AUTOMATIC FATOU PROPERTY OF LAW-INVARIANT RISK MEASURES

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ABSTRACT. In the paper we investigate automatic Fatou property of law-invariant risk measures on a rearrangement-invariant function space \mathcal{X} other than L^{∞} . The main result is the following characterization: Every real-valued, law-invariant, coherent risk measure on \mathcal{X} has the Fatou property at every random variable $X \in \mathcal{X}$ whose negative tails have vanishing norm (i.e., $\lim_n \|X \mathbf{1}_{\{X \leq -n\}}\| = 0$) if and only if \mathcal{X} satisfies the Almost Order Continuous Equidistributional Average (AOCEA) property, namely, $d(\mathcal{CL}(X), \mathcal{X}_a) = 0$ for any $X \in \mathcal{X}_+$, where $\mathcal{CL}(X)$ is the convex hull of all random variables having the same distribution as X and $\mathcal{X}_a = \{X \in \mathcal{X} :$ $\lim_n \|X \mathbf{1}_{\{|X| \geq n\}}\| = 0\}$. As a consequence, we show that under the AOCEA property, every real-valued, law-invariant, coherent risk measure on \mathcal{X} admits a tractable dual representation at every $X \in \mathcal{X}$ whose negative tails have vanishing norm. Furthermore, we show that the AOCEA property is satisfied by most classical model spaces, including Orlicz spaces, and therefore the foregoing results have wide applications.

1. INTRODUCTION

The axiomatic theory of risk measures has been an active research area ever since their introduction in the landmark paper by Artzner et al [3]. Many properties of risk measures have been investigated in depth, and rigorous debates have taken place as to which of these properties ought to be regarded as a natural part of the definition of risk measures. Among the properties considered, law invariance is regarded as highly relevant in practice as real-world computations of risk are often based on probability distributions of financial positions. As a matter of fact, most concrete risk measures (such as Value-at-risk, Expected shortfall, Haezendonck-Goovaerts risk measures) and special classes of risk measures (such as distortion and quantile-based risk measures) are indeed law invariant. Research on general law-invariant risk measures has been intense and has produced many profound results; see, e.g., Bellini et al [4, 5], Chen et

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al [11], Filipović and Svindland [19, 20], Gao et al [23], Jouini et al [29, 30], Krätschmer et al [31], Kusuoka [32], Liu et al [35], Wang and Zitikis [44], and Weber [45].

Below we provide a brief review of one specific direction of research that highlights the importance of law invariance and motivates the present paper. In the context of a convex risk measure, or more generally, a convex functional ρ defined on a function space \mathcal{X} , lower semicontinuity with respect to a weak topology $\sigma(\mathcal{X}, \mathcal{Y})$ determined by a dual space \mathcal{Y} leads to a dual representation of ρ , thanks to the well-known Fenchel-Moreau Duality:

(1.1)
$$\rho(X) = \sup_{Y \in \mathcal{Y}} \left(\langle X, Y \rangle - \rho^*(Y) \right), \quad X \in \mathcal{X},$$

where

$$\rho^*(Y) = \sup_{X \in \mathcal{X}} (\langle X, Y \rangle - \rho(X)), \quad Y \in \mathcal{Y}.$$

In general, such dual representations play an important role in optimization and portfolio selection. For practical purposes, it is particularly desirable to be able to take the dual space \mathcal{Y} as a space of functions (as opposed to abstract linear functionals). In this case, the representation (1.1) is generally deemed as tractable or manageable. A concrete, and more verifiable, alternative to topological lower semicontinuity is order lower semicontinuity. It is usually termed as the Fatou property in the actuarial and risk management literature. Specifically, a functional $\rho : \mathcal{X} \to (-\infty, \infty]$ defined on a function space \mathcal{X} is said to satisfy the *Fatou property* if

$$\rho(X) \leq \liminf \rho(X_k)$$
 whenever $X_k \to X$ a.s. and $|X_k| \leq Y$ for some $Y \in \mathcal{X}$.

In the first result of its kind, Delbaen [15] showed that if a proper convex functional ρ on L^{∞} satisfies the (easier to verify) Fatou property, then it is $\sigma(L^{\infty}, L^1)$ lower semicontinuous and thus enables a tractable dual representation with L^1 as dual space by the Fenchel-Moreau Duality. Biagini and Frittelli [8] and Delbaen and Owari [16] then drew attention to the following natural question:

if a proper convex functional ρ on a general function space \mathcal{X} satisfies the Fatou property, must it admit a tractable dual representation?

Gao et al [25] showed that in general the answer is no for the class of Orlicz spaces. However, it was later proved that, surprisingly, if ρ is additionally *law invariant* then the answer is yes! See Gao et al [23] for Orlicz spaces and Tantrawan and Leung [43] for general rearrangement-invariant spaces. These results highlight the superior behavior that law invariance brings to the study of risk measures. Additional developments on the Fatou property and tractable dual representations of law-invariant risk measures can be found in the recent works of Bellini et al [4, 5, 6], Chen et al [11], Filipović et al [20], Gao et al [27, 28], Liu et al [34] and Svindland [42]. Perhaps the most striking result demonstrating the power of law invariance in connection with the Fatou property is the following theorem.

Theorem (Jouini et al [29]). A real-valued, convex, decreasing, law-invariant functional on L^{∞} has the Fatou property. Consequently, it is $\sigma(L^{\infty}, L^1)$ lower semicontinuous and admits a dual representation via L^1 .

This theorem can be viewed as a result on automatic continuity since the Fatou property is a type of continuity property (order lower semicontinuity, to be precise). Automatic continuity has long been an interesting research topic and probably has its roots in the well-known fact that a real-valued convex function on an open interval is continuous. In infinite-dimensional spaces, Birkhoff's Theorem states that a positive linear functional on a Banach lattice is norm continuous. It was later extended to the following celebrated theorem for real-valued convex functionals.

Theorem (Ruszczyński and Shapiro [41]). A real-valued, convex, decreasing functional on a Banach lattice is norm continuous.

This result has drawn extensive attention in optimization, operations research and risk management in the past decade. We remark that Fatou property and norm continuity do not imply each other but Fatou property is stronger than norm lower semicontinuity. We refer the reader to Biagini and Frittelli [8], Farkas et al [18] and Munari [38] for further results on automatic norm continuity properties of risk measures.

The present paper aims at investigating automatic Fatou property of law-invariant risk measures on general model spaces. In recent years, study of risk measures has been extended from Lebesgue spaces to more general settings such as Orlicz spaces (see, e.g., Biagini and Frittelli [8], Cheridito and Li [12, 13], Krätschmer et al [31], Gao et al [23]) and other more general spaces (see, e.g., Bellini [4, 5], Chen et al [11], Drapeau and Kupper [14], Farkas et al [18], and Frittelli and Rosazza Gianin [22]). In this paper, we adopt general rearrangement-invariant function spaces as our model space (see the precise definition in the "Notation and Facts" subsection below). The reason is two-fold. Firstly, among all rearrangement-invariant spaces, we characterize precisely the spaces where the Fatou property automatically holds. This provides genuine insights into exactly what makes the Fatou property automatic. Secondly, the proofs on general model spaces are not more complicated than those on special model spaces such as Orlicz spaces. Quite on the contrary, working on special spaces often obscures the essential ingredients in the proofs with space-related technicalities.

The paper is structured as follows. Let \mathcal{X} be a rearrangement-invariant space other than L^{∞} over a non-atomic probability space (we refer to the next subsection for the reasons for excluding L^{∞}). In Section 2, Example 2.1 shows that there is always a realvalued law-invariant coherent risk measure on \mathcal{X} that fails the Fatou property at every $X \in \mathcal{X}$ whose negative tails do not have vanishing norm, i.e., $\lim_{n \to \infty} ||X \mathbf{1}_{\{X \leq -n\}}|| > 0$. Since automatic Fatou property for a general law-invariant coherent risk measure cannot be expected over the whole space in general, we seek to determine the spaces where the automatic Fatou property has maximal validity. The Main Theorem (Theorem 2.2) characterizes the spaces \mathcal{X} on which every real-valued, law-invariant, coherent risk measure is automatically Fatou at all $X \in \mathcal{X}$ whose negative tails have vanishing norm. The property of \mathcal{X} required, which we call Almost Order Continuous Equidistributional Average (AOCEA), asks that every nonnegative $X \in \mathcal{X}$ possesses averages of equidistributed copies that are arbitrarily close to the order continuous part \mathcal{X}_a of \mathcal{X} ; see the subsection below for definition of \mathcal{X}_a . A thorough analysis of the AOCEA property is carried out in Section 2. The results are then applied to prove the Main Theorem in Section 3. Since for a general law-invariant risk measure, the Fatou property is only valid on a proper subset of \mathcal{X} , topological lower semicontinuity and Fenchel-Moreau dual representation no longer follow directly from previously known results. Nevertheless, in Section 4, under the AOCEA property, we recover the result that every convex, decreasing, law-invariant functional $\rho: \mathcal{X} \to \mathbb{R}$ is $\sigma(\mathcal{X}, \mathcal{X}')$ lower semicontinuous at any $X \in \mathcal{X}$ whose negative tails have vanishing norm (Theorem 4.3); see the subsection below for definition of \mathcal{X}' . Furthermore, the dual representation formula is valid at such X's (Theorem 4.7).

1.1. Notation and Facts. Throughout the paper, $(\Omega, \mathcal{F}, \mathbb{P})$ stands for a non-atomic probability space. It is a standard fact that there exists a random vector on Ω with any prescribed joint distribution. In particular, if \mathbf{X}' is a random vector \mathbf{X} on $(\Omega, \mathcal{F}, \mathbb{P})$ that has the same distribution as \mathbf{X}' , which we write as $\mathbf{X} \sim \mathbf{X}'$. These notions and facts extend in a plain manner to finite measure spaces of the same measure. In particular, we will often consider a set $A \in \mathcal{F}$ endowed with the probability structure restricted from $(\Omega, \mathcal{F}, \mathbb{P})$. Given a (measurable) partition $\pi = (A_i)_{i \in I}$ of Ω , where Iis at most countable, we often define random variables on Ω by specifying its values on each A_i and then gluing the pieces together. A frequently used fact is as follows. Let $\pi' = (A'_i)_{i \in I}$ be a partition of Ω' for another probability space $(\Omega', \mathcal{F}', \mathbb{P}')$, such that $\mathbb{P}(A_i) = \mathbb{P}'(A'_i)$ for each $i \in I$. If \mathbf{X} and \mathbf{X}' are random vectors on Ω and Ω' , respectively, such that $\mathbf{X}|A_i \sim \mathbf{X}'|A'_i$ for any $i \in I$, then $\mathbf{X} \sim \mathbf{X}'$.

Let $L^0 := L^0(\Omega)$ be the space of all random variables on $(\Omega, \mathcal{F}, \mathbb{P})$ (to be precise, L^0 consists of equivalent classes of random variables modulo a.s. equality). Throughout the paper, \mathcal{X} stands for a *rearrangement-invariant (abbr., r.i.) space* over $(\Omega, \mathcal{F}, \mathbb{P})$, other than L^{∞} . By an r.i. space, we mean that $\mathcal{X} \neq \{0\}$ and \mathcal{X} is a Banach space of random variables in L^0 such that for any $X \in \mathcal{X}$,

- (1) if $Y \in L^0$ and $|Y| \le |X|$ a.s., then $Y \in \mathcal{X}$ and $||Y|| \le ||X||$,
- (2) if $Z \in L^0$ and $Z \sim X$, then $Z \in \mathcal{X}$ and ||Z|| = ||X||.

The classical L^p -spaces are all r.i. spaces. It is well known (see, e.g., [11, Appendix]) that $L^{\infty} \subset \mathcal{X} \subset L^1$ and there exists a constant C > 0 such that

(1.2)
$$||X||_{L^1} \le C||X|| \quad \text{for every } X \in \mathcal{X}.$$

Besides the definition, two notions about r.i. spaces are needed for the paper. Given an r.i. space \mathcal{X} , its associate space \mathcal{X}' is defined by

$$\mathcal{X}' = \{ Y \in L^0 : \mathbb{E}[|XY|] < \infty \text{ for any } X \in \mathcal{X} \}.$$

 \mathcal{X}' itself is also an r.i. space and it sits naturally as a closed subspace in the norm continuous dual \mathcal{X}^* of \mathcal{X} . In the literature, dual representations with respect to \mathcal{X}' are regarded as "tractable" (cf., e.g., [4, 5]). In the Banach lattice literature, \mathcal{X}' is just the order continuous dual \mathcal{X}_n^{\sim} . The order continuous part \mathcal{X}_a of \mathcal{X} is defined by

(1.3)
$$\mathcal{X}_a := \left\{ X \in \mathcal{X} : \lim_{\mathbb{P}(A) \to 0} \| X \mathbf{1}_A \| = 0 \right\}.$$

The members in \mathcal{X}_a are termed as order continuous in \mathcal{X} , or as having absolutely continuous norm in some literature. An r.i. space \mathcal{X} is said to be *order continuous* if $\mathcal{X}_a = \mathcal{X}$, or equivalently, if $\mathcal{X}' = \mathcal{X}^*$ ([37, Theorem 2.4.2]). It is well known that $(L^p)' = L^q$ for any $p, q \in [1, \infty]$ with $\frac{1}{p} + \frac{1}{q} = 1$ and that $L^p(1 \leq p < \infty)$ is order continuous but L^∞ is not. In fact, $(L^\infty)_a = \{0\}$.

