr}(\mathcal{X})$. □

Lemma 5.10. *Assume $\mathcal{X}$ is a subcategory of $\mathrm{mod}(\mathcal{C})$ satisfying condition (2a) in Theorem 5.8. The following are equivalent:*

1. *For all $F \in \mathcal{X}$ of projective dimension ≤ 1, the map $\mathrm{ev}_F \colon F \to F^{**}$ is an inflation in $\mathcal{X}$;*
2. *$\mathrm{Tr}(\mathcal{X})$ has no objects of grade 1.*

Proof. For each $F \in \mathcal{X}$ choose a projective presentation $\mathrm{Hom}_{\mathcal{C}}(-,Y) \xrightarrow{\mathrm{Hom}_{\mathcal{C}}(-,f)} \mathrm{Hom}_{\mathcal{C}}(-,X) \to F \to 0$ such that $\mathrm{Hom}_{\mathcal{C}}(-,f)$ is a monomorphism if F has projective dimension ≤ 1. Define $\mathrm{Tr}(F)$ via the exact sequence $\mathrm{Hom}_{\mathcal{C}}(X,-) \xrightarrow{\mathrm{Hom}_{\mathcal{C}}(f,-)} \mathrm{Hom}_{\mathcal{C}}(Y,-) \to \mathrm{Tr}(F)$. Then clearly F has projective dimension ≤ 1 if and only if $\operatorname{grade} \mathrm{Tr}(F) > 0$. Furthermore, ev_F is an inflation in $\mathcal{X}$ if and only if $\mathrm{Ext}^1_{\mathcal{C}^{\mathrm{op}}}(\mathrm{Tr}(F), \mathcal{C}^{\mathrm{op}}) \cong 0$ by Proposition 5.6 and Lemma 5.9.(2). Hence, ev_F is an inflation for any object $F \in \mathcal{X}$ of projective dimension ≤ 1 if and only if no object of the form $\mathrm{Tr}(F)$ with $F \in \mathcal{X}$ has grade 1. Since any object in $\mathrm{Tr}(\mathcal{X})$ is isomorphic in $\underline{\mathrm{mod}}(\mathcal{C}^{\mathrm{op}})$ to an object of the form $\mathrm{Tr}(F)$ with $F \in \mathcal{X}$, this is equivalent to $\mathrm{Tr}(\mathcal{X})$ having no objects of grade 1. □

Definition 5.11. Assume $\mathcal{X}$ is a subcategory of $\mathrm{mod}(\mathcal{C})$ satisfying condition (2a) and (2b) in Theorem 5.8, and let $f\colon X \to Y$ be a morphism in $\mathcal{C}$.

1. f is an $\mathcal{X}$-*inflation* if $\mathrm{Hom}_{\mathcal{C}}(-,f)\colon \mathrm{Hom}_{\mathcal{C}}(-,X) \to \mathrm{Hom}_{\mathcal{C}}(-,Y)$ is an inflation in $\mathcal{X}$;
2. f is an $\mathcal{X}$-*deflation* if $\mathrm{Hom}_{\mathcal{C}}(f,-)\colon \mathrm{Hom}_{\mathcal{C}}(Y,-) \to \mathrm{Hom}_{\mathcal{C}}(X,-)$ is an inflation in $\mathrm{Tr}(\mathcal{X})$;
3. f is $\mathcal{X}$-*admissible* if $\mathrm{Hom}_{\mathcal{C}}(-,f)\colon \mathrm{Hom}_{\mathcal{C}}(-,X) \to \mathrm{Hom}_{\mathcal{C}}(-,Y)$ is admissible in $\mathcal{X}$.

Proposition 5.12. *Assume $\mathcal{X}$ is a subcategory of* $\mathrm{mod}(\mathcal{C})$ *satisfying condition* (2a) *and* (2b) *in Theorem 5.8. Then there exists a unique conflation category $(\mathcal{C}, \mathbb{C})$ with inflations and deflations the $\mathcal{X}$-inflations and $\mathcal{X}$-deflations, respectively.*

Proof. Let f be an $\mathcal{X}$-inflation, and consider the exact sequences

$$0 \to \mathrm{Hom}_{\mathcal{C}}(-,X) \xrightarrow{\mathrm{Hom}_{\mathcal{C}}(-,f)} \mathrm{Hom}_{\mathcal{C}}(-,Y) \to F \to 0$$
$$0 \to F^* \to \mathrm{Hom}_{\mathcal{C}}(Y,-) \xrightarrow{\mathrm{Hom}_{\mathcal{C}}(f,-)} \mathrm{Hom}_{\mathcal{C}}(X,-) \to \mathrm{Tr}(F) \to 0.$$

