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RESEARCH ARTICLE

Autocovariance and autocorrelation structures of the generalised autoregressive moving average (GARMA(1,3;δ,1)) model

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Abstract

Generalized ARMA (GARMA) model is a new class of model that has been introduced to reveal some unknown features of certain time series data. The objective of this paper is to derive the autocovariance and autocorrelation structure of GARMA (1,3;δ,1) model in order to study the behavior of the model. It is shown that the results of this model can be reduced to the autocovariance and autocorrelation of the standard ARMA model as well as a special case. Numerical examples are used to illustrate the behavior of the autocovariance and autocorrelation at different δ values to show the various structures that the model can represent.

Keywords: Generalized ARMA model, GAR, GMA, autocovariance, autocorrelation

INTRODUCTION

A time series is a sequence of observations, recorded at successive time intervals. In time series analysis, frequency-domain methods and time-domain methods are employed to analyze time series data to extract meaningful characteristics of the data. In time series forecasting, models are used to predict future values based of previously observed values. Three such classes of models that depend linearly on previous observations or/and residuals, are the autoregressive (AR) model, the moving average (MA) models and the integrated (I) models (Gershenfeld, 1999). The various combinations of these models produce the autoregressive moving average (ARMA) and autoregressive integrated moving average (ARIMA) models.

A time series, $\{X_t, t = 0,1, \ldots\}$ is said to be strictly stationary if the joint distributions defined by $\{X_1, \ldots, X_n\}$ and $\{X_{1+h}, \ldots, X_{n+h}\}$ and do not change with time. A time series is weakly stationary when mean $\mu_t(t) = E(X_t)$ exists and constant for all t, variance $Var(X_t) < \infty$ for all t, and the autocovariance $\gamma_x(h) = Cov(X_{t+h}, X_t)$ depends only on the lag, h. Weak stationarity is also implied with strict stationarity together with first and second moments (Brockwell and Davis, 2002).

The ARMA(p, q) model takes the following form,

$$X_t - \emptyset_1 X_{t-1} - \cdots \emptyset_p X_{t-p} = Z_t + \theta_1 Z_{t-1} + \cdots \theta_q Z_{t-q}, \qquad (1)$$

in a more concise form,

$$\emptyset(B) X_t = \theta(B) Z_t \qquad (2)$$

where $\{Z_t\} \sim WN(0, \sigma^2)$ and the polynomials $\emptyset(z) = (1 + \emptyset_1 z - \cdots - \emptyset_p z^p)$ and $\theta(z) = (1 + \theta_1 z + \cdots + \theta_q z^q)$ have no common factors. B is the backward shift character, such that $BX_t = X_{t-1}$.

ARMA models are widely known to predict the behaviour of economic and industrial data. However ARMA type models could not be used to accommodate the changing frequency behaviour of time series data as it leads to a misclassification problem. In order to solve this problem, Peiris (2003) introduced the generalised autoregressive model of order one (GAR (1)); and Peiris et.al (2004) introduced the generalised moving average of order one (GMA(1)). Pillai et. al (2009, 2012) studied the generalization of the standard ARMA (1,1) model, denoted by GARMA$(1,1;\delta_1,\delta_2)$ and defined by

$$(1 - \alpha B)^{\delta_1} X_t = (1 - \beta B)^{\delta_2} Z_t \qquad (3)$$

where $-1 < \alpha, \beta < 1, \delta_1 \geq 0, \delta_2 \geq 0$.

Shitan and Peiris (2011) studied the behaviour of the GARMA$(1,1;\delta,1)$ process and the model is written as

$$(1 - \alpha B)^{\delta} X_t = (1 - \beta B) Z_t, \qquad (4)$$

where $-1 < \alpha, \beta < 1, \delta \geq 0$ [8].

Some properties of the second order of GARMA model denoted by GARMA$(1,2;\delta,1)$ was examined by Pillai and Shitan (2014) and the process is written as

$$(1-\alpha B)^{\delta}X_t=(1-\beta_1 B-\beta_2 B^2)Z_t, \quad (5)$$

where $-1<\alpha,\beta_1,\beta_2<1, \delta\geq 0$ and $\{Z_t\}\sim WN(0,\delta^2)$.

This paper focuses on the third order of the GARMA model, that is GARMA$(1,3;\delta,1)$ and defined as

$$(1-\alpha B)^{\delta}X_t=(1-\beta_1 B-\beta_2 B^2-\beta_3 B^3)Z_t, \quad (6)$$

where $|\alpha|,|\beta_i|<1, i=1,2,3$ and $\delta\geq 0$. Specifically the autocovariance and autocorrelation structure of the model will be derived explicitly.

