

AUTOMORPHISM GROUPS AND UNIQUENESS OF HOLOMORPHIC VERTEX OPERATOR ALGEBRAS OF CENTRAL CHARGE 24

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ABSTRACT. We describe the automorphism groups of all holomorphic vertex operator algebras of central charge 24 with non-trivial weight one Lie algebras by using their constructions as simple current extensions. We also confirm a conjecture of G. Höhn on the numbers of holomorphic vertex operator algebras of central charge 24 obtained as inequivalent simple current extensions of certain vertex operator algebras, which gives another proof of the uniqueness of holomorphic vertex operator algebras of central charge 24 with non-trivial weight one Lie algebras.

1. INTRODUCTION

The classification of (strongly regular) holomorphic vertex operator algebras (VOAs) of central charge 24 has been completed except for the characterization of the moonshine VOA; more precisely, the following are proved:

- (a) The weight one Lie algebra of a holomorphic VOA of central charge 24 is 0, 24-dimensional abelian or one of the 69 semisimple Lie algebras in [Sc93]; the list of these 71 Lie algebras is called *Schellekens' list*.
- (b) For any Lie algebra \mathfrak{g} in Schellekens' list, there exists a holomorphic VOA of central charge 24 whose weight one Lie algebra is isomorphic to \mathfrak{g} .
- (c) The isomorphism class of a holomorphic VOA V of central charge 24 with $V_1 \neq 0$ is uniquely determined by the weight one Lie algebra structure on V_1 .

Item (a) was proved in [Sc93, EMS20] (see [ELMS21] for another proof). Items (b) and (c) were proved by using case by case analysis (see [LS19] and [LS20b, Introduction]); several uniform approaches for (b) and (c) are also discussed in [Hö, MS22+, MS, HM22+, CLM22, LM].

For our purpose, we first recall a uniform approach proposed by [Hö] briefly. Let V be a holomorphic VOA of central charge 24. Then V_1 is a reductive Lie algebra of rank at most

2010 *Mathematics Subject Classification*. Primary 17B69, Secondary 20B25.

Key words and phrases. Automorphism groups, Holomorphic vertex operator algebras, Simple current extensions.

C.H. Lam was partially supported by a research grant AS-IA-107-M02 of Academia Sinica and MoST grants 107-2115-M-001-003-MY3 and 110-2115-M-001-011-MY3 of Taiwan.

H. Shimakura was partially supported by JSPS KAKENHI Grant Numbers JP19KK0065 and JP20K03505.

24 ([DM04b]); if $\text{rank } V_1 = 24$, then V is isomorphic to a Niemeier lattice VOA ([DM04b]) and that if $\text{rank } V_1 = 0$, equivalently, $V_1 = 0$, then it is conjectured in [FLM88] that V is isomorphic to the moonshine VOA. In this article, we assume that $0 < \text{rank } V_1 < 24$. Then V_1 is semisimple and the subVOA $\langle V_1 \rangle$ generated by V_1 has central charge 24, that means $\langle V_1 \rangle$ is a full subVOA of V ([DM04a]). Let \mathfrak{h} be a Cartan subalgebra of V_1 . The following item (d) was essentially proved in [Hö]; the necessary assumptions are confirmed in [La20a] (see also [ELMS21] for another proof):

- (d) The commutant $W = \text{Com}_V(\mathfrak{h})$ of \mathfrak{h} in V is isomorphic to the fixed-point subVOA $V_{\Lambda_g}^{\hat{g}}$ of the lattice VOA V_{Λ_g} with respect to a (standard) lift $\hat{g} \in \text{Aut}(V_{\Lambda_g})$ of an isometry $g|_{\Lambda_g}$ of Λ_g , where g is an isometry of the Leech lattice Λ in one of the following 10 conjugacy classes (as the notations in [ATLAS])

$$2A, 2C, 3B, 4C, 5B, 6F, 6G, 7B, 8E \text{ and } 10F,$$

and Λ_g is the coinvariant lattice of g (see Definition 2.1). In addition, the conjugacy class of g is uniquely determined by the Lie algebra structure of V_1 .

The commutant $\text{Com}_V(W)$ of W in V is isomorphic to a lattice VOA V_L . In fact, the lattice $L = L_{\mathfrak{g}}$, called the orbit lattice in [Hö], is also uniquely determined by the Lie algebra structure of $\mathfrak{g} = V_1$. Note that some non-isomorphic Lie algebras in Schellekens' list give isometric orbit lattices. Since both V_L and W have group-like fusion ([Do93, La20a]), V is a simple current extension of $V_L \otimes W$. In order to prove (b), it suffices to construct all holomorphic VOAs of central charge 24 as simple current extensions of $V_L \otimes W$, which was discussed in [Hö, Theorem 4.4] under some assumptions (see also [La20a]). In order to prove (c), it suffices to classify all holomorphic VOAs of central charge 24 as simple current extensions of $V_L \otimes W$ up to isomorphism, which can be proved by confirming the conjecture [Hö, Conjecture 4.8] on the number of inequivalent simple current extensions of $V_L \otimes W$ that form holomorphic VOAs of central charge 24.

Another question is to determine the automorphism groups of holomorphic VOAs of central charge 24. Our strategy is to describe the automorphism group of a VOA via its weight one Lie algebra. Let T be a VOA of CFT-type. Set

$$K(T) := \{g \in \text{Aut}(T) \mid g = \text{id on } T_1\},$$

the subgroup of $\text{Aut}(T)$ which acts trivially on T_1 . Let $\text{Aut}(T)|_{T_1}$ denote the restriction of $\text{Aut}(T)$ to T_1 . Then

$$\text{Aut}(T)|_{T_1} \cong \text{Aut}(T)/K(T) \subset \text{Aut}(T_1).$$

Recall that $\text{Aut}(T)$ contains the inner automorphism group $\text{Inn}(T)$, the normal subgroup generated by inner automorphisms $\{\exp(a_{(0)}) \mid a \in T_1\}$. Let $\text{Inn}(T)|_{T_1}$ denote the

restriction of $\text{Inn}(T)$ to T_1 , that is,

$$\text{Inn}(T)|_{T_1} \cong \text{Inn}(T)/(K(T) \cap \text{Inn}(T)).$$

Clearly, $\text{Inn}(T)|_{T_1}$ is isomorphic to the inner automorphism group $\text{Inn}(T_1)$ of T_1 .

Define

$$\text{Out}(T) := \text{Aut}(T)/\text{Inn}(T).$$

In Proposition 3.10, we will show that

$$K(T) \subset \text{Inn}(T)$$

when T is a holomorphic VOA of central charge 24 with $T_1 \neq 0$. For these cases,

$$\text{Out}(T) = \text{Aut}(T)/\text{Inn}(T) \cong \text{Aut}(T)|_{T_1}/\text{Inn}(T)|_{T_1},$$

and $\text{Out}(T)$ is a subgroup of the outer automorphism group $\text{Out}(T_1) \cong \text{Aut}(T_1)/\text{Inn}(T_1)$ of the weight one Lie algebra T_1 . Note that the inclusion $K(T) \subset \text{Inn}(T)$ does not hold in general; for example, the moonshine VOA V^\natural satisfies $K(V^\natural) = \text{Aut}(V^\natural) \neq 1$ and $\text{Inn}(V^\natural) = 1$. Therefore, for a holomorphic VOA T of central charge 24 with $T_1 \neq 0$, we have

$$\text{Aut}(T) \cong K(T) \cdot (\text{Inn}(T)|_{T_1} \cdot \text{Out}(T)),$$

where $A.B$ means a group G that contains a normal subgroup A with $G/A \cong B$. Hence $\text{Aut}(T)$ is roughly described by the groups $\text{Inn}(T)|_{T_1}$, $K(T)$ and $\text{Out}(T)$. Note that $\text{Inn}(T)|_{T_1} (\cong \text{Inn}(T_1))$ is well-studied.

For a holomorphic lattice VOA V_N associated with a Niemeier lattice N , the groups $K(V_N)$ and $\text{Out}(V_N)$ can be easily determined by the description of $\text{Aut}(V_N)$ in [DN99] (see Remark 3.1). For the 14 holomorphic VOAs $V_N^{\text{orb}(\theta)}$ obtained by applying the \mathbb{Z}_2 -orbifold construction to V_N and a lift θ of the -1 -isometry of N , the groups $K(V_N^{\text{orb}(\theta)})$ and $\text{Out}(V_N^{\text{orb}(\theta)})$ are determined in [Sh20] by using the explicit constructions. For some holomorphic VOAs V of central charge 24, $K(V)$ is determined in [LS20b] by using the module structure of V over the simple affine VOA generated by V_1 and its fusion product.

In this article, we assume (a), (b) and (d) and describe the orbit lattice L for each semisimple Lie algebra \mathfrak{g} in Schellekens' list of rank less than 24; this shows that the orbit lattice $L = L_{\mathfrak{g}}$ is uniquely determined by \mathfrak{g} , up to isometry. Note that the orbit lattices have been described in [Hö] by using Niemeier lattices. In addition, we determine the groups $K(V)$ and $\text{Out}(V)$ for all holomorphic VOAs V of central charge 24 with $0 < \text{rank } V_1 < 24$ by using $O(L)$, $\text{Aut}(W)$ and the constructions of V as simple current extensions of $V_L \otimes W$. Note that the automorphism groups $\text{Aut}(W)$ for all 10 VOAs W in (d) have been determined in [Gr98, Sh04, CLS18, La20b, La22+, BLS22+]. In particular, we prove the following (see Remark 3.1, Proposition 3.10 and Tables 6, 8, 10, 12, 13, 14, 16, 18, 22, 20 and 24):

Theorem 1.1. *Let V be a strongly regular holomorphic VOA of central charge 24 with $V_1 \neq 0$. Then $K(V) \subset \text{Inn}(V)$. Moreover, the group structures of $K(V)$ and $\text{Out}(V)$ are given as in Table 1. Here the genus symbol A, B, \dots, K in the table are used in [Hö]*

Table 1: $K(V)$ and $\text{Out}(V)$ for holomorphic VOAs V of central charge 24 with $V_1 \neq 0$

No.	Genus	V_1	$\text{Out}(V)$	$K(V)$	No.	Genus	V_1	$\text{Out}(V)$	$K(V)$
1	A	$U(1)^{24}$	$C_{0,0}$	\mathbb{C}^{24}	48	B	$C_{6,1}^2 B_{4,1}$	\mathbb{Z}_2	\mathbb{Z}_2
15		$A_{1,1}^{24}$	M_{24}	\mathbb{Z}_2^{12}	50		$D_{9,2} A_{7,1}$	\mathbb{Z}_2	\mathbb{Z}_8
24		$A_{2,1}^{12}$	$\mathbb{Z}_2.M_{12}$	\mathbb{Z}_3^6	52		$C_{8,1} F_{4,1}^2$	\mathbb{Z}_2	1
30		$A_{3,1}^8$	$\mathbb{Z}_2.AGL_3(2)$	\mathbb{Z}_4^4	53		$E_{7,2} B_{5,1} F_{4,1}$	1	\mathbb{Z}_2
37		$A_{4,1}^6$	$\mathbb{Z}_2.\mathfrak{S}_5$	\mathbb{Z}_5^3	56		$C_{10,1} B_{6,1}$	1	\mathbb{Z}_2
42		$D_{4,1}^6$	$\mathbb{Z}_3.\mathfrak{S}_6$	\mathbb{Z}_2^6	62		$B_{8,1} E_{8,2}$	1	\mathbb{Z}_2
43		$A_{5,1}^4 D_{4,1}$	$\mathbb{Z}_2.\mathfrak{S}_4$	$\mathbb{Z}_2^3 \times \mathbb{Z}_3^2$	6	C	$A_{2,3}^6$	\mathfrak{S}_6	\mathbb{Z}_3
46		$A_{6,1}^4$	$\mathbb{Z}_2.\mathfrak{A}_4$	\mathbb{Z}_7^2	17		$A_{5,3} D_{4,3} A_{1,1}^3$	\mathfrak{S}_3	\mathbb{Z}_2^3
49		$A_{7,1}^2 D_{5,1}^2$	$\mathbb{Z}_2.\mathbb{Z}_2^2$	$\mathbb{Z}_4 \times \mathbb{Z}_8$	27		$A_{8,3} A_{2,1}^2$	\mathbb{Z}_2	\mathbb{Z}_3^2
51		$A_{8,1}^3$	$\mathbb{Z}_2.\mathfrak{S}_3$	$\mathbb{Z}_9 \times \mathbb{Z}_3$	32		$E_{6,3} G_{2,1}^3$	\mathfrak{S}_3	1
54		$D_{6,1}^4$	\mathfrak{S}_4	\mathbb{Z}_2^4	34		$D_{7,3} A_{3,1} G_{2,1}$	1	\mathbb{Z}_4
55		$A_{9,1}^2 D_{6,1}$	$\mathbb{Z}_2.\mathbb{Z}_2$	$\mathbb{Z}_2^2 \times \mathbb{Z}_5$	45		$E_{7,3} A_{5,1}$	1	\mathbb{Z}_6
58		$E_{6,1}^4$	$\mathbb{Z}_2.\mathfrak{S}_4$	\mathbb{Z}_3^2	2	D	$A_{1,4}^{12}$	M_{12}	\mathbb{Z}_2
59		$A_{11,1} D_{7,1} E_{6,1}$	\mathbb{Z}_2	$\mathbb{Z}_3 \times \mathbb{Z}_4$	12		$B_{2,2}^6$	\mathfrak{S}_5	\mathbb{Z}_2
60		$A_{12,1}^2$	$\mathbb{Z}_2.\mathbb{Z}_2$	\mathbb{Z}_{13}	23		$B_{3,2}^4$	\mathfrak{A}_4	\mathbb{Z}_2
61		$D_{8,1}^3$	\mathfrak{S}_3	\mathbb{Z}_2^3	29		$B_{4,2}^3$	\mathfrak{S}_3	\mathbb{Z}_2
63		$A_{15,1} D_{9,1}$	\mathbb{Z}_2	\mathbb{Z}_8	41		$B_{6,2}^2$	\mathbb{Z}_2	\mathbb{Z}_2
64		$D_{10,1} E_{7,1}^2$	\mathbb{Z}_2	\mathbb{Z}_2^2	57		$B_{12,2}$	1	\mathbb{Z}_2
65		$A_{17,1} E_{7,1}$	\mathbb{Z}_2	$\mathbb{Z}_2 \times \mathbb{Z}_3$	13		$D_{4,4} A_{2,2}^4$	$2.\mathfrak{S}_4$	\mathbb{Z}_3^2
66		$D_{12,1}^2$	\mathbb{Z}_2	\mathbb{Z}_2^2	22		$C_{4,2} A_{4,2}^2$	\mathbb{Z}_4	\mathbb{Z}_5
67		A_{24}	\mathbb{Z}_2	\mathbb{Z}_5	36		$A_{8,2} F_{4,2}$	\mathbb{Z}_2	\mathbb{Z}_3
68		$E_{8,1}^3$	\mathfrak{S}_3	1	7	E	$A_{3,4}^3 A_{1,2}$	$\mathbb{Z}_2^2.\mathfrak{S}_3$	\mathbb{Z}_2
69		$D_{16,1} E_{8,1}$	1	\mathbb{Z}_2	18		$A_{7,4} A_{1,1}^3$	\mathbb{Z}_2	\mathbb{Z}_2^3
70		$D_{24,1}$	1	\mathbb{Z}_2	19		$D_{5,4} C_{3,2} A_{1,1}^2$	\mathbb{Z}_2	\mathbb{Z}_2^3
5	B	$A_{1,2}^{16}$	$AGL_4(2)$	\mathbb{Z}_2^5	28		$E_{6,4} A_{2,1} B_{2,1}$	1	\mathbb{Z}_6
16		$A_{3,2}^4 A_{1,1}^4$	$W(D_4)$	$\mathbb{Z}_2^3 \times \mathbb{Z}_4$	35		$C_{7,2} A_{3,1}$	1	\mathbb{Z}_2^2
25		$D_{4,2}^2 C_{2,1}^4$	$\mathbb{Z}_2 \times \mathfrak{S}_4$	\mathbb{Z}_2^3	9	F	$A_{4,5}^2$	\mathbb{Z}_2^2	1
26		$A_{5,2}^2 C_{2,1} A_{2,1}^2$	Dih_8	$\mathbb{Z}_3 \times \mathbb{Z}_6$	20		$D_{6,5} A_{1,1}^2$	1	\mathbb{Z}_2^2
31		$D_{5,2}^2 A_{3,1}^2$	Dih_8	\mathbb{Z}_4^2	8	G	$A_{5,6} B_{2,3} A_{1,2}$	\mathbb{Z}_2	\mathbb{Z}_2
33		$A_{7,2} C_{3,1}^2 A_{3,1}$	\mathbb{Z}_2^2	$\mathbb{Z}_2 \times \mathbb{Z}_4$	21		$C_{5,3} G_{2,2} A_{1,1}$	1	\mathbb{Z}_2
38		$C_{4,1}^4$	\mathfrak{S}_4	\mathbb{Z}_2	11	H	$A_{6,7}$	1	1
39		$D_{6,2} C_{4,1} B_{3,1}^2$	\mathbb{Z}_2	\mathbb{Z}_2^2	10	I	$D_{5,8} A_{1,2}$	1	\mathbb{Z}_2
40		$A_{9,2} A_{4,1} B_{3,1}$	\mathbb{Z}_2	\mathbb{Z}_{10}	3	J	$D_{4,12} A_{2,6}$	\mathfrak{S}_3	1
44		$E_{6,2} C_{5,1} A_{5,1}$	\mathbb{Z}_2	\mathbb{Z}_6	14		$F_{4,6} A_{2,2}$	1	\mathbb{Z}_3
47		$D_{8,2} B_{4,1}^2$	\mathbb{Z}_2	\mathbb{Z}_2^2	4	K	$C_{4,10}$	1	1

Remark 1.2. The former assertion $K(V) \subset \text{Inn}(V)$ of Theorem 1.1 is proved in Proposition 3.10 by using the fact that for W in (d), $\text{Aut}(W)$ acts faithfully on the set of isomorphism classes of irreducible W -modules (see Theorem 3.4).

Remark 1.3. We do *not* describe the embedding of $\text{Out}(V)$ into $\text{Out}(V_1)$. However, in Appendix A, we describe the subgroup $\text{Out}_1(V)$ of $\text{Out}(V)$ which preserves every simple ideal of V_1 , and the quotient $\text{Out}_2(V) := \text{Out}(V)/\text{Out}_1(V)$, which is a permutation group on the set of simple ideals of V_1 .

By using $O(L)$ and $\text{Aut}(W)$, we also prove the following (see Propositions 4.3, 5.7, 5.15, 5.18, 5.24 and 5.29). In particular, we confirm [Hö, Conjecture 4.8].

Theorem 1.4. *Let $g \in O(\Lambda)$ in one of the 10 conjugacy classes in (d). Set $W = V_{\Lambda_g}^{\hat{g}}$. Let L be an even lattice such that there exists a simple current extension of $V_L \otimes W$ which forms a holomorphic VOA V of central charge 24; in addition, $V_L = \text{Com}_V(W)$ and $W = \text{Com}_V(V_L)$. Then, we have the following results.*

- (1) *Assume that the conjugacy class of g is 2A, 3B, 4C, 5B, 6F, 7B, 8E or 10F. Then, there exists a unique holomorphic VOA of central charge 24 obtained as an inequivalent simple current extension of $V_L \otimes W$, up to isomorphism, for each possible L .*
- (2) *Assume that the conjugacy class of g is 2C. Then $L \cong \sqrt{2}D_{12}$ or $\sqrt{2}E_8\sqrt{2}D_4$. In addition, there exist exactly 6 (resp. 3) semisimple Lie algebras in Schellekens' list such that the associated orbit lattices are isometric to $\sqrt{2}D_{12}$ (resp. $\sqrt{2}E_8\sqrt{2}D_4$), and there exist exactly 6 (resp. 3) holomorphic VOAs of central charge 24 obtained as inequivalent simple current extensions of $V_{\sqrt{2}D_{12}} \otimes W$ (resp. $V_{\sqrt{2}E_8\sqrt{2}D_4} \otimes W$), up to isomorphism.*
- (3) *Assume that the conjugacy class of g is 6G. Then $L \cong \sqrt{6}D_4\sqrt{2}A_2$, and there exist exactly 2 semisimple Lie algebras in Schellekens' list such that the associated orbit lattices are isometric to $\sqrt{6}D_4\sqrt{2}A_2$. In addition, there exist exactly 2 holomorphic VOAs of central charge 24 obtained as inequivalent simple current extensions of $V_{\sqrt{6}D_4\sqrt{2}A_2} \otimes W$, up to isomorphism.*

Remark 1.5. The assumption on L and W in Theorem 1.4 is equivalent to the conditions that $(\mathcal{D}(L), q_L) \cong (\text{Irr}(W), -q_W)$ as quadratic spaces and the sum of the rank of L and the central charge of W is 24 (see Section 3.1).

