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AUTOMATIC CONTINUITY OF CONCAVE FUNCTIONS

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ABSTRACT. A necessary and sufficient condition is given that a semicontinuous, nonnegative, concave function on a finite dimensional closed convex set X necessarily be continuous at a point $x_0 \in X$. Application of this criterion at all points of X yields a characterization, due to Gale, Klee and Rockafellar, of convex polyhedra in terms of continuity of their convex functions.

Let V be a real vector space of dimension $n < \infty$. Let $X \subseteq V$ be a closed convex body. Let ϕ be a concave, nonnegative function on X. (Recall ϕ is concave if $-\phi$ is convex.) Define $G^-(\phi)$, the *subgraph* of ϕ, as the subset of $V \times \mathbf{R}$ specified by

$$G^-(\phi) = \{(x,t)\colon x \in X,\ 0 \leq t \leq \phi(x)\}. \tag{1}$$

If ϕ is not identically zero then $G^-(\phi)$ will be a convex body in $V \times \mathbf{R}$. We call ϕ *semicontinuous* if $G^-(\phi)$ is a closed subset of $V \times \mathbf{R}$. (This is usually called upper semicontinuity; since lower semicontinuity is not very important here, we let the "upper" be understood implicitly.) Observe that this is equivalent to the superlevel sets

$$L^+(\phi, s) = \{x \in X\colon s \leq \phi(x)\}, \qquad s \geq 0, \tag{2}$$

being closed. Observe also that the $L^+(\phi, s)$ are convex.

We say X is *polyhedral* if it is specified by a finite number of linear inequalities

$$X = \{v \in V\colon \lambda_i(v) \leq b_i, \lambda_i \in V^*, b_i \in \mathbf{R}, 1 \leq i \leq m\}. \tag{3}$$

In [**GKR**] (see also [**R**, §10]) it is shown that if ϕ is a nonnegative, concave, semicontinuous function on X, and X is polyhedral, then ϕ is in fact continuous. (Actually, in [**GKR**], convex functions are considered; but concave and convex are interchangeable here.) The purpose of this note is to refine the result by giving a pointwise criterion for automatic continuity. If our condition holds at all points of a convex set X, then X is close to being polyhedral. (More precisely it is boundedly polyhedral in the sense of [**GKR**]; see Proposition 3.)

With X as above, suppose that for some $t > 0$ we have a closed convex set of $Y \subseteq V \times [0,t]$ such that

$$\begin{aligned} &\text{(a) } Y \cap (V \times \{0\}) = X \times \{0\} \\ &\text{(b) If } (x,r) \in Y, \text{ then } (x,r') \in Y \text{ for } 0 \leq r' \leq r. \end{aligned} \tag{4}$$

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Then the recipe

(c) $\phi_Y(x) = \max\{r\colon (x,r) \in Y\}$, $x \in X$, defines a concave nonnegative function on X, and

(d) $Y = G^-(\phi_Y)$.

Denote by $\mathrm{SCNC}(X)$ the set of semicontinuous, nonnegative concave functions on X. It is straightforward to check that the sum of two functions in $\mathrm{SCNC}(X)$ is again in $\mathrm{SCNC}(X)$. Also a positive scalar multiple of an element in $\mathrm{SCNC}(X)$ is again an element. Thus $\mathrm{SCNC}(X)$ is a cone in the space of all real-valued functions on X. Also given a family $\{\phi_i\}_{i\in I}$ of functions in $\mathrm{SCNC}(X)$ (the index set I may be infinite), we may form their infimum

$$\text{(5)(a)} \qquad \inf\{\phi_i\}(x) = \inf\{\phi_i(x)\colon i \in I\}, \qquad x \in X.$$

It is easy to see that $\inf\{\phi_i\}$ is concave and nonnegative. We also clearly have

$$\text{(b)} \qquad G^-(\inf\{\phi_i\}) = \bigcap_i G^-(\phi_i)$$

so that $\inf\{\phi_i\}$ again belongs to $\mathrm{SCNC}(X)$.

Let $Z \subseteq X$ be an arbitrary subset of X, and let f be an arbitrary real-valued function on Z. Consider the set of ϕ in $\mathrm{SCNC}(X)$ such that ϕ dominates f on Z (i.e., $\phi(z) \geq f(z)$ for all $z \in Z$). Evidently, the infimum of such ϕ will again dominate f. Thus if there are any elements of $\mathrm{SCNC}(X)$ dominating f on Z, there is a minimum one. In particular, given a point $x_0 \in X$, there is a minimum element of $\mathrm{SCNC}(X)$ taking the value 1 at x_0.