We exclude L^{∞} from our consideration for two main reasons. Firstly, the results for L^{∞} are already established in [29]. Secondly, the results for L^{∞} and for other r.i. spaces hold for dramatically different reasons, which we briefly point out now. When $\mathcal{X} \neq L^{\infty}$, it is known that

$$L^{\infty} \subset \overline{L^{\infty}} = \mathcal{X}_a \subset \mathcal{X} \subset L^1,$$

where the closure of L^{∞} is taken with respect to the norm of \mathcal{X} . Therefore by (1.3), for any $X \in L^{\infty}$, $\lim_{\mathbb{P}(A)\to 0} ||X\mathbf{1}_A|| = 0$. This fact, which clearly fails in L^{∞} , will serve as a primary tool in our developments. It is the essential technical difference between our model space \mathcal{X} and L^{∞} . Finally, we remark that since $\mathcal{X} \neq L^{\infty}$,

(1.4)
$$X \in \mathcal{X}_a \iff \lim_n \|X \mathbf{1}_{\{|X| \ge n\}}\| = 0.$$

A function $\Phi : [0, \infty) \to [0, \infty)$ is called an Orlicz function if it is convex, increasing, non-constant, and $\Phi(0) = 0$. The Orlicz space $L^{\Phi} := L^{\Phi}(\Omega)$ is the space of all $X \in L^{0}$ such that

$$|X||_{\Phi} := \inf \left\{ \lambda > 0 : \mathbb{E} \left[\Phi \left(\frac{|X|}{\lambda} \right) \right] \le 1 \right\} < \infty.$$

 L^{Φ} with the Luxemburg norm $\|\cdot\|_{\Phi}$ is an r.i. space. Furthermore, $(L^{\Phi})' = L^{\Psi}$ if $\lim_{t\to\infty} \frac{\Phi(t)}{t} = \infty$ and $(L^{\Phi})' = L^{\infty}$ otherwise, where Ψ is the conjugate of Φ given by

$$\Psi(s) = \sup\{st - \Phi(t) : t \ge 0\} \text{ for all } s \ge 0.$$

The order continuous part of L^{Φ} is given by the space of all $X \in L^{\Phi}$ such that

$$\mathbb{E}\left[\Phi\left(\frac{|X|}{\lambda}\right)\right] < \infty \quad \text{for all } \lambda > 0.$$

This space is also called the Orlicz heart of L^{Φ} and is denoted by H^{Φ} . It is known that L^{Φ} is order continuous (i.e., $L^{\Phi} = H^{\Phi}$) iff Φ satisfies the Δ_2 -condition, i.e., there exist C > 0 and $t_0 > 0$ such that

$$\Phi(2t) \le C\Phi(t), \quad \forall \ t > t_0.$$

A functional $\rho: \mathcal{X} \to (-\infty, \infty]$ is called a coherent risk measure if it is

- (1) decreasing, i.e., $\rho(X_1) \leq \rho(X_2)$ whenever $X_1, X_2 \in \mathcal{X}$ satisfies $X_1 \geq X_2$,
- (2) subadditive, i.e., $\rho(X_1 + X_2) \leq \rho(X_1) + \rho(X_2)$ for any $X_1, X_2 \in \mathcal{X}$,
- (3) positive homogeneous, i.e., $\rho(\lambda X) = \lambda \rho(X)$ for any $X \in \mathcal{X}$ and any real number $\lambda \ge 0$,
- (4) cash invariant, i.e., $\rho(X + m\mathbf{1}) = \rho(X) m$ for any $X \in \mathcal{X}$ and $m \in \mathbb{R}$.

A functional $\rho : \mathcal{X} \to (-\infty, \infty]$ is said to be convex if $\rho(\lambda X_1 + (1 - \lambda)X_2) \leq \lambda \rho(X_1) + (1 - \lambda)\rho(X_2)$ for any $X_1, X_2 \in \mathcal{X}$ and $\lambda \in [0, 1]$. ρ is said to be law invariant if $\rho(X_1) = \rho(X_2)$ for any $X_1, X_2 \in \mathcal{X}$ with $X_1 \sim X_2$.

Given any $X \in \mathcal{X}$, a functional $\rho : \mathcal{X} \to (-\infty, \infty]$ is said to have the Fatou property at X if $\rho(X) \leq \liminf_n \rho(X_n)$ for any sequence (X_n) that order converges to X in \mathcal{X} . By order convergence in \mathcal{X} , we mean that $X_n \xrightarrow{a.s.} X$ and there exists $X_0 \in \mathcal{X}$ such that $|X_n| \leq X_0$ for all $n \in \mathbb{N}$. ρ is said to be $\sigma(\mathcal{X}, \mathcal{X}')$ lower semicontinuous at X if $\rho(X) \leq \liminf_{\alpha} \rho(X_{\alpha})$ for any net (X_{α}) that converges to X in $\sigma(\mathcal{X}, \mathcal{X}')$, or equivalently, if $\{\rho > \lambda\}$ is a $\sigma(\mathcal{X}, \mathcal{X}')$ -neighborhood of X for any real number λ such that $\rho(X) > \lambda$.

Finally, the convex hull of a set $\mathcal{A} \subset \mathcal{X}$ is denoted by $\operatorname{co}(\mathcal{A})$. The distance of $X \in \mathcal{X}$ and $\mathcal{A} \subset \mathcal{X}$ is given by $\operatorname{d}(X, \mathcal{A}) = \inf_{Y \in \mathcal{A}} ||X - Y||$; the distance of $\mathcal{A}, \mathcal{B} \subset \mathcal{X}$ is given by $\operatorname{d}(\mathcal{A}, \mathcal{B}) = \inf_{X \in \mathcal{A}, Y \in \mathcal{B}} ||X - Y||$. The positive and negative parts of $X \in \mathcal{X}$ are given by $X^+ := \max\{X, 0\}$ and $X^- := \max\{-X, 0\}$, respectively.

We refer the reader to [7] for facts and results on general r.i. spaces, to [17] for detailed information on Orlicz spaces, and to [2, 37] for relevant terminology and facts on Banach lattices and order structures.

2. The Main Result. Automatic Fatou Property

2.1. Formulation of the Main Result. We begin with the following example, which indicates that for the class of real-valued, law-invariant, coherent risk measures, automatic Fatou property cannot be expected at any random variable $X \in \mathcal{X}$ such that $X^- \notin \mathcal{X}_a$. Note that $X^- \notin \mathcal{X}_a \iff \lim_n ||X \mathbf{1}_{\{X \leq -n\}}|| > 0$.

Example 2.1. Let \mathcal{X} be an r.i. space over a non-atomic probability space such that $\mathcal{X} \neq L^{\infty}$. Consider the functional $\rho : \mathcal{X} \to \mathbb{R}$ given by

(2.1)
$$\rho(X) = \operatorname{d}(X^{-}, \mathcal{X}_{a}) - \mathbb{E}[X] = \inf_{Y \in \mathcal{X}_{a}} ||X^{-} - Y|| - \mathbb{E}[X].$$

We show that ρ is a law-invariant coherent risk measure that fails the Fatou property at any $X \in \mathcal{X}$ with $X^- \notin \mathcal{X}_a$.

(1) ρ is a coherent risk measure.

Since \mathcal{X}_a is a norm-closed subspace in \mathcal{X} , the quotient space $\mathcal{X}/\mathcal{X}_a$ is a Banach space with the quotient norm $||[X]||_q := d(X, \mathcal{X}_a)$, where [X] is the equivalent class of $X \in \mathcal{X}$ in $\mathcal{X}/\mathcal{X}_a$; see, e.g., [1, Theorem 1.11]. Moreover, since \mathcal{X}_a is an order ideal in \mathcal{X} (i.e., if $|Y| \leq |X|$ and $X \in \mathcal{X}_a$ then $Y \in \mathcal{X}_a$), $\mathcal{X}/\mathcal{X}_a$ with the quotient norm $||\cdot||_q$ and the quotient order $[X] \vee [Y] := [X \vee Y]$ is in fact a Banach lattice; see, e.g., [33, p. 3]. With these observations, we write

(2.2)
$$\rho(X) = \|[X^-]\|_q - \mathbb{E}[X]$$

Using (2.2), it is clear that ρ is decreasing and positive homogeneous. Let's show subadditivity of ρ . Take any $X_1, X_2 \in \mathcal{X}$. Since $0 \leq (X_1 + X_2)^- \leq X_1^- + X_2^-$,

$$[0] \le [(X_1 + X_2)^-] \le [X_1^- + X_2^-] = [X_1^-] + [X_2^-]$$

in the quotient space $\mathcal{X}/\mathcal{X}_a$. Therefore,

$$\left\| \left[(X_1 + X_2)^{-} \right] \right\|_q \le \left\| [X_1^{-}] + [X_2^{-}] \right\|_q \le \left\| [X_1^{-}] \right\|_q + \left\| [X_2^{-}] \right\|_q.$$

From this it follows easily that $\rho(X_1 + X_2) \leq \rho(X_1) + \rho(X_2)$. To complete the proof that ρ is a coherent risk measure, it remains to show cash invariance of ρ . Take any $X \in \mathcal{X}$ and $m \in \mathbb{R}$. Then $(X + m\mathbf{1})^- - X^- \in L^\infty \subset \mathcal{X}_a$. Hence

$$\|[(X+m\mathbf{1})^{-}]\|_{q} = \|[X^{-}]\|_{q}$$

It follows easily from (2.2) that $\rho(X + m\mathbf{1}) = \rho(X) - m$.

(2) ρ is law invariant.

For a real number $r \ge 0$, set $h(x) = x^- - x^- \wedge r$. If $X' \sim X$, then $h(X') \sim h(X)$ and ||h(X')|| = ||h(X)||. Thus for law invariance of ρ , it is enough to show that for any $X \in \mathcal{X}$,

(2.3)
$$\rho(X) = \inf_{r \in \mathbb{R}, r \ge 0} \|X^- - X^- \wedge r\mathbf{1}\| - \mathbb{E}[X].$$

Denote the right hand side of (2.3) by $\tilde{\rho}(X)$. Since $0 \leq X^- \wedge r\mathbf{1} \in L^{\infty} \subset \mathcal{X}_a$, by comparing the infima in (2.1) and (2.3), it is immediate that $\rho(X) \leq \tilde{\rho}(X)$. On the other hand, recall that L^{∞} is norm dense in \mathcal{X}_a . Thus it is easy to see from (2.1) that for any $X \in \mathcal{X}$,

$$\rho(X) = d(X^{-}, L^{\infty}) - \mathbb{E}[X] = \inf_{Y \in L^{\infty}} ||X^{-} - Y|| - \mathbb{E}[X].$$

For any $Y \in L^{\infty}$, since $Y \leq ||Y||_{\infty} \mathbf{1}$, $|X^{-} - Y| \geq (X^{-} - Y)^{+} \geq (X^{-} - ||Y||_{\infty} \mathbf{1})^{+} = X^{-} - X^{-} \wedge (||Y||_{\infty} \mathbf{1})$. Thus

$$\|X^{-} - Y\| - \mathbb{E}[X] \ge \|X^{-} - X^{-} \wedge (\|Y\|_{\infty} \mathbf{1})\| - \mathbb{E}[X] \ge \widetilde{\rho}(X).$$

Taking infimum over $Y \in L^{\infty}$ yields $\rho(X) \geq \tilde{\rho}(X)$. It follows that $\rho(X) = \tilde{\rho}(X)$. (3) ρ fails the Fatou property at every $X \in \mathcal{X}$ with $X^- \notin \mathcal{X}_a$.

Let $X \in \mathcal{X}$ be such that $X^- \notin \mathcal{X}_a$. Set $X_n = (X \vee (-n\mathbf{1})) \wedge n\mathbf{1}$ for $n \in \mathbb{N}$. Then $X_n \xrightarrow{a.s.} X$ and $|X_n| \leq |X|$ for any $n \in \mathbb{N}$. It suffices to show that $\rho(X) > \lim_n \rho(X_n)$. Since $X_n \in L^\infty \subset \mathcal{X}_a$, $d(X_n, \mathcal{X}_a) = 0$. By Dominated Convergence Theorem,

$$\rho(X_n) = \mathrm{d}(X_n, \mathcal{X}_a) - \mathbb{E}[X_n] = -\mathbb{E}[X_n] \to -\mathbb{E}[X].$$

However, as $X^- \notin \mathcal{X}_a$, $\rho(X) = d(X^-, \mathcal{X}_a) - \mathbb{E}[X] > -\mathbb{E}[X]$. Therefore, $\rho(X) > \lim_n \rho(X_n)$, as required.