The GARMA$(1,3;\delta,1)$model can be written as equation

$$X_t=\sum_{j=0}^{\infty}\varphi_j Z_{t-j}$$

where $\sum_{j=0}^{\infty}\varphi_j Z^j=\frac{\theta(z)}{\emptyset(z)}$, $|z|\leq 1$, with $\varphi(B)=(1-\alpha B)^{-\delta}(1-\beta_1 B-\beta_2 B^2-\beta_3 B^3)$. This is used to obtain the spectral density of this model as given by

$$f(w)=\frac{\sigma^2[(1+\beta_1^2+\beta_2^2+\beta_3^2)-2(\beta_1-\beta_1\beta_2-\beta_2\beta_3)\cos w-2(\beta_2-\beta_1\beta_3)\cos 2w-2\beta_3\cos 3w]}{2\pi(1-2\alpha\cos w+\alpha^2)^{\delta}} \quad (7)$$

where $-\pi\leq w\leq\pi$. The spectral density will be used in deriving the ACVF and hence the ACF structure of the GARMA$(1,3;\delta,1)$ model.

This paper is organized as follows: The following section presents the derived expressions for the ACVF and ACF of the model. The next section deals with a special case of GARMA$(1,3;\delta,1)$ model. This is followed by some numerical studies to illustrate the behaviour of the ACVF and the ACF of the model with different δ values. The final section concludes the study based on the behaviour of the ACVF and ACF structures of the GARMA$(1,3;\delta,1)$ model.

THE ACVF AND ACF OF THE GARMA$(1,3;\delta,1)$ MODEL

This section provides the autocorrelation and autocovariance structures of the GARMA$(1,3;\delta,1)$ model. First the variance, γ_0 is presented in the following proposition.

Proposition 1: The variance, γ_0 for the model in Equation (6) is given as

$$\gamma(0)=\sigma^2\Bigg[(1+\beta_1^2+\beta_2^2+\beta_3^2)F(\delta,\delta;1;\alpha^2) -2(\beta_1-\beta_1\beta_2-\beta_2\beta_3)\frac{\alpha\tau(1+\delta)F(\delta,1+\delta;2;\alpha^2)}{\tau(\delta)\tau(2)} -2(\beta_2-\beta_1\beta_3)\frac{\alpha^2\tau(\delta+2)F(\delta,2+\delta;3;\alpha^2)}{\tau(\delta)\tau(3)} -2\beta_3\frac{\alpha^3\tau(\delta+3)F(\delta,3+\delta;4;\alpha^2)}{\tau(\delta)\tau(4)}\Bigg] \quad (8)$$

where $F(\theta_1,\theta_2;\theta_3;\theta)$ is the hypergeometric function given by

$$F(\theta_1,\theta_2;\theta_3;\theta)=\sum_{j=0}^{\infty}\frac{\tau(\theta_1+j)\tau(\theta_2+j)\tau(\theta_3)\theta^j}{\tau(\theta_1)\tau(\theta_2)\tau(\theta_3+j)\tau(j+1)},$$

Proof : The spectral density given in Equation (7) gives

$$\gamma_0=\int_{-\pi}^{\pi}f(w)\,dw$$

$$=2\int_0^{\pi}\frac{\sigma^2\begin{bmatrix}(1+\beta_1^2+\beta_2^2+\beta_3^2)-2(\beta_1-\beta_1\beta_2-\beta_2\beta_3)\cos w\\-2(\beta_2-\beta_1\beta_3)\cos 2w-2\beta_3\cos 3w\end{bmatrix}}{2\pi(1-2\alpha\cos w+\alpha^2)^{\delta}}dw$$

$$=\frac{\sigma^2}{\pi}\Bigg[(1+\beta_1^2+\beta_2^2+\beta_3^2)\int_0^{\pi}\frac{1}{(1-2\alpha\cos w+\alpha^2)^{\delta}}dw -2(\beta_1-\beta_1\beta_2-\beta_2\beta_3)\int_0^{\pi}\frac{\cos w}{(1-2\alpha cosw+\alpha^2)^{\delta}}dw -2(\beta_2-\beta_1\beta_3)\int_0^{\pi}\frac{\cos 2w}{(1-2\alpha cosw+\alpha^2)^{\delta}}dw -2\beta_3\int_0^{\pi}\frac{\cos 3w}{(1-2\alpha cosw+\alpha^2)^{\delta}}dw\Bigg]$$