Since $F^* \cong \mathrm{Hom}_{\mathcal{C}}(Z,-)$ by Lemma 5.9.(1), we get that f admits a cokernel $f\colon Y \to Z$ in $\mathcal{C}$ which is an $\mathcal{X}$-deflation. Now since F has projective dimension ≤ 1, the map $\mathrm{ev}_F\colon F \to F^{**} \cong \mathrm{Hom}_{\mathcal{C}}(Z,-)$ is an inflation in $\mathcal{X}$ by Lemma 5.10. Hence the sequence $0 \to \mathrm{Hom}_{\mathcal{C}}(-,X) \to \mathrm{Hom}_{\mathcal{C}}(-,Y) \to \mathrm{Hom}_{\mathcal{C}}(-,Z)$ is exact, so f is the kernel of g. This together with the dual argument shows that the $\mathcal{X}$-inflations and $\mathcal{X}$-deflations makes $\mathcal{C}$ into a conflation category. □

Proposition 5.13. *Assume $\mathcal{X}$ is a subcategory of* $\mathrm{mod}(\mathcal{C})$ *satisfying condition* (2a) *and* (2b) *in Theorem 5.8. Then the conflation category in Proposition 5.12 is an exact category.*

Proof. Since the $\mathcal{X}$-inflations and $\mathcal{X}$-deflations clearly satisfy axioms (**L0**), (**L1**) and (**R0**), (**R1**) for an exact category, it only remains to show that axioms (**L2**) and (**R2**) hold. We only prove (**L2**) since (**R2**) is dual. To this end, let $f\colon X \to Y$ be an $\mathcal{X}$-inflation and let $g\colon X \to Z$ be a morphism in $\mathcal{C}$. Taking the pushout of $\mathrm{Hom}_{\mathcal{C}}(-,f)\colon \mathrm{Hom}_{\mathcal{C}}(-,X) \to \mathrm{Hom}_{\mathcal{C}}(-,Y)$ along $\mathrm{Hom}_{\mathcal{C}}(-,g)\colon \mathrm{Hom}_{\mathcal{C}}(-,X) \to \mathrm{Hom}_{\mathcal{C}}(-,Z)$, we get a commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{C}}(-,X) \arrow[r, tail, "\mathrm{Hom}_{\mathcal{C}}(-,f)"] \arrow[d, "\mathrm{Hom}_{\mathcal{C}}(-,g)"] & \mathrm{Hom}_{\mathcal{C}}(-,Y) \arrow[r, two heads] \arrow[d] & F \arrow[d, equal] \\
\mathrm{Hom}_{\mathcal{C}}(-,Z) \arrow[r, tail] & G \arrow[r, two heads] & F
\end{tikzcd}$$

where the rows are exact sequences. Since the left hand square is a pullback and a pushout square, we have an exact sequence

$$0 \to \mathrm{Hom}_{\mathcal{C}}(-,X) \xrightarrow{\begin{pmatrix}\mathrm{Hom}_{\mathcal{C}}(-,f)\\ \mathrm{Hom}_{\mathcal{C}}(-,g)\end{pmatrix}} \mathrm{Hom}_{\mathcal{C}}(-,Y) \oplus \mathrm{Hom}_{\mathcal{C}}(-,Z) \to G \to 0 \tag{3}$$

and hence G has projective dimension ≤ 1. Therefore by Lemma 5.10 the map $\mathrm{ev}_G\colon G \to G^{**}$ is an inflation in $\mathcal{X}$. Furthermore, $G^* \cong \mathrm{Hom}_{\mathcal{C}}(W,-)$ for some object $W \in \mathcal{C}$ by Lemma 5.9.(1). Let $h\colon Z \to W$ and $k\colon Y \to W$ be the unique morphisms in $\mathcal{C}$ so that $\mathrm{Hom}_{\mathcal{C}}(-,h)$ and $\mathrm{Hom}_{\mathcal{C}}(-,k)$ are equal to the composites $\mathrm{Hom}_{\mathcal{C}}(-,Z) \rightarrowtail G \xrightarrow{\mathrm{ev}_G} G^{**} \cong \mathrm{Hom}_{\mathcal{C}}(-,W)$ and $\mathrm{Hom}_{\mathcal{C}}(-,Y) \to G \xrightarrow{\mathrm{ev}_G} G^{**} \cong \mathrm{Hom}_{\mathcal{C}}(-,W)$,

respectively. Since $\mathrm{Hom}_{\mathcal{C}}(-,h)$ is a composite of inflations in $\mathcal{X}$, it follows that h is an $\mathcal{X}$-inflation. Now consider the commutative square

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "g"] & Y \arrow[d, "k"] \\ Z \arrow[r, "h"] & W. \end{tikzcd}$$