Example 2.1 tells us that, in looking for random variables X in an r.i. space at which all real-valued, law-invariant, coherent risk measures automatically satisfies the Fatou property, one must confine the search to those X's with $X^- \in \mathcal{X}_a$. Remarkably, in most classical r.i. spaces, all real-valued, law-invariant, coherent risk measures are indeed automatically Fatou at all such X. This is the case, for instance, in all L^p spaces, $1 \leq p \leq \infty$, all Orlicz spaces and Orlicz hearts and all order continuous r.i. spaces. In fact, the precise structural property on an r.i. space can be identified in order for this to happen. Let \mathcal{X} be an r.i. space over a non-atomic probability space other than L^{∞} . For $X \in \mathcal{X}$, define

$$\mathcal{CL}(X) = \operatorname{co}\{Y : Y \sim X\}.$$

We say that \mathcal{X} has the Almost Order Continuous Equidistributional Average (abbr., AOCEA) property if for any $X \in \mathcal{X}_+$,

$$d(\mathcal{CL}(X), \mathcal{X}_a) = 0.$$

Note that in the definition of the AOCEA property, one may replace the set $\mathcal{CL}(X)$ with the set

(2.4)
$$\mathcal{AL}(X) = \left\{ \frac{1}{n} \sum_{k=1}^{n} X_k : n \in \mathbb{N}, X_1, \dots, X_n \sim X \right\}.$$

This follows from the observation that the set of convex combinations with rational coefficients of elements of the set $\{Y : Y \sim X\}$ is norm dense in $\mathcal{CL}(X)$ and hence so is the set $\mathcal{AL}(X)$. Therefore, the AOCEA property says that every nonnegative random variable in \mathcal{X} possesses averages of equidistributed copies that are almost order continuous, i.e., arbitrarily close to \mathcal{X}_a .

We can now state the main result of the paper.

Theorem 2.2. Let \mathcal{X} be an r.i. space over a non-atomic probability space $(\Omega, \mathcal{F}, \mathbb{P})$ other than L^{∞} . The following statements are equivalent:

- (1) Every law-invariant, coherent risk measure $\rho : \mathcal{X} \to \mathbb{R}$ has the Fatou property at 0.
- (2) Every convex, decreasing, law invariant functional $\rho : \mathcal{X} \to \mathbb{R}$ has the Fatou property at any $X \in \mathcal{X}$ such that $X^- \in \mathcal{X}_a$.
- (3) \mathcal{X} satisfies the AOCEA property.

A detailed analysis of the AOCEA property is given in the next subsection. Proof of Theorem 2.2 will be presented in Section 3.

2.2. An in-depth look at the AOCEA property. In this part, we provide a detailed investigation of the AOCEA property. The outcome of the investigation will also lend significant aid to the proof of Theorem 2.2 itself. The main aspects of the property are revealed in the following proposition. Given a sequence $(A_n)_{n=1}^{\infty}$ of measurable sets, we write $A_n \downarrow \emptyset$ if $A_n \supseteq A_{n+1}$ for all $n \ge 1$ and $\bigcap_{n=1}^{\infty} A_n = \emptyset$.

Proposition 2.3. Let \mathcal{X} be an r.i. space over a non-atomic probability space $(\Omega, \mathcal{F}, \mathbb{P})$ other than L^{∞} . The following statements are equivalent:

(1) \mathcal{X} satisfies the AOCEA property, i.e., for any $X \in \mathcal{X}_+$,

$$d(\mathcal{CL}(X), \mathcal{X}_a) = 0, \text{ where } \mathcal{CL}(X) = co\{Y : Y \sim X\}.$$

(2) For any $A \in \mathcal{F}$ with $\mathbb{P}(A) > 0$, any $X \in \mathcal{X}$ supported in A, and any $\varepsilon > 0$, there exist random variables $(X_i)_{i=1}^k$, all supported in A, with $X_i \sim X$ for $i = 1, \ldots, k$, a convex combination $\sum_{i=1}^k \lambda_i X_i$, and $V \in \mathcal{X}_a$, also supported in A, such that

$$\left\|\sum_{i=1}^{k} \lambda_i X_i - V\right\| < \varepsilon.$$

- (3) For any $X \in \mathcal{X}_+$, any sequence of measurable sets $(A_n)_{n=1}^{\infty}$ with $A_n \downarrow \emptyset$, and any $\varepsilon > 0$, there exist $n_1, \ldots, n_k \in \mathbb{N}$, random variables $(Z_i)_{i=1}^k$ and a convex combination $Z = \sum_{i=1}^k \lambda_i Z_i$ such that $Z_i \sim X \mathbf{1}_{A_{n_i}}$ for $i = 1, \ldots, k$ and $||Z|| < \varepsilon$.
- (4) For any $X \in \mathcal{X}$, any $A \in \mathcal{F}$ with $\mathbb{P}(A) > 0$, any sequence of measurable sets $(A_n)_{n=1}^{\infty}$ with $A_n \downarrow \emptyset$, and any $\varepsilon > 0$, there exist $n_1, \ldots, n_k \in \mathbb{N}$, random

variables $(Z_i)_{i=1}^k$ and a convex combination $Z = \sum_{i=1}^k \lambda_i Z_i$ such that $Z_i \sim X \mathbf{1}_{A_{n_i}}$ for $i = 1, \ldots, k$, all Z_i 's are supported in A, and $||Z|| < \varepsilon$.

(2) is the "localized" version of (1). In particular, taking $A = \Omega$ in (2) yields (1) for all random variables, not necessarily nonnegative. Similarly, one can compare (3) and (4). The four equivalent formulations are each useful in their own way. (1) is succinct and aesthetically pleasing; (3) is easier to verify in practice. (2) will be used for establishing automatic $\sigma(\mathcal{X}, \mathcal{X}')$ lower semicontinuity in Subsection 4.1; (4) will be used in the proof of Theorem 2.2 (2) \Longrightarrow (1) in Subsection 3.1.

The proof of this proposition is, however, rather involved. We put it into Appendix A in order to facilitate the accessibility of the main results of the paper on automatic Fatou property and tractable dual representations of law-invariant risk measures.

The AOCEA property is satisfied by most classical r.i. spaces. First of all, it trivially holds for an order continuous r.i. space \mathcal{X} because $\mathcal{X} = \mathcal{X}_a$. Therefore, Lebesgue spaces L^p $(1 \leq p < \infty)$ and Orlicz hearts all satisfy the property. The next proposition shows that Orlicz spaces, which have been widely used as model spaces in the recent literature, also satisfy the property.

Proposition 2.4. An Orlicz space L^{Φ} has the AOCEA property.

Proof. We verify (3) of Proposition 2.3. Let $X \in (L^{\Phi})_+$, $A_n \downarrow \emptyset$ and $\varepsilon > 0$ be given. Since $\mathbb{P}(A_n) \to 0$, by passing to a subsequence, we may assume that $\sum_{n=1}^{\infty} \mathbb{P}(A_n) < 1$. By non-atomicity of \mathbb{P} , there exists a disjoint sequence $(B_n)_{n=1}^{\infty}$ of measurable sets such that $\mathbb{P}(B_n) = \mathbb{P}(A_n)$ for any $n \in \mathbb{N}$. Using non-atomicity again, we obtain, for each $n \in \mathbb{N}$, a random variable Z_n , supported in B_n , such that $Z_n \sim X \mathbf{1}_{A_n}$.

Let $\eta > 0$ be such that $\mathbb{E}[\Phi(\eta X)] < \infty$. Choose $k \in \mathbb{N}$ so that $k\eta \varepsilon \geq 1$. By Dominated Convergence Theorem, $\lim_n \mathbb{E}[\Phi(\eta X \mathbf{1}_{A_n})] = 0$. Thus we can pick n_1, \ldots, n_k such that

$$\sum_{i=1}^{k} \mathbb{E}[\Phi(\eta X \mathbf{1}_{A_{n_i}})] \le 1.$$

Set $Z = \frac{1}{k} \sum_{i=1}^{k} Z_{n_i}$. Then $\mathbb{E} \left[\Phi\left(\frac{Z}{\varepsilon}\right) \right] = \mathbb{E} \left[\Phi\left(\frac{\eta Z}{\eta \varepsilon}\right) \right]$ $= \sum_{i=1}^{k} \mathbb{E} \left[\Phi\left(\frac{\eta Z_{n_i}}{k\eta \varepsilon}\right) \right] \text{ (since } Z_{n_i}\text{'s are disjoint)}$ $\leq \sum_{i=1}^{k} \mathbb{E} [\Phi(\eta X \mathbf{1}_{A_{n_i}})] \leq 1.$

Hence, $||Z|| \leq \varepsilon$. This completes the verification of condition (3) of Proposition 2.3. \Box

Combining Proposition 2.4 with Theorem 2.2, we obtain the following result.

Corollary 2.5. A real-valued, convex, decreasing, law-invariant functional on an Orlicz space L^{Φ} has the Fatou property at any $X \in L^{\Phi}$ such that $X^{-} \in H^{\Phi}$.

It should be noted that the AOCEA property is not universally satisfied by all r.i. spaces. A brief example of an r.i. space failing it is presented at Appendix B.

We also refer to Chen et al [10] for other interesting applications of the AOCEA property, e.g., regarding collapse to the mean of law-invariant linear functionals. See also Bellini et al [5] for more results on collapse to the mean.

At this point, we make a slight digression to discuss the set $\mathcal{CL}(X) = co\{Y : Y \sim X\}$ that appears in the AOCEA property. It has been drawing attention in some recent works. Its relation with the well-studied set $\{Y \in \mathcal{X} : Y \preceq_{cx} X\}$ is investigated in Bellini et al [4]. Recall that for random variables $X, Y \in L^1, Y \preceq_{cx} X$ means that $\mathbb{E}[f(Y)] \leq \mathbb{E}[f(X)]$ for every convex function $f : \mathbb{R} \to \mathbb{R}$, whenever the expectations exist. It was proved in [4] that

$$\overline{\mathcal{CL}(X)}^{\sigma(\mathcal{X},\mathcal{X}')} = \{ Y \in \mathcal{X} : Y \preceq_{cx} Y \},\$$

where the left hand side is the closure of $\mathcal{CL}(X)$ in the $\sigma(\mathcal{X}, \mathcal{X}')$ weak topology. This identity implies in particular the coincidence of law invariance and Schur convexity for certain convex functionals; see [4] for details. More results and applications about the set $\mathcal{CL}(X)$ can also be found in Gao et al [26]. Sets of a similar fashion as the set $\mathcal{AL}(X)$ in (2.4) has also appeared in Mao and Wang [36] in their study of risk aggregation. We believe that more appealing aspects and applications of the set $\mathcal{CL}(X)$ will come forth in the literature.

3. Proof of the Main Result

In this section, we prove the main result of the paper, Theorem 2.2. The implication $(2) \implies (1)$ is obvious. In the following two subsections, we prove the implications $(3) \implies (2)$ and $(1) \implies (3)$, respectively.

3.1. Proof of $(3) \Longrightarrow (2)$ in Theorem 2.2. The following technical lemma reduces the Fatou property to a much simpler form.

Lemma 3.1. Let \mathcal{X} be an r.i. space over a probability space. Let $\rho : \mathcal{X} \to (-\infty, \infty]$ be a decreasing functional and $X \in \mathcal{X}$. The following are equivalent:

- (1) ρ has the Fatou property at X;
- (2) Let $Y \in \mathcal{X}$ be such that $Y \ge X$, (c_n) be real numbers such that $c_n \downarrow 0$, and (A_n) be measurable sets such that $A_n \downarrow \emptyset$. For any $n \in \mathbb{N}$, put $Y_n = X \mathbf{1}_{A_n^c} + c_n \mathbf{1}_{A_n^c} + Y \mathbf{1}_{A_n}$. Then $\rho(X) = \lim_n \rho(Y_n)$.

Proof. Assume that (1) holds. Let Y, (c_n) , (A_n) and (Y_n) be as given in (2). Since $Y_n \ge X\mathbf{1}_{A_n^c} + 0 + X\mathbf{1}_{A_n} = X$, $\rho(Y_n) \le \rho(X)$ for any $n \ge 1$. In particular, $\limsup_n \rho(Y_n) \le \rho(X)$. Since $X \le Y_n \le Y\mathbf{1}_{A_n^c} + c_1\mathbf{1} + Y\mathbf{1}_{A_n} = Y + c_1\mathbf{1}$, (Y_n) is dominated in \mathcal{X} . It is easy to see that $Y_n \xrightarrow{a.s.} X$. Thus by the assumption (1), $\rho(X) \le \liminf_n \rho(Y_n)$. It follows that $\rho(X) = \lim_n \rho(Y_n)$. This proves that (1) \Longrightarrow (2).