According to Peiris (2003),

$$\int_0^{\pi}\frac{\cos kw}{1-2\alpha\cos w+\alpha^2}dw=\frac{\pi\alpha^k\tau(k+\delta)F(\delta,k+\delta;k+1;\alpha^2)}{\tau(\delta)\tau(k+1)}$$

for $\delta>1$. Therefore

$$=\sigma^2\Bigg[(1+\beta_1^2+\beta_2^2+\beta_3^2)F(\delta,\delta;1;\alpha^2) -2(\beta_1-\beta_1\beta_2-\beta_2\beta_3)\frac{\alpha\tau(1+\delta)F(\delta,1+\delta;2;\alpha^2)}{\tau(\delta)\tau(2)} -2(\beta_2-\beta_1\beta_3)\frac{\alpha^2\tau(\delta+2)F(\delta,2+\delta;3;\alpha^2)}{\tau(\delta)\tau(3)} -2\beta_3\frac{\alpha^3\tau(\delta+3)F(\delta,3+\delta;4;\alpha^2)}{\tau(\delta)\tau(4)}\Bigg]$$

This completes the proof for γ_0.

Proposition 2:
The autocovariance function at lag k, γ_k for the model in Equation (6) is given as

$$\gamma_k=\frac{\sigma^2}{\tau(\delta)}\Bigg[(1+\beta_1^2+\beta_2^2+\beta_3^2)\frac{\alpha^k\tau(k+\delta)F(\delta,k+\delta;k+1;\alpha^2)}{\tau(k+1)} -(\beta_1-\beta_1\beta_2-\beta_2\beta_3)\frac{\alpha^{k+1}\tau(k+1+\delta)F(\delta,k+1+\delta;k+2;\alpha^2)}{\tau(k+2)} -(\beta_1-\beta_1\beta_2-\beta_2\beta_3)\frac{\alpha^{k-1}\tau(k-1+\delta)F(\delta,k-1+\delta;k;\alpha^2)}{\tau(k)} -(\beta_2-\beta_1\beta_3)\frac{\alpha^{k+2}\tau(k+2+\delta)F(\delta,k+2+\delta;k+3;\alpha^2)}{\tau(k+3)} -(\beta_2-\beta_1\beta_3)\frac{\alpha^{|k-2|}\tau(|k-2|+\delta)F(\delta,|k-2|+\delta;|k-2|+1;\alpha^2)}{\tau(|k-2|+1)} -\beta_3\frac{\alpha^{k+3}\tau(k+3+\delta)F(\delta,k+3+\delta;k+4;\alpha^2)}{\tau(k+4)} -\beta_3\frac{\alpha^{|k-3|}\tau(|k-3|+\delta)F(\delta,|k-3|+\delta;|k-3|+1;\alpha^2)}{\tau(|k-3|+1)}\Bigg] \quad (9)$$

for $k\geq 1$.

Proof : Using the spectral density given in Equation (7) and $\gamma_k=\int_{-\pi}^{\pi}e^{iwk}f(w)\,dw$,

$$\gamma_k=2\int_0^{\pi}\cos kw\,f(w)\,dw$$

$$=2\int_0^{\pi}\cos kw\frac{\sigma^2\begin{bmatrix}(1+\beta_1^2+\beta_2^2+\beta_3^2)\\-2(\beta_1-\beta_1\beta_2-\beta_2\beta_3)\cos w\\-2(\beta_2-\beta_1\beta_3)\cos 2w-2\beta_3\cos 3w\end{bmatrix}}{2\pi(1-2\alpha\cos w+\alpha^2)^{\delta}}dw$$

$$= \frac{\sigma^2}{\pi}\Bigg[(1+\beta_1^2+\beta_2^2+\beta_3^2)\int_0^\pi \frac{\cos kw}{(1-2\alpha\cos w+\alpha^2)^\delta}dw - (\beta_1-\beta_1\beta_2 - \beta_2\beta_3)\int_0^\pi \frac{2\cos kw\cos w}{(1-2\alpha\cos w+\alpha^2)^\delta}dw - (\beta_2-\beta_1\beta_3)\int_0^\pi \frac{2\cos kw\cos 2w}{(1-2\alpha\cos w+\alpha^2)^\delta}dw - \beta_3\int_0^\pi \frac{2\cos kw\cos 3w}{(1-2\alpha\cos w+\alpha^2)^\delta}dw\Bigg]$$

Since $2\cos(kw)\cos(w) = \cos(k+1)w + \cos(k-1)w$,

$$= \frac{\sigma^2}{\pi}\Bigg[(1+\beta_1^2+\beta_2^2+\beta_3^2)\int_0^\pi \frac{\cos kw}{(1-2\alpha\cos w+\alpha^2)^\delta}dw - (\beta_1-\beta_1\beta_2 - \beta_2\beta_3)\int_0^\pi \frac{\cos(k+1)w+\cos(k-1)w}{(1-2\alpha\cos w+\alpha^2)^\delta}dw - (\beta_2 - \beta_1\beta_3)\int_0^\pi \frac{\cos(k+2)w+\cos(k-2)w}{(1-2\alpha\cos w+\alpha^2)^\delta}dw - \beta_3\int_0^\pi \frac{\cos(k+3)w+\cos(k-3)w}{(1-2\alpha\cos w+\alpha^2)^\delta}dw\Bigg]$$