It follows that a semisimple Lie algebra \mathfrak{g} in Schellekens' list of rank less than 24 determines a unique equivalence class of a simple current extension of $V_L \otimes W$ which forms a holomorphic VOA V of central charge 24 with $V_1 \cong \mathfrak{g}$. Hence Theorem 1.4 and the characterization of Niemeier lattice VOAs in [DM04b] give another proof of (c) (see [Hö, Section 4.3]).

Let us explain the main ideas for determining the groups $K(V)$ and $\text{Out}(V)$ for holomorphic VOAs V of central charge 24 with $0 < \text{rank } V_1 < 24$. As we mentioned above, V is a simple current extension of $V_L \otimes W$. Recall from [Do93] (resp. [La20a]) that all irreducible V_L -modules (resp. irreducible W -modules) are simple current modules. Hence the set $\text{Irr}(V_L)$ (resp. $\text{Irr}(W)$) of their isomorphism classes has group-like fusion, that is, it forms an abelian group under the fusion product. In addition, the map q_{V_L} (resp. q_W) from $\text{Irr}(V_L)$ (resp. $\text{Irr}(W)$) to \mathbb{Q}/\mathbb{Z} defined by conformal weights modulo \mathbb{Z} is a quadratic form ([EMS20]). It is well-known that $(\text{Irr}(V_L), q_{V_L})$ is isometric to the quadratic space $(\mathcal{D}(L), q_L)$ on the discriminant group $\mathcal{D}(L) = L^*/L$ with the quadratic form $q_L(v + L) = \langle v|v \rangle/2 + \mathbb{Z}$. Since V is holomorphic, there exists a bijection φ from $\mathcal{D}(L)$ to $\text{Irr}(W)$ such that for any $\lambda + L \in \mathcal{D}(L)$, $V_{\lambda+L} \otimes \varphi(\lambda + L)$ appears as a $V_L \otimes W$ -submodule of V . Note that φ is an isometry from $(\mathcal{D}(L), q_L)$ to $(\text{Irr}(W), -q_W)$. Then $S_\varphi = \{(V_{\lambda+L}, \varphi(\lambda + L)) \mid \lambda + L \in \mathcal{D}(L)\}$ is a maximal totally isotropic subspace of $(\text{Irr}(V_L), q_{V_L}) \oplus (\text{Irr}(W), q_W)$.

Since the group $K(V)$ acts trivially on the (fixed) Cartan subalgebra \mathfrak{h} , it preserves $V_L \otimes W$ and S_φ . In addition, the restriction of $K(V)$ to V_L is a subgroup of the inner automorphisms associated with \mathfrak{h} and preserves every element in $\text{Irr}(V_L)$. Hence the restriction of $K(V)$ to W also preserves every element in $\text{Irr}(W)$ via the isometry φ ; since the action of $\text{Aut}(W)$ on $\text{Irr}(W)$ is faithful, the restriction of $K(V)$ to W must be the identity. Note that the subgroup which acts trivially on $V_L \otimes W$ is the dual S_φ^* of S_φ , which is contained in $\text{Inn}(V)$. Hence $K(V)$ is contained in $\text{Inn}(V)$. In addition, we describe $K(V)$ in terms of L and the root lattice of V_1 (Proposition 3.12).

By the transitivity of $\text{Inn}(V)$ on Cartan subalgebras of V_1 , $\text{Out}(V)$ can be obtained as the quotient of the stabilizer $\text{Stab}_{\text{Aut}(V)}(\mathfrak{h})$ of the (fixed) Cartan subalgebra \mathfrak{h} in $\text{Aut}(V)$ by the normal subgroup $\text{Stab}_{\text{Inn}(V)}(\mathfrak{h}) = \text{Stab}_{\text{Aut}(V)}(\mathfrak{h}) \cap \text{Inn}(V)$. Since $\text{Stab}_{\text{Aut}(V)}(\mathfrak{h})$ preserves $V_L \otimes W$ and normalizes S_φ^* , the restriction of $\text{Stab}_{\text{Aut}(V)}(\mathfrak{h})$ to $V_L \otimes W$ is $\text{Stab}_{\text{Aut}(V_L \otimes W)}(\mathfrak{h}) \cap \text{Stab}_{\text{Aut}(V_L \otimes W)}(S_\varphi)$. By using $\text{Aut}(V_L)$ and S_φ , we see that it acts on $(\mathcal{D}(L), q_L)$ as $\overline{O}(L) \cap \varphi^*(\overline{\text{Aut}}(W))$, where $\mu_L : O(L) \rightarrow O(\mathcal{D}(L), q_L)$ (resp. $\mu_W : \text{Aut}(W) \rightarrow O(\text{Irr}(W), q_W)$) is the canonical group homomorphism, $\overline{O}(L)$ (resp. $\overline{\text{Aut}}(W)$) is the image of μ_L (resp. μ_W) and $\varphi^*(\overline{\text{Aut}}(W)) = \varphi^{-1} \overline{\text{Aut}}(W) \varphi$. Then $\text{Stab}_{\text{Aut}(V)}(\mathfrak{h})$ acts on \mathfrak{h} as $\mu_L^{-1}(\overline{O}(L) \cap \varphi^*(\overline{\text{Aut}}(W)))$. Clearly, $\text{Stab}_{\text{Inn}(V)}(\mathfrak{h})$ acts on \mathfrak{h} as the Weyl group $W(V_1)$ of V_1 . Thus $\text{Out}(V) \cong \mu_L^{-1}(\overline{O}(L) \cap \varphi^*(\overline{\text{Aut}}(W)))/W(V_1)$ (Proposition 3.17). For each \mathfrak{g} in Schellekens' list with $0 < \text{rank } \mathfrak{g} < 24$, we describe $L = L_{\mathfrak{g}}$ and $O(L)$ explicitly. In addition, by using the structures of the groups $\overline{O}(L)$, $\varphi^*(\overline{\text{Aut}}(W)) (\cong \text{Aut}(W))$ and $O(\mathcal{D}(L), q_L)$, we determine $\overline{O}(L) \cap \varphi^*(\overline{\text{Aut}}(W))$, which gives the shape of $\text{Out}(V)$. In our calculations, we use the fact that except for the case $2C$, the index $|O(\text{Irr}(W), q_W)|$:

$|\overline{\text{Aut}}(W)|$ is at most 4, which implies that $\text{Out}(V)$ has small index in $O(L)/W(V_1)$ (see (3.15) and Lemma 3.18).

Our strategy for the uniqueness has been discussed in [Hö] (see Proposition 4.2); we compute the number $|\varphi^*(\overline{\text{Aut}}(W)) \backslash O(\mathcal{D}(L), q_L) / \overline{O}(L)|$ of double cosets for given W and L , which gives the number of holomorphic VOAs obtained as inequivalent simple current extensions of $V_L \otimes W$. In fact, we verify that this number is 1 if the conjugacy class is neither $2C$ nor $6G$, and compute the numbers for $2C$ and $6G$ by using the group structures of $O(L)$, $\text{Aut}(W)$ and $O(\mathcal{D}(L), q_L)$. Then we obtain Theorem 1.4.

The organization of the article is as follows. In Section 2, we review some basic notions for integral lattices and vertex operator algebras. In Section 3, we view holomorphic VOAs as simple current extensions of $V_L \otimes W$ and study some stabilizers. We will also describe the groups $\text{Out}(V)$ and $K(V)$. In Section 4, we discuss the number of inequivalent simple current extensions of $V_L \otimes W$ that form holomorphic VOAs. In Section 5, for each W mentioned in (d) and the semisimple Lie algebra \mathfrak{g} in Schellekens' list with $0 < \text{rank } \mathfrak{g} < 24$, we describe the orbit lattice $L = L_{\mathfrak{g}}$ and determine the groups $K(V)$ and $\text{Out}(V)$ explicitly. In Appendix A, we describe the subgroup $\text{Out}_1(V)$ of $\text{Out}(V)$ and the quotient $\text{Out}_2(V) := \text{Out}(V) / \text{Out}_1(V)$.

Some calculations on lattices and finite groups are done by MAGMA ([BCP97]).

ACKNOWLEDGMENTS

The authors would like to thank the reviewers for giving them careful suggestions and comments. They also thank Brandon Rayhaun for useful discussion.

Conflict of interest statement. The authors have no competing interests to declare that are relevant to the content of this article.

Notations.

2_+^{1+2n}	an extra special 2-group of order 2^{1+2n} of plus type.
$A.B$	a group G that contains a normal subgroup A with $G/A \cong B$.
$\overline{\text{Aut}}(T)$	the subgroup of $O(\text{Irr}(T), q_T)$ induced by $\text{Aut}(T)$, i.e., $\overline{\text{Aut}}(T) = \text{Im } \mu_T$.
$\text{Aut}_0(T)$	the subgroup of $\text{Aut}(T)$ which acts trivially on $\text{Irr}(T)$.
$\text{Com}_T(X)$	the commutant of a subset X in a VOA T .
$\mathcal{D}(H)$	the discriminant group of an even lattice H , i.e., $\mathcal{D}(H) = H^*/H$.
$\text{Inn}(T)$	the inner automorphism group of a VOA T of CFT-type, i.e., the subgroup generated by $\{\exp(a_{(0)}) \mid a \in T_1\}$.
$\text{Irr}(T)$	the set of the isomorphism classes of irreducible modules over a VOA T .
$K(T)$	the subgroup of $\text{Aut}(T)$ which acts trivially on T_1 for a VOA T of CFT-type.
H^g	the fixed-point sublattice of a lattice H by an isometry g .
H_g	the coinvariant lattice of $g \in O(H)$, i.e., $H_g = \{x \in H \mid \langle x H^g \rangle = 0\}$.

$L = L_{\mathfrak{g}}$	the orbit lattice associated with \mathfrak{g} , i.e., $V_{L_{\mathfrak{g}}} \cong \text{Com}_V(\text{Com}_V(\mathfrak{h}))$, where V is a holomorphic VOA of $c = 24$ with $V_1 \cong \mathfrak{g}$ and \mathfrak{h} is a Cartan subalgebra of \mathfrak{g} .
μ_H	the group homomorphism $\mu_H : O(H) \rightarrow O(\mathcal{D}(H), q_H)$ for an even lattice H .
μ_T	the group homomorphism $\mu_T : \text{Aut}(T) \rightarrow O(\text{Irr}(T), q_T)$ for certain VOA T .
$O(H)$	the isometry group of a lattice H .
$O(X, q)$	the isometry group of a quadratic space (X, q) .
$\text{Out}(\mathfrak{g})$	$\text{Out}(\mathfrak{g}) = \text{Aut}(\mathfrak{g})/\text{Inn}(\mathfrak{g})$, the group of outer automorphisms of a Lie algebra \mathfrak{g} .
$\text{Out}(T)$	$\text{Out}(T) := \text{Aut}(T)/\text{Inn}(T)$ for a VOA T of CFT-type.
$\overline{O}(H)$	the subgroup of $O(\mathcal{D}(H), q_H)$ induced by $O(H)$, i.e., $\overline{O}(H) = \text{Im } \mu_H$
$O_0(H)$	the subgroup of $O(H)$ which acts trivially on $\mathcal{D}(H)$ for an even lattice H .
p^n	an elementary abelian p -group of order p^n .
p^{n+m}	a p -group G that contains a normal subgroup p^n with $G/p^n \cong p^m$.
$P_{\mathfrak{g}}$	$P_{\mathfrak{g}} = \bigoplus_{i=1}^s \frac{\sqrt{\ell}}{\sqrt{k_i}} Q^i \subset U_{\mathfrak{g}} = \sqrt{\ell} L_{\mathfrak{g}}^*$, where $\mathfrak{g} = \bigoplus_{i=1}^s \mathfrak{g}_i$ is the direct sum of simple ideals, k_i is the level of \mathfrak{g}_i , Q^i is the root lattice of \mathfrak{g}_i and ℓ is the level of $L_{\mathfrak{g}}$.
q_H	the quadratic form on $\mathcal{D}(H)$, $q_H(v + H) = \langle v v \rangle/2 + \mathbb{Z}$.
q_T	the quadratic form on $\text{Irr}(T)$ defined as conformal weights modulo \mathbb{Z} .
$Q_{\mathfrak{g}}$	$Q_{\mathfrak{g}} = \bigoplus_{i=1}^s \sqrt{k_i} Q_{long}^i \subset L_{\mathfrak{g}}$, where $\mathfrak{g} = \bigoplus_{i=1}^s \mathfrak{g}_i$ is the direct sum of simple ideals, k_i is the level of \mathfrak{g}_i and Q_{long}^i is the lattice spanned by long roots of \mathfrak{g}_i .
$R(H)$	the root system of a lattice H (see Section 2.1).
$\rho(M)$	the conformal weight of an irreducible module M over a VOA.
\mathfrak{S}_n	the symmetric group of degree n .
$\text{Stab}_G(X)$	the stabilizer of X in a group G .
$U = U_{\mathfrak{g}}$	$U = \sqrt{\ell} L^*$, where ℓ is the level of the orbit lattice $L = L_{\mathfrak{g}}$.
T^σ	the set of fixed-points of an automorphism σ of a VOA T .
$W(R), W(\mathfrak{g})$	the Weyl group of a root system R or a semisimple Lie algebra \mathfrak{g} .
$X_{n,k}$	(the type of) a simple Lie algebra whose type is X_n and level is k .

2. PRELIMINARY

In this section, we review some basic terminology and notation for integral lattices and vertex operator algebras.

2.1. Lattices. By a *lattice*, we mean a free abelian group of finite rank with a rational valued, positive-definite symmetric bilinear form $\langle | \rangle$. A lattice H is *integral* if $\langle H|H \rangle \subset \mathbb{Z}$ and it is *even* if $\langle x|x \rangle \in 2\mathbb{Z}$ for any $x \in H$. Note that an even lattice is integral. Let H^* denote the dual lattice of a lattice H , that is, $H^* = \{v \in \mathbb{Q} \otimes_{\mathbb{Z}} H \mid \langle v|H \rangle \subset \mathbb{Z}\}$. If H is integral, then $H \subset H^*$; $\mathcal{D}(H)$ denotes the discriminant group H^*/H .

An *isometry* of a lattice H is a linear isomorphism $g \in GL(\mathbb{Q} \otimes_{\mathbb{Z}} H)$ such that $g(H) = H$ and $\langle gx|gy \rangle = \langle x|y \rangle$ for all $x, y \in H$. Let $O(H)$ denote the group of all isometries of H , which we call the *isometry group* of H . Note that $O(H) = O(H^*)$.

Let H be an even lattice. Let $q_H : \mathcal{D}(H) \rightarrow \mathbb{Q}/\mathbb{Z}$ denote the quadratic form on $\mathcal{D}(H)$ defined by $q_H(v + H) = \langle v|v \rangle/2 + \mathbb{Z}$ for $v + H \in \mathcal{D}(H)$, and let

$$\mu_H : O(H) \rightarrow O(\mathcal{D}(H), q_H)$$

denote the canonical group homomorphism, where

$$O(\mathcal{D}(H), q_H) = \{g \in \text{Aut}(\mathcal{D}(H)) \mid q_H(gx) = q_H(x) \text{ for all } x \in \mathcal{D}(H)\}.$$

The group $\overline{O}(H)$ denotes the subgroup of $O(\mathcal{D}(H), q_H)$ induced by $O(H)$, and $O_0(H)$ denotes the subgroup of $O(H)$ which acts trivially on $\mathcal{D}(H)$, that is,

$$\overline{O}(H) = \text{Im } \mu_H, \quad O_0(H) = \text{Ker } \mu_H.$$

Definition 2.1. Let H be a lattice and $g \in O(H)$. Let H^g denote the fixed-point sublattice of g , that is, $H^g = \{x \in H \mid gx = x\}$. The *coinvariant lattice* of g is defined to be

$$H_g = \{x \in H \mid \langle x|y \rangle = 0 \text{ for all } y \in H^g\}.$$

Clearly, the restriction of g to H_g is fixed-point free on H_g .

Next we recall the definition of a root system from [Hum72]. A subset Φ of \mathbb{R}^n is called a *root system* in \mathbb{R}^n if Φ satisfies (R1)–(R4) below:

- (R1) $|\Phi| < \infty$ and Φ spans \mathbb{R}^n ;
- (R2) If $\alpha \in \Phi$, then $\mathbb{Z}\alpha \cap \Phi = \{\pm\alpha\}$;
- (R3) If $\alpha \in \Phi$, then the reflection $\sigma_\alpha : \beta \mapsto \beta - 2(\langle \beta|\alpha \rangle / \langle \alpha|\alpha \rangle)\alpha$ leaves Φ invariant;
- (R4) If $\alpha, \beta \in \Phi$, then $\langle \beta|\alpha \rangle / \langle \alpha|\alpha \rangle \in \mathbb{Z}$.

The *root lattice* L_Φ of Φ is the lattice spanned by roots. If Φ is irreducible, of type A_n , D_n or E_n and $\langle \alpha|\alpha \rangle = 2$ for all $\alpha \in \Phi$, then we often denote L_Φ just by Φ .

Let L be a positive-definite rational lattice. An element $\alpha \in L$ is *primitive* if $L/\mathbb{Z}\alpha$ has no torsion. A primitive element $\alpha \in L$ is called a *root* of L if the reflection σ_α in the ambient space of L is in $O(L)$. The set $R(L)$ of roots is an abstract root system in the ambient space of the sublattice $L_{R(L)}$ of L spanned by $R(L)$. Hence the general theory of root system applies to $R(L)$ and $R(L)$ decomposes into irreducible components of type A_n , B_n , C_n , D_n or G_2 , F_4 , E_6 , E_7 , E_8 .

For $\ell \in \mathbb{Z}_{>0}$ and a lattice H , we denote $\sqrt{\ell}H = \{\sqrt{\ell}x \mid x \in H\}$. The *level* of an even lattice H is defined to be the smallest positive integer ℓ such that $\sqrt{\ell}H^*$ is again even. The following can be obtained from [Sch06, Propositions 2.1 and 2.2].

Lemma 2.2. *Let H be an even lattice of level ℓ .*

- (1) Let α be a root of H with $\langle \alpha | \alpha \rangle = 2k$. Then $k \mid \ell$ and $\alpha \in H \cap kH^*$.
(2) Assume that ℓ is prime. Then

$$R(H) = \{v \in H \mid \langle v | v \rangle = 2\} \cup \{v \in \ell H^* \mid \langle v | v \rangle = 2\ell\}.$$

For a root system Φ , the *Weyl group* $W(\Phi)$ is the subgroup of $O(L_\Phi)$ generated by reflections associated with elements in Φ . The following lemma is well-known:

Lemma 2.3. *There are isomorphisms of the Weyl groups of root systems and the isometry groups of the root lattices:*

$$\begin{aligned} W(B_4) &\cong W(C_4) \cong W(D_4).2, & W(B_n) &\cong W(C_n) \cong O(D_n), & (n \geq 2, n \neq 4), \\ W(F_4) &\cong O(D_4) \cong W(D_4).\mathfrak{S}_3, & W(G_2) &\cong O(A_2), \end{aligned}$$

where $D_2 = A_1^2$ and $D_3 = A_3$.

2.2. Vertex operator algebras. Throughout this article, all VOAs are defined over the field \mathbb{C} of complex numbers.

A *vertex operator algebra* (VOA) $(T, Y, \mathbf{1}, \omega)$ is a \mathbb{Z} -graded vector space $T = \bigoplus_{m \in \mathbb{Z}} T_m$ over the complex field \mathbb{C} equipped with a linear map

$$Y(a, z) = \sum_{i \in \mathbb{Z}} a_{(i)} z^{-i-1} \in (\text{End}(T))[[z, z^{-1}]], \quad a \in T,$$

the *vacuum vector* $\mathbf{1} \in T_0$ and the *conformal vector* $\omega \in T_2$ satisfying certain axioms ([Bo86, FLM88]). Note that the operators $L(m) = \omega_{(m+1)}$, $m \in \mathbb{Z}$, satisfy the Virasoro relation:

$$[L(m), L(n)] = (m - n)L(m + n) + \frac{1}{12}(m^3 - m)\delta_{m+n,0}c \text{id}_T,$$

where $c \in \mathbb{C}$ is called the *central charge* of T , and $L(0)$ acts by the multiplication of scalar m on T_m .

A linear automorphism σ of a VOA T is called a (VOA) *automorphism* of T if

$$\sigma\omega = \omega \quad \text{and} \quad \sigma Y(v, z) = Y(\sigma v, z)\sigma \quad \text{for all } v \in T.$$

The group of all (VOA) automorphisms of T is denoted by $\text{Aut}(T)$.

A *vertex operator subalgebra* (or a *subVOA*) of a VOA T is a graded subspace of T which has a structure of a VOA such that the operations and its grading agree with the restriction of those of T and they share the vacuum vector. In addition, if they also share the conformal vector, then the subVOA is said to be *full*. For an automorphism σ of a VOA T , let T^σ denote the fixed-point set of σ , i.e.,

$$T^\sigma = \{v \in T \mid \sigma v = v\},$$

which is a full subVOA of T . For a subset X of a VOA T , the *commutant* $\text{Com}_T(X)$ of X in T is the subalgebra of T which commutes with X ([FZ92]). Note that the double commutant $\text{Com}_T(\text{Com}_T(X))$ contains X .