Assume that (2) holds. Suppose otherwise that (1) fails. Then there exists a sequence (X_n) that order converges to X in \mathcal{X} but $\rho(X) > \liminf_n \rho(X_n) := \lambda \in [-\infty, \infty)$. By switching to a subsequence of (X_n) , we may assume that

$$(3.1) \qquad \qquad \rho(X_n) \to \lambda < \rho(X)$$

For any $k \in \mathbb{N}$, by Egoroff's Theorem, there exist $n_k \in \mathbb{N}$ and a measurable set B_k such that $\mathbb{P}(B_k) \leq \frac{1}{2^k}$ and $|X_{n_k} - X| \leq \frac{1}{k}$ on B_k^c . Set $A_k = \bigcup_{m \geq k} B_m$. Then (A_k) is a decreasing sequence of measurable sets such that $\mathbb{P}(A_k) \leq \sum_{m \geq k} \mathbb{P}(B_m) \leq \sum_{m \geq k} \frac{1}{2^m} = \frac{1}{2^{k-1}} \to 0$ and thus $\mathbb{P}(\bigcap_{k=1}^{\infty} A_k) = 0$. Without loss of generality, we may assume that $A_k \downarrow \emptyset$. Clearly, $|X_{n_k} - X| \leq \frac{1}{k}$ on A_k^c for any $k \geq 1$. Since (X_n) order converges to Xin \mathcal{X} , there exists $Y \in \mathcal{X}$ such that $X_n \leq Y$ for all $n \in \mathbb{N}$. Since $X_n \xrightarrow{a.s.} X, X \leq Y$. Set $Y_k = X \mathbf{1}_{A_k^c} + \frac{1}{k} \mathbf{1}_{A_k^c} + Y \mathbf{1}_{A_k}$ for $k \in \mathbb{N}$. Then

$$X_{n_{k}} = X_{n_{k}} \mathbf{1}_{A_{k}^{c}} + X_{n_{k}} \mathbf{1}_{A_{k}} \le \left(X + \frac{1}{k}\right) \mathbf{1}_{A_{k}^{c}} + Y \mathbf{1}_{A_{k}} = Y_{k}$$

and hence $\rho(Y_k) \leq \rho(X_{n_k})$. By assumption (2),

$$\rho(X) = \lim_{k} \rho(Y_k) \le \lim_{k} \rho(X_{n_k}) = \lim_{n} \rho(X_n) = \lambda$$

contradicting (3.1). This proves that $(2) \Longrightarrow (1)$.

Proof of Theorem 2.2 (3) \Longrightarrow (2). Assume that \mathcal{X} satisfies the AOCEA property. Let $\rho : \mathcal{X} \to \mathbb{R}$ be a convex, decreasing, law-invariant functional. Let $X \in \mathcal{X}$ be such that $X^- \in \mathcal{X}_a$. We apply the previous lemma to establish the Fatou property at X. Suppose otherwise that (2) in Lemma 3.1 fails. Then there exist $\mathcal{X} \ni Y \ge X$, $c_n \downarrow 0$ and $A_n \downarrow \emptyset$ such that $\rho(Y_n) \not\to \rho(X)$, where $Y_n = (X + c_n \mathbf{1})\mathbf{1}_{A_n^c} + Y\mathbf{1}_{A_n}$ for $n \in \mathbb{N}$. Since $Y_n \ge X$, $\rho(Y_n) \le \rho(X)$. Thus there exist $\varepsilon > 0$ and a subsequence (Y_{n_k}) such that $\rho(Y_{n_k}) < \rho(X) - \varepsilon$ for all k. Replacing (Y_n) with (Y_{n_k}) , we may assume that

(3.2)
$$\rho(Y_n) < \rho(X) - \varepsilon$$
 for all $n \in \mathbb{N}$.

We aim at contradicting (3.2). Recall from [41, Proposition 3.1] that ρ is norm continuous. Thus there exists $\delta > 0$ such that

(3.3)
$$\rho(X+V) > \rho(X) - \varepsilon \quad \text{if } V \in \mathcal{X} \text{ and } \|V\| \le \delta.$$

Take a real number r > 0 such that $\mathbb{P}(B) > 0$, where $B = \{|X| \le r\}$. Since $\mathbf{1} \in L^{\infty} \subset \mathcal{X}_a$, there exists $\eta > 0$ such that

(3.4)
$$\|\mathbf{1}_C\| \le \frac{\delta}{6r} \quad \text{if } \mathbb{P}(C) \le \eta.$$

Similarly, since $X^- \in \mathcal{X}_a$ and $\mathbb{P}(A_n) \to 0$, $||X^- \mathbf{1}_{A_n}|| \to 0$. Thus by passing to subsequences if necessary, we may assume that $\mathbb{P}(A_1) < \mathbb{P}(B)$ and

$$\sum_{n=1}^{\infty} \mathbb{P}(A_n) \le \min \left\{ \eta, \mathbb{P}(B \setminus A_1) \right\},$$
$$\|X^{-} \mathbf{1}_{A_n}\| \le \frac{\delta}{6} \quad \text{for all } n \in \mathbb{N},$$
$$c_n \|\mathbf{1}\| \le \frac{\delta}{6} \quad \text{for all } n \in \mathbb{N}.$$

By Proposition 2.3(4) and passing to a subsequence of (A_n) if necessary, there exist random variables $(Z_n)_{n=1}^k$, all supported in $B \setminus A_1$, and a convex combination $\sum_{n=1}^k \lambda_n Z_n$ such that

$$Z_n \sim Y \mathbf{1}_{A_n} \text{ for } n = 1, \dots, k, \text{ and } \left\| \sum_{n=1}^k \lambda_n Z_n \right\| < \frac{\delta}{6}$$

For n = 1, ..., m, since $\{Z_n \neq 0\} \subset B \setminus A_1$ and $A_n \subset A_1$, $\{Z_n \neq 0\} \cap A_n = \emptyset$. Moreover, $\mathbb{P}(Z_n \neq 0) = \mathbb{P}(Y \mathbf{1}_{A_n} \neq 0) \leq \mathbb{P}(A_n)$. Take $B_n \subset A_n$ such that $\mathbb{P}(B_n) = \mathbb{P}(A_n) - \mathbb{P}(Z_n \neq 0)$. Hence, we get a partition of Ω :

(3.5)
$$\Omega = \left(A_n^c \cap \{Z_n = 0\}\right) \cup \left(A_n \setminus B_n\right) \cup \left(B_n \cup \{Z_n \neq 0\}\right).$$

On the other hand, we have another partition of Ω as follows:

(3.6)
$$\Omega = \left(A_n^c \cap \{Z_n = 0\}\right) \cup \{Z_n \neq 0\} \cup A_n.$$

Since $\{Z_n \neq 0\}$ has the same probability as $A_n \setminus B_n$, we can take a random variable W_n , supported on $A_n \setminus B_n$, such that $W_n \sim (X + c_n \mathbf{1}) \mathbf{1}_{\{Z_n \neq 0\}}$. Now for any $n = 1, \ldots, k$, define Y'_n piecewise according to the partition of Ω in (3.5):

$$Y'_n = (X + c_n \mathbf{1}) \mathbf{1}_{A_n^c \cap \{Z_n = 0\}} + W_n + Z_n.$$

Comparing with Y_n along the partition in (3.6), we see that $Y'_n \sim Y_n$.

Recall that $\{Z_n \neq 0\} \subset B \setminus A_1 \subset B$. Thus by the choice of B, $|X \mathbf{1}_{\{Z_n \neq 0\}}| \leq r \mathbf{1}_{\{Z_n \neq 0\}}$. By (3.4) and the fact that $\mathbb{P}(Z_n \neq 0) \leq \mathbb{P}(A_n) \leq \eta$, we have

$$||X\mathbf{1}_{\{Z_n\neq 0\}}|| \le r ||\mathbf{1}_{\{Z_n\neq 0\}}|| \le r \frac{\delta}{6r} = \frac{\delta}{6}$$

It follows that

$$||W_n|| \le ||X\mathbf{1}_{\{Z_n \ne 0\}}|| + c_n ||\mathbf{1}|| \le \frac{\delta}{3}$$
 for $n = 1, \dots, k$.

Recall that $\{Z_n \neq 0\}$ and A_n are disjoint. Thus rewrite

$$Y'_{n} = X - X^{+} \mathbf{1}_{A_{n}} + X^{-} \mathbf{1}_{A_{n}} - X \mathbf{1}_{\{Z_{n} \neq 0\}} + c_{n} \mathbf{1}_{A_{n}^{c} \cap \{Z_{n} = 0\}} + W_{n} + Z_{n}.$$

Then

$$\sum_{n=1}^{k} \lambda_n Y'_n = X - \sum_{n=1}^{k} \lambda_n X^+ \mathbf{1}_{A_n} + V,$$

where

$$V = \sum_{n=1}^{k} \lambda_n (X^{-} \mathbf{1}_{A_n} - X \mathbf{1}_{\{Z_n \neq 0\}} + c_n \mathbf{1}_{A_n^c \cap \{Z_n = 0\}} + W_n + Z_n).$$

We have

$$\|V\| \le \left\|\sum_{n=1}^{k} \lambda_n Z_n\right\| + \sum_{n=1}^{k} \lambda_n \left(\|X^{-} \mathbf{1}_{A_n}\| + \|X\mathbf{1}_{\{Z_n \neq 0\}}\| + c_n \|\mathbf{1}\| + \|W_n\|\right) \le \delta.$$

Hence by monotonicity of ρ and (3.3),

(3.7)
$$\rho\left(\sum_{n=1}^{k}\lambda_{n}Y_{n}'\right) \ge \rho(X+V) > \rho(X) - \varepsilon$$

Finally,

$$\sum_{n=1}^{k} \lambda_n \rho(Y_n) = \sum_{n=1}^{k} \lambda_n \rho(Y'_n) \text{ (law invariance)}$$
$$\geq \rho(\sum_{n=1}^{k} \lambda_n Y'_n) \text{ (convexity)}$$
$$> \rho(X) - \varepsilon.$$

Hence, there exists n such that $\rho(Y_n) > \rho(X) - \varepsilon$, contradicting (3.2).

Remark 3.2. The reader may re-examine the role of real-valuedness of ρ in the proof of (3) \implies (2). It is only used to ensure norm lower semicontinuity of ρ at X; see (3.3) and (3.7). Therefore, one sees that the following statement is also equivalent to the AOCEA property when $\mathcal{X} \neq L^{\infty}$ and can be added to Theorem 2.2:

(1') Every convex, decreasing, law-invariant functional $\rho : \mathcal{X} \to (-\infty, \infty]$ has the Fatou property at any $X \in \mathcal{X}$, where ρ is norm lower semicontinuous and $X^- \in \mathcal{X}_a$.

3.2. **Proof of (1)** \implies (3) in Theorem 2.2. Throughout this subsection, assume that \mathcal{X} is an r.i. space over a non-atomic probability space and $\mathcal{X} \neq L^{\infty}$. If \mathcal{X} fails the AOCEA property, we aim to construct a law-invariant coherent risk measure $\rho : \mathcal{X} \to \mathbb{R}$ that fails the Fatou property at 0.

The following discretization lemma will be useful in the course of the construction. Let f be a positive linear functional on \mathcal{X} , i.e., $f(X) \ge 0$ for any $X \ge 0$. Clearly, f is positive iff f is increasing, i.e., $f(X_1) \ge f(X_2)$ whenever $X_1 \ge X_2$. By Birkhoff's Theorem ([2, Theorem 4.3]), f is bounded on \mathcal{X} . Therefore, if $X' \sim X \in \mathcal{X}$, then $f(X') \le ||f|| ||X'|| = ||f|| ||X||$. It follows that for any $X \in \mathcal{X}$,

$$\sup\{f(X'): X' \sim X\} \in \mathbb{R}.$$

Lemma 3.3. Let f be a positive linear functional on \mathcal{X} . For any $X \in \mathcal{X}$,

(3.8)
$$\sup\{f(X'): X' \sim X\} = \sup\{f(Z): Z \sim U \le X, U \in \mathcal{X}, U \text{ is discrete}\}$$

Proof. Take any $X \in \mathcal{X}$. Denote the left and right hand sides of (3.8) by $\phi_1(X)$ and $\phi_2(X)$, respectively. Also, put

$$\phi_3(X) = \sup\{f(Z) : Z \sim U \le X, U \in \mathcal{X}\}.$$

We first show that $\phi_1(X) = \phi_3(X)$. If $X' \sim X$, then $X' \sim X \leq X$, implying that $\phi_1(X) \leq \phi_3(X)$. Let $Z \in \mathcal{X}$ be such that $Z \sim U \leq X$ for some $U \in \mathcal{X}$. By Lemma A.1, there exists a random variable X' such that $Z \leq X' \sim X$. Clearly, $X' \in \mathcal{X}$. Since f is increasing, $f(Z) \leq f(X') \leq \phi_1(X)$. Taking supremum over Z, we obtain $\phi_3(X) \leq \phi_1(X)$. It follows that $\phi_1(X) = \phi_3(X)$, as desired.

