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AUTOMATIC DISCONTINUITY OF INTERTWINING OPERATORS

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1. Introduction

Throughout this paper, we suppose that T and R are continuous linear operators on the Banach spaces X and Y, respectively. One of the basic problems in the theory of automatic continuity is the determination of conditions under which a linear transformation $S: X \to Y$ which satisfies $RS = ST$ is continuous or is discontinuous. Johnson and Sinclair [**4**], [**6**], [**11**; pp. 24–30] have given a variety of conditions on R and T which guarantee that all such S are automatically continuous. In this paper we consider the converse problem and find conditions on the range $S(X)$ which guarantee that S is automatically discontinuous. The construction of such automatically discontinuous S is then accomplished by a simple modification of a technique of Sinclair's [**10**; pp. 260–261], [**11**; pp. 21–23].

It is convenient to turn X and Y into modules over the ring of complex polynomials, $C[z]$, by defining $px = p(T)x$ and $py = p(R)y$, when p is a polynomial and x and y are elements of X and Y [**11**; pp. 20–21] [**7**; pp. 34–38]. Thus a linear operator $S: X \to Y$ satisfies $RS = ST$ precisely when S is a module homomorphism, and a linear subspace M of Y is R-invariant precisely when M is a submodule. A linear subspace D of Y is said to be R-*divisible* if $(R-\lambda)D = D$ for all complex numbers λ. Thus D is R-divisible if and only if it is a divisible $C[z]$-module [**11**; p. 20].

Suppose that D is an R-divisible subspace of Y, that $RS = ST$, and that $S(X)$ is an infinite-dimensional subspace of D. Our main result, Theorem 2.1, essentially says that if D is not "too large" then S must be discontinuous. A companion result, Theorem 2.2, says that S is discontinuous if $S(X)$ has countable codimension in D. Both theorems are proved by showing that the restriction of R to $S(X)$ cannot be the restriction of R to the range of a bounded operator. The proofs are similar in spirit to [**6**; Lemma 2.4, p. 534], [**5**; Method 2, p. 914], [**1**; Theorem 2.6, p. 1432]. In Theorem 2.3, we show how Sinclair's technique for constructing discontinuous S [**10**; pp. 260–261] can be modified to construct S satisfying the hypothesis of Theorems 2.1 and 2.2.

In Section 3, we prove some related results. In particular, in Theorem 3.1, we give necessary and sufficient conditions for the existence of finite-rank discontinuous S satisfying $RS = ST$.

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2. Subspaces of R-divisible spaces

Every divisible $C[z]$-module D is isomorphic to a direct sum of indecomposable divisible submodules. Each summand is either torsion free, and thus $C[z]$-isomorphic to the rational functions $C(z)$; or it is $(z-\lambda)$-primary for some fixed λ and is isomorphic to a fixed module [**7**; Theorem 4, p. 10]. This $(z-\lambda)$-primary module can be realised as the $C[z]$-module given by a linear transformation R for which the null-space $N(R-\lambda)^n$ has codimension one in $N(R-\lambda)^{n+1}$ for each non-negative integer n [**7**; p. 37]. While the decomposition of D is not unique, the number of free summands and the number of $(z-\lambda)$-primary summands for each λ depends only on D [**7**; Remark (b), p. 11]. We will say that D is of *countable type* if the number of summands in each isomorphism class is countable.

In proving our major result, Theorem 2.1 below, we show that if R is a bounded operator on a Banach space, then no R-divisible submodule of countable type can have an infinite-dimensional submodule which is the range of a bounded operator.

Theorem 2.1. *Suppose that T and R are bounded linear operators on the Banach spaces X and Y, respectively, and that $S: X \to Y$ is a linear transformation of infinite rank which satisfies $RS = ST$. If $S(X)$ is a subspace of an R-divisible module, D, of countable type, then S must be discontinuous.*

Proof. We will show that, for all λ, $(R-\lambda)S(X)$ has countable codimension in $S(X)$. This will show that $S(X)$ cannot be the range of a bounded operator. For if $S(X)$ is the range of a bounded operator, it can be given a norm under which it becomes a Banach space continuously embedded in Y [**1**; Formula (3.1), p. 1433]. The restriction, R', of R to $S(X)$ is then continuous by the closed graph theorem [**1**; p. 1433]. Since no operator range can have countably infinite codimension in another operator range [**1**; p. 1434], each $(R-\lambda)S(X)$ has finite codimension in $S(X)$. Hence, if λ belongs to the boundary of the spectrum of R', then λ must be a pole of finite rank of R' [**1**; Theorem (5.2), p. 1438] [**8**; Theorem (2.9), p. 205]. Since poles are isolated points of the spectrum, the spectrum of R' must consist entirely of finitely many poles of finite rank; but this contradicts the infinite-dimensionality of $S(X)$.

To finish the proof, we suppose that M is a submodule of an arbitrary divisible $C[z]$-module, D, of countable type, and that λ is a complex number. We must show that $(z-\lambda)M$ has countable codimension in M when M is considered as a complex vector space. We break the proof into three cases.