$$= \frac{\sigma^2}{\pi}\Bigg[(1+\beta_1^2+\beta_2^2+\beta_3^2)\int_0^\pi \frac{\cos kw}{(1-2\alpha\cos w+\alpha^2)^\delta}dw - (\beta_1-\beta_1\beta_2 - \beta_2\beta_3)\int_0^\pi \frac{\cos(k+1)w}{(1-2\alpha\cos w+\alpha^2)^\delta}dw - (\beta_1-\beta_1\beta_2 - \beta_2\beta_3)\int_0^\pi \frac{\cos(k-1)w}{(1-2\alpha\cos w+\alpha^2)^\delta}dw - (\beta_2-\beta_1\beta_3)\int_0^\pi \frac{\cos(k+2)w}{(1-2\alpha\cos w+\alpha^2)^\delta}dw - (\beta_2-\beta_1\beta_3)\int_0^\pi \frac{\cos(k-2)w}{(1-2\alpha\cos w+\alpha^2)^\delta}dw - \beta_3\int_0^\pi \frac{\cos(k+3)w}{(1-2\alpha\cos w+\alpha^2)^\delta}dw - \beta_3\int_0^\pi \frac{\cos(k-3)w}{(1-2\alpha\cos w+\alpha^2)^\delta}dw\Bigg]$$

$$= \sigma^2\Bigg[(1+\beta_1^2+\beta_2^2+\beta_3^2)\frac{\alpha^k\tau(k+\delta)\,F(\delta,k+\delta;k+1;\alpha^2)}{\tau(\delta)\tau(k+1)} - (\beta_1-\beta_1\beta_2-\beta_2\beta_3)\frac{\alpha^{k+1}\tau(k+1+\delta)F(\delta,k+1+\delta;k+2;\alpha^2)}{\tau(\delta)\tau(k+2)} - (\beta_1-\beta_1\beta_2-\beta_2\beta_3)\frac{\alpha^{k-1}\tau(k-1+\delta)F(\delta,k-1+\delta;k;\alpha^2)}{\tau(\delta)\tau(k)} - (\beta_2-\beta_1\beta_3)\frac{\alpha^{k+2}\tau(k+2+\delta)F(\delta,k+2+\delta;k+3;\alpha^2)}{\tau(\delta)\tau(k+3)} - (\beta_2 - \beta_1\beta_3)\frac{\alpha^{|k-2|}\tau(|k-2|+\delta)F(\delta,|k-2|+\delta;|k-2|+1;\alpha^2)}{\tau(\delta)\tau(|k-2|+1)} - \beta_3\frac{\alpha^{k+3}\tau(k+3+\delta)F(\delta,k+3+\delta;k+4;\alpha^2)}{\tau(\delta)\tau(k+4)} - \beta_3\frac{\alpha^{|k-3|}\tau(|k-3|+\delta)F(\delta,|k-3|+\delta;|k-3|+1;\alpha^2)}{\tau(\delta)\tau(|k-3|+1)}\Bigg]$$

for $k \geq 1$. This complete the proof for γ_k.

By definition, the autocorrelation, ρ_k takes the form

$$\rho_k = \frac{\gamma(k)}{\gamma(0)}\,. \tag{10}$$

SPECIAL CASE

It can be seen that when $\beta_3 = 0$, the variance expressions takes the form

$$\gamma(0) = \sigma^2\Bigg[(1+\beta_1^2+\beta_2^2+0)F(\delta,\delta;1;\alpha^2) - 2(\beta_1-\beta_1\beta_2 - \beta_2.0)\frac{\alpha\tau(1+\delta)F(\delta,1+\delta;2;\alpha^2)}{\tau(\delta)\tau(2)} - 2(\beta_2-\beta_1.0)\frac{\alpha^2\tau(\delta+2)F(\delta,2+\delta;3;\alpha^2)}{\tau(\delta)\tau(3)} - 2.0\frac{\alpha^3\tau(\delta+3)F(\delta,3+\delta;4;\alpha^2)}{\tau(\delta)\tau(4)}\Bigg]$$

$$= \sigma^2\Bigg[(1+\beta_1^2+\beta_2^2)F(\delta,\delta;1;\alpha^2) - 2(\beta_1-\beta_1\beta_2)\frac{\alpha\tau(1+\delta)F(\delta,1+\delta;2;\alpha^2)}{\tau(\delta)\tau(2)} - 2(\beta_2)\frac{\alpha^2\tau(\delta+2)F(\delta,2+\delta;3;\alpha^2)}{\tau(\delta)\tau(3)}$$

$$= \sigma^2\Bigg[(1+\beta_1^2+\beta_2^2)F(\delta,\delta;1;\alpha^2) - 2\beta_1(1-\beta_2)\frac{\alpha\tau(1+\delta)F(\delta,1+\delta;2;\alpha^2)}{\tau(\delta)\tau(2)} - (\beta_2)\frac{\alpha^2\tau(\delta+2)F(\delta,2+\delta;3;\alpha^2)}{\tau(\delta)} \tag{11}$$

The variance structure obtained in Equation (11) is the variance structure of GARMA(1,2; δ, 1) as seen in Pillai and Shitan (2014).