Proposition 1: Given $x_0 \in X$, define a function $E_X(x_0, x)$ on X by

$$\text{(6)} \qquad \begin{aligned} E_X(x_0,x) &= \sup\{(t-1)/t\colon x_0 + t(x-x_0) \in X\}, \qquad x \in X, \\ &= \sup\{s \in [0,1]\colon x = sx_0 + (1-s)z \text{ for some } z \in X\}. \end{aligned}$$

Then $E_X(x_0,\cdot)$ is the minimum among elements of $\mathrm{SCNC}(X)$ taking the value 1 at x_0.

REMARK. In pictorial terms we may describe the (closure of the) graph of $E_X(x_0,\cdot)$ as the surface of the cone with base $X \times \{0\}$ and vertex $(x_0, 1)$.

PROOF. In $V \times \mathbf{R}$, let $C(X,x_0)$ denote the closed convex hull of the points $(x,0)$, $x \in X$, and the point $(x_0,1)$. Since X is convex, the convex hull of $X \times \{0\}$ and $(x_0,1)$ is the set $\{(sx_0 + (1-s)y, s)\colon y \in X,\ 0 \leq s \leq 1\}$ and $C(X,x_0)$ will be the closure of this set. Suppose $x \neq x_0$, and

$$(x,r) = (sx_0 + (1-s)y, s).$$

Then $r = s < 1$, and

$$y = x_0 + (1-s)^{-1}(x - x_0)$$

belongs to X. Setting $t = (1-s)^{-1}$ we have

$$r = s = 1 - t^{-1} = (t-1)/t.$$

From the convexity of X it is clear that if (x,r) is in $C(X,x_0)$, then so is (x,r') for $0 \leq r' \leq r$. Hence $C(X,x_0)$ satisfies conditions (4)(a)(b), and comparing (4)(c)(d) with (6) shows

$$C(X,x_0) = G^-(E_X(x_0,\cdot)).$$

Furthermore, if ϕ is any function in $\mathrm{SCNC}(X)$ such that $\phi(x_0) \geq 1$, then obviously $G^-(\phi) \supseteq C(X,x_0)$, whence $\phi(x) \geq E_X(x_0,x)$. This proves Proposition 1.

Given a point x_0 in X, we say X is *conical at* x_0 if there exist
(i) a neighborhood U of x_0 in V and,
(ii) a closed convex cone $C \subseteq V$,
such that

$$X \cap U = (C + x_0) \cap U. \tag{7}$$

That is, near x_0, the set X looks like a translated cone. Note that C need not be a proper, also called pointed, cone. In particular, we could take $C = V$. Thus X is conical at all of its interior points.

PROPOSITION 2. (a) *If* $E_X(x_0, \cdot)$ (*cf. formula* (6)) *is continuous at* x_0*, then all functions in* SCNC(X) *are continuous at* x_0.
(b) *The function* $E_X(x_0, \cdot)$ *is continuous at* x_0 *if and only if* X *is conical at* x_0.

PROOF. (a) Suppose $E_X(x_0, \cdot)$ is continuous at x_0. Then given $\varepsilon > 0$, there is a neighborhood U of x_0 such that $E_X(x_0, x) > 1 - \varepsilon$ for $x \in U \cap X$. Consider $\phi \in \mathrm{SCNC}(X)$. By semicontinuity the superlevel set $L^+(\phi, \phi(x_0) + \varepsilon)$ (cf. (2)) is closed, and since it does not contain x_0, the set $U'' = V - L^+(\phi, \phi(x_0) + \varepsilon)$ is a neighborhood of x_0. If $\phi(x_0) = 0$, then since $\phi \geq 0$, we see $|\phi(x) - \phi(x_0)| < \varepsilon$ on $U'' \cap X$, so ϕ is continuous at x_0. If $\phi(x_0) > 0$, then it suffices to show $\phi(x)/\phi(x_0)$ is continuous at x_0. Hence we may assume $\phi(x_0) = 1$. Then on the neighborhood $U \cap U'' \cap X$ of x in X we have $1 + \varepsilon > \phi(x) > E_X(x_0, x) > 1 - \varepsilon$. Hence again ϕ is continuous at x_0.

(b) Let U be an open convex neighborhood of the origin in V, with compact closure $\overline{U}$. Then any neighborhood of x_0 contains a set of the form $x_0 + \delta U$ for a suitably small number $\delta > 0$. Let $\partial U = \overline{U} - U$ be the boundary of U. If C is any closed convex cone in V then we have

$$C = \bigcup_{s \geq 0} s(C \cap \partial U).$$

Suppose $E_X(x_0, \cdot)$ is continuous at x_0. Then we can find $\delta > 0$ such that $E_X(x_0, x) > \frac{1}{2}$ for $x \in (x_0 + \delta U) \cap X$. Set

$$B = (x_0 + \delta(\partial U)) \cap X, \qquad C = \bigcup_{s \geq 0} s(B - x_0).$$