Case 1. D is a torsion module.

Then M is also a torsion module, so by the primary decomposition theorem [**7**; Theorem 1, p. 5] we can express M as the direct sum of its $(z-\lambda)$-primary component M_λ and a submodule N for which $(z-\lambda)N = N$. M_λ is contained in the $(z-\lambda)$-primary component of D, which, by hypothesis, has countable dimension as a complex vector space. So $(z-\lambda)M_\lambda$ must have countable codimension in M_λ, forcing $(z-\lambda)M$ to have countable codimension in M.

Case 2. D is torsion free.

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To say that D is a torsion-free divisible $C[z]$-module is equivalent to saying that D is a vector space over the field $C(z)$ of rational functions. Let $\{x_i\}$ be a maximal $C(z)$-linearly-independent subset of M. Let N be the $C[z]$-module generated by $\{x_i\}$ and let P be the $C(z)$ vector space generated by $\{x_i\}$. Clearly $N \subseteq M \subseteq P$, so that M/N is a $C[z]$-submodule of P/N. By hypothesis $\{x_i\}$ is countable, so by [**7**; Remark (b), p. 6] P/N is a divisible torsion $C[z]$-module of countable type. Hence, by Case 1, $(z-\lambda)M+N$ has countable codimension in M as a complex vector space. Since N has countable dimension as a complex vector space, $(z-\lambda)M$ has countable codimension in M.

Case 3. D is an arbitrary divisible module.

Let A be the torsion module of D. By Case 1, $(z-\lambda)(M\cap A)$ has countable codimension in $M\cap A$. Also $M/M\cap A$ is isomorphic to the submodule $(M+A)/A$ of D/A; so, by Case 2, $(z-\lambda)M+(M\cap A)$ has countable codimension in M. Hence $(z-\lambda)M=(z-\lambda)M+(z-\lambda)(M\cap A)$ has countable codimension in $(z-\lambda)M+(M\cap A)$, and therefore in M. This completes the proof of the theorem.

We now prove the analogue of Theorem 2.1 for submodules of small codimension in D.

Theorem 2.2. *Suppose that T and R are bounded operators on X and Y and that $S: X\to Y$ is an infinite-rank linear transformation which satisfies $RS=ST$. If $S(X)$ is a subspace of countable codimension in an R-divisible subspace D, then S must be discontinuous.*

Proof. For each complex number λ, $(R-\lambda)D=D$; so that $(R-\lambda)S(X)$ has countable codimension in $S(X)$. Hence, just as in the proof of Theorem 2.1, $S(X)$ would have to be finite-dimensional if it were the range of a bounded operator.

The proofs of the above two theorems essentially show that no "small" or "large" submodule of an R-divisible subspace can be the range of a bounded operator. On the other hand if Y does have a non-zero R-divisible subspace then its maximal R-divisible subspace, D, (which exists by [**7**; Theorem 3, p. 97]) always contains an infinite-dimensional submodule which is the range of a bounded operator. For Y contains a torsion-free R-divisible subspace [**11**; Theorem 3.3, p. 265]; so that if B is the algebra of bounded operators which commute with R, then there is a y in D for which By is an infinite-dimensional subspace of D. Notice that if we let T be the identity on B and $S: B\to Y$ be defined by $S(A)=Ax$, then S is a bounded operator satisfying $RS=ST$. By Theorem 2.1, we know that D cannot be of countable type. In fact, Thomas [**12**; Proposition 1.23] has shown that D contains a torsion-free R-divisible subspace which has uncountable dimension as a vector space over $C(z)$.

It is now easy to modify Sinclair's construction of discontinuous intertwining operators [**10**; pp. 260–261] to construct S which are discontinuous by virtue of Theorems 2.1 and 2.2.

Theorem 2.3. *Suppose that T and R are bounded linear operators on the Banach spaces X and Y and that T is not algebraic. If V is a linear subspace of the R-divisible space D and if V has dimension no greater than the power of the continuum, then there is* *a linear transformation* $S: X \to Y$ *for which* $RS = ST$ *and* $V \subseteq S(X) \subseteq D$. *If* V *has infinite dimension, and either* D *is of countable type or* V *has countable codimension in* D, *then* S *must be discontinuous.*

Proof. By [**12**; Corollary 1.22], X has a free submodule M whose rank has the power of the continuum. Thus we can construct a module homomorphism S_1 from M to D with $V \subseteq S_1(M)$. Since D is divisible it is injective [**3**; Theorem 1.7, p. 6] so that we can extend S_1 to a module homomorphism, S, from X to D [**3**; p. 5]. Since S is a module homomorphism, $RS = ST$; and by construction $V \subseteq S(X) \subseteq D$. The last statement in the theorem follows from Theorems 2.1 and 2.2.