Apparently, $\phi_3(X) \ge \phi_2(X)$. To see the reverse inequality, take any random variables $Z, U \in \mathcal{X}$ such that $Z \sim U \le X$. For any real number a > 1, put

$$Z' = \sum_{n \in \mathbb{Z}} a^n \mathbf{1}_{\{a^n < Z \le a^{n+1}\}} - \sum_{n \in \mathbb{Z}} a^{n+1} \mathbf{1}_{\{-a^{n+1} \le Z < -a^n\}},$$
$$U' = \sum_{n \in \mathbb{Z}} a^n \mathbf{1}_{\{a^n < U \le a^{n+1}\}} - \sum_{n \in \mathbb{Z}} a^{n+1} \mathbf{1}_{\{-a^{n+1} \le U < -a^n\}}.$$

Then U' is discrete and $Z' \sim U' \leq U \leq X$. Moreover, $Z \geq Z' \geq \frac{1}{a}Z^+ - aZ^-$, so that $Z' \in \mathcal{X}$. Hence

$$\phi_2(X) \ge f(Z') \ge \frac{1}{a}f(Z^+) - af(Z^-).$$

Letting $a \downarrow 1$ and taking supremum over Z, we obtain $\phi_2(X) \ge \phi_3(X)$. It follows that $\phi_2(X) = \phi_3(X) = \phi_1(X)$, completing the proof.

Lemma 3.4. Let f be a positive linear functional on \mathcal{X} . Define $\phi : \mathcal{X} \to \mathbb{R}$ by

$$\phi(X) = \sup\{f(X') : X' \sim X\}$$

Then ϕ is law invariant, increasing, positive homogeneous and subadditive on \mathcal{X} . If in addition f vanishes on L^{∞} , then $\phi(X + m\mathbf{1}) = \phi(X)$ for all $X \in \mathcal{X}$ and $m \in \mathbb{R}$.

Proof. Clearly ϕ is law invariant and positive homogeneous. Suppose that $X_1 \leq X_2$ in \mathcal{X} and take any $X' \sim X_1$. By Lemma A.1, there exists a random variable X'_2 such that $X' \leq X'_2 \sim X_2$. Since f is increasing, $f(X') \leq f(X'_2) \leq \phi(X_2)$. Taking supremum over X' gives $\phi(X_1) \leq \phi(X_2)$. This proves that ϕ is increasing. Next, let us show that ϕ is subadditive. Consider any any $U, V \in \mathcal{X}$. By Lemma 3.3, it suffices to show that if $X \in \mathcal{X}$ is a discrete random variable such that $X \sim Z \leq U + V$, then $f(X) \leq \phi(U) + \phi(V)$. Let $\{a_i\}_{i \in I}$ be the (at most countable) set of values $a \in \mathbb{R}$ such that $\mathbb{P}(X = a) > 0$. Then $\mathbb{P}(Z = a_i) = \mathbb{P}(X = a_i)$. On $\{X = a_i\}$, find $U'|_{\{X=a_i\}} \sim U|_{\{Z=a_i\}}$, and put

$$V'|_{\{X=a_i\}} := a_i - U'|_{\{X=a_i\}} \sim a_i - U|_{\{Z=a_i\}}$$

Glue over *i* to obtain random variables U' and V'. Clearly, $U' \sim U$, $V' \sim Z - U \leq V$, and X = U' + V'. Hence, $f(X) = f(U') + f(V') \leq \phi(U) + \phi(V)$ by Lemma 3.3. This establishes subadditivity of ϕ .

Finally, consider $X \in \mathcal{X}$, $m \in \mathbb{R}$ and $Z \sim X + m\mathbf{1}$. Then $Z - m\mathbf{1} \sim X$. Since f vanishes on L^{∞} , $f(Z) = f(Z - m\mathbf{1}) \leq \phi(X)$. Taking supremum over Z gives $\phi(X + m\mathbf{1}) \leq \phi(X)$. The same inequality gives

$$\phi(X) = \phi(X + m\mathbf{1} + (-m)\mathbf{1}) \le \phi(X + m\mathbf{1}).$$

- $m\mathbf{1}) = \phi(X).$

With Lemma 3.4 at hand, we now present the proof of $(1) \Longrightarrow (3)$ in Theorem 2.2.

Proof of $(1) \Longrightarrow (3)$ in Theorem 2.2. Suppose that \mathcal{X} fails the AOCEA property. Take $X_0 \in \mathcal{X}_+$ such that $d(\mathcal{CL}(X_0), \mathcal{X}_a) > 0$, where $\mathcal{CL}(X_0) = co\{X : X \sim X_0\}$. We will show that condition (1) of Theorem 2.2 fails.

First, we claim that $d(\mathcal{CL}(X_0), co(\mathcal{X}_a \cup (-\mathcal{X}_+))) > 0$. Indeed, since \mathcal{X}_a and $-\mathcal{X}_+$ are both convex, a general element of $co(\mathcal{X}_a \cup (-\mathcal{X}_+))$ is of the form $\alpha V - (1 - \alpha)W$, where $0 \leq \alpha \leq 1$, $V \in \mathcal{X}_a$ and $W \in \mathcal{X}_+$. Using the notation in Example 2.1, for any $Z \in \mathcal{CL}(X_0), [Z + (1 - \alpha)W] \geq [Z] \geq [0]$ in the quotient space $\mathcal{X}/\mathcal{X}_a$. Hence

$$\begin{aligned} \left\| Z - \left(\alpha V - (1 - \alpha) W \right) \right\| &\geq \left\| [Z - (\alpha V - (1 - \alpha) W)] \right\|_q \\ &= \left\| [Z + (1 - \alpha) W] \right\|_q \\ &\geq \| [Z] \|_q = \mathrm{d}(Z, \mathcal{X}_a). \end{aligned}$$

This proves that $d(\mathcal{CL}(X_0), co(\mathcal{X}_a \cup (-\mathcal{X}_+))) \ge d(\mathcal{CL}(X_0), \mathcal{X}_a) > 0$, as claimed.

Let \mathcal{B} denote the open unit ball of \mathcal{X} . By the claim, there exists r > 0 such that $\mathcal{CL}(X_0)$ and $\operatorname{co}(\mathcal{X}_a \cup (-\mathcal{X}_+)) + r\mathcal{B}$ are disjoint (convex) sets in \mathcal{X} . Since the latter set

Therefore, $\phi(X +$

is an open set, the Hahn-Banach Separation Theorem ([40, Theorem 3.4]) says that there is a nonzero linear functional $f \in \mathcal{X}^*$ such that

$$\sup\{f(X): X \in \operatorname{co}(\mathcal{X}_a \cup (-\mathcal{X}_+)) + r\mathcal{B}\} \le \inf\{f(Z): Z \in \mathcal{CL}(X_0)\}.$$

In particular, $\sup\{f(X) : X \in \mathcal{X}_a\} < \infty$ and $\sup\{f(X) : X \leq 0\} < \infty$. Consequently, since \mathcal{X}_a is a linear space and $\{X \leq 0\}$ is a cone, f = 0 on \mathcal{X}_a and $f(X) \leq 0$ if $X \leq 0$. It follows from the latter conclusion that f is positive. Furthermore, since $f \neq 0$,

$$0 = f(0) \leq \sup\{f(X) : X \in \operatorname{co}(\mathcal{X}_a \cup (-\mathcal{X}_+))\} < \sup\{f(X) : X \in \operatorname{co}(\mathcal{X}_a \cup (-\mathcal{X}_+))\} + r ||f|| = \sup\{f(X) : X \in \operatorname{co}(\mathcal{X}_a \cup (-\mathcal{X}_+)) + r\mathcal{B}\} \leq \inf\{f(Z) : Z \in \mathcal{CL}(X_0)\} := \beta.$$

Define $\phi : \mathcal{X} \to \mathbb{R}$ by $\phi(X) = \sup\{f(X') : X' \sim X\}$ for $X \in \mathcal{X}$. Set

$$\rho(X) = \phi(-X) - \mathbb{E}[X], \text{ for any } X \in \mathcal{X}.$$

Invoking Lemma 3.4, one sees that ρ is a law-invariant coherent risk measure on \mathcal{X} .

It remains to show that ρ fails the Fatou property at 0. For any $n \geq 1$, let $X_n = X_0 \mathbf{1}_{\{X_0 \geq n\}}$. Then (X_n) order converges to 0. Let's compute $\rho(X_n)$. Suppose that $Z \sim -X_n$. Then $-Z \geq 0$ and $\mathbb{P}(-Z > 0) = \mathbb{P}(X_0 \geq n)$ so that there exists a random variable W, supported on $\{-Z > 0\}^c = \{Z = 0\}$, such that $W \sim -X_0 \mathbf{1}_{\{X_0 < n\}}$. Clearly, $-(Z + W) \sim X_0$, and by (3.9), $f(-(Z + W)) \geq \beta > 0$. Since $W \in L^{\infty} \subset \mathcal{X}_a$, f(W) = 0, implying that $f(Z) \leq -\beta < 0$. Thus $\phi(-X_n) \leq -\beta < 0$, and consequently, $\rho(X_n) \leq -\beta - \mathbb{E}(X_n)$ for all $n \in \mathbb{N}$. It follows that

$$\liminf_{n} \rho(X_n) \le -\beta < 0 = \rho(0).$$

Thus ρ fails the Fatou property at 0, i.e., condition (1) of Theorem 2.2 fails.

4. Automatic Representations

In this section, building on Theorem 2.2 and the techniques developed for the proof, we obtain automatic $\sigma(\mathcal{X}, \mathcal{X}')$ lower semicontinuity and corresponding dual representations of law-invariant risk measures.

4.1. Automatic $\sigma(\mathcal{X}, \mathcal{X}')$ lower semicontinuity. The following lemma is a refinement of Proposition 2.3(2) and hence its conclusion is another equivalent formulation of the AOCEA property. Recall that for a (measurable) partition $\pi = \{C_1, \ldots, C_k\}$ of Ω ,

$$\mathbb{E}[X|\pi] := \sum_{j=1}^{k} \frac{\mathbb{E}[X\mathbf{1}_{C_j}]}{\mathbb{P}(C_j)} \mathbf{1}_{C_j}, \quad \text{for any } X \in L^1.$$

Lemma 4.1. Let \mathcal{X} be an r.i. space over a non-atomic probability space $(\Omega, \mathcal{F}, \mathbb{P})$ other than L^{∞} . Suppose that \mathcal{X} satisfies the AOCEA property. Let $X \in \mathcal{X}$, a finite partition $\pi = \{C_1, \ldots, C_k\}$ of Ω and $\varepsilon > 0$ be given. Then there exist $X_1, \ldots, X_N \in \mathcal{X}$ and a convex combination $X' = \sum_{i=1}^N \lambda_i X_i$ such that

$$X_i|_{C_j} \sim X|_{C_j} \quad \text{for any } 1 \le i \le N \text{ and } 1 \le j \le k,$$
$$d(X', \mathcal{X}_a) < \varepsilon.$$

In particular, $X_i \sim X$ for each i = 1, ..., N and $\mathbb{E}[X'|\pi] = \mathbb{E}[X|\pi]$.

Proof. We demonstrate the proof for k = 2; the same argument applies for other values of k. Write $\pi = \{A, B\}$. Let $X \in \mathcal{X}$ and $\varepsilon > 0$ be given. By Proposition 2.3(2), there exist random variables $(X'_i)_{i=1}^m$, a convex combination $\sum_{i=1}^m \alpha_i X'_i$, and a random variable $V' \in \mathcal{X}_a$ such that all X'_i 's are supported in $A, X'_i \sim X \mathbf{1}_A$ for each $i = 1, \ldots, m$, and

$$\left\|\sum_{i=1}^{m} \alpha_i X'_i - V'\right\| < \frac{\varepsilon}{2}.$$

Similarly, there exist random variables $(X''_j)_{j=1}^l$, a convex combination $\sum_{j=1}^l \beta_j X''_j$, and a random variable $V'' \in \mathcal{X}_a$ such that all X''_j 's are supported in $B, X''_j \sim X \mathbf{1}_B$ for each $j = 1, \ldots, l$, and

$$\left\|\sum_{j=1}^{l}\beta_j X_j'' - V''\right\| < \frac{\varepsilon}{2}.$$

For any i = 1, ..., m and j = 1, ..., l, put $X_{ij} = X'_i + X''_j$. Then

$$X_{ij}|_A = X'_i|_A \sim X|_A$$
 and $X_{ij}|_B = X''_j|_B \sim X|_B$.

This implies in particular that $X_{ij} \sim X$ and $\mathbb{E}[X_{ij}|\pi] = \mathbb{E}[X|\pi]$. The convex combination $X' := \sum_{1 \leq i \leq m, 1 \leq j \leq l} \alpha_i \beta_j X_{ij} = \sum_{i=1}^m \alpha_i X'_i + \sum_{j=1}^l \beta_j X''_j$ and the random variable $V := V' + V'' \in \mathcal{X}_a$ clearly satisfy $\mathbb{E}[X'|\pi] = \mathbb{E}[X|\pi]$ and

$$\|X' - V\| < \varepsilon$$

This completes the proof by reordering X_{ij} 's into $(X_i)_{i=1}^N$.