It only remains to show that this is a pushout square. But this is true since applying $(-)^*$ to (3) gives the left exact sequence

$$0 \to \mathrm{Hom}_{\mathcal{C}}(W,-) \xrightarrow{\begin{pmatrix} \mathrm{Hom}_{\mathcal{C}}(k,-) \\ \mathrm{Hom}_{\mathcal{C}}(h,-) \end{pmatrix}} \mathrm{Hom}_{\mathcal{C}}(Y,-) \oplus \mathrm{Hom}_{\mathcal{C}}(Z,-) \xrightarrow{(\,\mathrm{Hom}_{\mathcal{C}}(f,-)\ \mathrm{Hom}_{\mathcal{C}}(g,-)\,)} \mathrm{Hom}_{\mathcal{C}}(X,-).$$

□

Proposition 5.14. *Assume $\mathcal{X}$ is a subcategory of* $\mathrm{mod}(\mathcal{C})$ *satisfying condition* (2a) *and* (2b) *in Theorem 5.8, and let $(\mathcal{C},\mathbb{C})$ be the exact category given by the $\mathcal{X}$-inflations and $\mathcal{X}$-deflations. The following hold:*

1. *The $\mathcal{X}$-admissible morphisms in $\mathcal{C}$ are precisely the admissible morphisms in $(\mathcal{C},\mathbb{C})$;*
2. $\mathrm{mod}_{\mathsf{adm}}(\mathbb{C}) = \mathcal{X}$.

Proof. 1. Let $f\colon X \to Y$ be a morphism in $\mathcal{C}$. If f is admissible in $(\mathcal{C},\mathbb{C})$, then it can be written as a composite $f = f_2 \circ f_1$ where f_1 is an $\mathcal{X}$-deflation and f_2 is an $\mathcal{X}$-inflation. By definition $\mathrm{Hom}_{\mathcal{C}}(f_1,-)$ is an inflation in $\mathrm{Tr}(\mathcal{X})$, and hence $\mathrm{Hom}_{\mathcal{C}}(-,f_1)$ must be admissible in $\mathcal{X}$. Since $\mathrm{Hom}_{\mathcal{C}}(-,f_2)$ is an inflation in $\mathcal{X}$, the composite

$$\mathrm{Hom}_{\mathcal{C}}(-,f) = \mathrm{Hom}_{\mathcal{C}}(-,f_2) \circ \mathrm{Hom}_{\mathcal{C}}(-,f_1)$$

is admissible in $\mathcal{X}$. Hence f is $\mathcal{X}$-admissible.

Now assume f is $\mathcal{X}$-admissible. First note that f has a kernel which is an $\mathcal{X}$-inflation and a cokernel which is an $\mathcal{X}$-deflation, since $\mathrm{Hom}_{\mathcal{C}}(-,f)$ and $\mathrm{Hom}_{\mathcal{C}}(-,f)$ are admissible in $\mathcal{X}$ with kernels being representable functors. Therefore f has a coimage $\mathrm{coim}(f)$ obtained by taking the cokernel of the inclusion $\ker(f) \rightarrowtail X$, and an image $\mathrm{im}(f)$ obtained by taking the kernel of the projection $Y \twoheadrightarrow \mathrm{coker}(f)$. Hence we get a canonical map $\mathrm{coim}(f) \to \mathrm{im}(f)$, and it suffices to show that this is an isomorphism. To this end, let $g\colon X \twoheadrightarrow \mathrm{coim}(f)$ denote the given $\mathcal{X}$-deflation, and let $h\colon \mathrm{coim}(f) \to Y$ denote the composite $\mathrm{coim}(f) \to \mathrm{im}(f) \to Y$. Consider the commutative diagram with exact rows

$$\begin{tikzcd} & \mathrm{Hom}_{\mathcal{C}}(Y,-) \arrow[r, "\mathrm{Hom}_{\mathcal{C}}(f,-)"] \arrow[d, "\mathrm{Hom}_{\mathcal{C}}(h,-)"] & \mathrm{Hom}_{\mathcal{C}}(X,-) \arrow[r] \arrow[d, equal] & F \arrow[r] \arrow[d, "p"] & 0 \\ 0 \arrow[r] & \mathrm{Hom}_{\mathcal{C}}(\mathrm{coim}(f),-) \arrow[r, "\mathrm{Hom}_{\mathcal{C}}(g,-)"] & \mathrm{Hom}_{\mathcal{C}}(X,-) \arrow[r] & G \arrow[r] & 0. \end{tikzcd}$$