3. Finite-rank and multilinear operators

In the previous section we considered conditions under which an infinite rank intertwining operator whose range was a subspace of an R-divisible space was automatically discontinuous. We did this by showing that certain infinite-dimensional submodules M of R-divisible subspaces could not be given norms under which they are continuously embedded in Y. In fact, the two problems are equivalent; for if M is a submodule which is continuously embedded in Y, then the inclusion map intertwines R with its restriction to M.

In this section we consider three questions suggested by the results in the previous section. In Theorem 3.1, we give necessary and sufficient conditions for the existence of discontinuous finite-rank intertwining operators. In Theorem 3.2, we give conditions under which an R-divisible subspace cannot be the span of the range of a bounded multilinear operator. And, in Theorem 3.3, we show that no non-zero intertwining operator can have a subspace of a free module as its range.

In Theorem 3.1, we will need the following definition from [**11**; Definition 3.1, p. 19]. The complex number λ is a *critical eigenvalue* of the pair (T, R) of bounded operators whenever λ is an eigenvalue of R, and the range of $T-\lambda$ has infinite codimension.

Theorem 3.1. *Suppose that* T *and* R *are bounded linear operators on the Banach spaces* X *and* Y. *Then every finite rank linear transformation* $S: X \to Y$ *which satisfies* $RS = ST$ *is continuous if and only if* (T, R) *has no critical eigenvalue.*

Proof. Johnson [**11**; Lemma 3.2, p. 19], [**4**; pp. 88–89] has shown that if (T, R) has a critical eigenvalue then there is a rank one discontinuous operator intertwining T and R.

Suppose, conversely, that there is a discontinuous finite rank linear transformation $S: X \to Y$ for which $RS = ST$. The null-space $N(S)$ is not closed. Hence, by restricting S and T if necessary, we can assume that $N(S)$ is a dense subspace of finite codimension in X. The restriction of R to $S(X)$ and the map induced by T on $X/N(S)$ are similar linear transformations on finite dimensional space and therefore have a common eigenvalue λ.

Clearly λ is an eigenvalue of R. Also $(T-\lambda)X + N(S)$ is a proper subspace of X. If λ were not a critical eigenvalue of (T, R) then $(T-\lambda)(X)$ would be a closed subspace of finite codimension in X [**11**; Lemma 3.3, p. 20]. Hence $(T-\lambda)X+N(S)$ would be a proper closed subspace of X, which contradicts the density of $N(S)$. This completes the proof of the theorem.

We now consider the R-divisibility of the linear span of the range of a multilinear operator.

Theorem 3.2. *Suppose that R is a bounded operator on Y and that M is the span of the range of a bounded multilinear operator from a product of Banach spaces to Y. If the spectrum $\sigma(R)$ is totally disconnected and if $(R-\lambda)M \supseteq M$ for all λ in the spectrum of R, then $M=0$.*

Proof. Fix a point λ in $\sigma(R)$. There is a sequence of open sets U_n with intersection $\{\lambda\}$ whose boundary does not intersect $\sigma(R)$ [**9**; Corollary 1, p. 83]. Let P_n be the spectral projection associated with the characteristic function of U_n. Since the spectral radius of $P_n(R-\lambda)$ approaches 0 and since $(R-\lambda)M \supseteq M$, there is a U_n, which we will call $U(\lambda)$, for which $P_n(M)=0$ [**2**; Theorem (2.2)].

Since $\sigma(R)$ is compact, it has a finite subset $\{\lambda_1, \lambda_2, \ldots, \lambda_k\}$ for which $U(\lambda_1)$, $U(\lambda_2), \ldots, U(\lambda_k)$ covers $\sigma(R)$. If $P(\lambda_i)$ is the spectral projection associated with $U(\lambda_i)$ then

$$(I-P(\lambda_1))(I-P(\lambda_2)) \ldots (I-P(\lambda_k))=0.$$

But if $x \in M$, $(I-P(\lambda_i))x=x$ for all i. This completes the proof.

The above proof used the fact that R had totally disconnected spectrum only to obtain a basis for the topology of $\sigma(R)$ to which we could associate a sufficiently well-behaved family of projections. Thus the theorem would hold, with essentially the same proof, if R were a spectral operator or if R satisfied [**6**; Condition 4.1, pp. 537–538].

Our final results considers subspaces of free modules rather than of divisible modules.

Theorem 3.3. *Suppose that T and R are bounded linear operators on X and Y and that $S: X \to Y$ is a linear transformation for which $RS=ST$. If $S(X)$ is a subspace of a free $C[z]$-module, then $S=0$.*

Proof. Since every submodule of a free $C[z]$-module is itself free [**3**; Theorem 1.4, p. 4], $X/N(S)$ is free. Every free module is projective [**3**; Proposition 1.1, p. 3], so X is the direct sum of $N(S)$ and a free module F. Since F is free, $(T-\lambda)F \neq F$ for any λ. Since F is complemented, this implies that $T-\lambda$ is never onto. But this is impossible if T is a bounded operator.

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