From Equation (9), when $\beta_3 = 0$,

$$\begin{aligned}\gamma_k &= \frac{\sigma^2}{\tau(\delta)}\Bigg[(1+\beta_1^2+\beta_2^2+0)\frac{\alpha^k\tau(k+\delta)\,F(\delta,k+\delta;k+1;\alpha^2)}{\tau(k+1)} \\ &- (\beta_1-\beta_1\beta_2-\beta_2.0)\frac{\alpha^{k+1}\tau(k+1+\delta)F(\delta,k+1+\delta;k+2;\alpha^2)}{\tau(k+2)} \\ &- (\beta_1-\beta_1\beta_2-\beta_2.0)\frac{\alpha^{k-1}\tau(k-1+\delta)F(\delta,k-1+\delta;k;\alpha^2)}{\tau(k)} \\ &- (\beta_2-\beta_1.0)\frac{\alpha^{k+2}\tau(k+2+\delta)F(\delta,k+2+\delta;k+3;\alpha^2)}{\tau(k+3)} \\ &- (\beta_2 - \beta_1.0)\frac{\alpha^{|k-2|}\tau(|k-2|+\delta)F(\delta,|k-2|+\delta;|k-2|+1;\alpha^2)}{\tau(|k-2|+1)} \\ &- .0\frac{\alpha^{k+3}\tau(k+3+\delta)F(\delta,k+3+\delta;k+4;\alpha^2)}{\tau(k+4)} \\ &- .0\frac{\alpha^{|k-3|}\tau(|k-3|+\delta)F(\delta,|k-3|+\delta;|k-3|+1;\alpha^2)}{\tau(|k-3|+1)}\Bigg] \\ &= \frac{\sigma^2}{\tau(\delta)}\Bigg[(1+\beta_1^2+\beta_2^2)\frac{\alpha^k\tau(k+\delta)\,F(\delta,k+\delta;k+1;\alpha^2)}{\tau(k+1)} \\ &- \beta_1(1-\beta_2)\frac{\alpha^{k+1}\tau(k+1+\delta)F(\delta,k+1+\delta;k+2;\alpha^2)}{\tau(k+2)} \\ &- \beta_1(1-\beta_2)\frac{\alpha^{k-1}\tau(k-1+\delta)F(\delta,k-1+\delta;k;\alpha^2)}{\tau(k)} \\ &- \beta_2\frac{\alpha^{k+2}\tau(k+2+\delta)F(\delta,k+2+\delta;k+3;\alpha^2)}{\tau(k+3)} \\ &- \beta_2\frac{\alpha^{|k-2|}\tau(|k-2|+\delta)F(\delta,|k-2|+\delta;|k-2|+1;\alpha^2)}{\tau(|k-2|+1)}\Bigg] \qquad (12)\end{aligned}$$

for $k \geq 1$. The autocovariance structure obtained in Equation (12) is similar to the autocovariance structure of GARMA$(1,2;\delta,1)$.

Hence, the variance and autocovariance of GARMA$(1,3;\delta,1)$ greatly reduces to the autocovariance of GARMA$(1,2;\delta,1)$when $\beta_3 = 0$. Furthermore, when $\delta = 1$, Propositions 1 and 2 takes the form of the variance and autocovariance of ARMA(1,3).

NUMERICAL EXAMPLES

This section presents the numerical results of the variance using Proposition 1 and the autocorrelation (ACF) using Proposition 2 of the GARMA$(1,3;\delta,1)$model for selected values of the parameters.

Table 1 shows the computed results of the variance using Proposition 1 when $\alpha = 0.9$, $\beta_1 = 0.3$, $\beta_2 = 0.2$, $\beta_3 = 0.2$ and $\sigma^2 = 1$ with δ ranging from 0.2 to 1.8. It it can be seen that the variance is reducing for $\delta = 0.2$ to $\delta = 0.4$ and increasing for $\delta = 0.6$ to $\delta = 1.8$.

Table 1 The values of $\gamma(0)$ ($\alpha = 0.9, \beta_1 = 0.3, \beta_2 = 0.2, \beta_3 = 0.2, \sigma^2 = 1$).