Let $M = \bigoplus_{m \in \mathbb{C}} M_m$ be a module over a VOA T (see [FHL93] for the definition). If M is irreducible, then there exists unique $\rho(M) \in \mathbb{C}$ such that $M = \bigoplus_{m \in \mathbb{Z}_{\geq 0}} M_{\rho(M)+m}$ and $M_{\rho(M)} \neq 0$. The number $\rho(M)$ is called the *conformal weight* of M . Let $\text{Irr}(T)$ denote the set of isomorphism classes of irreducible T -modules. We often identify an irreducible module with its isomorphism class without confusion.

A VOA is said to be *rational* if the admissible module category is semisimple. (See [DLM00] for the definition of admissible modules.) A rational VOA is said to be *holomorphic* if it itself is the only irreducible module up to isomorphism. A VOA T is of *CFT-type* if $T_0 = \mathbb{C}\mathbf{1}$ (note that $T_i = 0$ for all $i < 0$ if $T_0 = \mathbb{C}\mathbf{1}$), and is *C_2 -cofinite* if the co-dimension in T of the subspace spanned by $\{u_{(-2)}v \mid u, v \in T\}$ is finite. If T is rational and C_2 -cofinite, then $\rho(M) \in \mathbb{Q}$ for any $M \in \text{Irr}(T)$ ([DLM00, Theorem 1.1]). A module over a VOA is said to be *self-contragredient* if it is isomorphic to its contragredient module (see [FHL93]). A VOA is said to be *strongly regular* if it is rational, C_2 -cofinite, self-contragredient and of CFT-type. Note that a strongly regular VOA is simple. A simple VOA T of CFT-type is said to satisfy the *positivity condition* if $\rho(M) \in \mathbb{R}_{>0}$ for all $M \in \text{Irr}(T)$ with $M \not\cong T$.

Let T be a VOA and let M be a T -module. For $\sigma \in \text{Aut}(T)$, let $M \circ \sigma$ denote the σ -conjugate module, i.e., $M \circ \sigma = M$ as a vector space and its vertex operator is $Y_{M \circ \sigma}(u, z) = Y_M(\sigma u, z)$ for $u \in T$. If M is irreducible, then so is $M \circ \sigma$. Hence $\text{Aut}(T)$ acts on $\text{Irr}(T)$ as follows: for $\sigma \in \text{Aut}(T)$, $M \mapsto M \circ \sigma$. Note that $\rho(M) = \rho(M \circ \sigma)$ for $\sigma \in \text{Aut}(T)$ and $M \in \text{Irr}(T)$.

Let T be a strongly regular VOA. Then the *fusion products* \boxtimes are defined on irreducible T -modules ([HL95]). Note that the action of $\text{Aut}(T)$ on $\text{Irr}(T)$ above also preserves the fusion products. An irreducible T -module M^1 is called a *simple current module* if for any irreducible T -module M^2 , the fusion product $M^1 \boxtimes M^2$ is also an irreducible T -module. If all irreducible T -modules are simple current modules, then $\text{Irr}(T)$ has an abelian group structure under the fusion products; in this case, we say that T has *group-like fusion*.

Theorem 2.4 ([EMS20, Theorem 3.4, Proposition 3.5]). *Let T be a strongly regular VOA. Assume that T has group-like fusion and satisfies the positivity condition. Let*

$$q_T : \text{Irr}(T) \rightarrow \mathbb{Q}/\mathbb{Z}, \quad M \mapsto \rho(M) \pmod{\mathbb{Z}}.$$

Then q_T is a quadratic form on the abelian group $\text{Irr}(T)$ and the associated bilinear form is non-degenerate.

Remark 2.5. We call a finite abelian group with a quadratic form a *quadratic space*.

Let T be a strongly regular VOA satisfying the assumption of Theorem 2.4. Then, we obtain the canonical group homomorphism

$$(2.1) \quad \mu_T : \text{Aut}(T) \rightarrow O(\text{Irr}(T), q_T),$$

where $O(\text{Irr}(T), q_T) = \{f \in \text{Aut}(\text{Irr}(T)) \mid q_T(W) = q_T(f(W)) \text{ for all } W \in \text{Irr}(T)\}$ is the orthogonal group of the quadratic space $(\text{Irr}(T), q_T)$. The group $\overline{\text{Aut}}(T)$ denotes the subgroup of $O(\text{Irr}(T), q_T)$ induced by $\text{Aut}(T)$, and $\text{Aut}_0(T)$ denotes the subgroup of $\text{Aut}(T)$ which acts trivially on $\text{Irr}(T)$, that is,

$$\overline{\text{Aut}}(T) = \text{Im } \mu_T, \quad \text{Aut}_0(T) = \text{Ker } \mu_T.$$

Let T^0 be a strongly regular VOA. Let $\{T^\alpha \mid \alpha \in D\}$ be a set of inequivalent irreducible T^0 -modules indexed by a finite abelian group D . A simple VOA $T_D = \bigoplus_{\alpha \in D} T^\alpha$ is called a *simple current extension* of T^0 if every T^α is a simple current module. Note that $T^\alpha \boxtimes_{T^0} T^\beta \cong T^{\alpha+\beta}$ and that the simple VOA structure of T_D is uniquely determined by its T^0 -module structure, up to isomorphism ([DM04b, Proposition 5.3]). Two simple current extensions T_D and T_E of T^0 are *equivalent* if there exists an isomorphism $\sigma : T_D \rightarrow T_E$ such that $\sigma(T^0) = T^0$, equivalently, there exists $\tau \in \text{Aut}(T^0)$ such that $T_D \cong T_E \circ \tau$ as T^0 -modules.

2.3. Automorphisms of lattice VOAs. Let H be an even lattice and let V_H be the lattice VOA associated with H (see [FLM88] for detail). It is well-known ([Do93]) that V_H is strongly regular, has group-like fusion and satisfies the positivity condition. In addition, $\text{Irr}(V_H) = \{V_{\lambda+H} \mid \lambda + H \in \mathcal{D}(H)\}$ and $(\text{Irr}(V_H), q_{V_H}) \cong (\mathcal{D}(H), q_H)$ as quadratic spaces (see [Do93]).

Let $\hat{H} = \{\pm e^\alpha \mid \alpha \in H\}$ be a central extension of H by $\{\pm 1\}$ satisfying $e^\alpha e^\beta = (-1)^{(\alpha|\beta)} e^\beta e^\alpha$ for $\alpha, \beta \in H$. Note that such a central extension is unique up to isomorphism. Let $\text{Aut}(\hat{H})$ be the set of all automorphisms of \hat{H} . For $\varphi \in \hat{H}$, we define the element $\iota(\varphi) \in \text{Aut}(H)$ by $\varphi(e^\alpha) \in \{\pm e^{\iota(\varphi)(\alpha)}\}, \alpha \in H$. Set $O(\hat{H}) = \{\varphi \in \text{Aut}(\hat{H}) \mid \iota(\varphi) \in O(H)\}$. It was proved in [FLM88, Proposition 5.4.1] that there exists an exact sequence:

$$(2.2) \quad 1 \rightarrow \text{Hom}(H, \mathbb{Z}_2) \rightarrow O(\hat{H}) \xrightarrow{\iota} O(H) \rightarrow 1.$$

We also identify $O(\hat{H})$ as a subgroup of $\text{Aut}(V_H)$ as in [DN99, Section 2.4]. Note that $\text{Hom}(H, \mathbb{Z}_2) = \{\exp(2\pi\sqrt{-1}\alpha_{(0)}) \mid \alpha \in (H^*/2)/H^*\}$ in $\text{Aut}(V_H)$.

For $g \in O(H)$, an element $\tau \in O(\hat{H})$ with $\iota(\tau) = g$ is called a *standard lift* of g if τ acts trivially on the subVOA V_{H^g} . Note that a standard lift of g always exists and standard lifts of g are conjugate in $\text{Aut}(V_H)$ ([EMS20, Proposition 7.1] or [LS20a, Proposition 4.6]); we often denote a standard lift of g by \hat{g} . If g is fixed-point free on H , then we have $|\hat{g}| = |g|$ ([EMS20, Proposition 7.4]).

Recall from [DN99, Theorem 2.1] that

$$(2.3) \quad \text{Aut}(V_H) = \text{Inn}(V_H)O(\hat{H}).$$

Set $\mathfrak{h} = \text{Span}_{\mathbb{C}}\{h(-1)\mathbf{1} \mid h \in H\}$. Then \mathfrak{h} is a Cartan subalgebra of the reductive Lie algebra $(V_H)_1$. By [DN99, Lemmas 2.3 and 2.5], we have

$$(2.4) \quad \{\sigma \in \text{Aut}(V_H) \mid \sigma = \text{id on } \mathfrak{h}\} = \{\exp(a_{(0)}) \mid a \in \mathfrak{h}\}$$

and

$$(2.5) \quad \text{Stab}_{\text{Aut}(V_H)}(\mathfrak{h}) = \{\sigma \in \text{Aut}(V_H) \mid \sigma(\mathfrak{h}) = \mathfrak{h}\} = \{\exp(a_{(0)}) \mid a \in \mathfrak{h}\}O(\hat{H}).$$

It follows from (2.4), (2.5) and $\ker \iota \subset \{\exp(a_{(0)}) \mid a \in \mathfrak{h}\}$ (cf. (2.2)) that

$$(2.6) \quad \text{Stab}_{\text{Aut}(V_H)}(\mathfrak{h})_{|\mathfrak{h}} \cong \text{Stab}_{\text{Aut}(V_H)}(\mathfrak{h})/\{\exp(a_{(0)}) \mid a \in \mathfrak{h}\} \cong O(H).$$

The explicit action of $\text{Aut}(V_H)$ on $\text{Irr}(V_H)$ via the conjugation in Section 2.2 is well-known (cf. [LS20a, Lemma 2.11] and [Sh04, Proposition 2.9]):

Lemma 2.6. (1) For $\sigma \in \text{Inn}(V_H)$ and $M \in \text{Irr}(V_H)$, we have $M \circ \sigma \cong M$, that is,

$$\text{Inn}(V_H) \subset \text{Aut}_0(V_H).$$

(2) For $\sigma \in O(\hat{H})$, we have $V_{\lambda+H} \circ \sigma \cong V_{(\iota\sigma)^{-1}(\lambda)+H}$ for any $\lambda + H \in \mathcal{D}(H)$.

By (2.2), (2.3) and Lemma 2.6, we have the following.

Lemma 2.7. $\text{Aut}_0(V_H) = \text{Inn}(V_H)\iota^{-1}(O_0(H))$ and $\overline{\text{Aut}}(V_H) \cong O(H)/O_0(H) \cong \overline{O}(H)$.

3. HOLOMORPHIC VOAs OF CENTRAL CHARGE 24 AS SIMPLE CURRENT EXTENSIONS

Let V be a (strongly regular) holomorphic VOA of central charge 24. By [DM04a, DM04b], V satisfies one of the following:

- (i) $V_1 = 0$;
- (ii) V is isomorphic to a Niemeier lattice VOA;
- (iii) V_1 is a semisimple Lie algebra whose Lie rank $\text{rank } V_1$ is less than 24.

Note that in (ii) and (iii), the subVOA generated by V_1 is a full subVOA ([DM04a, Proposition 4.1]). In this section, we assume (iii), i.e., $0 < \text{rank } V_1 < 24$, and explain how to determine $K(V)$ and $\text{Out}(V)$.

Remark 3.1. It is conjectured that if (i) holds, then V is isomorphic to the moonshine VOA V^\natural ([FLM88]). Note that $K(V^\natural)(= \text{Aut}(V^\natural))$ is the Monster simple group and $\text{Inn}(V^\natural) = 1$, which shows $K(V^\natural) \not\subset \text{Inn}(V^\natural)$ and $\text{Out}(V^\natural) = \text{Aut}(V^\natural)/\text{Inn}(V^\natural) \cong \text{Aut}(V^\natural)$.

If (ii) holds, then $K(V)$ and $\text{Out}(V)$ are easily determined by (2.3); indeed, $K(V_\Lambda) \cong \mathbb{C}^{24}$ and $\text{Out}(V_\Lambda) = O(\Lambda)$ for the Leech lattice Λ and $K(V_N) \cong N/Q$ and $\text{Out}(V_N) \cong O(N)/W(Q)$ for a Niemeier lattice N with the root lattice $Q \neq \{0\}$. By (2.4), $K(V_N) \subset \text{Inn}(V_N)$ for any Niemeier lattice N .

3.1. Commutant of a Cartan subalgebra. Let V be a holomorphic VOA of central charge 24 with $0 < \text{rank } V_1 < 24$. Set $\mathfrak{g} = V_1$ and let \mathfrak{h} be a Cartan subalgebra of V_1 . Set $W = \text{Com}_V(\mathfrak{h})$. Then $W_1 = 0$. Recall from [DM06a, Corollary 5.8] that the double commutant of a Cartan subalgebra in a simple affine VOA at positive level is a lattice VOA. Since the subVOA generated by V_1 is a tensor product of simple affine VOAs at positive level ([DM06a, Theorem 1.1]), the double commutant $\text{Com}_V(\text{Com}_V(\mathfrak{h}))$ contains a lattice VOA as a full subVOA; there exists an even lattice L such that

$$\text{Com}_V(\text{Com}_V(\mathfrak{h})) \cong V_L.$$

In fact, L is uniquely determined by the Lie algebra structure of \mathfrak{g} , which will be verified by the explicit description of L in Section 5 (cf. [Hö, ELMS21]); $L = L_{\mathfrak{g}}$ is called the *orbit lattice* in [Hö]. Hence V contains $V_L \otimes W$ as a full subVOA, which shows

$$(3.1) \quad \text{rank } L + c_W = 24,$$

where c_W is the central charge of W . Note that the injective map from $V_L \otimes W$ to V is given by $a \otimes b \mapsto a_{(-1)}b$ for $a \in V_L$ and $b \in W$. By [Mi15, CM] and [CKLR19, Section 4.3], W is also strongly regular. In addition, by [ELMS21, Lemma 5.2], W satisfies the positivity condition; indeed, W contains a full subVOA isomorphic to the tensor product of parafermion VOAs ([DR17]), which satisfies the positivity condition.

It then follows from [Lin17, CKM22] that W has group-like fusion and

$$(3.2) \quad (\text{Irr}(V_L), q_{V_L}) \cong (\text{Irr}(W), -q_W)$$

as quadratic spaces. Note that $O(\text{Irr}(W), q_W) = O(\text{Irr}(W), -q_W)$ as groups. The VOA W was essentially identified in [Hö, Theorem 4.7] (cf. [HM22+, Theorem 4.2]) as follows; note that the necessary assumptions are confirmed in [La20a].

Theorem 3.2. *The VOA W is isomorphic to the orbifold VOA $V_{\Lambda_g}^{\hat{g}}$ for an isometry g of the Leech lattice Λ , where g belongs to one of 10 conjugacy classes $2A, 2C, 3B, 4C, 5B, 6F, 6G, 7B, 8E$ and $10F$, and \hat{g} is a (standard) lift of $g|_{\Lambda_g} \in O(\Lambda_g)$. In addition, the conjugacy class g is uniquely determined by the structure of V_1 .*

Remark 3.3. Theorem 3.2 can also be proved by using the fact that any holomorphic VOA of central charge 24 is constructed from the Leech lattice VOA by a cyclic orbifold construction ([ELMS21, Theorem 6.3]).

Theorem 3.4. [Gr98, Sh04, La20b, La22+, BLS22+] *Let $g \in O(\Lambda)$ whose conjugacy class is one of 10 cases in Theorem 3.2. Then the automorphism group of $W \cong V_{\Lambda_g}^{\hat{g}}$ has the shape as in Table 3 (see [Wi09] for the notation of classical groups). In addition, the group homomorphism μ_W in (2.1) is injective and the index of $\overline{\text{Aut}}(W) (\cong \text{Aut}(W))$ in $O(\text{Irr}(W), q_W)$ is given as in Table 3.*

Remark 3.5. The shapes of some groups in Table 3 are recalculated by MAGMA; they are more precise than the original shapes in the references. We adopt the genus symbol B, C, \dots, K of $(\text{Irr}(W), -q_W)$ and quadratic space structures from [Hö, Table 4].

Table 3: VOAs $W = V_{\Lambda_g}^{\hat{g}}$ for $g \in O(\Lambda)$

Genus	Class	rank Λ_g	$(\text{Irr}(W), -q_W)$	$\text{Aut}(W)(\cong \overline{\text{Aut}}(W))$	$O(\text{Irr}(W), q_W)$	index
B	$2A$	8	2_{II}^{+10}	$\text{GO}_{10}^+(2)$	$\text{GO}_{10}^+(2)$	1
C	$3B$	12	3^{-8}	$\text{P}\Omega_8^-(3).2$	$\text{GO}_8^-(3)$	2
D	$2C$	12	$2_{\text{II}}^{-10}4_{\text{II}}^{-2}$	$2_+^{1+20}.\mathfrak{S}_{12} \times \mathfrak{S}_3$	$2_+^{1+20}.\text{GO}_{10}^-(2) \times \mathfrak{S}_3$	$2^{11} \cdot 3 \cdot 17$
E	$4C$	14	$2_2^{+2}4_{\text{II}}^{+6}$	$2^{21}.\text{GO}_7(2)$	$2^{22}.\text{GO}_7(2)$	2
F	$5B$	16	5^{+6}	$2.\text{P}\Omega_6^+(5).2$	$\text{GO}_6^+(5)$	2
G	$6E$	16	$2_{\text{II}}^{+6}3^{-6}$	$\text{GO}_6^+(2) \times \text{GO}_6^+(3)$	$\text{GO}_6^+(2) \times \text{GO}_6^+(3)$	1
H	$7B$	18	7^{-5}	$\text{P}\Omega_5(7).2$	$\text{GO}_5(7)$	2
I	$8E$	18	$2_5^{+1}4_1^{+1}8_{\text{II}}^{+4}$	$2^{11+9}.\mathfrak{S}_6$	$2^{12+9}.\mathfrak{S}_6$	2
J	$6G$	18	$2_{\text{II}}^{+4}4_{\text{II}}^{-2}3^{+5}$	$2_+^{1+8}:(\mathfrak{S}_3) \times \text{P}\Omega_5(3).2$	$2_+^{1+8}:(\text{GO}_4^+(2) \times \mathfrak{S}_3) \times \text{GO}_5(3)$	4
K	$10F$	20	$2_{\text{II}}^{-2}4_{\text{II}}^{-2}5^{+4}$	$2_+^{1+4}:(2 \times \mathfrak{S}_3) \times \text{GO}_4^+(5)$	$2_+^{1+4}:(\mathfrak{S}_3 \times \mathfrak{S}_3) \times \text{GO}_4^+(5)$	3

The following properties of $\overline{\text{Aut}}(W)(\cong \text{Aut}(W))$ will be used later.

Lemma 3.6. *Assume that $g \in O(\Lambda)$ belongs to one of 10 conjugacy classes in Theorem 3.2. Set $W = V_{\Lambda_g}^{\hat{g}}$.*

- (1) *If the conjugacy class of g is neither $2C$, $6G$ nor $10F$, then $\overline{\text{Aut}}(W)$ is a normal subgroup of $O(\text{Irr}(W), q_W)$.*
- (2) *If the conjugacy class of g is $3B, 4C, 6G, 7B$ or $8E$, then $\overline{\text{Aut}}(W)$ does not contain the -1 -isometry of the abelian group $\text{Irr}(W)$.*

Proof. (1) is obvious from the indexes in Table 3.

Assume that $\overline{\text{Aut}}(W)$ contains the -1 -isometry σ ; we view σ as an element of $\text{Aut}(W)$. Then for any $M \in \text{Irr}(W)$, $M \circ \sigma$ is the contragredient module M' of M . Recall that the fusion products in $\text{Irr}(W)$ are determined in [La20a]. In particular, $V_{\lambda+\Lambda_g}' \cong V_{-\lambda+\Lambda_g}$ as W -modules for any $\lambda + \Lambda_g \in \mathcal{D}(\Lambda_g)$. Then, $V_{\Lambda_g} \circ \sigma \cong V_{\Lambda_g}$, which shows that σ can be lifted to an automorphism of V_{Λ_g} ([Sh04, Theorem 3.3]); we fix such an automorphism of V_{Λ_g} and use the same symbol σ . In addition, $V_{\lambda+\Lambda_g} \circ \sigma \cong V_{-\lambda+\Lambda_g}$ as V_{Λ_g} -modules for any $\lambda + \Lambda_g \in \mathcal{D}(\Lambda_g)$. By (2.2), (2.3) and Lemma 2.6, there exists $f \in O(\Lambda_g)$ of order 2 such that $\sigma \in \text{Inn}(V_{\Lambda_g})\iota^{-1}(f)$ and $f = -1$ on $\mathcal{D}(\Lambda_g)$. By the fusion products in $\text{Irr}(W)$, the σ -conjugate modules of irreducible \hat{g}^i -twisted V_{Λ_g} -modules are irreducible \hat{g}^{-i} -twisted V_{Λ_g} -modules. Hence $f g f^{-1} = g^{-1}$. Then $-f$ is an element in $O_0(\Lambda_g)$ of order 2 and $(-f)g(-f)^{-1} = g^{-1}$. It follows from $\Lambda^* = \Lambda$ that for any element $\lambda + \Lambda_g \in \mathcal{D}(\Lambda_g)$ there exists $\xi + \Lambda^g \in \mathcal{D}(\Lambda^g)$ such that $(\lambda + \Lambda_g, \xi + \Lambda^g)$ appears in $\Lambda/(\Lambda^g \oplus \Lambda_g)$. Since $g|_{\Lambda^g}$ act trivially on $\mathcal{D}(\Lambda^g)$ and $g \in O(\Lambda)$, we see that g preserves every element in $\Lambda/(\Lambda^g \oplus \Lambda_g)$. Hence $g \in O_0(\Lambda_g)$. Thus, $O_0(\Lambda_g)$ contains the subgroup $\langle f, g \rangle$ isomorphic to the dihedral group of order $2|g|$.