Then C is a cone (a union of rays), and clearly

$$(C + x_0) \cap (x_0 + \delta U) \subseteq X \cap (x_0 + \delta U). \tag{8}$$

For if $x \in C + x_0$, then $x = x_0 + sb$, $b \in B$, $s \geq 0$; and if $x \in x_0 + \delta U$, then $s < 1$. Hence $x = (1 - s)x_0 + s(x_0 + b) \in X$, since X is convex. I claim that in fact the inclusion (8) is an equality. To verify this, consider a point v in $(x_0 + \delta U) \cap X$. Assume $y \neq x_0$. For suitable $t \geq 1$ the point $z = x_0 + t(y - x_0)$ will be in $x_0 + \delta(\partial U)$. If we show $z \in X$, the claim will be established. Suppose $z \notin X$. Since X is closed and convex, there is a number a, $0 < a < 1$ such that the points $z_r = x_0 + r(z - x_0)$ are in X for $r \leq a$, and are not in X for $r > a$. We see then that $E_X(x_0, z_a) = 0$. But since clearly $z_a \in X \cap (x_0 + \delta U)$, this contradicts our choice of δ. Thus inclusion (8) is an equality, and X is conical at x_0.

Conversely, suppose X is conical at x_0. Let U be a convex neighborhood of the origin, and C a closed convex cone such that

$$(9) \qquad \text{(a)} \quad (x_0 + U) \cap X = x_0 + (C \cap U).$$

Then for $0 < a \leq 1$, the set

$$\text{(b)} \qquad U'_a = (x_0 + aU) \cap X = x_0 + a(C \cap U)$$

will be a neighborhood of x_0 in X. Taking t in formula (6) to be $\frac{1}{a}$ we see that $E_X(x_0, x) \geq 1 - a$ if $x \in U'_a$. Hence $E_X(x_0, \cdot)$ is continuous at x_0. This proves Proposition 2.

The connection of the above two results with automatic continuity is provided by the following result. Given a point $x_0 \in X$, we say X is *polyhedral at* x_0 if there is a polyhedral closed convex subset $P_{x_0} \subseteq X$ such that P_{x_0} contains a neighborhood of x_0 in X. We say S is *locally polyhedral* if X is polyhedral at each of its points. We say X is *semilocally* polyhedral or *boundedly* polyhedral if any compact subset $C \subseteq X$ is contained in a polyhedral subset $P \subseteq X$.

This definition may seem superficially different from the definition of boundedly polyhedral in [**GKR**, p. 867], but it is easily seen to be equivalent.

PROPOSITION 3. *The following are equivalent*:
(i) *X is conical at each of its points.*
(ii) *X is locally polyhedral.*
(iii) *X is semilocally polyhedral.*
(iv) *All $\phi \in \mathrm{SCNC}(X)$ are continuous.*

REMARKS. (a) The implication (ii) $\Rightarrow$ (i) has a local version: if X is polyhedral at x_0, then X is conical at x_0; the implication (i) $\Rightarrow$ (ii) has no such local version.

(b) The implication (i) $\Rightarrow$ (ii) can be deduced from [**K**] (see especially Theorems 4.1 and 4.7), but we give a short proof.

(c) The equivalence (iii) $\Leftrightarrow$ (iv) amounts more or less to the equivalence $(BP) \Leftrightarrow$ S of Theorem 2 of [**GKR**].

PROOF. The implication (iii) $\Rightarrow$ (ii) is trivial. The implication (ii) $\Rightarrow$ (i) is routine; we omit its proof. The equivalence (i) $\Leftrightarrow$ (iv) follows from Proposition 2. The implication (ii) $\Rightarrow$ (iii) is Proposition 2.17 of [**K**].

We prove (i) $\Rightarrow$ (ii) by induction on $\dim X = \dim V$. If $\dim V = 2$, it is immediate since closed convex cones in 2-space are polyhedral. It follows directly from the definitions that if X is conical at every point, and $A \subseteq V$ is an affine subspace, then $X \cap A$ is conical at every point. Hence, if $\dim A < \dim V$ we may assume $A \cap X$ is locally polyhedral. If the neighborhood U in the proof of Proposition (2b) (see inclusion (8)) is chosen so that its closure $\overline{U}$ is polyhedral, then (using (ii) $\Rightarrow$ (iii)) we see that the intersection of X with each codimension one face of $x_0 + \delta\overline{U}$ will be polyhedral. Hence the set B is polyhedral (in the sense that it is a finite union of convex polyhedra; it may not be convex), and in particular has a finite number of extreme points. By (8) (which, we recall, is an equality, not just an inclusion) we see that $X \cap (x_0 + \delta\overline{U})$ is the convex hull of (the extreme points of) B and of x_0, and so is polyhedral.

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