δ	0.2	0.4	0.6	0.8	1.0	1.4	
$\gamma(0)$	1.0804	1.0435	1.0745	1.2103	1.5347	3.7230	

Table 2 and 3 show the tabulated values of the ACF function of GARMA$(1,3;\delta,1)$model, using Proposition 2 and $\rho_k = \frac{\gamma_0}{\gamma_k}$, with k ranging from 1 to 20 and δ ranging from 0.2 to 1.8. Table 2 shows the computed results of the ACF when $\alpha = 0.9$, $\beta_1 = 0.3$, $\beta_2 = 0.2$, $\beta_3 = 0.2$ and $\sigma^2 = 1$, whereas Table 3 shows the computed results of the ACF when $\alpha = 0.9$, $\beta_1 = 0.3$, $\beta_2 = 0.1$, $\beta_3 = 0.1$ and $\sigma^2 = 1$.

Table 2 The values of ρ_k ($\alpha = 0.9, \beta_1 = 0.3, \beta_2 = 0.2, \beta_3 = 0.2, \sigma^2 = 1$).

k	δ						
	0.2	0.4	0.6	0.8	1.0	1.4	1.8
0	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
1	-0.0588	0.0744	0.2304	0.4049	0.5820	0.8498	0.9578
2	-0.1164	-0.0816	-0.0017	0.1330	0.3153	0.6860	0.8876
3	-0.1802	-0.1691	-0.1200	-0.0142	0.1535	0.5551	0.8146
4	-0.0238	-0.0396	-0.0313	0.0228	0.1381	0.4830	0.7519
5	-0.0091	-0.0160	-0.0086	0.0322	0.1243	0.4288	0.6954
6	-0.0043	-0.0071	0.0003	0.0342	0.1119	0.3842	0.6436
7	-0.0022	-0.0031	0.0043	0.0335	0.1007	0.3461	0.5957
8	-0.0012	-0.0010	0.0060	0.0317	0.0906	0.3128	0.5512
9	-0.0006	0.0000	0.0067	0.0294	0.0816	0.2833	0.5098
10	-0.0003	0.0006	0.0067	0.0270	0.0734	0.2569	0.4712
11	-0.0001	0.0009	0.0065	0.0246	0.0661	0.2332	0.4353
12	-0.0000	0.0010	0.0062	0.0223	0.0595	0.2118	0.4019
13	0.0001	0.0011	0.0057	0.0202	0.0535	0.1925	0.3708
14	0.0001	0.0010	0.0053	0.0182	0.0482	0.1749	0.3419
15	0.0001	0.0010	0.0048	0.0164	0.0433	0.1590	0.3150
16	0.0001	0.0009	0.0044	0.0148	0.0390	0.1445	0.2901
17	0.0001	0.0009	0.0039	0.0133	0.0351	0.1314	0.2671
18	0.0001	0.0008	0.0035	0.0119	0.0316	0.1194	0.2457
19	0.0001	0.0007	0.0032	0.0107	0.0284	0.1085	0.2259
20	0.0001	0.0006	0.0029	0.0096	0.0256	0.0986	0.2076

Table 3 The values of ρ_k($\alpha = 0.9, \beta_1 = 0.3, \beta_2 = 0.1, \beta_3 = 0.1, \sigma^2 = 1$).

k	δ						
	0.2	0.4	0.6	0.8	1.0	1.4	1.8
0	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
1	-0.1056	0.0618	0.2698	0.4972	0.6995	0.9219	0.9808
2	-0.0459	0.0180	0.1472	0.3387	0.5501	0.8437	0.9480
3	-0.0815	-0.0377	0.0656	0.2371	0.4455	0.7719	0.9090
4	-0.0020	0.0199	0.0894	0.2225	0.4009	0.7150	0.8684
5	0.0035	0.0248	0.0847	0.2003	0.3608	0.6627	0.8264
6	0.0044	0.0237	0.0762	0.1787	0.3247	0.6135	0.7837
7	0.0043	0.0213	0.0675	0.1590	0.2923	0.5673	0.7409
8	0.0038	0.0188	0.0595	0.1414	0.2630	0.5238	0.6984
9	0.0034	0.0164	0.0523	0.1256	0.2367	0.4829	0.6567
10	0.0029	0.0143	0.0460	0.1117	0.2131	0.4447	0.6161
11	0.0025	0.0125	0.0404	0.0994	0.1918	0.4090	0.5769
12	0.0022	0.0108	0.0356	0.0884	0.1726	0.3757	0.5391
13	0.0018	0.0094	0.0313	0.0787	0.1553	0.3449	0.5030
14	0.0016	0.0082	0.0276	0.0701	0.1398	0.3162	0.4685
15	0.0014	0.0072	0.0243	0.0625	0.1258	0.2898	0.4358
16	0.0012	0.0063	0.0215	0.0557	0.1132	0.2653	0.4048
17	0.0011	0.0055	0.0190	0.04971	0.1019	0.2427	0.3756
18	0.0009	0.0048	0.0168	0.0444	0.0917	0.2219	0.3481
19	0.0008	0.0042	0.0149	0.0396	0.0825	0.2028	0.3223
20	0.0007	0.0037	0.0132	0.0353	0.0743	0.1852	0.2981