By using MAGMA, one can verify the following: if the conjugacy class of g is $3B$, $4C$ or $6G$, then $O_0(\Lambda_g)$ is the cyclic group $\langle g \rangle$; if the conjugacy class of g is $7B$, then $O_0(\Lambda_g)$ has order 21; if the conjugacy class of g is $8E$, then $O_0(\Lambda_g)$ has order 16 but it is not the dihedral group of order 16. Hence we obtain (2). \square

Remark 3.7. If the conjugacy class of g is $5B$, then $O_0(\Lambda_g)$ is a dihedral group of order 10 ([GL11]); in fact, $\text{Aut}(W)$ contains the -1 -isometry ([La20b]).

Lemma 3.8. *Let $g \in O(\Lambda)$ whose conjugacy class is one of the 10 cases in Theorem 3.2. Set $\ell = 2|g|$ if the conjugacy class of g is $2C$, $6G$ or $10F$, and set $\ell = |g|$ otherwise. Set $W = V_{\Lambda_g}^{\hat{g}}$. Let H be an even lattice satisfying $(\mathcal{D}(H), q_H) \cong (\text{Irr}(W), -q_W)$ as quadratic spaces. Then H has level ℓ . Moreover, if the conjugacy class of g is $2A$, $3B$, $5B$, $6E$ or $7B$ and the rank of H is $24 - \text{rank } \Lambda_g$, then $\sqrt{\ell}H^*$ also has level ℓ .*

Proof. By the classification of irreducible W -modules (see [La20a]), one can see that ℓ is the minimal positive integer such that $q_W(\text{Irr}(W)) \subset (1/\ell)\mathbb{Z}_{\geq 0}$. It then follows from $(\mathcal{D}(H), q_H) \cong (\text{Irr}(W), -q_W)$ that ℓ is also the minimal positive integer satisfying $q_H(\mathcal{D}(H)) \subset (1/\ell)\mathbb{Z}_{\geq 0}$. Hence ℓ is the minimal positive integer so that $\sqrt{\ell}H^*$ is even, and H has level ℓ .

Assume that the conjugacy class of g is $2A$, $3B$, $5B$, $6E$ or $7B$ and the rank of H is $24 - \text{rank } \Lambda_g$. Since $\sqrt{\ell}(\sqrt{\ell}H^*)^* = H$, the latter assertion follows from the fact that $(1/\sqrt{n})H$ is not even if $n \in \mathbb{Z}_{>1}$. Indeed, if $(1/\sqrt{n})H$ is even for $n \in \mathbb{Z}_{\geq 1}$, then $(1/\sqrt{n})H \subset \sqrt{n}H^*$, and hence $n^{\text{rank } H}$ divides $|H^*/H| = |\text{Irr}(W)|$. By Table 3, the only possibility is $n = 1$. \square

Remark 3.9. In Lemma 3.8, ℓ is equal to $|\hat{g}|$ for the (standard) lift $\hat{g} \in O(\hat{\Lambda})$ of g (cf. [EMS20, Proposition 7.4]).

3.2. The group $K(V)$. Let V be a holomorphic VOA of central charge 24 with $0 < \text{rank } V_1 < 24$. Let \mathfrak{h} be a Cartan subalgebra of V_1 . Set $W = \text{Com}_V(\mathfrak{h})$ and $V_L = \text{Com}_V(W)$ as in Section 3.1. In this subsection, we describe the group $K(V)$, defined in the introduction, in terms of V_1 and L .

Recall that $V_L \otimes W$ has group-like fusion. Hence V is a simple current extension of $V_L \otimes W$. Since V is holomorphic, for any irreducible V_L -module $V_{\lambda+L}$, there exists a unique irreducible W -module X such that $V_{\lambda+L} \otimes X$ appears as an irreducible $V_L \otimes W$ -submodule of V with multiplicity one; let φ be the bijection from $\mathcal{D}(L)$ to $\text{Irr}(W)$ defined by the following decomposition of V as a $V_L \otimes W$ -module:

$$(3.3) \quad V \cong \bigoplus_{\lambda+L \in \mathcal{D}(L)} V_{\lambda+L} \otimes \varphi(\lambda+L).$$

Then φ is a group isomorphism and $\rho(V_{\lambda+L}) + \rho(\varphi(\lambda + L)) \in \mathbb{Z}$, which shows that φ is an isometry of quadratic spaces from $(\mathcal{D}(L), q_L)$ to $(\text{Irr}(W), -q_W)$. Set

$$(3.4) \quad S_\varphi = \{(V_{\lambda+L}, \varphi(\lambda + L)) \mid \lambda + L \in \mathcal{D}(L)\} \subset \text{Irr}(V_L) \times \text{Irr}(W).$$

Since V is holomorphic, S_φ is a maximal totally isotropic subspace of $(\text{Irr}(V_L), q_{V_L}) \oplus (\text{Irr}(W), q_W)$. Here a vector is isotropic if the value of the form is zero and a totally isotropic subspace is a subspace consisting of isotropic vectors. Note that $S_\varphi \cong \mathcal{D}(L)$ as groups. We then view V as a simple current extension of $V_L \otimes W$ graded by S_φ ; $V = \bigoplus_{M \in S_\varphi} M$. Here $(V_{\lambda+L}, \varphi(\lambda + L)) \in S_\varphi$ is regarded as an irreducible $V_L \otimes W$ -module $V_{\lambda+L} \otimes \varphi(\lambda + L)$. Hence the dual $S_\varphi^* = \text{Hom}(S_\varphi, \mathbb{C}^\times)$ of S_φ acts faithfully on V as an automorphism group. More precisely, by (3.4), we have

$$(3.5) \quad S_\varphi^* = \{\exp(2\pi\sqrt{-1}v_{(0)}) \mid v + L \in \mathcal{D}(L)\}.$$

In addition, by [Sh04, Theorem 3.3], we obtain

$$(3.6) \quad S_\varphi^* = \{\sigma \in \text{Aut}(V) \mid \sigma = id \text{ on } V_L \otimes W\}.$$

Proposition 3.10. $\{\sigma \in \text{Aut}(V) \mid \sigma = id \text{ on } \mathfrak{h}\} = \{\exp(a_{(0)}) \mid a \in \mathfrak{h}\}$. In particular, $K(V) \subset \text{Inn}(V)$.

Proof. Clearly, $\{\sigma \in \text{Aut}(V) \mid \sigma = id \text{ on } \mathfrak{h}\} \supset \{\exp(a_{(0)}) \mid a \in \mathfrak{h}\}$.

Let $\sigma \in \text{Aut}(V)$ such that $\sigma = id$ on \mathfrak{h} . Then σ preserves the commutant and the double commutant of \mathfrak{h} , that is, σ preserves both V_L and W . Since $\sigma|_{V_L}$ acts trivially on $\mathfrak{h} \subset (V_L)_1$, we have $\sigma|_{V_L} \in \{\exp(a_{(0)}) \mid a \in \mathfrak{h}\}$ by (2.4). By Lemma 2.6 (1), $\sigma|_{V_L}$ acts trivially on $\text{Irr}(V_L)$, and hence $\sigma(V_{\lambda+L} \otimes \varphi(\lambda + L)) = V_{\lambda+L} \otimes \varphi(\lambda + L)$ for all $\lambda + L \in \mathcal{D}(L)$. Since φ is a bijection from $\mathcal{D}(L)$ to $\text{Irr}(W)$, $\sigma|_W \in \text{Aut}(W)$ also acts trivially on $\text{Irr}(W)$. By Theorem 3.4, the action of $\text{Aut}(W)$ on $\text{Irr}(W)$ is faithful. Hence we have $\sigma|_W = id$. It follows from (3.6) that $\sigma \in S_\varphi^* \{\exp(a_{(0)}) \mid a \in \mathfrak{h}\}$. By (3.5), we have $\sigma = \exp(u_{(0)})$ for some $u \in \mathfrak{h}$. \square

Remark 3.11. If V is isomorphic to a Niemeier lattice VOA, then $K(V) \subset \text{Inn}(V)$ by Remark 3.1. Hence for any holomorphic VOA V of central charge 24 with $V_1 \neq 0$, we have $K(V) \subset \text{Inn}(V)$, which proves the first assertion of Theorem 1.1.

Let $V_1 = \mathfrak{g} = \bigoplus_{i=1}^s \mathfrak{g}_i$, where \mathfrak{g}_i are simple ideals, and let k_i be the level of \mathfrak{g}_i . Note that $k_i \in \mathbb{Z}_{>0}$ ([DM06a]). The norm of roots in \mathfrak{g} is normalized so that $\langle \alpha | \alpha \rangle = 2$ for any long roots α .

Proposition 3.12. Let Q^i be the root lattice of \mathfrak{g}_i and set $\tilde{Q} = \bigoplus_{i=1}^s \frac{1}{\sqrt{k_i}} Q^i$. Then

$$K(V) = \{\exp(2\pi\sqrt{-1}v_{(0)}) \mid v + L \in \tilde{Q}^*/L\}$$

and it is isomorphic to L^*/\tilde{Q} as a group.

Proof. By (3.3) and $\mathfrak{h} \subset (V_L)_1$, for $x \in \mathfrak{h}$, $\exp(2\pi\sqrt{-1}x_{(0)}) = id$ on V if and only if $x \in L$. By Proposition 3.10, we have $K(V) \subset \{\exp(2\pi\sqrt{-1}v_{(0)}) \mid v + L \in \mathfrak{h}/L\}$.

Recall from [DM06a] that the subVOA generated by V_1 is isomorphic to $\bigotimes_{i=1}^s L_{\widehat{\mathfrak{g}}_i}(k_i, 0)$, where $L_{\widehat{\mathfrak{g}}_i}(k_i, 0)$ is the simple affine VOA associated with \mathfrak{g}_i at level k_i . It was proved in [DR17] that $L_{\widehat{\mathfrak{g}}_i}(k_i, 0)$ is a simple current extension of $V_{\sqrt{k_i}Q_{long}^i} \otimes K(\mathfrak{g}_i, k_i)$ as follows:

$$(3.7) \quad L_{\widehat{\mathfrak{g}}_i}(k_i, 0) \cong \bigoplus_{\lambda \in (1/\sqrt{k_i})Q^i/\sqrt{k_i}Q_{long}^i} V_{\lambda + \sqrt{k_i}Q_{long}^i} \otimes M^{0,\lambda},$$

where Q_{long}^i is the sublattice of the root lattice Q^i spanned by long roots, $K(\mathfrak{g}_i, k_i)$ is the parafermion VOA and $M^{0,\lambda}$ are certain irreducible $K(\mathfrak{g}_i, k_i)$ -modules.

By (3.7), for $v \in \mathfrak{h}$, $\exp(2\pi\sqrt{-1}v_{(0)}) = id$ on V_1 if and only if $v \in \tilde{Q}^*$. Hence $K(V) = \{\exp(2\pi\sqrt{-1}v_{(0)}) \mid v + L \in \tilde{Q}^*/L\}$. Clearly, this group is isomorphic to the dual L^*/\tilde{Q} of \tilde{Q}^*/L . \square

Remark 3.13. For a short root β in the root lattice Q^i of \mathfrak{g}_i , we have $\langle \beta | \beta \rangle = 2/r_i$, where r_i is the lacing number of \mathfrak{g}_i . Hence Q^i is not necessarily even.

Later, we use the sublattice

$$(3.8) \quad Q_{\mathfrak{g}} = \bigoplus_{i=1}^s \sqrt{k_i}Q_{long}^i \subset L.$$

Note that the ranks of both $Q_{\mathfrak{g}}$ and L are equal to $\dim \mathfrak{h}$.

3.3. The group $\text{Out}(V)$. Let V be a holomorphic VOA of central charge 24 with $0 < \text{rank } V_1 < 24$. Let \mathfrak{h} be a Cartan subalgebra of V_1 . Set $W = \text{Com}_V(\mathfrak{h})$ and $V_L = \text{Com}_V(W)$ as in Section 3.1. In this subsection, we describe $\text{Out}(V)$, defined in the introduction, in terms of V_1 and L .

As discussed in the previous section, V is a simple current extension $V = \bigoplus_{M \in S_\varphi} M$. Hence the fixed-point subVOA of S_φ^* is

$$(3.9) \quad V^{S_\varphi^*} = \{v \in V \mid \sigma v = v \text{ for all } \sigma \in S_\varphi^*\} = V_L \otimes W.$$

It follows that the normalizer of S_φ^* in $\text{Aut}(V)$ is given by

$$(3.10) \quad N_{\text{Aut}(V)}(S_\varphi^*) = \{\sigma \in \text{Aut}(V) \mid \sigma(V_L \otimes W) = V_L \otimes W\}.$$

By [Sh04, Theorem 3.3], we obtain

$$(3.11) \quad N_{\text{Aut}(V)}(S_\varphi^*)/S_\varphi^* \cong \text{Stab}_{\text{Aut}(V_L \otimes W)}(S_\varphi) = \{\sigma \in \text{Aut}(V_L \otimes W) \mid S_\varphi \circ \sigma = S_\varphi\}.$$

Recall that \mathfrak{h} is the fixed Cartan subalgebra of V_1 . Set

$$\text{Stab}_{\text{Aut}(V)}(\mathfrak{h}) = \{\sigma \in \text{Aut}(V) \mid \sigma(\mathfrak{h}) = \mathfrak{h}\}, \quad \text{Stab}_{\text{Inn}(V)}(\mathfrak{h}) = \text{Stab}_{\text{Aut}(V)}(\mathfrak{h}) \cap \text{Inn}(V).$$

Lemma 3.14. (1) $\text{Aut}(V) = \text{Inn}(V)\text{Stab}_{\text{Aut}(V)}(\mathfrak{h})$;

- (2) $\text{Out}(V) \cong \text{Stab}_{\text{Aut}(V)}(\mathfrak{h})/\text{Stab}_{\text{Inn}(V)}(\mathfrak{h})$;
(3) $N_{\text{Aut}(V)}(S_\varphi^*) = \text{Inn}(V_L)\text{Stab}_{\text{Aut}(V)}(\mathfrak{h})$.

Proof. Let $\sigma \in \text{Aut}(V)$. Since all Cartan subalgebras of V_1 are conjugate under $\text{Inn}(V_1)$, there exists $\tau \in \text{Inn}(V)$ such that $\tau\sigma(\mathfrak{h}) = \mathfrak{h}$. Hence $\tau\sigma \in \text{Stab}_{\text{Aut}(V)}(\mathfrak{h})$, which proves (1). Clearly, the assertion (1), Lemma 3.10 and the definition of $\text{Out}(V)$ imply (2).

It follows from $\text{Com}_V(\mathfrak{h}) = W$ and $\text{Com}_V(W) = V_L$ that $\text{Stab}_{\text{Aut}(V)}(\mathfrak{h})$ preserves $V_L \otimes W$. Hence by (3.10), $\text{Stab}_{\text{Aut}(V)}(\mathfrak{h}) \subset N_{\text{Aut}(V)}(S_\varphi^*)$. In addition, by Lemma 2.6, $\text{Inn}(V_L)$ preserves S_φ . Hence by (3.11), we have $\text{Inn}(V_L) \subset N_{\text{Aut}(V)}(S_\varphi^*)$. Thus $\text{Inn}(V_L)\text{Stab}_{\text{Aut}(V)}(\mathfrak{h}) \subset N_{\text{Aut}(V)}(S_\varphi^*)$.

Let $\sigma \in N_{\text{Aut}(V)}(S_\varphi^*)$. By (3.10), σ preserves $V_L \otimes W$, and therefore also $(V_L \otimes W)_1 = (V_L)_1 \otimes \mathbf{1}$. Since \mathfrak{h} is a Cartan subalgebra of $(V_L)_1$, there exists $\tau \in \text{Inn}(V_L)$ such that $\tau\sigma(\mathfrak{h}) = \mathfrak{h}$. Hence $\sigma \in \text{Inn}(V_L)\text{Stab}_{\text{Aut}(V)}(\mathfrak{h})$. \square

Lemma 3.15. $\text{Stab}_{\text{Aut}(V)}(\mathfrak{h})/S_\varphi^* \cong \text{Stab}_{\text{Aut}(V_L \otimes W)}(S_\varphi) \cap \text{Stab}_{\text{Aut}(V_L \otimes W)}(\mathfrak{h})$.

Proof. By Lemma 3.14 (3), $\text{Stab}_{\text{Aut}(V)}(\mathfrak{h}) \subset N_{\text{Aut}(V)}(S_\varphi^*)$. By (3.11), $\text{Stab}_{\text{Aut}(V)}(\mathfrak{h})/S_\varphi^* \subset \text{Stab}_{\text{Aut}(V_L \otimes W)}(S_\varphi)$. Hence $\text{Stab}_{\text{Aut}(V)}(\mathfrak{h})/S_\varphi^* \subset \text{Stab}_{\text{Aut}(V_L \otimes W)}(S_\varphi) \cap \text{Stab}_{\text{Aut}(V_L \otimes W)}(\mathfrak{h})$. For $\sigma \in \text{Stab}_{\text{Aut}(V_L \otimes W)}(S_\varphi) \cap \text{Stab}_{\text{Aut}(V_L \otimes W)}(\mathfrak{h})$, by (3.6) and (3.11), there exists $\tilde{\sigma} \in N_{\text{Aut}(V)}(S_\varphi^*)$ such that $\tilde{\sigma}|_{V_L \otimes W} = \sigma$ and $\tilde{\sigma}(\mathfrak{h}) = \mathfrak{h}$. Hence $\sigma \in \text{Stab}_{\text{Aut}(V)}(\mathfrak{h})/S_\varphi^*$. \square

It follows from $(V_L \otimes W)_1 = (V_L)_1 \otimes \mathbf{1}$ and $\text{Com}_{V_L \otimes W}((V_L)_1 \otimes \mathbf{1}) = \mathbf{1} \otimes W$ that

$$\text{Aut}(V_L \otimes W) \cong \text{Aut}(V_L) \times \text{Aut}(W).$$

Hence we obtain the group homomorphism

$$(3.12) \quad \text{Aut}(V_L \otimes W) \rightarrow O(\text{Irr}(V_L), q_{V_L}) \times O(\text{Irr}(W), -q_W), \quad \sigma \mapsto (\mu_{V_L}(\sigma|_{V_L}), \mu_W(\sigma|_W)).$$

Here we view $\mu_W(\sigma|_W) \in O(\text{Irr}(W), -q_W)$ via $O(\text{Irr}(W), q_W) = O(\text{Irr}(W), -q_W)$. By the injectivity of μ_W (Theorem 3.4),

$$\text{Aut}_0(W) = 1, \quad \overline{\text{Aut}}(W) \cong \text{Aut}(W);$$

we often identify $\overline{\text{Aut}}(W)$ with $\text{Aut}(W)$. Hence the kernel of the homomorphism (3.12) is $\text{Aut}_0(V_L) \times 1$. By (3.4) and (3.11), we have

$$\text{Stab}_{\text{Aut}(V_L \otimes W)}(S_\varphi) \cong (\text{Aut}_0(V_L) \times 1) \cdot \{(k, \varphi k \varphi^{-1}) \mid k \in \overline{\text{Aut}}(V_L), \varphi k \varphi^{-1} \in \overline{\text{Aut}}(W)\}.$$

We now identify $(\text{Irr}(V_L), q_{V_L})$ with $(\mathcal{D}(L), q_L)$. Note that $\overline{\text{Aut}}(V_L) \cong \overline{O}(L)$ (see Lemma 2.7). Considering the restriction of $\text{Stab}_{\text{Aut}(V_L \otimes W)}(S_\varphi)$ to V_L , we have

$$(3.13) \quad \text{Stab}_{\text{Aut}(V_L \otimes W)}(S_\varphi) \cong \text{Aut}_0(V_L) \cdot (\overline{O}(L) \cap \varphi^*(\overline{\text{Aut}}(W))),$$

where

$$\varphi^*(\overline{\text{Aut}}(W)) = \varphi^{-1}(\overline{\text{Aut}}(W))\varphi \subset O(\mathcal{D}(L), q_L).$$

By Lemma 2.7, (2.5) and (3.13), we have

$$\begin{aligned} & \text{Stab}_{\text{Aut}(V_L \otimes W)}(S_\varphi) \cap \text{Stab}_{\text{Aut}(V_L \otimes W)}(\mathfrak{h}) \\ & \cong \{\exp(a_{(0)}) \mid a \in \mathfrak{h}\} \iota^{-1}(O_0(L)) \cdot (\overline{O}(L) \cap \varphi^*(\overline{\text{Aut}}(W))). \end{aligned}$$

By (2.2) and (2.6),

$$(3.14) \quad \begin{aligned} & (\text{Stab}_{\text{Aut}(V_L \otimes W)}(S_\varphi) \cap \text{Stab}_{\text{Aut}(V_L \otimes W)}(\mathfrak{h}))|_{\mathfrak{h}} \cong O_0(L) \cdot (\overline{O}(L) \cap \varphi^*(\overline{\text{Aut}}(W))) \\ & \cong \mu_L^{-1}(\overline{O}(L) \cap \varphi^*(\overline{\text{Aut}}(W))). \end{aligned}$$

Let $W(V_1)$ denote the Weyl group of the semisimple Lie algebra V_1 .