The following lemma was proved for L^{∞} in [29] and for Orlicz spaces in [23]. But new techniques are needed to extend it to general r.i. spaces.

Lemma 4.2. Let \mathcal{X} be an r.i. space over a non-atomic probability space $(\Omega, \mathcal{F}, \mathbb{P})$ other than L^{∞} . Suppose that \mathcal{X} satisfies the AOCEA property. Let \mathcal{C} be a normclosed, convex, law-invariant set in \mathcal{X} . Let $\pi = \{C_1, \dots, C_k\}$ be a finite partition of Ω . Then $\mathbb{E}[X|\pi] \in \mathcal{C}$ for any $X \in \mathcal{C}$. *Proof.* Let $X \in \mathcal{C}$ and $\varepsilon > 0$ be given. We demonstrate the proof for k = 2; the other cases can be proved similarly. Write $\pi = \{A, B\}$. We obtain the convex combination $X' = \sum_{i=1}^{N} \lambda_i X_i$ as in Lemma 4.1. Then

$$\mathbb{E}[X'|\pi] = \mathbb{E}[X|\pi]$$
 and $||X' - V|| < \varepsilon$

for some $V \in \mathcal{X}_a$. For each *i*, since $X_i \sim X$, $X_i \in \mathcal{C}$ by law-invariance of \mathcal{C} . Thus $X' \in \mathcal{C}$ by convexity of \mathcal{C} . Recall that simple functions are norm dense in \mathcal{X}_a . Thus we may assume that V is simple. Since $\lim_{\mathbb{P}(C)\to 0} ||\mathbf{1}_C|| = 0$, we may further assume, by perturbation if necessary, that

$$V\mathbf{1}_{A} = \sum_{j=1}^{m_{1}} a_{j}\mathbf{1}_{A_{j}}, \text{ where } \mathbb{P}(A_{j}) = \frac{\mathbb{P}(A)}{m_{1}} \text{ for } 1 \leq j \leq m_{1},$$
$$V\mathbf{1}_{B} = \sum_{l=1}^{m_{2}} b_{l}\mathbf{1}_{B_{l}}, \text{ where } \mathbb{P}(B_{l}) = \frac{\mathbb{P}(B)}{m_{2}} \text{ for } 1 \leq l \leq m_{2},$$

where A_j 's and B_l 's form a partition of A and B, respectively. For any permutation τ on $\{1, \dots, m_1\}$ and any permutation σ on $\{1, \dots, m_2\}$, let $V_{(\tau, \sigma)}$ be the random variable defined by

$$V_{(\tau,\sigma)}|_{A_j} = a_{\tau(j)}, \quad j = 1, \cdots, m_1,$$

 $V_{(\tau,\sigma)}|_{B_l} = b_{\sigma(l)}, \quad l = 1, \cdots, m_2,$

Using non-atomicity and the fact that $\mathbb{P}(A_j) = \mathbb{P}(A_{\tau(j)})$ and $\mathbb{P}(B_l) = \mathbb{P}(B_{\sigma(l)})$, we can also find a random variable $X_{(\tau,\sigma)}$ such that

$$X_{(\tau,\sigma)}|_{A_j} \sim X'|_{A_{\tau(j)}}, \quad j = 1, \cdots, m_1$$
$$X_{(\tau,\sigma)}|_{B_l} \sim X'|_{B_{\sigma(l)}}, \quad l = 1, \cdots, m_2.$$

Clearly, $X_{(\tau,\sigma)} \sim X'$ so that $X_{(\tau,\sigma)} \in \mathcal{C}$. By convexity of \mathcal{C} ,

$$\frac{1}{m_1!m_2!}\sum_{\tau,\sigma}X_{(\tau,\sigma)}\in\mathcal{C}.$$

Moreover, $X_{(\tau,\sigma)}|_{A_j} - V_{(\tau,\sigma)}|_{A_j} = X_{(\tau,\sigma)}|_{A_j} - a_{\tau(j)} \sim X'|_{A_{\tau(j)}} - a_{\tau(j)} = X'|_{A_{\tau(j)}} - V|_{A_{\tau(j)}}$. Similarly, $X_{(\tau,\sigma)}|_{B_l} - V_{(\tau,\sigma)}|_{B_l} \sim X'|_{B_{\sigma(l)}} - V|_{B_{\sigma(l)}}$. Thus

(4.1)
$$X_{(\tau,\sigma)} - V_{(\tau,\sigma)} \sim X' - V.$$

In particular, $||X_{(\tau,\sigma)} - V_{(\tau,\sigma)}|| = ||X' - V|| < \varepsilon$, so that

$$\left\|\frac{1}{m_1!m_2!}\sum_{\tau,\sigma} X_{(\tau,\sigma)} - \frac{1}{m_1!m_2!}\sum_{\tau,\sigma} V_{(\tau,\sigma)}\right\| \le \frac{1}{m_1!m_2!}\sum_{\tau,\sigma} \|X_{(\tau,\sigma)} - V_{(\tau,\sigma)}\| < \varepsilon.$$

Furthermore,

$$\frac{1}{m_1!m_2!}\sum_{\tau,\sigma}V_{(\tau,\sigma)} = \mathbb{E}[V|\pi]$$

and

$$\begin{aligned} \left\| \mathbb{E}[V|\pi] - \mathbb{E}[X'|\pi] \right\| &= \left\| \frac{1}{\mathbb{P}(A)} \mathbb{E}[(V - X')\mathbf{1}_A] \mathbf{1}_A + \frac{1}{\mathbb{P}(B)} \mathbb{E}[(V - X')\mathbf{1}_B] \mathbf{1}_B \right\| \\ &\leq \left(\frac{1}{\mathbb{P}(A)} + \frac{1}{\mathbb{P}(B)} \right) \|V - X'\|_{L^1} \|\mathbf{1}\| \leq C \|V - X'\| < C\varepsilon, \end{aligned}$$

where C is a constant depending only on \mathcal{X} and π ; cf. (1.2). Hence, in view of $\mathbb{E}[X'|\pi] = \mathbb{E}[X|\pi]$, we have

$$\begin{aligned} \left\| \frac{1}{m_1!m_2!} \sum_{\tau,\sigma} X_{(\tau,\sigma)} - \mathbb{E}[X|\pi] \right\| \\ \leq \left\| \frac{1}{m_1!m_2!} \sum_{\tau,\sigma} X_{(\tau,\sigma)} - \frac{1}{m_1!m_2!} \sum_{\tau,\sigma} V_{(\tau,\sigma)} \right\| + \left\| \frac{1}{m_1!m_2!} \sum_{\tau,\sigma} V_{(\tau,\sigma)} - \mathbb{E}[V|\pi] \right\| \\ + \left\| \mathbb{E}[V|\pi] - \mathbb{E}[X'|\pi] \right\| \\ \leq (C+1)\varepsilon. \end{aligned}$$

Since $\frac{1}{m_1!m_2!} \sum_{\tau,\sigma} X_{(\tau,\sigma)} \in \mathcal{C}$ and \mathcal{C} is norm closed, we get $\mathbb{E}[X|\pi] \in \mathcal{C}$.

The critical idea in the proof is that while we easily swap V around to average to $\mathbb{E}[V|\pi]$, we need to swap X in a way that the key property (4.1) is maintained.

The proof of the theorem below is standard, once one is furnished with Lemma 4.2. We include a proof for the sake of completeness.

Theorem 4.3. Let \mathcal{X} be an r.i. space over a non-atomic probability space $(\Omega, \mathcal{F}, \mathbb{P})$ other than L^{∞} . Suppose that \mathcal{X} satisfies the AOCEA property. Let $\rho : \mathcal{X} \to \mathbb{R}$ be convex, decreasing, and law invariant. Then ρ is $\sigma(\mathcal{X}, \mathcal{X}')$ lower semicontinuous at every $X \in \mathcal{X}$ such that $X^- \in \mathcal{X}_a$.

Proof. Let $X \in \mathcal{X}$ be such that $X^- \in \mathcal{X}_a$. By [39, Lemma 2], there is a sequence (π_k) of finite partitions of Ω such that $\mathbb{E}[X|\pi_k] \xrightarrow{o} X$ in \mathcal{X} . Since ρ has the Fatou property at X by Theorem 2.2,

$$\rho(X) \le \liminf_{k} \rho(\mathbb{E}[X|\pi_k]).$$

Let $X_{\alpha} \xrightarrow{\sigma(\mathcal{X},\mathcal{X}')} X$. For any $A \in \mathcal{F}$, since $\mathbf{1}_A \in L^{\infty} \subset \mathcal{X}'$, $\mathbb{E}[X_{\alpha}\mathbf{1}_A] \to \mathbb{E}[X\mathbf{1}_A]$. Thus by the definition of $\mathbb{E}[\cdot|\pi]$, one sees that for any $k \in \mathbb{N}$, $\mathbb{E}[X_{\alpha}|\pi_k] \to \mathbb{E}[X|\pi_k]$ in the norm topology of \mathcal{X} . Recall from [41, Proposition 3.1] that ρ is norm continuous. It follows that

$$\rho(\mathbb{E}[X|\pi_k]) = \lim_{\alpha} \rho(\mathbb{E}[X_{\alpha}|\pi_k]).$$

For each α , the set $\{\rho \leq \rho(X_{\alpha})\}$ is norm closed, by norm continuity of ρ again. Since it is also convex and law invariant and contains X_{α} , Lemma 4.2 implies that

$$\mathbb{E}[X_{\alpha}|\pi_{k}] \in \{\rho \leq \rho(X_{\alpha})\}, \text{ i.e., } \rho(\mathbb{E}[X_{\alpha}|\pi_{k}]) \leq \rho(X_{\alpha}) \text{ for any } k \geq 1. \text{ Therefore,} \\ \rho(\mathbb{E}[X|\pi_{k}]) = \lim_{\alpha} \rho(\mathbb{E}[X_{\alpha}|\pi_{k}]) \leq \liminf_{\alpha} \rho(X_{\alpha}).$$

Taking lim inf over k, we obtain $\rho(X) \leq \liminf_{\alpha} \rho(X_{\alpha})$.

4.2. Automatic Dual Representations. The well-known Fenchel-Moreau Duality asserts that if $\rho : \mathcal{X} \to (-\infty, \infty]$ is proper (i.e., not identically ∞), convex, and $\sigma(\mathcal{X}, \mathcal{X}')$ lower semicontinuous everywhere, then ρ has a dual representation via the dual space \mathcal{X}' at every $X \in \mathcal{X}$. In our framework, ρ , however, only has $\sigma(\mathcal{X}, \mathcal{X}')$ lower semicontinuity locally, not everywhere, and as a result, the Fenchel-Moreau Duality cannot be applied directly. Fortunately, the classical proof in [9, Section 1.4] can be modified to recover the dual representation theorem locally. We include the complete proof here for the convenience of the reader.

Let \mathcal{Y} be a locally convex topological vector space and let \mathcal{Y}^* be its continuous dual. Let $\rho : \mathcal{Y} \to (-\infty, \infty]$ be a proper, convex functional. Define the conjugate functional $\rho^* : \mathcal{Y}^* \to (-\infty, \infty]$ by

$$\rho^*(F) = \sup_{Y \in \mathcal{Y}} (F(Y) - \rho(Y)), \quad F \in \mathcal{Y}^*.$$

Lemma 4.4. Suppose that $\rho : \mathcal{Y} \to \mathbb{R}$ is convex and (topologically) lower semicontinuous at $Y_0 \in \mathcal{Y}$. For any real number $\lambda_0 < \rho(Y_0)$, there exist $F \in \mathcal{Y}^*$ and a real number k > 0 such that

$$F(Y_0) + k\lambda_0 < \inf \left\{ F(Y) + k\rho(Y) : Y \in \mathcal{Y} \right\}.$$

Proof. Choose $\lambda \in \mathbb{R}$ such that $\lambda_0 < \lambda < \rho(Y_0)$. Since ρ is lower semicontinuous at Y_0 and \mathcal{Y} is locally convex, there exists a convex open neighborhood \mathcal{O} of Y_0 such that $\mathcal{O} \subseteq \{\rho > \lambda\}$. Let $\mathcal{A} = \mathcal{O} \times (-\infty, \lambda)$. Then \mathcal{A} is an open convex set in $\mathcal{Y} \times \mathbb{R}$. It is disjoint with the convex set $\mathcal{C}_{\rho} := \{(Y, \mu) \in \mathcal{Y} \times \mathbb{R} : \rho(Y) \leq \mu\}$. By the Hahn-Banach Separation Theorem ([40, Theorem 3.4]), there exist a nonzero linear functional $(F, k) \in (\mathcal{Y} \times \mathbb{R})^* = \mathcal{Y}^* \times \mathbb{R}$ and $\alpha \in \mathbb{R}$ such that

(4.2)

$$\sup\{F(Y) + k\mu : Y \in \mathcal{O}, \mu \in \mathbb{R}, \mu < \lambda\}$$

$$\leq \alpha \leq \inf\{F(Y) + k\mu : Y \in \mathcal{Y}, \mu \in \mathbb{R}, \mu \geq \rho(Y)\}.$$

Fix $Y \in \mathcal{O}$. By the first inequality in (4.2), $F(Y) + k\mu \leq \alpha$ for all $\mu < \lambda$. Hence $k \geq 0$. If k = 0, then $F \neq 0$. Since ρ is real-valued, $(Y, \rho(Y)) \in \mathcal{C}_{\rho}$ for all $Y \in \mathcal{X}$. Thus the second inequality in (4.2) implies that $\inf\{F(Y) : Y \in \mathcal{Y}\} \geq \alpha$. This is impossible, since F is linear and nonzero. Thus k > 0. Choose sufficiently small $\varepsilon > 0$ such that $\lambda_0 + \varepsilon < \lambda$. Then $(Y_0, \lambda_0 + \varepsilon) \in \mathcal{O} \times (-\infty, \lambda)$. Hence

$$F(Y_0) + k\lambda_0 < F(Y_0) + k(\lambda_0 + \varepsilon) \le \alpha$$

$$\le \inf\{F(Y) + k\mu : Y \in \mathcal{Y}, \mu \in \mathbb{R}, \mu \ge \rho(Y)\} = \inf\{F(Y) + k\rho(Y) : Y \in \mathcal{Y}\}.$$

Lemma 4.5. If $\rho : \mathcal{Y} \to \mathbb{R}$ is convex and lower semicontinuous at some $Y_0 \in \mathcal{Y}$, then ρ^* is proper.