Table 4 The values of ρ_k($\alpha = 0.9, \beta_1 = 0.2, \beta_2 = 0.5, \beta_3 = 0.1, \sigma^2 = 1$).

k	δ						
	0.2	0.4	0.6	0.8	1.0	1.4	1.8
0	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
1	0.0478	0.1469	0.2604	0.3886	0.5267	0.7815	0.9261
2	-0.3443	-0.2932	-0.2071	-0.0778	0.0962	0.5059	0.8007
3	-0.1118	-0.1332	-0.1285	-0.082	0.0203	0.3672	0.6966
4	-0.0229	-0.045	-0.0564	-0.0416	0.0183	0.2934	0.6132
5	-0.0104	-0.0232	-0.0321	-0.0248	0.0164	0.2409	0.5433
6	-0.0057	-0.0137	-0.0202	-0.0158	0.0148	0.2025	0.4833
7	-0.0035	-0.0088	-0.0135	-0.0105	0.0133	0.1724	0.4313
8	-0.0022	-0.0059	-0.0094	-0.0071	0.012	0.1483	0.3856
9	-0.0015	-0.0041	-0.0067	-0.0049	0.0108	0.1284	0.3455
10	-0.0011	-0.003	-0.0049	-0.0033	0.0097	0.1118	0.3099
11	-0.0008	-0.0022	-0.0037	-0.0029	0.0087	0.0978	0.2783
12	-0.0006	-0.0016	-0.0027	-0.0015	0.0079	0.0859	0.2501
13	-0.0004	-0.0012	-0.0021	-0.001	0.0071	0.0756	0.225
14	-0.0003	-0.0009	-0.0016	-0.0006	0.0064	0.0668	0.2025
15	-0.0002	-0.0007	-0.0012	-0.0003	0.0057	0.0591	0.1824
16	-0.0002	-0.0006	-0.0009	-0.0001	0.0052	0.0524	0.1644
17	-0.0001	-0.0004	-0.0007	0	0.0046	0.0465	0.1482
18	-0.0001	-0.0003	-0.0006	0.0001	0.0042	0.0414	0.13336
19	-0.0001	-0.0003	-0.0004	0.0002	0.0038	0.0368	0.1205
20	-0.0001	-0.0002	-0.0003	0.0002	0.0034	0.0328	0.1087

The tabulated values in Table 2, 3 and 4 are used to plot the ACF graphs given in Figures 1, 2 and 3 respectively.

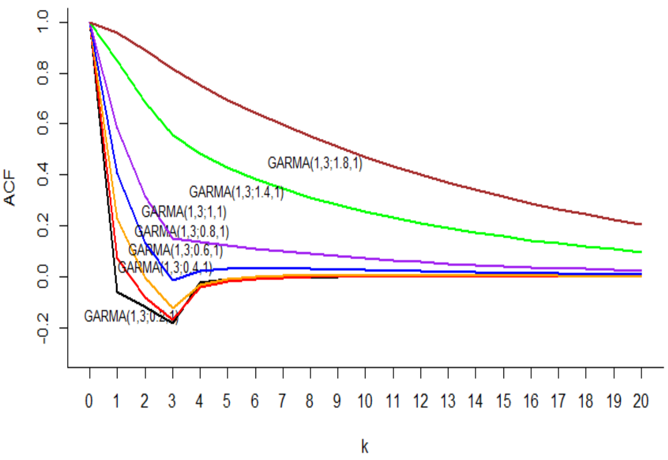

Figure 1 ACF Plot of $\text{GARMA}(1,3;\delta,1)$ when $\alpha = 0.9$, $\beta_1 = 0.3$, $\beta_2 = 0.2$, $\beta_3 = 0.2$ and $\sigma^2 = 1$.