Lemma 3.16. (1) $\text{Stab}_{\text{Inn}(V)}(\mathfrak{h})/\{\exp(a_{(0)}) \mid a \in \mathfrak{h}\} \cong W(V_1)$.
(2) $\text{Stab}_{\text{Aut}(V)}(\mathfrak{h})/\{\exp(a_{(0)}) \mid a \in \mathfrak{h}\} \cong \mu_L^{-1}(\overline{O}(L) \cap \varphi^*(\overline{\text{Aut}}(W)))$.

Proof. Since V_1 is a semisimple Lie algebra, $\text{Stab}_{\text{Inn}(V)}(\mathfrak{h})$ acts on \mathfrak{h} as $W(V_1)$. Hence (1) follows from Proposition 3.10. Combining Proposition 3.10 and (3.14), we obtain (2). \square

By Lemmas 3.14 (2) and 3.16, we obtain the following:

Proposition 3.17. $\text{Out}(V) \cong \mu_L^{-1}(\overline{O}(L) \cap \varphi^*(\overline{\text{Aut}}(W)))/W(V_1)$.

As a corollary, we obtain

$$(3.15) \quad \begin{aligned} |O(L)/W(V_1) : \text{Out}(V)| &= |O(L) : \mu_L^{-1}(\overline{O}(L) \cap \varphi^*(\overline{\text{Aut}}(W)))| \\ &= |\overline{O}(L) : (\overline{O}(L) \cap \varphi^*(\overline{\text{Aut}}(W)))|. \end{aligned}$$

Moreover, we obtain the following:

Lemma 3.18. *Assume that the conjugacy class of $g \in O(\Lambda)$ is neither $2C$, $6G$ nor $10F$ and that $\overline{O}(L)$ and $\varphi^*(\overline{\text{Aut}}(W))$ generate $O(\mathcal{D}(L), q_L)$. Then*

$$|O(L)/W(V_1) : \text{Out}(V)| = |O(\text{Irr}(W), q_W) : \overline{\text{Aut}}(W)|.$$

In particular,

- (1) *if the conjugacy class of g is $2A$ or $6E$, then $\text{Out}(V) \cong O(L)/W(V_1)$;*
- (2) *if the conjugacy class of g is $3B$, $4C$, $7B$ or $8E$, then $\text{Out}(V) \cong O(L)/\langle W(V_1), -1 \rangle$.*

Proof. By Lemma 3.6 (1), $\overline{\text{Aut}}(W)$ is normal in $O(\text{Irr}(W), q_W)$. The equation (3.15) and the group isomorphism theorem show

$$|O(L)/W(V_1) : \text{Out}(V)| = |O(\mathcal{D}(L), q_L) : \varphi^*(\overline{\text{Aut}}(W))| = |O(\text{Irr}(W), q_W) : \overline{\text{Aut}}(W)|.$$

The assertion (1) follows from $O(\text{Irr}(W), q_W) = \overline{\text{Aut}}(W)$ in Table 3. The assertion (2) follows from Lemma 3.6 (2), Table 3 and the fact that the -1 -isometry in $O(L)$ gives the -1 -isometry in $\overline{O}(L)$. \square

3.4. Weight one Lie algebra structures and orbit lattices. Let V be a strongly regular holomorphic VOA of central charge 24 with $0 < \text{rank } V_1 < 24$. Then $\mathfrak{g} = V_1$ is semisimple. Let $\mathfrak{g} = \bigoplus_{i=1}^s \mathfrak{g}_{i,k_i}$, where \mathfrak{g}_{i,k_i} is a simple ideal with level $k_i \in \mathbb{Z}_{>0}$.

Remark 3.19. The level k_i of a simple ideal \mathfrak{g}_i of V_1 is determined by the following formula in [Sc93, DM04a]:

$$(3.16) \quad \frac{h_i^\vee}{k_i} = \frac{\dim V_1 - 24}{24},$$

where h_i^\vee is the dual Coxeter number of \mathfrak{g}_i .

Let \mathfrak{h} be a Cartan subalgebra of V_1 . By Theorem 3.2, $W = \text{Com}_V(\mathfrak{h}) \cong V_{\Lambda_g}^{\hat{g}}$ for some $g \in O(\Lambda)$ belonging to the 10 conjugacy classes. In addition, $\text{Com}_V(W) \cong V_L$ for some even lattice L . In this subsection, we describe some properties of L by using $\mathfrak{g} = V_1$.

Set $\ell = 2|g|$ if $g \in 2C, 6G, 10F$ and $\ell = |g|$ otherwise (cf. Remark 3.9). By Lemma 3.8, L has level ℓ , and $\sqrt{\ell}L^*$ is an even lattice.

Proposition 3.20. *Let Q^i be the root lattice of \mathfrak{g}_i ; here the norm of roots in \mathfrak{g}_i is normalized so that $\langle \alpha | \alpha \rangle = 2$ for any long roots α (cf. Remark 3.13). Then the even lattice $U = \sqrt{\ell}L^*$ contains*

$$(3.17) \quad P_{\mathfrak{g}} = \bigoplus_{i=1}^s \frac{\sqrt{\ell}}{\sqrt{k_i}} Q^i$$

and $\text{rank } U = \text{rank } P_{\mathfrak{g}}$. Moreover, if the vector $v = \frac{\sqrt{\ell}}{\sqrt{k_i}} \beta$ associated with a root β of \mathfrak{g}_i is primitive in U , then v is a root of U .

Proof. Recall that the ratio of the normalized killing form on \mathfrak{g}_i and the bilinear form $\langle | \rangle$ on $L \subset \mathfrak{h}$ is k_i . Hence $\bigoplus_{i=1}^s (1/\sqrt{k_i})Q^i$ is the set of weights for \mathfrak{h} of the subVOA generated by V_1 with respect to the bilinear form $\langle | \rangle$ (see also (3.7)). By (3.3), we have $(1/\sqrt{k_i})Q^i \subset L^*$, which shows the former assertion (cf. the proof of Proposition 3.12).

Set $r_\beta = 1$ (resp. $r_\beta = r_i$) if β is long (resp. short), where r_i is the lacing number of \mathfrak{g}_i . Then $r_\beta \beta$ belongs to the even lattice Q_{long}^i generated by long roots of Q^i , and $\sqrt{k_i} r_\beta \beta \in \sqrt{k_i} Q_{long}^i \subset L$ (see (3.8)). In addition, $\langle v | v \rangle / 2 = \ell / (k_i r_\beta)$. Hence

$$v = \frac{\ell}{k_i r_\beta} \frac{1}{\sqrt{\ell}} \sqrt{k_i} r_\beta \beta \in \frac{\langle v | v \rangle}{2} \frac{1}{\sqrt{\ell}} L = \frac{\langle v | v \rangle}{2} U^*.$$

Thus the reflection σ_v preserves U , and v is a root of U . □

Remark 3.21. The lattice $P_{\mathfrak{g}}$ is equal to $\sqrt{\ell} \tilde{Q}$, where \tilde{Q} is defined in Proposition 3.12.

By the classification of irreducible W -modules (cf. [La20a]), we obtain the following lemma:

Lemma 3.22. (cf. [La20a]) *Assume that the conjugacy class of g is $2A, 3B, 5B$ or $7B$. Let M be an irreducible W -module. If M is not isomorphic to W , then $\rho(M) \geq (\ell - 1)/\ell$ and $\rho(M) \in (1/\ell)\mathbb{Z}$. Moreover, if $\rho(M) \in (\ell - 1)/\ell + \mathbb{Z}$, then $\rho(M) = (\ell - 1)/\ell$.*

Proposition 3.23. *Assume that the conjugacy class of g is $2A, 3B, 5B$ or $7B$. Then $U = \sqrt{\ell}L^*$ is a level ℓ lattice. Moreover, the root system of U and the root system of the semisimple Lie algebra $V_1 = \mathfrak{g}$ have the same type. In particular, the sublattice $P_{\mathfrak{g}}$ in (3.17) of U is generated by roots of U .*

Proof. By Lemma 3.8, U has level ℓ . Since $W_1 = 0$ and ℓ is prime, by (3.3) and Lemma 3.22, we have

$$(3.18) \quad V_1 = (V_L)_1 \otimes \mathbf{1} \oplus \bigoplus_{\min(\lambda+L)=2/\ell} (V_{\lambda+L})_{1/\ell} \otimes \varphi(\lambda+L)_{1-1/\ell},$$

where $\min(\lambda+L) = \min\{\langle x|x \rangle \mid x \in \lambda+L\}$. Then the roots of V_1 with respect to \mathfrak{h} are given by

$$\{\alpha \in L \mid \langle \alpha|\alpha \rangle = 2\} \cup \{\alpha \in L^* \mid \langle \alpha|\alpha \rangle = 2/\ell\}.$$

We can rewrite it as follows:

$$\{\alpha \in \ell U^* \subset U \mid \langle \alpha|\alpha \rangle = 2\ell\} \cup \{\alpha \in U \mid \langle \alpha|\alpha \rangle = 2\}.$$

By Lemma 2.2, this set is $R(U)$. Since ℓ is prime, for any root $\beta \in Q^i$, the vector $(\sqrt{\ell}/\sqrt{k_i})\beta$ is primitive in $U = \sqrt{\ell}L^*$. Hence by Proposition 3.20, the root systems of V_1 and $R(U)$ have the same type. The last assertion also follows from Proposition 3.20. \square

3.5. Schellekens' list and isometries of the Leech lattice. Let V be a holomorphic VOA of central charge 24 such that $0 < \text{rank } V_1 < 24$. Set $\mathfrak{g} = V_1$. Then \mathfrak{g} is one of 46 semisimple Lie algebras in Schellekens' list ([Sc93]). Let \mathfrak{h} be a Cartan subalgebra of \mathfrak{g} . By Theorem 3.2, $W = \text{Com}_V(\mathfrak{h}) \cong V_{\Lambda_g}^{\hat{g}}$ for some $g \in O(\Lambda)$ belonging to the 10 conjugacy classes. In addition, the conjugacy class of g is uniquely determined by \mathfrak{g} , which is summarized in Table 4 (see also [Hö, Tables 6–15] and [ELMS21, Table 2]). Here the symbol $X_{n,k}$ denotes (the type of) a simple Lie algebra whose type is X_n and level is k .

Table 4: Weight one Lie algebras of holomorphic VOAs of central charge 24 associated with $V_{\Lambda_g}^{\hat{g}}$ for $g \in O(\Lambda)$

Genus	Class	# of L	# of V_1	Weight one Lie algebra structures
B	$2A$	17	17	$A_{1,2}^{16}, A_{3,2}^4 A_{1,1}^4, D_{4,2}^2 B_{2,1}^4, A_{5,2}^2 C_{2,1} A_{2,1}^2, D_{5,2}^2 C_{2,1} A_{2,1}^2, A_{7,2} C_{3,1}^2 A_{3,1},$ $C_{4,1}^4, D_{6,2} C_{4,1} B_{3,1}^2, A_{9,2} A_{4,1} B_{3,1}, E_{6,2} C_{5,1} A_{5,1}, D_{8,2} B_{4,1}^2, C_{6,1}^2 B_{4,1},$ $D_{9,2} A_{7,1}, C_{8,1} F_{4,1}^2, E_{7,2} B_{5,1} F_{4,1}, C_{10,1} B_{6,1}, B_{8,1} E_{8,2}$

C	$3B$	6	6	$A_{2,3}^6, A_{5,3}D_{4,3}A_{1,1}^3, A_{8,3}A_{2,1}^2, E_{6,3}G_{2,1}^3, D_{7,3}A_{3,1}G_{2,1}, E_{7,3}A_{5,1}$
D	$2C$	2	9	$A_{1,4}^{12}, B_{2,2}^6, B_{3,2}^4, B_{4,2}^3, B_{6,2}^2, B_{12,2}, D_{4,4}A_{2,2}^4, C_{4,2}A_{4,2}^2, A_{8,2}F_{4,2}$
E	$4C$	5	5	$A_{3,4}^3A_{1,2}, D_{5,4}C_{3,2}A_{1,1}^2, A_{7,4}A_{1,1}^3, E_{6,4}A_{2,1}B_{2,1}, C_{7,2}A_{3,1}$
F	$5B$	2	2	$A_{4,5}^2, D_{6,5}A_{1,1}^2$
G	$6E$	2	2	$A_{5,6}B_{2,3}A_{1,2}, C_{5,3}G_{2,2}A_{1,1}$
H	$7B$	1	1	$A_{6,7}$
I	$8E$	1	1	$D_{5,8}A_{1,2}$
J	$6G$	1	2	$D_{4,12}A_{2,6}, F_{4,6}A_{2,2}$
K	$10F$	1	1	$C_{4,10}$

Remark 3.24. It would be possible to classify orbit lattices by the rank and the quadratic space structure on the discriminant group; in fact, the number of isometric classes of orbit lattices is given in [Hö, Table 4] (see Table 4). We will explicitly describe the orbit lattice $L_{\mathfrak{g}}$ corresponding to \mathfrak{g} in Section 5. Note that the orbit lattices have been described in [Hö] by using Niemeier lattices.

Remark 3.25. By Table 4, we observe

$$\ell = \text{lcm}(\{r_1k_1, r_2k_2, \dots, r_s k_s\}),$$

where r_i is the lacing number of \mathfrak{g}_i and $\ell = 2|g|$ if $g \in 2C, 6G, 10F$ and $\ell = |g|$ otherwise (See also Remark 3.9).

4. INEQUIVALENT SIMPLE CURRENT EXTENSIONS

Let W be one of the 10 VOAs in Theorem 3.2 and let L be an even lattice satisfying (3.1) and (3.2). In this subsection, we determine the number of holomorphic VOAs of central charge 24 obtained as inequivalent simple current extensions of $V_L \otimes W$ based on the arguments in [Hö].

Let \mathcal{O} be the set of all isometries from $(\mathcal{D}(L), q_L)$ to $(\text{Irr}(W), -q_W)$. For $\psi \in \mathcal{O}$,

$$V_\psi = \bigoplus_{\lambda+L \in \mathcal{D}(L)} V_{\lambda+L} \otimes \psi(\lambda+L)$$

has a holomorphic VOA structure of central charge 24 as a simple current extension of $V_L \otimes W$ ([EMS20, Theorem 4.2]). Define $S_\psi = \{(V_{\lambda+L}, \psi(\lambda+L)) \mid \lambda+L \in \mathcal{D}(L)\}$ as in (3.4).

Let $f \in \overline{\text{Aut}}(W)$, $h \in \overline{\mathcal{O}}(L)$ and $\psi \in \mathcal{O}$. Then $f \circ \psi \circ h$ also belongs to \mathcal{O} and $S_{f \circ \psi \circ h} \circ (h, f^{-1}) = S_\psi$. Hence (h, f^{-1}) induces an isomorphism between the holomorphic VOAs V_ψ and $V_{f \circ \psi \circ h}$. Conversely, we assume that $\psi, \psi' \in \mathcal{O}$ satisfy $V_\psi \cong V_{\psi'}$ as simple current extensions of $V_L \otimes W$, that is, there exists an isomorphism $\xi : V_\psi \rightarrow V_{\psi'}$ such that $\xi(V_L \otimes W) = V_L \otimes W$. Then S_ψ and $S_{\psi'}$ are conjugate by the restriction of ξ to $V_L \otimes W$.

Note that $\text{Aut}(V_L \otimes W) \cong \text{Aut}(V_L) \times \text{Aut}(W)$ and $\overline{\text{Aut}}(V_L)$ is identified with $\overline{\mathcal{O}}(L)$ (see Lemma 2.7). Therefore, the number of holomorphic VOAs obtained by inequivalent simple current extensions $\{V_\varphi \mid \varphi \in \mathcal{O}\}$ of $V_L \otimes W$ is equal to the number of double cosets in

$$\overline{\text{Aut}}(W) \backslash \mathcal{O} / \overline{\mathcal{O}}(L).$$

Remark 4.1. In general, inequivalent simple current extensions may become isomorphic VOAs. Fortunately, in our cases, this does not happen; see Propositions 4.3, 5.7, 5.15, 5.18, 5.24 and 5.29.

Now fix an isometry $i \in \mathcal{O}$. Then $i^*(h) = i^{-1} \circ h \circ i \in O(\mathcal{D}(L), q_L)$ for any $h \in O(\text{Irr}(W), -q_W)$. We consider the double cosets in $i^*(\overline{\text{Aut}}(W)) \backslash O(\mathcal{D}(L), q_L) / \overline{\mathcal{O}}(L)$. Note that $i \circ f \in \mathcal{O}$ for any $f \in O(\mathcal{D}(L), q_L)$. Conversely, $i^{-1} \circ \psi \in O(\mathcal{D}(L), q_L)$ for any $\psi \in \mathcal{O}$. Therefore, i induces a bijective map between \mathcal{O} and $O(\mathcal{D}(L), q_L)$, which gives the following:

Proposition 4.2. [Hö, Theorem 2.7] *Let $\psi, \psi' \in \mathcal{O}$. Then ψ and ψ' are in the same double coset of $\overline{\text{Aut}}(W) \backslash \mathcal{O} / \overline{\mathcal{O}}(L)$ if and only if $i^{-1} \circ \psi$ and $i^{-1} \circ \psi'$ are in the same double coset of $i^*(\overline{\text{Aut}}(W)) \backslash O(\mathcal{D}(L), q_L) / \overline{\mathcal{O}}(L)$. In particular, the number of inequivalent simple current extensions in $\{V_\varphi \mid \varphi \in \mathcal{O}\}$ is equal to $|i^*(\overline{\text{Aut}}(W)) \backslash O(\mathcal{D}(L), q_L) / \overline{\mathcal{O}}(L)|$.*

The following proposition proves the conjecture [Hö, Conjecture 4.8] for six conjugacy classes. The other four cases will be discussed in Section 5. Some cases were discussed in [Hö, Remark 4.9].

Proposition 4.3. *Let $g \in O(\Lambda)$ such that $W \cong V_{\Lambda_g}^g$. Assume that the conjugacy class of g is 2A, 3B, 4C, 6E, 7B or 8E. Then, for each L satisfying (3.1) and (3.2), there exists exactly one holomorphic VOA of central charge 24 obtained as a simple current extension of $V_L \otimes W$, up to isomorphism.*

Proof. By Proposition 4.2, it suffices to show that $|i^*(\overline{\text{Aut}}(W)) \backslash O(\mathcal{D}(L), q_L) / \overline{\mathcal{O}}(L)| = 1$, that is, $i^*(\overline{\text{Aut}}(W)) \overline{\mathcal{O}}(L) = O(\mathcal{D}(L), q_L)$.

If the conjugacy class of g is 2A or 6E, then the assertion is obvious since $i^*(\overline{\text{Aut}}(W)) \cong \text{Aut}(W) \cong O(\mathcal{D}(L), q_L)$ by Table 3.

If the conjugacy class of g is 3B, 4C, 7B or 8E, then $|O(\mathcal{D}(L), q_L) : i^*(\overline{\text{Aut}}(W))| = 2$ by Table 3. In addition, the -1 -isometry of $\mathcal{D}(L)$ belongs to $\overline{\mathcal{O}}(L)$ but it does not belong to $i^*(\overline{\text{Aut}}(W))$ by Proposition 3.6 (2). Hence we obtain the desired result. \square

The following lemma, which will be used to determine the number of double cosets, is probably well-known.

Lemma 4.4. *Let G be a finite group and let G_1, G_2 be subgroups of G . Suppose $N_G(G_2) = G_2$. Then a, a' are in the same double coset of $G_2 \backslash G / G_1$ if and only if $b^{-1} a^{-1} G_2 a b = a'^{-1} G_2 a'$ for some $b \in G_1$.*

Proof. Suppose $a' \in G_2aG_1$. Then $a' = a_2aa_1$ for some $a_1 \in G_1$ and $a_2 \in G_2$. Then $a'^{-1}G_2a' = a_1^{-1}a^{-1}a_2^{-1}G_2a_2aa_1 = a_1^{-1}(a^{-1}G_2a)a_1$.