Since f and g are $\mathcal{X}$-admissible, F and G are in $\mathrm{Tr}(\mathcal{X})$. By the snake lemma p is an epimorphism and $\mathrm{coker}(\mathrm{Hom}_{\mathcal{C}}(h,-)) \cong \ker(p)$. Since $\mathrm{Tr}(\mathcal{X})$ is closed under kernels of epimorphisms, $\ker(p)$ is contained in $\mathrm{Tr}(\mathcal{X})$, and hence h is $\mathcal{X}$-admissible. Let $i\colon \ker(h) \to \mathrm{coim}(f)$ denote the inclusion of the kernel, and let $j\colon E \to X$ denote the pullback of i along g. Then we get a commutative diagram

$$\begin{tikzcd} \ker(f) \arrow[r, tail] \arrow[d, equal] & E \arrow[r, two heads, "k"] \arrow[d, tail, "j"] & \ker(h) \arrow[d, tail, "i"] \\ \ker(f) \arrow[r, tail] \arrow[d, equal] & X \arrow[r, two heads, "g"] \arrow[d, equal] & \mathrm{coim}(f) \arrow[d, "h"] \\ \ker(f) \arrow[r, tail] & X \arrow[r, "f"] & Y \end{tikzcd}$$

where the two upper rows are conflations in $(\mathcal{C}, \mathbb{C})$. Hence $f \circ j = 0$, so j factors through $\ker(f)$. It follows that i must be 0, so $\ker(h) \cong 0$ since i is an inflation in $(\mathcal{C}, \mathbb{C})$. Hence h is an $\mathcal{X}$-inflation. Since h can be written as a composite $\operatorname{coim}(f) \to \operatorname{im}(f) \to Y$ and $\mathcal{C}$ is weakly idempotent complete, the map $\operatorname{coim}(f) \to \operatorname{im}(f)$ must be an $\mathcal{X}$-inflation by [24, Proposition 7.6]. Dually, one shows that the map $\operatorname{coim}(f) \to \operatorname{im}(f)$ is an $\mathcal{X}$-deflation. Since a map in an exact category which is both an inflation and a deflation must be an isomorphism, the claim follows.

2. This follows immediately from part (1). □

Proof of Theorem 5.8. If $(\mathcal{C}, \mathbb{C})$ is an exact category, then $\operatorname{mod}_{\mathsf{adm}}(\mathbb{C})$ and $\operatorname{Tr}(\operatorname{mod}_{\mathsf{adm}}(\mathbb{C})) = \operatorname{mod}_{\mathsf{adm}}(\mathbb{C}^{\mathrm{op}})$ are resolving by Proposition 5.4. Also any $F \in \operatorname{mod}_{\mathsf{adm}}(\mathbb{C})$ and $G \in \operatorname{mod}_{\mathsf{adm}}(\mathbb{C}^{\mathrm{op}})$ of grade > 0 must satisfy

$$\operatorname{Ext}^1_{\operatorname{Mod}(\mathcal{C})}(F, \operatorname{Hom}_{\mathcal{C}}(-, X)) = 0 \quad \text{and} \quad \operatorname{Ext}^1_{\operatorname{Mod}(\mathcal{C}^{\mathrm{op}})}(F, \operatorname{Hom}_{\mathcal{C}}(X, -)) = 0$$

for all $X \in \mathcal{C}$ by Proposition 3.14.(3). Therefore F and G must have grade > 1. This shows that $\operatorname{mod}_{\mathsf{adm}}(\mathbb{C})$ satisfies (2a) and (2b) in Theorem 5.8. The fact that the association $\mathbb{C} \mapsto \operatorname{mod}_{\mathsf{adm}}(\mathbb{C})$ is a bijection follows from Proposition 5.12, Proposition 5.13 and Proposition 5.14. □

We end this section by considering exact structures on an abelian category $\mathcal{A}$.

Corollary 5.15. *Let $\mathcal{A}$ be an abelian category. Then there exists a bijection between the following:*

1. *Exact structures on $\mathcal{A}$;*
2. *Resolving subcategory $\mathcal{X}$ of $\operatorname{mod}(\mathcal{A})$ for which $\operatorname{Tr}(\mathcal{X})$ is a resolving subcategory of $\operatorname{mod}(\mathcal{A}^{\mathrm{op}})$.*

Proof. Since $\mathcal{A}$ is abelian, $\operatorname{mod}(\mathcal{A}) = \mathcal{P}^2(\mathcal{A})$ and $\operatorname{mod}(\mathcal{A}^{\mathrm{op}}) = \mathcal{P}^2(\mathcal{A}^{\mathrm{op}})$, and $\operatorname{mod}(\mathcal{A})$ and $\operatorname{mod}(\mathcal{A}^{\mathrm{op}})$ have no objects of grade 1. Hence, the claim follows from Theorem 5.8. □

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