Proof. (Modified from [9]). Choose $\lambda_0 \in \mathbb{R}$ such that $\lambda_0 < \rho(Y_0)$. By Lemma 4.4, there exist $F \in \mathcal{Y}^*$ and k > 0 such that

$$m := \inf\{F(Y) + k\rho(Y) : Y \in \mathcal{Y}\} \in \mathbb{R}.$$

Hence

$$\frac{-F}{k}(Y) - \rho(Y) \le \frac{-m}{k} \text{ for all } Y \in \mathcal{Y}.$$

By the definition of ρ^* , it follows that $\rho^*(\frac{-F}{k}) < \infty$.

Now, let \mathcal{Y} be a vector space and $\mathcal{Y}^{\#}$ be a vector space of linear functionals on \mathcal{Y} separating points of \mathcal{Y} . Then \mathcal{Y} with the topology $\sigma(\mathcal{Y}, \mathcal{Y}^{\#})$ is a locally convex topological vector space, and its continuous dual \mathcal{Y}^* is just $\mathcal{Y}^{\#}$. Assume that $\rho : \mathcal{Y} \to \mathbb{R}$ is convex and $\sigma(\mathcal{Y}, \mathcal{Y}^{\#})$ lower semicontinuous at some $Y_0 \in \mathcal{Y}$. By Lemma 4.5, ρ^* is proper. We can thus define $\rho^{**} : \mathcal{Y} \to (-\infty, \infty]$ by

$$\rho^{**}(Y) = \sup_{F \in \mathcal{Y}^{\#}} (F(Y) - \rho^{*}(F)).$$

Proposition 4.6. If $\rho : \mathcal{Y} \to \mathbb{R}$ is convex and $\sigma(\mathcal{Y}, \mathcal{Y}^{\#})$ lower semicontinuous at $Y_0 \in \mathcal{Y}$, then $\rho^{**}(Y_0) = \rho(Y_0)$.

Proof. (Modified from [9]). By the definitions of ρ^* and ρ^{**} , it is clear that $\rho^{**}(Y_0) \leq \rho(Y_0)$. Assume by way of contradiction that $\rho^{**}(Y_0) < \rho(Y_0)$. By Lemma 4.4, there are $F \in \mathcal{Y}^{\#}$ and k > 0 such that

$$F(Y_0) + k\rho^{**}(Y_0) < \inf\{F(Y) + k\rho(Y) : Y \in \mathcal{Y}\}.$$

Then

(4.3)
$$\frac{-F}{k}(Y_0) - \rho^{**}(Y_0) > \sup\left\{\frac{-F}{k}(Y) - \rho(Y) : Y \in \mathcal{Y}\right\} = \rho^*\left(\frac{-F}{k}\right).$$

In particular, $\rho^*\left(\frac{-F}{k}\right) < \infty$. But then by (4.3),

$$\rho^{**}(Y_0) < \frac{-F}{k}(Y_0) - \rho^*\left(\frac{-F}{k}\right).$$

contradicting the definition of $\rho^{**}(Y_0)$.

Taking $\mathcal{Y} = \mathcal{X}$ and $\mathcal{Y}^{\#} = \mathcal{X}'$ in Proposition 4.6 and applying Theorem 4.3, we obtain the following automatic dual representation theorem.

Theorem 4.7. Let \mathcal{X} be an r.i. space over a non-atomic probability space $(\Omega, \mathcal{F}, \mathbb{P})$ other than L^{∞} . Suppose that \mathcal{X} satisfies the AOCEA property. Let $\rho : \mathcal{X} \to \mathbb{R}$ be convex, decreasing, and law invariant. Then for any $X \in \mathcal{X}$ with $X^- \in \mathcal{X}_a$,

(4.4)
$$\rho(X) = \sup_{Y \in \mathcal{X}'} \left(\mathbb{E}[XY] - \rho^*(Y) \right),$$

where $\rho^*(Y) = \sup_{X \in \mathcal{X}} (\mathbb{E}[XY] - \rho(X))$ for any $Y \in \mathcal{X}'$.

We end the paper with two more equivalent formulations of the AOCEA property.

Remark 4.8. It is well known that if ρ has the dual representation (4.4) at some $X \in \mathcal{X}$, then it is $\sigma(\mathcal{X}, \mathcal{X}')$ lower semicontinuous at X. It is also well known that if ρ is $\sigma(\mathcal{X}, \mathcal{X}')$ lower semicontinuous at some $X \in \mathcal{X}$, then it has the Fatou property at X. Thus we can add two further equivalent statements in Theorem 2.2:

- (1") Every convex, decreasing, law-invariant functional $\rho : \mathcal{X} \to \mathbb{R}$ is $\sigma(\mathcal{X}, \mathcal{X}')$ lower semicontinuous at every $X \in \mathcal{X}$ such that $X^- \in \mathcal{X}_a$.
- (1") Every convex, decreasing, law-invariant functional $\rho : \mathcal{X} \to \mathbb{R}$ has the representation (4.4) at every $X \in \mathcal{X}$ such that $X^- \in \mathcal{X}_a$.

Appendix A. Proof of Proposition 2.3

Lemma A.1. Let X_1, X_2 be random variables on a non-atomic probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and X'_1 be a random variable on a non-atomic probability space $(\Omega', \mathcal{F}', \mathbb{P}')$. If $X'_1 \sim X_1 \geq X_2$, then there exists a random variable X'_2 on Ω' such that

$$X_1' \ge X_2' \sim X_2$$

The conclusion still holds if both " \geq " are replaced by " \leq ".

Proof. Suppose that $X'_1 \sim X_1 \geq X_2$. Let F_i and q_i be the CDF and quantile of X_i , respectively. Then q_1 is also a quantile function of X'_1 . Since $X_1 \geq X_2$, $F_1 \leq F_2$, and thus $q_1 \geq q_2$. Recall that there exists a random variable with uniform distribution on (0, 1) such that $X'_1 = q_1(U)$ ([21, Lemma A.32]). Thus

$$X_1' = q_1(U) \ge q_2(U) \sim X_2,$$

by [21, Lemma A.23]. Thus it is enough to let $X'_2 = q_2(U)$.

Proof of Proposition 2.3. Assume that (4) holds. We show that (2) holds. Let $A \in \mathcal{F}$ with $\mathbb{P}(A) > 0, X \in \mathcal{X}$ supported in A, and $\varepsilon > 0$ be given. Put $A_n = \{|X| \ge n\}$ for $n \in \mathbb{N}$. Then $A_n \downarrow \emptyset$. By the assumption (4), there exist natural numbers $(n_i)_{i=1}^k$ and random variables $(Z_i)_{i=1}^k$ such that $Z_i \sim X \mathbf{1}_{A_{n_i}}$ for $i = 1, \ldots, k$, all Z_i 's are supported in A. Moreover, a convex combination $\sum_{i=1}^k \lambda_i Z_i$ satisfies

$$\left\|\sum_{i=1}^{k} \lambda_i Z_i\right\| < \varepsilon$$

Since $\{X\mathbf{1}_{A_{n_i}} \neq 0\} = \{|X| \ge n_i\} = A_{n_i},$

 $Z_i|_{\{Z_i \neq 0\}} \sim X|_{A_{n_i}}.$

Since Z_i is supported in A, $\{Z_i \neq 0\} \subset A$; since X is supported in A, $A_{n_i} \subset A$. Thus it is immediate to see that $\mathbb{P}(A \setminus \{Z_i \neq 0\}) = \mathbb{P}(A \setminus A_{n_i})$. By non-atomicity, we can find a random variable W_i such that

 $W_i|_{A\setminus\{Z_i\neq 0\}} \sim X|_{A\setminus A_{n_i}}$ and W_i vanishes off $A\setminus\{Z_i\neq 0\}$.

For i = 1, ..., k, set $X_i = Z_i + W_i$. Clearly, $X_i \sim X$ for all i and X_i 's are supported in A. Moreover, since X is bounded by n_i on $A_{n_i}^c$, W_i is bounded by n_i . In particular, $W_i \in L^{\infty} \subset \mathcal{X}_a$. Thus $V := \sum_{i=1}^k \lambda_i W_i \in \mathcal{X}_a$. Clearly, V is supported in A and $\|\sum_{i=1}^k \lambda_i X_i - V\| = \|\sum_{i=1}^k \lambda_i Z_i\| < \varepsilon$. This proves that (4) \Longrightarrow (2).

Taking $A = \Omega$, the same argument gives $(3) \Longrightarrow (1)$. Since $(2) \Longrightarrow (1)$ and $(4) \Longrightarrow (3)$ are obvious, we have $(4) \Longrightarrow (2) \Longrightarrow (1)$ and $(4) \Longrightarrow (3) \Longrightarrow (1)$. To complete the proof, we show that $(1) \Longrightarrow (3) \Longrightarrow (4)$.

Assume that (1) holds. We show that (3) holds. Let $X \in \mathcal{X}_+$, $A_n \downarrow \emptyset$ and $\varepsilon > 0$ be given. By the assumption (1), there is a convex combination $\sum_{i=1}^{m_1} \alpha_i X_i$ such that $X_i \sim X$ for $1 \leq i \leq m_1$ and $\|\sum_{i=1}^{m_1} \alpha_i X_i - V\| < 1$ for some $V \in \mathcal{X}_a$. In view of $|a^+ - b^+| \leq |a - b|$ and $\sum_{i=1}^{m_1} \alpha_i X_i \geq 0$, we may replace V with V^+ so that $V \geq 0$. For any $i = 1, \ldots, m_1$, by $X_i \sim X \geq X \mathbf{1}_{A_i}$ and Lemma A.1, there exists a random variable Z_i such that

$$X_i \ge Z_i \sim X \mathbf{1}_{A_i}.$$

Set

$$U_1 := V \land \Big(\sum_{i=1}^{m_1} \alpha_i Z_i\Big).$$

Clearly, $0 \leq U_1 \leq V$ so that $U_1 \in \mathcal{X}_a$. In view of $a - a \wedge b = (a - b)^+$, since $\sum_{i=1}^{m_1} \alpha_i Z_i \leq \sum_{i=1}^{m_1} \alpha_i X_i$, we have

$$\left\|\sum_{i=1}^{m_1} \alpha_i Z_i - U_1\right\| = \left\|\left(\sum_{i=1}^{m_1} \alpha_i Z_i - V\right)^+\right\| \le \left\|\left(\sum_{i=1}^{m_1} \alpha_i X_i - V\right)^+\right\| < 1.$$

Applying the same arguments to the sequence $\{A_{m_1+n}\}_{n=1}^{\infty}$, we obtain $m_2 > m_1$, $(Z_i)_{i=m_1+1}^{m_2}$, and $U_2 \in \mathcal{X}_a$ such that $Z_i \sim X \mathbf{1}_{A_i}$ for $i = m_1+1, \ldots, m_2$ and $\left\|\sum_{i=m_1+1}^{m_2} \alpha_i Z_i - U_2\right\| < \frac{1}{2}$. Repeating this process, we get $(Z_i)_{i=1}^{\infty}$, convex combinations $(\sum_{i=m_{j-1}+1}^{m_j} \alpha_i Z_i)$, and $(U_j) \subset \mathcal{X}_a$ such that

$$Z_i \sim X \mathbf{1}_{A_i}$$
 for each $i \in \mathbb{N}$,

(A.1)
$$\left\|\sum_{i=m_{j-1}+1}^{m_j} \alpha_i Z_i - U_j\right\| < \frac{1}{j} \quad \text{for each } j \in \mathbb{N}.$$

For a random variable W, let W^* be its decreasing rearrangement given by $W^*(t) = \inf \{\lambda > 0 : \mathbb{P}(|W| > \lambda) \le t\}, t \in (0, 1)$. Let $W \in \mathcal{X}'$. By the Hardy-Littlewood Inequality ([7, Chp 2, Theorem 2.2]),

$$|\mathbb{E}[WZ_n]| \le \int_0^1 W^*(Z_n)^* \, \mathrm{d}t \le \int_0^1 W^*X^* \mathbf{1}_{[0,\mathbb{P}(A_n)]} \, \mathrm{d}t \to 0,$$

since $W^*X^* \in L^1$ ([7, Chp 2, Theorem 2.6]) and $\mathbb{P}(A_n) \to 0$. Thus as $j \to \infty$,

(A.2)
$$\mathbb{E}\Big[W\sum_{i=m_{j-1}+1}^{m_j}\alpha_i Z_i\Big] \to 0.$$

Since W acts a bounded linear functional on \mathcal{X} , it follows from (A.1) and (A.2) that

$$\lim_{j} \mathbb{E}[WU_j] = 0.$$

That is, $U_j \xrightarrow{\sigma(\mathcal{X}, \mathcal{X}')} 0$. Recall from [24, Lemma 3.3] that $(\mathcal{X}_a)^* = \mathcal{X}'$. Thus (U_i) converges to 0 weakly in \mathcal{X}_a .