Conversely, we suppose $a'^{-1}G_2a' = a_1^{-1}a^{-1}G_2aa_1$ for some $a_1 \in G_1$. Then $G_2 = a'a_1^{-1}a^{-1}G_2aa_1a'^{-1}$. Since $N_G(G_2) = G_2$, we have $aa_1a'^{-1} = a_2 \in G_2$ and $a' = a_2^{-1}aa_1$ as desired. \square

Remark 4.5. Under the same assumptions as in Lemma 4.4, the number of double cosets of $G_2 \backslash G / G_1$ is equal to the number of G_1 -orbits on the set $\{a^{-1}G_2a \mid a \in G\}$ of all subgroups of G conjugate to G_2 by conjugation.

5. AUTOMORPHISM GROUPS OF HOLOMORPHIC VOAs OF CENTRAL CHARGE 24

Let V be a (strongly regular) holomorphic VOA of central charge 24 with $0 < \text{rank } V_1 < 24$. Fix a Cartan subalgebra \mathfrak{h} of V_1 . By Theorem 3.2, $W = \text{Com}_V(\mathfrak{h}) \cong V_{\Lambda_g}^{\hat{g}}$ for some $g \in O(\Lambda)$ belonging to the 10 conjugacy classes. Note that $\text{Com}_V(W)$ is a lattice VOA V_L and the conjugacy class of g is uniquely determined by the Lie algebra structure of V_1 (see Table 4). In addition, V is a simple current extension of $V_L \otimes W$.

In this section, by using the Lie algebra structure of $\mathfrak{g} = V_1$ in Schellekens' list, we describe the orbit lattice L explicitly, which implies that $L = L_{\mathfrak{g}}$ is uniquely determined by \mathfrak{g} , up to isometry. For each \mathfrak{g} , we also determine the group structures of $K(V)$ and $\text{Out}(V)$ based on the case-by-case analysis on W and $L_{\mathfrak{g}}$. For the conjugacy classes of g that we have not dealt with in Proposition 4.3, we also determine the number of holomorphic VOAs obtained as inequivalent simple current extensions of $V_L \otimes W$.

Remark 5.1. Based on a similar method, some partial results for the conjugacy classes 2A, 3B, 5B and 7B and 2C were obtained in [LS17] and [HS14], respectively.

Remark 5.2. In the tables of this section, \mathfrak{S}_n , \mathfrak{A}_n and Dih_n denote the symmetric group of degree n , the alternating group of degree n and the dihedral group of order n , respectively.

5.1. Conjugacy class 2A (Genus B). Assume that g belongs to the conjugacy class 2A of $O(\Lambda)$. Then $O(\text{Irr}(W), q_W) \cong GO_{10}^+(2) \cong \Omega_{10}^+(2) \cdot 2$. By Table 3, $\text{Aut}(W) (\cong \overline{\text{Aut}}(W))$ has the shape $GO_{10}^+(2)$, which is the full orthogonal group $O(\text{Irr}(W), q_W)$.

Since the central charge of W is 8, L is an even lattice of rank 16 such that $(\mathcal{D}(L), q_L) \cong (\text{Irr}(W), -q_W)$. Then $\mathcal{D}(L) \cong \mathbb{Z}_2^{10}$. Set $U = \sqrt{2}L^*$. Then $\mathcal{D}(U) \cong \mathbb{Z}_2^6$, and by Proposition 3.23, U is a level 2 lattice. Such lattices U were classified in [SV01]. Furthermore that can now be verified easily using MAGMA. More precisely, it was proved in [SV01, Theorem 2] (see also [HS14, Remark 3.12]) that there exist exactly 17 level 2 lattices of rank 16 with determinant 2^6 up to isometry and they are uniquely determined by their root system (see Table 5). Their isometry groups are determined by MAGMA as in Table 5. Hence there

are 17 possible lattices for $L = \sqrt{2}U^*$; indeed, they satisfy $(\mathcal{D}(L), q_L) \cong (\text{Irr}(W), -q_W)$. Note that $O(L) = O(U)$.

Table 5: Level 2 lattices of rank 16 for the case $2A$

$R(U_{\mathfrak{g}})$	$U_{\mathfrak{g}}/P_{\mathfrak{g}}$	$O(U_{\mathfrak{g}})/W(R(U_{\mathfrak{g}}))$	$O(U_{\mathfrak{g}})$
A_1^{16}	\mathbb{Z}_2^5	$\text{AGL}_4(2)$	$W(A_1) \wr \text{AGL}_4(2)$
$A_3^4(\sqrt{2}A_1)^4$	$\mathbb{Z}_2^3 \times \mathbb{Z}_4$	$W(D_4)$	$(W(A_3)^4 \times W(A_1)^4) \cdot W(D_4)$
$D_4^2 C_2^4$	\mathbb{Z}_2^3	$2 \times \mathfrak{S}_4$	$(W(D_4)^2 \times W(C_2)^4) \cdot (2 \times \mathfrak{S}_4)$
$A_5^2(\sqrt{2}A_2)^2 C_2$	$\mathbb{Z}_3 \times \mathbb{Z}_6$	Dih_8	$(W(A_5)^2 \times W(A_2)^2 \times W(C_2)) \cdot \text{Dih}_8$
$A_7(\sqrt{2}A_3)C_3^2$	$\mathbb{Z}_2 \times \mathbb{Z}_4$	\mathbb{Z}_2^2	$(W(A_7) \times W(A_3) \times W(C_3)^2) \cdot \mathbb{Z}_2^2$
$D_5^2(\sqrt{2}A_3)^2$	\mathbb{Z}_4^2	Dih_8	$(W(D_5)^2 \times W(A_3)^2) \cdot \text{Dih}_8$
C_4^4	\mathbb{Z}_2	\mathfrak{S}_4	$W(C_4) \wr \mathfrak{S}_4$
$D_6 C_4 B_3^2$	\mathbb{Z}_2^2	\mathbb{Z}_2	$(W(D_6) \times W(C_4) \times W(B_3)^2) \cdot \mathbb{Z}_2$
$A_9(\sqrt{2}A_4)B_3$	\mathbb{Z}_{10}	\mathbb{Z}_2	$(W(A_9) \times W(A_4) \times W(B_3)) \cdot \mathbb{Z}_2$
$E_6(\sqrt{2}A_5)C_5$	\mathbb{Z}_6	\mathbb{Z}_2	$(W(E_6) \times W(A_5) \times W(C_5)) \cdot \mathbb{Z}_2$
$D_8 B_4^2$	\mathbb{Z}_2^2	\mathbb{Z}_2	$W(D_8) \times W(B_4) \wr \mathbb{Z}_2$
$C_6^2 B_4$	\mathbb{Z}_2	\mathbb{Z}_2	$W(C_6) \wr 2 \times W(B_4)$
$D_9(\sqrt{2}A_7)$	\mathbb{Z}_8	\mathbb{Z}_2	$(W(D_9) \times W(A_7)) \cdot \mathbb{Z}_2$
$C_8 F_4^2$	1	\mathbb{Z}_2	$W(C_8) \times W(F_4) \wr \mathbb{Z}_2$
$E_7 B_5 F_4$	\mathbb{Z}_2	1	$W(E_7) \times W(B_5) \times W(F_4)$
$C_{10} B_6$	\mathbb{Z}_2	1	$W(C_{10}) \times W(B_6)$
$E_8 B_8$	\mathbb{Z}_2	1	$W(B_8) \times W(E_8)$

Let \mathfrak{g} be one of the 17 Lie algebras in Table 4 corresponding to $2A$. By Proposition 3.23, the root system $R(U)$ of $U = \sqrt{2}L^*$ is uniquely determined by \mathfrak{g} as in Table 6. As we mentioned, U is also uniquely determined by \mathfrak{g} ; we set $L_{\mathfrak{g}} = L$ and $U_{\mathfrak{g}} = U$. Let $P_{\mathfrak{g}}$ be the sublattice of $U_{\mathfrak{g}}$ generated by $R(U_{\mathfrak{g}})$ as in (3.17) (see also Proposition 3.23).

Proposition 5.3. *Assume that the conjugacy class of g is $2A$.*

- (1) $K(V) \cong U_{\mathfrak{g}}/P_{\mathfrak{g}}$.
- (2) $\text{Out}(V) \cong O(U_{\mathfrak{g}})/W(R(U_{\mathfrak{g}}))$.

Proof. By Proposition 3.12, we have $K(V) \cong L_{\mathfrak{g}}^*/\tilde{Q}$. It follows from the definition of $P_{\mathfrak{g}}$ that $P_{\mathfrak{g}} \cong \sqrt{2}\tilde{Q}$ (cf. Remark 3.21), which proves (1). By Proposition 3.23, we obtain $W(V_1) \cong W(R(U_{\mathfrak{g}}))$. Hence (2) follows from Lemma 3.18 (1). \square

By the proposition above and Table 5, we obtain the group structures of $K(V)$ and $\text{Out}(V)$ for all 17 cases, which are summarized in Table 6.

Table 6: $K(V)$ and $\text{Out}(V)$ for the case $2A$

Genus	No.	$\mathfrak{g} = V_1$	$R(U_{\mathfrak{g}})$	$\text{Out}(V_1)$	$\text{Out}(V)$	$K(V)$
B	5	$A_{1,2}^{16}$	A_1^{16}	\mathfrak{S}_{16}	$\text{AGL}_4(2)$	\mathbb{Z}_2^5
	16	$A_{3,2}^4 A_{1,1}^4$	$A_3^4 (\sqrt{2} A_1)^4$	$(\mathbb{Z}_2 \wr \mathfrak{S}_4) \times \mathfrak{S}_4$	$W(D_4)$	$\mathbb{Z}_2^3 \times \mathbb{Z}_4$
	25	$D_{4,2}^2 C_{2,1}^4$	$D_4^2 C_2^4$	$(\mathfrak{S}_3 \wr \mathfrak{S}_2) \times \mathfrak{S}_4$	$\mathbb{Z}_2 \times \mathfrak{S}_4$	\mathbb{Z}_2^3
	26	$A_{5,2}^2 C_{2,1} A_{2,1}^2$	$A_5^2 C_2 (\sqrt{2} A_2)^2$	$(\mathbb{Z}_2 \wr \mathfrak{S}_2) \times (\mathbb{Z}_2 \wr \mathfrak{S}_2)$	Dih_8	$\mathbb{Z}_3 \times \mathbb{Z}_6$
	31	$D_{5,2}^2 A_{3,1}^2$	$D_5^2 (\sqrt{2} A_3)^2$	$(\mathbb{Z}_2 \wr \mathfrak{S}_2) \times (\mathbb{Z}_2 \wr \mathfrak{S}_2)$	Dih_8	\mathbb{Z}_4^2
	33	$A_{7,2} C_{3,1}^2 A_{3,1}$	$A_7 C_3^2 (\sqrt{2} A_3)$	$\mathbb{Z}_2 \times \mathfrak{S}_2 \times \mathbb{Z}_2$	\mathbb{Z}_2^2	$\mathbb{Z}_2 \times \mathbb{Z}_4$
	38	$C_{4,1}^4$	C_4^4	\mathfrak{S}_4	\mathfrak{S}_4	\mathbb{Z}_2
	39	$D_{6,2} C_{4,1} B_{3,1}^2$	$D_6 C_4 B_3^2$	$\mathbb{Z}_2 \times \mathfrak{S}_2$	\mathbb{Z}_2	\mathbb{Z}_2^2
	40	$A_{9,2} A_{4,1} B_{3,1}$	$A_9 (\sqrt{2} A_4) B_3$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	\mathbb{Z}_2	\mathbb{Z}_{10}
	44	$E_{6,2} C_{5,1} A_{5,1}$	$E_6 C_5 (\sqrt{2} A_5)$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	\mathbb{Z}_2	\mathbb{Z}_6
	47	$D_{8,2} B_{4,1}^2$	$D_8 B_4^2$	$\mathbb{Z}_2 \times \mathfrak{S}_2$	\mathbb{Z}_2	\mathbb{Z}_2^2
	48	$C_{6,1}^2 B_{4,1}$	$C_6^2 B_4$	\mathfrak{S}_2	\mathbb{Z}_2	\mathbb{Z}_2
	50	$D_{9,2} A_{7,1}$	$D_9 (\sqrt{2} A_7)$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	\mathbb{Z}_2	\mathbb{Z}_8
	52	$C_{8,1} F_{4,1}^2$	$C_8 F_4^2$	\mathbb{Z}_2	\mathbb{Z}_2	1
	53	$E_{7,2} B_{5,1} F_{4,1}$	$E_7 B_5 F_4$	\mathbb{Z}_2	1	\mathbb{Z}_2
	56	$C_{10,1} B_{6,1}$	$C_{10} B_6$	1	1	\mathbb{Z}_2
	62	$B_{8,1} E_{8,2}$	$B_8 E_8$	1	1	\mathbb{Z}_2

Remark 5.4. The groups $K(V)$ and $\text{Out}(V)$ have been determined in [Sh20] if

$$\mathfrak{g} \in \{A_{1,2}^{16}, A_{3,2}^4 A_{1,1}^4, D_{4,2}^2 B_{2,1}^4, D_{5,2}^2 A_{3,1}^2, C_{4,1}^4, D_{6,2} B_{3,1}^2 C_{4,1}, D_{8,2} B_{4,1}^2, D_{9,2} A_{7,1}\}$$

by using the explicit construction of V .

5.2. Conjugacy class $3B$ (Genus C). Assume that g belongs to the conjugacy class $3B$ of $O(\Lambda)$. Then $O(\text{Irr}(W), q_W) \cong GO_{\bar{8}}(3) \cong 2 \times \text{P}\Omega_{\bar{8}}(3).2$. By Table 3, $\text{Aut}(W) (\cong \overline{\text{Aut}}(W))$ has the shape $\text{P}\Omega_{\bar{8}}(3).2$, which is an index 2 subgroup of $O(\text{Irr}(W), q_W)$.

Since the central charge of W is 12, L is an even lattice of rank 12 such that $(\mathcal{D}(L), q_L) \cong (\text{Irr}(W), -q_W)$. Then $\mathcal{D}(L) \cong \mathbb{Z}_3^8$. Set $U = \sqrt{3}L^*$. Then $\mathcal{D}(U) \cong \mathbb{Z}_3^4$, and by Proposition 3.23, U is a level 3 lattice. Such lattices U were classified in [SV01]. Furthermore that can now be verified easily using MAGMA. More precisely, it was proved in [SV01, Theorem 3] that there exist exactly 6 level 3 lattices of rank 12 with determinant 3^4 up to isometry, and they are uniquely determined by their root system (see Table 7). Since $O(U)$ is a subgroup of the automorphism group of the root system $R(U)$, its shape is easily determined as in Table 7. Hence there are 6 possible lattices for $L = \sqrt{3}U^*$; indeed, they satisfy $(\mathcal{D}(L), q_L) \cong (\text{Irr}(W), -q_W)$. Note that $O(L) = O(U)$.

Table 7: Level 3 lattices of rank 12 for the case 3B

$R(U_{\mathfrak{g}})$	$U_{\mathfrak{g}}/P_{\mathfrak{g}}$	$O(U_{\mathfrak{g}})/W(R(U_{\mathfrak{g}}))$	$O(U_{\mathfrak{g}})$
A_2^6	\mathbb{Z}_3	$\mathbb{Z}_2 \times \mathfrak{S}_6$	$(W(A_2) \wr \mathfrak{S}_6) \cdot \mathbb{Z}_2$
$A_5 D_4 (\sqrt{3} A_1)^3$	\mathbb{Z}_2^3	Dih_{12}	$(W(A_5) \times W(D_4) \times W(A_1)^3) \cdot \text{Dih}_{12}$
$A_8 (\sqrt{3} A_2)^2$	\mathbb{Z}_3^2	\mathbb{Z}_2^2	$(W(A_8) \times W(A_2)^2) \cdot \mathbb{Z}_2^2$
$E_6 G_2^3$	1	$\mathbb{Z}_2 \times \mathfrak{S}_3$	$(W(E_6) \times W(G_2) \wr \mathfrak{S}_3) \cdot \mathbb{Z}_2$
$D_7 (\sqrt{3} A_3) G_2$	\mathbb{Z}_4	\mathbb{Z}_2	$(W(D_7) \times W(A_3) \times W(G_2)) \cdot \mathbb{Z}_2$
$E_7 (\sqrt{3} A_5)$	\mathbb{Z}_6	\mathbb{Z}_2	$(W(E_7) \times W(A_5)) \cdot \mathbb{Z}_2$

Let \mathfrak{g} be one of the 6 Lie algebras in Table 4 corresponding to 3B. By Proposition 3.23, the root system of $U = \sqrt{3}L^*$ is uniquely determined by \mathfrak{g} as in Table 8. As we mentioned, U is also uniquely determined by \mathfrak{g} ; we set $L_{\mathfrak{g}} = L$ and $U_{\mathfrak{g}} = U$. Let $P_{\mathfrak{g}}$ be the sublattice of $U_{\mathfrak{g}}$ generated by $R(U_{\mathfrak{g}})$ as in (3.17) (see also Proposition 3.23).

Proposition 5.5. *Assume that the conjugacy class of g is 3B.*

- (1) $K(V) \cong U_{\mathfrak{g}}/P_{\mathfrak{g}}$.
- (2) $\text{Out}(V) \cong O(U_{\mathfrak{g}})/\langle W(R(U_{\mathfrak{g}})), -1 \rangle$.

Proof. By Proposition 3.12, we have $K(V) \cong L_{\mathfrak{g}}^*/\tilde{Q}$. It follows from the definition of $P_{\mathfrak{g}}$ that $P_{\mathfrak{g}} \cong \sqrt{3}\tilde{Q}$ (cf. Remark 3.21), which proves (1). By Proposition 3.23, we obtain $W(V_1) = W(R(U_{\mathfrak{g}}))$. Hence (2) follows from Proposition 3.17 and Lemma 3.18 (2). \square

The group structures of $K(V)$ and $\text{Out}(V)$ are summarized in Table 8.

 Table 8: $K(V)$ and $\text{Out}(V)$ for the case 3B

Genus	No.	$\mathfrak{g} = V_1$	$R(U_{\mathfrak{g}})$	$\text{Out}(V_1)$	$\text{Out}(V)$	$K(V)$
C	6	$A_{2,3}^6$	A_2^6	$\mathbb{Z}_2 \wr \mathfrak{S}_6$	\mathfrak{S}_6	\mathbb{Z}_3
	17	$A_{5,3} D_{4,3} A_{1,1}^3$	$A_5 D_4 (\sqrt{3} A_1)^3$	$\mathbb{Z}_2 \times \mathfrak{S}_3 \times \mathfrak{S}_3$	\mathfrak{S}_3	\mathbb{Z}_2^3
	27	$A_{8,3} A_{2,1}^2$	$A_8 (\sqrt{3} A_2)^2$	$\mathbb{Z}_2 \times (\mathbb{Z}_2 \wr \mathfrak{S}_2)$	\mathbb{Z}_2	\mathbb{Z}_3^2
	32	$E_{6,3} G_{2,1}^3$	$E_6 G_2^3$	$\mathbb{Z}_2 \times \mathfrak{S}_3$	\mathfrak{S}_3	1
	34	$D_{7,3} A_{3,1} G_{2,1}$	$D_7 (\sqrt{3} A_3) G_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	1	\mathbb{Z}_4
	45	$E_{7,3} A_{5,1}$	$E_7 (\sqrt{3} A_5)$	\mathbb{Z}_2	1	\mathbb{Z}_6

5.3. Conjugacy class 5B (Genus F). Assume that g belongs to the conjugacy class 5B of $O(\Lambda)$. Then $O(\text{Irr}(W), q_W) \cong GO_6^+(5) \cong 2 \cdot \text{P}\Omega_6^+(5) \cdot 2^2$. By Table 3, $\text{Aut}(W) (\cong \overline{\text{Aut}(W)})$ has the shape $2 \cdot \text{P}\Omega_6^+(5) \cdot 2$, which is an index 2 subgroup of $O(\text{Irr}(W), q_W)$ not isomorphic to $SO_6^+(5)$.

Since the central charge of W is 16, L is an even lattice of rank 8 such that $(\mathcal{D}(L), q_L) \cong (\text{Irr}(W), -q_W)$. Then $\mathcal{D}(L) \cong \mathbb{Z}_5^6$. Set $U = \sqrt{5}L^*$. Then $\mathcal{D}(U) \cong \mathbb{Z}_5^2$, and by Proposition 3.23, U is a level 5 lattice. Note that $O(L) = O(U)$.

By Table 4, the Lie algebra structure of $\mathfrak{g} = V_1$ is $A_{4,5}^2$ or $D_{6,5}A_{1,1}^2$. By Proposition 3.23, the root lattice $P_{\mathfrak{g}}$ of U is isometric to A_4^2 or $D_6(\sqrt{5}A_1^2)$, respectively. It is easy to see that U is uniquely determined as an overlattice of $P_{\mathfrak{g}}$; we set $U_{\mathfrak{g}} = U$ and $L_{\mathfrak{g}} = L$. Since $O(U)$ is a subgroup of the automorphism group of the root system $R(U)$, its shape is easily determined as in Table 9.