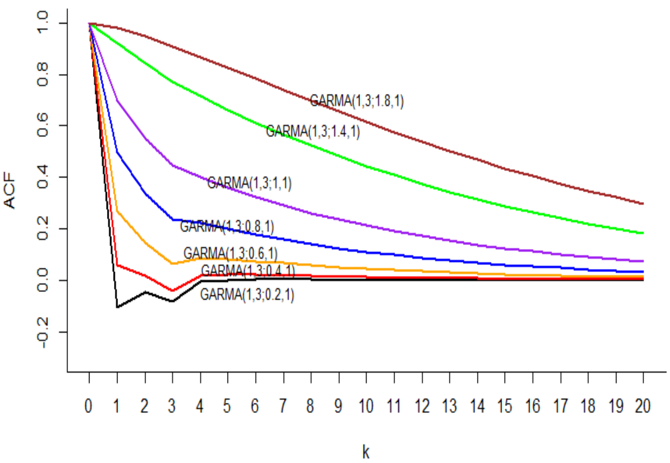

Figure 2 ACF Plot of $\text{GARMA}(1,3;\delta,1)$ when $\alpha = 0.9$, $\beta_1 = 0.3$, $\beta_2 = 0.1$, $\beta_3 = 0.1$ and $\sigma^2 = 1$.

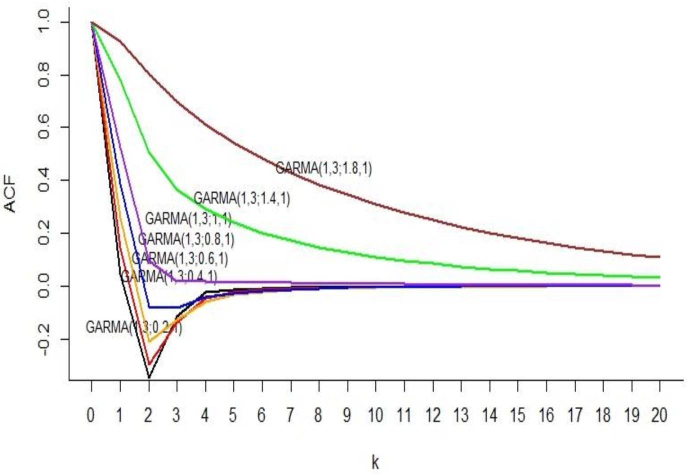

Figure 3 ACF Plot of $\text{GARMA}(1,3;\delta,1)$ when $\alpha = 0.9$, $\beta_1 = 0.2$, $\beta_2 = 0.5$, $\beta_3 = 0.1$ and $\sigma^2 = 1$.

Figure 1 shows that the ACF values increase when the δ values increase. In general, when δ values are 0.2, 0.4 and 0.6, the ACF values are negative in nature. The ACF values are positive when δ 0.8, 1.0, 1.4 and 1.8 is. The ACF values are also decreasing gradually when δ a 1.4 and 1.8. In addition, when $\delta > 1$, the ACF of the corresponding model is above the standard $\text{ARMA}(1,3)$ model, that is $\text{GARMA}(1,3;1,1)$.

Figure 2 also shows that when δ values increase, the ACF values increase as well. The ACF values are negative in nature when δ is 0.2 and 0.4. When δ is 0.6, 0.8, 1.0, 1.4 and 1.8, all the ACF values are positive in nature.

Similar to Figure 1 and Figure 2, Figure 3 shows that the ACF values increase when the δ values increase. The ACF values in Figure 3 are negative when δ values are 0.2, 0.4, 0.6 and 0.8. The positive ACF values can be seen when δ values are 1.0, 1.4 and 1.8.

The observations in Figure 1, Figure 2 and Figure 3 are not seen in $\text{GARMA}(1,2;\delta,1)$, as in Pillai (2012), as there were mixed positive and negative values of ACF for the same δ values. As in the case of Figure 1, Figure 2 and Figure 3 also show that when δ is 1.4 and 1.8, the ACF values are decreasing gradually. Gradual decrease of ACF in $\text{GARMA}(1,2;\delta,1)$ can only be seen when $\delta = 1.8$. In addition, when $\delta > 1$, ACF of the corresponding models are above the standard ARMA(1,3) model. Compared with the ACF values of $\text{GARMA}(1,2;\delta,1)$, the ACF values of the $\text{GARMA}(1,3;\delta,1)$ model start to decrease more gradually from a lower δ value.

From the above figures, we notice that the $\text{GARMA}(1,3;\delta,1)$ model can represent various structures of the ACF functions. Therefore, this model might serve as an alternative for modeling time series data.

CONCLUSION

The univariate ARMA model can be extended to a class of GARMA models. The objective of this paper was to derive the variance and the autocovariance properties of the $\text{GARMA}(1,3;\delta,1)$ as was shown in the propositions. The special cases presented also shows that the $\text{GARMA}(1,3;\delta,1)$ model can be reduced to the $\text{ARMA}(1,3)$ structure. From the numerical studies, it shows that the $\text{GARMA}(1,3;\delta,1)$ model can be used to represent various structures of the ACF functions. The results of this study contribute to the theory of the general order GARMA model.

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