Let $\varepsilon > 0$. Take $j_0 > \frac{2}{\varepsilon}$. By Mazur's Theorem, $0 \in \overline{\operatorname{co}(U_j)_{j \ge j_0}}^{\|\cdot\|}$. Thus there is a convex combination $\sum_{j=j_0}^{j_1} \beta_j U_j$ such that

$$\left\|\sum_{i=j_0}^{j_1}\beta_j U_j\right\| < \frac{\varepsilon}{2}$$

Let

(A.3)
$$Z = \sum_{j=j_0}^{j_1} \beta_j \Big(\sum_{i=m_{j-1}+1}^{m_j} \alpha_i Z_i \Big).$$

Then $Z \in co(Z_i)_{i=m_{j_0-1}+1}^{m_{j_1}}$ and

$$||Z|| \leq \sum_{j=j_0}^{j_1} \beta_j \left\| \sum_{i=m_{j-1}+1}^{m_j} \alpha_i Z_i - U_j \right\| + \left\| \sum_{j=j_0}^{j_1} \beta_j U_j \right\|$$
$$\leq \sum_{j=j_0}^{j_1} \beta_j \frac{1}{j} + \frac{\varepsilon}{2} < \sum_{j=j_0}^{j_1} \beta_j \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Finally, rewrite $Z = \sum_{i=1}^{k} \lambda_i Z_i$, where $k = m_{j_1}$ and $\lambda_i = 0$ if Z_i is not involved in defining Z in (A.3). This proves that (3) holds.

Now we show that (3) \Longrightarrow (4). Let $X \in \mathcal{X}$, $A \in \mathcal{F}$ with $\mathbb{P}(A) > 0$, $A_n \downarrow \emptyset$, and $\varepsilon > 0$ be given. By passing to a subsequence, we may assume without loss of generality that $\sum_{n=1}^{\infty} \mathbb{P}(A_n) < \frac{\mathbb{P}(A)}{2}$. Divide A as a disjoint union $B \cup C$, where $\mathbb{P}(B) = \mathbb{P}(C) = \frac{\mathbb{P}(A)}{2}$. Applying (3) to |X|, we find $Z'_i \sim |X| \mathbf{1}_{A_{n_i}}$, $i = 1, \ldots, k$, such that a convex combination satisfies $\|\sum_{i=1}^k \lambda_i Z'_i\| < \frac{\varepsilon}{2}$. Since

$$\mathbb{P}((Z'_1,\ldots,Z'_k)\neq 0) = \mathbb{P}\left(\bigcup_{i=1}^k \{Z'_i\neq 0\}\right) \le \sum_{i=1}^k \mathbb{P}(A_i) \le \mathbb{P}(B),$$

by non-atomicity, we can find a random vector (S_1, \ldots, S_k) such that

$$(S_1,\ldots,S_k)$$
 is supported in B , $(S_1,\ldots,S_k) \sim (Z'_1,\ldots,Z'_k)$.

In particular, $\sum_{i=1}^{k} \lambda_i S_i \sim \sum_{i=1}^{k} \lambda_i Z'_i$ so that $\|\sum_{i=1}^{k} \lambda_i S_i\| < \frac{\varepsilon}{2}$. Since $S_i \sim |X| \mathbf{1}_{A_{n_i}} \ge X^+ \mathbf{1}_{A_{n_i}}$, by Lemma A.1, there exists a random variable Q_i such that

$$S_i \ge Q_i \sim X^+ \mathbf{1}_{A_{n_i}}.$$

Then $0 \leq \sum_{i=1}^{k} \lambda_i Q_i \leq \sum_{i=1}^{k} \lambda_i S_i$, so that $\|\sum_{i=1}^{k} \lambda_i Q_i\| < \frac{\varepsilon}{2}$. Clearly, all Q_i 's are supported in B. Similarly, we obtain $R_i \sim X^- \mathbf{1}_{A_{n_i}}$, $i = 1, \ldots, k$, such that all R_i 's are supported in C and $\|\sum_{i=1}^{k} \lambda_i R_i\| < \frac{\varepsilon}{2}$. Since Q_i is supported in B and R_i is supported in C, $Q_i - R_i \sim X \mathbf{1}_{A_{n_i}}$ and $Q_i - R_i$ is supported in A. Finally,

$$\left\|\sum_{i=1}^{k} \lambda_{i}(Q_{i}-R_{i})\right\| \leq \left\|\sum_{i=1}^{k} \lambda_{i}Q_{i}\right\| + \left\|\sum_{i=1}^{k} \lambda_{i}R_{i}\right\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

The proof of (3) \Longrightarrow (4) is completed by setting $Z_i = Q_i - R_i$.

APPENDIX B. AN R.I. SPACE FAILING THE AOCEA PROPERTY

Endow (0,1) with the Lebesgue measure \mathbb{P} . For a random variable X on (0,1), let X^* be the decreasing rearrangement of X defined by

$$X^*(t) = \inf\{s > 0 : \mathbb{P}(|X| > s) \le t\}, \ t \in (0, 1).$$

We refer to [7, Chapter 2] for detailed properties of decreasing rearrangement.

Example B.1. Let \mathcal{X} be the space of all random variables X on (0, 1) such that

$$||X|| := \sup_{n \in \mathbb{N}} n2^n \int_0^{\frac{1}{2^n \cdot n!}} X^* \, \mathrm{d}t < \infty.$$

Then \mathcal{X} is an r.i. space over (0,1) that fails the AOCEA property.

Proof. Note that $\mathbf{1}^* = \mathbf{1}$. Thus it is easy to see that $\mathbf{1} \in \mathcal{X}$ and hence $\mathcal{X} \neq \{0\}$. For each $n \in \mathbb{N}$, define τ_n on \mathcal{X} by $\tau_n(X) = n2^n \int_0^{\frac{1}{2^n \cdot n!}} X^* dt$. We have the inequality

$$\int_0^s (X+Y)^* \, \mathrm{d}t \le \int_0^s X^* \, \mathrm{d}t + \int_0^s Y^* \, \mathrm{d}t$$

for any $X, Y \in L^1(0, 1)$ and all $s \in (0, 1)$; see, e.g., [7, p. 54]. Thus each τ_n , and hence $\|\cdot\|$, satisfies the triangle inequality. It is then clear that τ_n is a seminorm on \mathcal{X} . Moreover, as X^* is a decreasing function and $X^* \sim |X|$,

$$||X||_{1} = ||X^{*}||_{1} \le 2^{n} \cdot n! \int_{0}^{\frac{1}{2^{n} \cdot n!}} X^{*} \, \mathrm{d}t \le 2^{n} \cdot n! \, ||X^{*}||_{1} = 2^{n} \cdot n! \, ||X||_{1}$$

So each τ_n is in fact a lattice norm on \mathcal{X} that is equivalent to the L^1 -norm. In particular, $\|\cdot\|$ is a lattice norm on \mathcal{X} . Law-invariance of $\|\cdot\|$ is obvious. To see that \mathcal{X} is an r.i. space, it suffices to show the completeness of $\|\cdot\|$. Let $(X_k)_{k=1}^{\infty}$ be a norm Cauchy sequence in \mathcal{X} . Since each $\tau_n \leq \|\cdot\|$, (X_k) is Cauchy in τ_n -norm for all n. By equivalence of τ_n with L^1 -norm, there exists $X \in L^1$ such that (X_k) converges to Xwith respect to τ_n for all n. In particular,

$$\sup_{n} \tau_n(X) \le \sup_{n} \sup_{k} \tau_n(X_k) \le \sup_{k} ||X_k|| < \infty.$$

Thus $X \in \mathcal{X}$. Given $\varepsilon > 0$, choose $k_0 \in \mathbb{N}$ so that $||X_k - X_j|| \le \varepsilon$ if $k, j \ge k_0$. If $k \ge k_0$, then for any $n \in \mathbb{N}$,

$$\tau_n(X_k - X) = \lim_j \tau_n(X_k - X_j) \le \limsup_j \|X_k - X_j\| \le \varepsilon_j$$

Hence $||X_k - X|| \leq \varepsilon$ for any $k \geq k_0$. This shows that (X_k) converges to X in $|| \cdot ||$ -norm and completes the proof that \mathcal{X} is an r.i. space.

Next, we show that $\mathcal{X} \neq L^{\infty}$ and \mathcal{X} fails the AOCEA property. For convenience, set $c_n = \frac{1}{2^{n} \cdot (n+1)!}$ for all $n \in \mathbb{N}$. Let X be the function

$$X = \sum_{n=1}^{\infty} n! \mathbf{1}_{[c_{n+1}, c_n)}$$

Then $X \ge 0$ and it is decreasing so that $X^* = X$ a.s. For any $m \ge 2$,

$$\int_0^{\frac{1}{2^m \cdot m!}} X^* \, \mathrm{d}t \le \frac{(m-1)!}{2^m \cdot m!} + \sum_{n=m}^\infty n! c_n = \frac{1}{m2^m} + \sum_{n=m}^\infty \frac{1}{2^n(n+1)} \le \frac{3}{m2^m}$$

It follows that $X \in \mathcal{X}$ and that $\mathcal{X} \neq L^{\infty}$. Now suppose that there exists $Y \in co\{Z : Z \sim X\}$ and $U \in \mathcal{X}_a$ so that $||Y - U|| < \frac{1}{4}$. There exists $m_0 \in \mathbb{N}$ so that $||U\mathbf{1}_A|| < \frac{1}{4}$ if $\mathbb{P}(A) \leq \frac{1}{2^{m_0} \cdot m_0!}$. Thus, if $m \geq m_0$, then $m2^m \int_0^{\frac{1}{2^m \cdot m!}} U^* < \frac{1}{4}$. Write Y as a convex combination $\sum_{j=1}^m b_j Z_j$, where $Z_j \sim X$ for all j. We may assume that $m \geq m_0 \geq 2$. Choose measurable sets A_j , $1 \leq j \leq m$, so that $\mathbb{P}(A_j) = \frac{1}{m2^m \cdot m!}$ and that $\mathbb{E}[Z_j\mathbf{1}_{A_j}] = \int_0^{\frac{1}{m2^m \cdot m!}} Z_j^* \, \mathrm{d}t$. Then

$$\mathbb{E}[Z_j \mathbf{1}_{A_j}] \ge \int_0^{c_m} X^* \, \mathrm{d}t \ge m! \, c_m = \frac{1}{(m+1)2^m}$$

Let $A = \bigcup_{j=1}^{m} A_j$. Then $\mathbb{P}(A) \leq \frac{1}{2^m \cdot m!}$. Therefore,

$$\int_{0}^{\frac{1}{2^{m} \cdot m!}} Y^{*} \geq \mathbb{E}[Y \mathbf{1}_{A}] \geq \sum_{j=1}^{m} b_{j} \mathbb{E}[Z_{j} \mathbf{1}_{A_{j}}] \geq \frac{1}{(m+1)2^{m}}.$$

Thus

$$\frac{1}{2} \le m2^m \int_0^{\frac{1}{2^m \cdot m!}} Y^* \, \mathrm{d}t \le m2^m \Big[\int_0^{\frac{1}{2^m \cdot m!}} (Y - U)^* \, \mathrm{d}t + \int_0^{\frac{1}{2^m \cdot m!}} U^* \, \mathrm{d}t \Big] \\ \le \|Y - U\| + \frac{1}{4}.$$

Hence $||Y - U|| \ge \frac{1}{4}$, contrary to the choice of Y and U.

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