Table 9: Level 5 lattices of rank 8 for the case 5B

$\mathfrak{g} = V_1$	$R(U_{\mathfrak{g}})$	$U_{\mathfrak{g}}/P_{\mathfrak{g}}$	$O(U_{\mathfrak{g}})/W(R(U_{\mathfrak{g}}))$	$O(U_{\mathfrak{g}})$
$A_{4,5}^2$	A_4^2	1	Dih_8	$(2 \times W(A_4)) \wr \mathfrak{S}_2$
$D_{6,5}A_{1,1}^2$	$D_6(\sqrt{5}A_1^2)$	\mathbb{Z}_2^2	\mathfrak{S}_2	$(W(D_6) \times W(A_1)^2) \cdot 2$

Lemma 5.6. *The subgroups $\overline{O}(L_{\mathfrak{g}})$ and $\varphi^*(\overline{\text{Aut}}(W))$ generate $O(\mathcal{D}(L_{\mathfrak{g}}), q_{L_{\mathfrak{g}}})$.*

Proof. Let φ be the isometry from $(\mathcal{D}(L_{\mathfrak{g}}), q_{L_{\mathfrak{g}}})$ to $(\text{Irr}(W), -q_W)$ satisfying (3.3). Recall that $\varphi^*(\overline{\text{Aut}}(W))$ is an index 2 subgroup of $O(\text{Irr}(W), q_W) \cong GO_6^+(5)$ not isomorphic to $SO_6^+(5)$ and that $O(\text{Irr}(W), q_W) \cong O(\text{Irr}(W), -q_W)$. If the root system $R(U_{\mathfrak{g}})$ is A_4^2 or $D_6(\sqrt{5}A_1)^2$, then $\overline{O}(L_{\mathfrak{g}}) \cong (2 \times W(A_4)) \wr \mathfrak{S}_2 \cong \text{GO}_3(5) \wr \mathfrak{S}_2$ or $\mathbb{Z}_2^6 : \mathfrak{S}_6 \cong GO_1(5) \wr \mathfrak{S}_6$, respectively. In both cases, $\overline{O}(L_{\mathfrak{g}})$ is a maximal subgroup of $O(\mathcal{D}(L_{\mathfrak{g}}), q_{L_{\mathfrak{g}}})$ (cf. [Wi09, Theorem 3.12]), and hence $\overline{O}(L_{\mathfrak{g}})$ and $\varphi^*(\overline{\text{Aut}}(W))$ generate $O(\mathcal{D}(L_{\mathfrak{g}}), q_{L_{\mathfrak{g}}})$. \square

By Proposition 4.2 and Lemma 5.6, we obtain the following:

Proposition 5.7. *Assume that the conjugacy class of g is 5B. Then, for each L , there exists exactly one holomorphic VOA of central charge 24 obtained as a simple current extension of $V_L \otimes W$, up to isomorphism.*

Proposition 5.8. *Assume that the conjugacy class of g is 5B.*

- (1) $K(V) \cong U_{\mathfrak{g}}/P_{\mathfrak{g}}$.
- (2) $\text{Out}(V)$ have the shapes in Table 10.

Proof. By Proposition 3.12, we have $K(V) \cong L_{\mathfrak{g}}^*/\tilde{Q}$. It follows from the definition of $P_{\mathfrak{g}}$ that $P_{\mathfrak{g}} \cong \sqrt{5}\tilde{Q}$ (cf. Remark 3.21), which proves (1).

Next, we determine $\text{Out}(V)$. By Proposition 3.23, we obtain $W(V_1) = W(R(U_{\mathfrak{g}}))$. By Proposition 3.17 and Lemmas 3.18 and 5.6, we have $|O(L_{\mathfrak{g}})/W(R(U_{\mathfrak{g}})) : \text{Out}(V)| = 2$. Hence $|\text{Out}(V)| = 4$ or 1 if $R(U_{\mathfrak{g}}) \cong A_4^2$ or $D_6(\sqrt{5}A_1)^2$, respectively.

Assume that $R(U_{\mathfrak{g}}) \cong A_4^2$. Note that $\overline{O}(L_{\mathfrak{g}}) \cong O(L_{\mathfrak{g}})$ and that $\overline{O}(L_{\mathfrak{g}}) \cap \varphi^*(\overline{\text{Aut}}(W))$ contains $W(R(U_{\mathfrak{g}})) \cong \mathfrak{S}_5 \times \mathfrak{S}_5$ as a subgroup. Checking possible index 2 subgroups of

$\overline{O}(L_{\mathfrak{g}})$ obtained as $\overline{O}(L_{\mathfrak{g}}) \cap \varphi^*(\overline{\text{Aut}}(W))$, one can verify that $\overline{O}(L_{\mathfrak{g}}) \cap \varphi^*(\overline{\text{Aut}}(W))$ has the shape $2 \times (\mathfrak{S}_5 \wr \mathfrak{S}_2)$ by using MAGMA. Hence $\text{Out}(V) \cong \mathbb{Z}_2^2$ by Proposition 3.17. \square

The group structures of $K(V)$ and $\text{Out}(V)$ are summarized in Table 10.

Table 10: $K(V)$ and $\text{Out}(V)$ for the case 5B

Genus	No.	$\mathfrak{g} = V_1$	$R(U_{\mathfrak{g}})$	$\text{Out}(V_1)$	$\text{Out}(V)$	$K(V)$
F	9	$A_{4,5}^2$	A_4^2	$\mathbb{Z}_2 \wr \mathfrak{S}_2$	\mathbb{Z}_2^2	1
	20	$D_{6,5}A_{1,1}^2$	$D_6(\sqrt{5}A_1^2)$	$\mathbb{Z}_2 \times \mathfrak{S}_2$	1	\mathbb{Z}_2^2

Remark 5.9. The even lattice U is a level 5 lattice of rank 8 (see Proposition 3.23) and $(\mathcal{D}(U), q_U)$ is a 2-dimensional quadratic space over \mathbb{Z}_5 of plus type. By using these properties, one could prove that the root system of U is A_4^2 or $D_6(\sqrt{5}A_1^2)$.

5.4. Conjugacy class 7B (Genus H). Assume that g belongs to the conjugacy class 7B of $O(\Lambda)$. Then $O(\text{Irr}(W), q_W) \cong GO_5(7) \cong 2 \times \text{P}\Omega_5(7).2$. By Table 3, $\text{Aut}(W) (\cong \overline{\text{Aut}}(W))$ has the shape $\text{P}\Omega_5(7).2$, which is an index 2 subgroup of $O(\text{Irr}(W), q_W)$.

By Table 4, the Lie algebra structure of $\mathfrak{g} = V_1$ is $A_{6,7}$. Since the central charge of W is 18, L is an even lattice of rank 6 such that $(\mathcal{D}(L), q_L) \cong (\text{Irr}(W), -q_W)$. Then $\mathcal{D}(L) \cong \mathbb{Z}_7^5$. Set $U = \sqrt{7}L^*$. Then $\mathcal{D}(U) \cong \mathbb{Z}_7$ and by Proposition 3.23, U is a level 7 lattice. In addition, the root system $R(U)$ of U is A_6 . Hence $U = U_{\mathfrak{g}} \cong P_{\mathfrak{g}} \cong A_6$, and $L = L_{\mathfrak{g}} \cong \sqrt{7}A_6^*$. The isometry group of $U_{\mathfrak{g}}$ is summarized in Table 11.

Table 11: Level 7 lattice of rank 6 for the case 7B

$\mathfrak{g} = V_1$	$R(U_{\mathfrak{g}})$	$U_{\mathfrak{g}}/P_{\mathfrak{g}}$	$O(U_{\mathfrak{g}})/W(R(U_{\mathfrak{g}}))$	$O(U_{\mathfrak{g}})$
$A_{6,7}$	A_6	1	\mathbb{Z}_2	$\mathbb{Z}_2 \times W(A_6)$

Proposition 5.10. *Assume that the conjugacy class of g is 7B.*

- (1) $K(V) \cong U_{\mathfrak{g}}/P_{\mathfrak{g}}$.
- (2) $\text{Out}(V) = 1$

Proof. By Proposition 3.12, we have $K(V) \cong L_{\mathfrak{g}}^*/\tilde{Q}$. It follows from the definition of $P_{\mathfrak{g}}$ that $P_{\mathfrak{g}} \cong \sqrt{7}\tilde{Q}$ (cf. Remark 3.21), which proves (1). By Proposition 3.23, we have $W(V_1) = W(R(U_{\mathfrak{g}}))$. Hence (2) follows from Proposition 3.17 and Lemma 3.18 (2). \square

The group structures of $K(V)$ and $\text{Out}(V)$ are summarized in Table 12.

Table 12: $K(V)$ and $\text{Out}(V)$ for the case 7B

Genus	No.	$\mathfrak{g} = V_1$	$R(U_{\mathfrak{g}})$	$\text{Out}(V_1)$	$\text{Out}(V)$	$K(V)$
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H	11	$A_{6,7}$	A_6	\mathbb{Z}_2	1	1
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Remark 5.11. The lattice U is an even lattice of rank 6 with $\mathcal{D}(U) \cong \mathbb{Z}_7$. By using this property, one could prove that $U \cong A_6$.

5.5. Conjugacy class $2C$ (Genus D). Assume that g belongs to the conjugacy class $2C$ of $O(\Lambda)$. Then $O(\text{Irr}(W), q_W) \cong 2_+^{1+20} \cdot (GO_{10}^-(2) \times \mathfrak{S}_3)$. By Table 3, $\text{Aut}(W) (\cong \overline{\text{Aut}}(W))$ has the shape $2_+^{1+20} \cdot (\mathfrak{S}_{12} \times \mathfrak{S}_3)$. Since \mathfrak{S}_{12} is a maximal subgroup of $GO_{10}^-(2)$ ([ATLAS]), $\text{Aut}(W)$ is also a maximal subgroup of $O(\text{Irr}(W), q_W)$ and it is self-normalizing.

Remark 5.12. Set $\Omega = \{1, 2, \dots, 12\}$ and $\mathcal{X} = \{A \subset \Omega \mid |A| \equiv 0 \pmod{2}\} / \{\emptyset, \Omega\}$. Then \mathcal{X} is a 10-dimensional vector space over \mathbb{F}_2 by the symmetric difference. In addition, \mathcal{X} has the quadratic form of minus type defined by $A \mapsto |A|/2 \pmod{2}$ (cf. [ATLAS, page 147]). Since \mathfrak{S}_{12} naturally acts on \mathcal{X} and it preserves the quadratic form, we obtain $\mathfrak{S}_{12} \subset GO_{10}^-(2)$.

It was proved in [HS14, Theorem 2.8] that $(\text{Irr}(W), q_W) \cong (\mathcal{D}(\sqrt{2}D_{12}), q_{\sqrt{2}D_{12}})$. Since the central charge of W is 12, the rank of L is 12. Note that $(\text{Irr}(W), -q_W) \cong (\text{Irr}(W), q_W)$ and that $(\mathcal{D}(L), q_L) \cong (\text{Irr}(W), -q_W)$.

Lemma 5.13. *Let H be an even lattice of rank 12 such that $(\mathcal{D}(H), q_H) \cong (\mathcal{D}(\sqrt{2}D_{12}), q_{\sqrt{2}D_{12}})$ as quadratic spaces. Then $H \cong \sqrt{2}D_{12}$ or $\sqrt{2}E_8\sqrt{2}D_4$.*

Proof. It follows from $\mathcal{D}(H) \cong \mathbb{Z}_2^{10} \times \mathbb{Z}_4^2$ and $\text{rank } H = 12$ that $(1/2)H \subset H^*$. In addition, it also follows from $q_{\sqrt{2}D_{12}}((1/2)(\sqrt{2}D_{12})) \subset \mathbb{Z}$ that $q_H((1/2)H) \subset \mathbb{Z}$. Hence $(1/\sqrt{2})H$ is even. Since $\mathcal{D}((1/\sqrt{2})H) \cong \mathbb{Z}_2^2$ and the rank of $(1/\sqrt{2})H$ is 12, there exists an odd unimodular lattice of rank 12 whose even sublattice is $(1/\sqrt{2})H$. Since any odd unimodular lattice of rank 12 is isometric to \mathbb{Z}^{12} or $E_8\mathbb{Z}^4$ (cf. [CS99]), we have $(1/\sqrt{2})H \cong D_{12}$ or E_8D_4 \square

By this lemma, we have $L \cong \sqrt{2}D_{12}$ or $L \cong \sqrt{2}E_8\sqrt{2}D_4$. We will discuss each case in the following subsections.

5.5.1. Case $L \cong \sqrt{2}D_{12}$. In this case, μ_L is injective, that is, $\overline{O}(L) \cong O(L) \cong 2^{12} \cdot \mathfrak{S}_{12}$. Recall that $O(\mathcal{D}(L), q_L) \cong O(\text{Irr}(W), q_W) \cong 2_+^{1+20} \cdot (GO_{10}^-(2) \times \mathfrak{S}_3)$. Note that $\overline{O}(L) \cap O_2(O(\mathcal{D}(L), q_L)) \cong 2^{11}$, where $O_2(O(\mathcal{D}(L), q_L)) \cong 2_+^{1+20}$ is the maximal normal 2-subgroup of $O(\mathcal{D}(L), q_L)$. Fix an isometry $i : (\mathcal{D}(L), q_L) \rightarrow (\text{Irr}(W), -q_W)$. By Lemma 4.4 and Remark 4.5, the number of double cosets of

$$i^*(\overline{\text{Aut}}(W)) \backslash O(\mathcal{D}(L), q_L) / \overline{O}(L)$$

is equal to the number of $2^{12} \cdot \mathfrak{S}_{12}$ -orbits on the set of all $O(\text{Irr}(W), q_W)$ -conjugates of the subgroup $2_+^{1+20} \cdot (\mathfrak{S}_{12} \times \mathfrak{S}_3)$. Since $2_+^{1+20} \cdot \mathfrak{S}_3$ is a normal subgroup of $O(\text{Irr}(W), q_W)$ and

the quotient of $O(\text{Irr}(W), q_W)$ by this normal subgroup is $GO_{10}^-(2)$, this number is also equal to the number of \mathfrak{S}_{12} -orbits on its conjugates in $GO_{10}^-(2)$. By Remark 5.12, there is a natural embedding $\mathfrak{S}_{12} \subset GO_{10}^-(2)$, and by [ATLAS, page 147] (cf. [HS14, Remark 2.9]), there exist six \mathfrak{S}_{12} -orbits on its conjugates in $GO_{10}^-(2)$. Hence we obtain the following lemma.

Lemma 5.14. *There exist exactly 6 double cosets in $i^*(\overline{\text{Aut}}(W)) \backslash O(\mathcal{D}(L), q_L) / \overline{O}(L)$.*

By Proposition 4.2 and the lemma above, we obtain 6 holomorphic VOAs of central charge 24 as inequivalent simple current extensions. In fact, their weight one Lie algebras are non-isomorphic, which is discussed in [HS14, Remark 2.12] and [Hö, Table 8] (see Table 13 for the Lie algebra structures). Hence we obtain the following:

Proposition 5.15. *Assume that the conjugacy class of g is $2C$ and that $L \cong \sqrt{2}D_{12}$. Then there exist exactly 6 holomorphic VOAs of central charge 24 obtained as inequivalent simple current extensions of $V_L \otimes W$, up to isomorphism.*

In [ATLAS, Page 147], the shapes of the 6 subgroups of $GO_{10}^-(2)$ obtained as the intersection of two maximal subgroups isomorphic to \mathfrak{S}_{12} are described. These groups appear as the quotient of $\overline{O}(L) \cap \varphi^*(\overline{\text{Aut}}(W))$ by $O_2(\overline{O}(L)) \cong 2^{12}$ for isometries φ from $(\mathcal{D}(L), q_L)$ to $(\text{Irr}(W), -q_W)$. By Proposition 3.17, we obtain $\text{Out}(V)$ as in Table 13. For any weight one Lie algebra structure in Table 13, we have $\tilde{Q} \cong (1/\sqrt{2})\mathbb{Z}^{12}$. By Proposition 3.12 and $L^* \cong (1/\sqrt{2})D_{12}^*$, we have $K(V) \cong \mathbb{Z}_2$.

Table 13: $K(V)$ and $\text{Out}(V)$ for the case $2C$ and $L \cong \sqrt{2}D_{12}$

Genus	No.	V_1	$\overline{O}(L) \cap \varphi^*(\overline{\text{Aut}}(W))$	$W(V_1)$	$\text{Out}(V_1)$	$\text{Out}(V)$	$K(V)$
D	2	$A_{1,4}^{12}$	$2^{12} \cdot M_{12}$	$W(A_1)^{12}$	\mathfrak{S}_{12}	M_{12}	\mathbb{Z}_2
	12	$B_{2,2}^6$	$2^{12} \cdot (\mathbb{Z}_2^6 : \mathfrak{S}_5)$	$W(B_2)^6$	\mathfrak{S}_6	\mathfrak{S}_5	\mathbb{Z}_2
	23	$B_{3,2}^4$	$2^{12} \cdot (\mathfrak{S}_3 \wr \mathfrak{A}_4)$	$W(B_3)^4$	\mathfrak{S}_4	\mathfrak{A}_4	\mathbb{Z}_2
	29	$B_{4,2}^3$	$2^{12} \cdot (\mathfrak{S}_4 \wr \mathfrak{S}_3)$	$W(B_4)^3$	\mathfrak{S}_3	\mathfrak{S}_3	\mathbb{Z}_2
	41	$B_{6,2}^2$	$2^{12} \cdot (\mathfrak{S}_6 \wr 2)$	$W(B_6)^2$	\mathfrak{S}_2	\mathbb{Z}_2	\mathbb{Z}_2
	57	$B_{12,2}$	$2^{12} \cdot \mathfrak{S}_{12}$	$W(B_{12})$	1	1	\mathbb{Z}_2

Remark 5.16. For the cases in Table 13, the groups $K(V)$ and $\text{Out}(V)$ have been determined in [Sh20] by using the explicit construction of V .

5.5.2. *Case $L \cong \sqrt{2}E_8\sqrt{2}D_4$.* In this case, $O(L) \cong O(D_4) \times O(E_8) \cong ((2^{1+4} \cdot \mathfrak{S}_3) : \mathfrak{S}_3) \times 2.GO_8^+(2)$. In addition, $O_0(L)$ is generated by the -1 -isometry of $\sqrt{2}E_8$ and $\overline{O}(L) \cong$

$O(D_4) \times (O(E_8)/\langle -1 \rangle) \cong ((2^{1+4} \cdot \mathfrak{S}_3) : \mathfrak{S}_3) \times GO_8^+(2)$. We can rewrite as $\overline{O}(L) \cong 2^{1+4} \cdot ((\mathfrak{S}_3 \times GO_8^+(2)) \times \mathfrak{S}_3)$, which corresponds to the shape of $O(\text{Irr}(W), q_W) \cong 2_+^{1+20} \cdot (GO_{10}^-(2) \times \mathfrak{S}_3)$.

Fix an isometry $i : (\mathcal{D}(L), q_L) \rightarrow (\text{Irr}(W), -q_W)$. By Lemma 4.4, the number of double cosets is equal to the number of $2^{1+4} \cdot ((\mathfrak{S}_3 \times GO_8^+(2)) \times \mathfrak{S}_3)$ -orbits on the set of conjugates of $2_+^{1+20} \cdot (\mathfrak{S}_{12} \times \mathfrak{S}_3)$. It follows from $O_2(O(\text{Irr}(W), q_W)) \cong 2_+^{1+20}$ that the number is also equal to the number of $(\mathfrak{S}_3 \times GO_8^+(2))$ -orbits on the set of conjugates of the subgroup $\mathfrak{S}_{12} \subset GO_{10}^-(2)$.

Lemma 5.17. *There exist exactly 3 double cosets in $i^*(\overline{\text{Aut}}(W)) \backslash O(\mathcal{D}(L), q_L) / \overline{O}(L)$.*

Proof. Recall from Remark 5.12 the construction of a 10-dimensional quadratic space \mathcal{X} over \mathbb{F}_2 of minus type with natural embedding $\mathfrak{S}_{12} \subset GO_{10}^-(2)$. It is well-known (cf. [ATLAS, Wi09]) that the stabilizer in $GO_{10}^-(2)$ of a non-singular 2-space of minus-type is a maximal subgroup of the shape $\mathfrak{S}_3 \times GO_8^+(2)$. By the definition of the quadratic form in Remark 5.12, non-singular vectors of \mathcal{X} are 2-sets or 6-sets modulo $\{\emptyset, \Omega\}$. Then there exist exactly three orbits \mathcal{Q}_i ($i = 1, 2, 3$) of non-singular 2-spaces of minus type in \mathcal{X} under the action of \mathfrak{S}_{12} . Here the non-zero vectors of $\mathcal{Q}_1, \mathcal{Q}_2, \mathcal{Q}_3$ are three 2-sets, three 6-sets, or one 2-set and two 6-sets, respectively. One can then deduce that there exist exactly 3 $(\mathfrak{S}_3 \times GO_8^+(2))$ -orbits on the set of conjugates of the subgroup $\mathfrak{S}_{12} \subset GO_{10}^-(2)$. \square

By Proposition 4.2 and this lemma, we obtain 3 holomorphic VOAs of central charge 24 as inequivalent simple current extensions. In fact, their weight one Lie algebras are non-isomorphic, which is discussed in [HS14, Remark 2.12] and [Hö, Table 8] (see Table 14 for the Lie algebra structures.) Hence we obtain the following:

Proposition 5.18. *Assume that the conjugacy class of g is $2C$ and that $L \cong \sqrt{2}E_8\sqrt{2}D_4$. Then there exist exactly 3 holomorphic VOAs of central charge 24 obtained as inequivalent simple current extensions of $V_L \otimes W$, up to isomorphism.*

Table 14: $K(V)$ and $\text{Out}(V)$ for case $2C$ and $L \cong \sqrt{2}E_8\sqrt{2}D_4$

Genus	No.	V_1	$\mu_L^{-1}(\overline{O}(L) \cap \varphi^*(\overline{\text{Aut}}(W)))$	$W(V_1)$	$\text{Out}(V_1)$	$\text{Out}(V)$	$K(V)$
D	13	$D_{4,4}A_{2,2}^4$	$2 \cdot ((W(D_4) \times W(A_2)^4) \cdot \mathfrak{S}_4)$	$W(D_4) \times W(A_2)^4$	$\mathfrak{S}_3 \times \mathbb{Z}_2 \wr \mathfrak{S}_4$	$2 \cdot \mathfrak{S}_4$	\mathbb{Z}_3^2
	22	$C_{4,2}A_{4,2}^2$	$W(C_4) \times 2 \cdot (W(A_4)^2) \cdot 2$	$W(C_4) \times W(A_4)^2$	$\mathbb{Z}_2 \wr \mathfrak{S}_2$	\mathbb{Z}_4	\mathbb{Z}_5
	36	$A_{8,2}F_{4,2}$	$2 \cdot (W(A_8) \times W(F_4))$	$W(A_8) \times W(F_4)$	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}_3

Let \mathcal{Q}_i ($i = 1, 2, 3$) be non-singular 2-spaces of minus type in \mathcal{X} given in the proof of Lemma 5.17. Then the stabilizers of $\mathcal{Q}_1, \mathcal{Q}_2$ and \mathcal{Q}_3 in \mathfrak{S}_{12} are $\mathfrak{S}_3 \times \mathfrak{S}_9, \mathfrak{S}_3 \wr \mathfrak{S}_4$ and $\mathfrak{S}_2 \times \mathfrak{S}_5 \wr \mathfrak{S}_2$, respectively. Let φ_i be an isometry from $(\mathcal{D}(L), q_L)$ to $(\text{Irr}(W), -q_W)$ associated with \mathcal{Q}_i . Then $\overline{O}(L) \cap \varphi_i^*(\overline{\text{Aut}}(W))$ has the shapes $2^{1+4} \cdot (\mathfrak{S}_3 \times \mathfrak{S}_9) \cdot \mathfrak{S}_3, 2^{1+4} \cdot (\mathfrak{S}_3 \wr \mathfrak{S}_4) \cdot \mathfrak{S}_3$ and $2^{1+4} \cdot (\mathfrak{S}_2 \times \mathfrak{S}_5 \wr \mathfrak{S}_2) \cdot \mathfrak{S}_3$, respectively. By the shapes of these groups, the corresponding

Lie algebra structures of V_1 are $A_{8,2}F_{4,2}$, $D_{4,4}A_{2,2}^4$ and $C_{4,2}A_{4,2}^2$, respectively. Note that the Weyl groups of the simple ideals of type $F_{4,2}$, $D_{4,4}$ and $C_{4,2}$ act as the diagram automorphism group \mathfrak{S}_3 on $\sqrt{2}D_4 \subset L$, respectively. The group $\mu_L^{-1}(\overline{O}(L) \cap \varphi_i^*(\overline{\text{Aut}}(W)))$ is a central extension of $\overline{O}(L) \cap \varphi_i^*(\overline{\text{Aut}}(W))$ by $O_0(L) \cong \mathbb{Z}_2$. Since $\mu_L^{-1}(\overline{O}(L) \cap \varphi_i^*(\overline{\text{Aut}}(W)))$ contains $W(V_1)$, we can rewrite the shapes of $\mu_L^{-1}(\overline{O}(L) \cap \varphi_i^*(\overline{\text{Aut}}(W)))$ as $2.(W(A_8) \times W(F_4))$, $2.(W(D_4) \times W(A_2)^4).\mathfrak{S}_4$ and $2.(W(C_4) \times (W(A_4)^2).2)$, respectively. Note that for the case $C_{4,2}A_{4,2}^2$, the subgroup $2.(W(A_4)^2).2$ is the stabilizer in $O(E_8)$ of the sublattice A_4^2 of E_8 , and its quotient by $W(A_4)^2$ is isomorphic to \mathbb{Z}_4 . Hence we obtain the shape of $\text{Out}(V)$ as in Table 14 by Proposition 3.17.

Recall that $L^* \cong (1/\sqrt{2})D_4^*(1/\sqrt{2})E_8$. If the Lie algebra structure of V_1 is $A_{8,2}F_{4,2}$, $C_{4,2}A_{4,2}^2$ or $D_{4,4}A_{2,2}^4$, then \tilde{Q} is isometric to $(1/\sqrt{2})A_8(1/\sqrt{2})D_4^*$, $(1/2)D_4(1/\sqrt{2})A_4^2$ and $(1/2)D_4(1/\sqrt{2})A_2^4$, respectively. Note that $(1/\sqrt{2})D_4^* \cong (1/2)D_4$. By Proposition 3.12, we obtain $K(V) \cong L^*/\tilde{Q}$ as in Table 14.

Proposition 5.19. *Assume that g belongs to the conjugacy class $2C$. The shapes of the groups $K(V)$ and $\text{Out}(V)$ are given as in Tables 13 and 14.*

5.6. Conjugacy class $4C$ (Genus E). Assume that g belongs to the conjugacy class $4C$ of $O(\Lambda)$. Then $O(\text{Irr}(W), q_W) \cong 2^{22}.GO_7(2)$. By Table 3, $\text{Aut}(W) (\cong \overline{\text{Aut}}(W))$ has the shape $2^{21}.GO_7(2)$, which is an index 2 subgroup of $O(\text{Irr}(W), q_W)$. Note also that the Lie algebra structure of $\mathfrak{g} = V_1$ is given as in Table 4.

Since the central charge of W is 14, the rank of L is 10. By (3.8) and Proposition 3.20, we have $Q_{\mathfrak{g}} \subset L \subset \sqrt{4}P_{\mathfrak{g}}^*$. It follows from Table 3 and $\mathcal{D}(L) \cong \text{Irr}(W)$ that $\mathcal{D}(L) \cong \mathbb{Z}_2^2 \times \mathbb{Z}_4^6$. For each \mathfrak{g} , its Lie algebra structure gives the lattices $Q_{\mathfrak{g}}$ and $\sqrt{4}P_{\mathfrak{g}}^*$ as in Table 15. Then one can easily see that there exists a unique even lattice L up to isometry satisfying $\mathcal{D}(L) \cong \mathbb{Z}_2^2 \times \mathbb{Z}_4^6$ and $Q_{\mathfrak{g}} \subset L \subset \sqrt{4}P_{\mathfrak{g}}^*$; see Table 15 for the explicit description. We set $L_{\mathfrak{g}} = L$. The isometry groups of $L_{\mathfrak{g}}$ are also summarized in Table 16.

Remark 5.20. Let us explain the meaning of ‘‘Glue’’ in the tables. Let $Q_{\mathfrak{g}} = \bigoplus_{i=1}^s c_i R_i$, where $c_i \in \mathbb{R}$ and R_i are irreducible root lattices. In our cases, $L_{\mathfrak{g}}$ is a sublattice of $\bigoplus_{i=1}^s c_i R_i^*$; we associate $L_{\mathfrak{g}}/Q_{\mathfrak{g}}$ to a subgroup of $\bigoplus_{i=1}^s (R_i^*/R_i)$ via the inclusion $L_{\mathfrak{g}}/Q_{\mathfrak{g}} \subset (\bigoplus_{i=1}^s c_i R_i^*) / (\bigoplus_{i=1}^s c_i R_i) \cong \bigoplus_{i=1}^s (R_i^*/R_i)$. In the tables, based on the isomorphisms $A_m^*/A_m \cong \mathbb{Z}_{m+1}$, $D_{2m+1}^*/D_{2m+1} \cong \mathbb{Z}_4$, $D_{2m}^*/D_{2m} \cong \mathbb{Z}_2^2 = \langle b, c \rangle$ and $E_6^*/E_6 \cong \mathbb{Z}_3$, the generators of $L_{\mathfrak{g}}/Q_{\mathfrak{g}}$ is described as a subgroup of $\bigoplus_{i=1}^s (R_i^*/R_i)$ in ‘‘Glue’’. Here b, c are chosen so that they are permuted by the diagram automorphism of order 2.

By Proposition 3.17 and Lemma 3.18 (2), we have $\text{Out}(V) \cong O(L_{\mathfrak{g}})/\langle W(V_1), -1 \rangle$. The group $K(V)$ is determined by Proposition 3.12. These structures are summarized in Table 16.

Proposition 5.21. *Assume that g belongs to the conjugacy class $4C$. Then the shapes of the groups $K(V)$ and $\text{Out}(V)$ are given as in Table 16.*

Table 15: Even lattices of rank 10 for the case $4C$.

$\mathfrak{g} = V_1$	$Q_{\mathfrak{g}}$	$\sqrt{4}P_{\mathfrak{g}}^*$	$L_{\mathfrak{g}}/Q_{\mathfrak{g}}$	Glue	$O(L_{\mathfrak{g}})$
$A_{3,4}^3 A_{1,2}$	$(2A_3)^3 \sqrt{2}A_1$	$(2A_3^*)^3 \sqrt{2}A_1^*$	\mathbb{Z}_4^3	$(100; 1), (010; 1)$ $(001; 1)$	$O(A_3) \wr \mathfrak{S}_3 \times W(A_1)$
$A_{7,4} A_{1,1}^3$	$2A_7 A_1^3$	$2A_7^* (A_1^*)^3$	\mathbb{Z}_8	$(1; 100)$	$O(A_7) \times W(A_1) \times W(A_1) \wr \mathfrak{S}_2$
$D_{5,4} C_{3,2} A_{1,1}^2$	$2D_5 \sqrt{2} A_1^3 A_1^2$	$2D_5^* 2A_3^* (A_1^*)^2$	$\mathbb{Z}_4 \times \mathbb{Z}_2$	$(1; 111; 00)$ $(0; 111; 11)$	$O(D_5) \times O(A_3) \times W(A_1) \wr \mathfrak{S}_2$
$E_{6,4} A_{2,1} B_{2,1}$	$2E_6 A_2 A_1^2$	$2E_6^* A_2^* A_1^2$	\mathbb{Z}_3	$(1; 1; 00)$	$(W(E_6) \times W(A_1)) \cdot 2 \times W(B_2)$
$C_{7,2} A_{3,1}$	$\sqrt{2} A_1^7 A_3$	$2D_7^* A_3^*$	\mathbb{Z}_2	$(1^7; 2)$	$O(D_7) \times O(A_3)$

Table 16: $K(V)$ and $\text{Out}(V)$ for the case $4C$

No.	$\mathfrak{g} = V_1$	$W(V_1)$	$\text{Out}(V_1)$	$\text{Out}(V)$	$K(V)$
7	$A_{3,4}^3 A_{1,2}$	$W(A_3)^3 \times W(A_1)$	$\mathbb{Z}_2 \wr \mathfrak{S}_3$	$\mathbb{Z}_2^2 : \mathfrak{S}_3$	\mathbb{Z}_2
18	$A_{7,4} A_{1,1}^3$	$W(A_7) \times W(A_1)^3$	$\mathbb{Z}_2 \times \mathfrak{S}_3$	\mathbb{Z}_2	\mathbb{Z}_2^3
19	$D_{5,4} C_{3,2} A_{1,1}^2$	$W(D_5) \times W(C_3) \times W(A_1)^2$	$\mathbb{Z}_2 \times \mathfrak{S}_2$	\mathbb{Z}_2	\mathbb{Z}_2^3
28	$E_{6,4} A_{2,1} B_{2,1}$	$W(E_6) \times W(A_2) \times W(B_2)$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	1	\mathbb{Z}_6
35	$C_{7,2} A_{3,1}$	$W(C_7) \times W(A_3)$	\mathbb{Z}_2	1	\mathbb{Z}_2^2

5.7. Conjugacy class $6E$ (Genus G). Assume that g belongs to the conjugacy class $6E$ of $O(\Lambda)$. Then $O(\text{Irr}(W), q_W) \cong GO_6^+(2) \times GO_6^+(3)$. By Table 3, $\text{Aut}(W) (\cong \overline{\text{Aut}}(W))$ is isomorphic to the full orthogonal group $O(\text{Irr}(W), q_W)$. Note also that the Lie algebra structure of $\mathfrak{g} = V_1$ is given as in Table 4.

Since the central charge of W is 16, the rank of L is 8. By (3.8) and Proposition 3.20, we have $Q_{\mathfrak{g}} \subset L \subset \sqrt{6}P_{\mathfrak{g}}^*$. It follows from Table 3 and $\mathcal{D}(L) \cong \text{Irr}(W)$ that $\mathcal{D}(L) \cong \mathbb{Z}_2^6 \times \mathbb{Z}_3^6$. For each \mathfrak{g} , its Lie algebra structure gives the lattices $Q_{\mathfrak{g}}$ and $\sqrt{6}P_{\mathfrak{g}}^*$ as in Table 17. Then one can easily see that there exists a unique even lattice L up to isometry satisfying $\mathcal{D}(L) \cong \mathbb{Z}_2^6 \times \mathbb{Z}_3^6$ and $Q_{\mathfrak{g}} \subset L \subset \sqrt{6}P_{\mathfrak{g}}^*$; see Table 17 for the explicit description. Set $L_{\mathfrak{g}} = L$. The isometry groups of $L_{\mathfrak{g}}$ are also summarized in Table 17.

The group $\text{Out}(V)$ is determined by Lemma 3.18 (1), and the group $K(V)$ is determined by Proposition 3.12. These structures are summarized in Table 18.

Proposition 5.22. *Assume that g belongs to the conjugacy class $6E$. Then the shapes of the groups $K(V)$ and $\text{Out}(V)$ are given as in Tables 18.*

Table 17: Even lattices of rank 8 for the case $6E$

$\mathfrak{g} = V_1$	$Q_{\mathfrak{g}}$	$\sqrt{6}P_{\mathfrak{g}}^*$	$L_{\mathfrak{g}}/Q_{\mathfrak{g}}$	Glue	$O(L_{\mathfrak{g}})$
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$A_{5,6}B_{2,3}A_{1,2}$	$\sqrt{6}A_5\sqrt{3}A_1^2\sqrt{2}A_1$	$\sqrt{6}A_5^*(\sqrt{3}A_1^*)^2\sqrt{2}A_1^*$	$\mathbb{Z}_6 \times \mathbb{Z}_2$	(1; 00; 1) (0; 11; 1)	$O(A_5) \times O(A_1^2) \times W(A_1)$
$C_{5,3}G_{2,2}A_{1,1}$	$\sqrt{3}A_1^5\sqrt{2}A_2A_1$	$\sqrt{6}D_5^*\sqrt{2}A_2^*A_1^*$	\mathbb{Z}_2	(1 ⁵ ; 0; 1)	$O(A_5) \times O(A_1^2) \times O(A_1)$

Table 18: $K(V)$ and $\text{Out}(V)$ for the case $6E$

No.	$\mathfrak{g} = V_1$	$W(V_1)$	$\text{Out}(V_1)$	$\text{Out}(V)$	$K(V)$
8	$A_{5,6}B_{2,3}A_{1,2}$	$W(A_5) \times W(B_2) \times W(A_1)$	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}_2
21	$C_{5,3}G_{2,2}A_{1,1}$	$W(C_5) \times W(G_2) \times W(A_1)$	1	1	\mathbb{Z}_2

5.8. **Conjugacy class $6G$ (Genus J).** Assume that g belongs to the conjugacy class $6G$ of $O(\Lambda)$. Then $O(\text{Irr}(W), q_W) \cong 2_+^{1+8}:(GO_4^+(2) \times \mathfrak{S}_3) \times GO_5(3)$. Here $GO_4^+(2) \cong \mathfrak{S}_3 \wr \mathbb{Z}_2$ and $GO_5(3) \cong 2 \times \text{P}\Omega_5(3).2$. By Table 3, $\text{Aut}(W) (\cong \overline{\text{Aut}}(W))$ has the shape $2_+^{1+8}:(\mathfrak{S}_3 \times \mathfrak{S}_3 \times \mathfrak{S}_3) \times \text{P}\Omega_5(3).2$, which is an index 4 subgroup of $O(\text{Irr}(W), q_W)$.

Remark 5.23. Let $O_2(\text{Irr}(W))$ be the Sylow 2-subgroup of $\text{Irr}(W)$ of shape $2^4.4^2$. Then $O(O_2(\text{Irr}(W)), q_W) \cap \text{Aut}(W) \cong 2_+^{1+8}:(3 \times \mathfrak{S}_3 \times \mathfrak{S}_3)$, which is computed by MAGMA. Hence we can rewrite $\text{Aut}(W) \cong (2_+^{1+8}:(3 \times \mathfrak{S}_3 \times \mathfrak{S}_3) \times \text{P}\Omega_5(3).2).2$ with respect to the shape of $O(\text{Irr}(W), q_W)$. In fact, $\text{Aut}(W)$ is not normal in $O(\text{Irr}(W), q_W)$ by MAGMA.

By Table 4, the Lie algebra structure of $\mathfrak{g} = V_1$ is $F_{4,6}A_{2,2}$ or $D_{4,12}A_{2,6}$. By (3.8) and Proposition 3.20, we have $Q_{\mathfrak{g}} \subset L \subset \sqrt{12}P_{\mathfrak{g}}^*$. It follows from Table 3 and $\mathcal{D}(L) \cong \text{Irr}(W)$ that $\mathcal{D}(L) \cong \mathbb{Z}_2^4 \times \mathbb{Z}_4^2 \times \mathbb{Z}_3^5$. In both cases, we have $L \cong \sqrt{6}D_4\sqrt{2}A_2 \cong \sqrt{12}D_4^*\sqrt{6}A_2^*$ and $O(L) \cong O(D_4) \times O(A_2)$ (see Table 19). Note that $\overline{O}(L) \cong O(L)$.

Let $i : (\mathcal{D}(L), q_L) \rightarrow (\text{Irr}(W), -q_W)$ be an isometry. By the possible Lie algebra structures of \mathfrak{g} , there exist at least two non-isomorphic holomorphic VOAs obtained as inequivalent extensions of $V_L \otimes W$. Since $\overline{\text{Aut}}(W)$ is an index 4 subgroup of $O(\text{Irr}(W), q_W)$, we have

$$2 \leq |i^*(\overline{\text{Aut}}(W)) \setminus O(\mathcal{D}(L), q_L)/\overline{O}(L)| \leq 4$$

by Proposition 4.2. By Lemma 3.6 (2), the -1 -isometry is not in $i^*(\overline{\text{Aut}}(W))$. Clearly it is in $\overline{O}(L)$. Hence,

$$|i^*(\overline{\text{Aut}}(W)) \setminus O(\mathcal{D}(L), q_L)/\overline{O}(L)| = 2.$$

By Proposition 4.2, we obtain the following:

Proposition 5.24. *Assume that the conjugacy class of g is $6G$. Then there exist exactly two holomorphic VOAs of central charge 24 obtained as inequivalent simple current extensions of $V_L \otimes W$, up to isomorphism.*

By the argument above, $i^*(\overline{\text{Aut}}(W))$ is an index 2 subgroup of the group generated by $i^*(\overline{\text{Aut}}(W))$ and $\overline{O}(L)$. Thus $\overline{O}(L) \cap i^*(\overline{\text{Aut}}(W))$ is an index 2 subgroup of $\overline{O}(L)$, and by

Lemma 3.6 (2), $\overline{O}(L) \cap i^*(\overline{\text{Aut}}(W)) \cong \overline{O}(L)/\langle -1 \rangle$. By Proposition 3.17 and $\overline{O}(L) \cong O(L)$, we have $\text{Out}(V) \cong O(L)/\langle W(V_1), -1 \rangle$. The group $K(V)$ is determined by Proposition 3.12. These group structures are summarized in Table 20.

Proposition 5.25. *Assume that g belongs to the conjugacy class $6G$. Then the shapes of the groups $K(V)$ and $\text{Out}(V)$ are given as in Table 20.*

Table 19: Even lattice of rank 6 for the case $6G$

$\mathfrak{g} = V_1$	$Q_{\mathfrak{g}}$	$\sqrt{12}P_{\mathfrak{g}}^*$	$L/Q_{\mathfrak{g}}$	Glue	$O(L)$
$D_{4,12}A_{2,6}$	$\sqrt{12}D_4\sqrt{6}A_2$	$\sqrt{12}D_4^*\sqrt{6}A_2^*$	$\mathbb{Z}_2^2 \times \mathbb{Z}_3$	$(b; 0), (c; 0), (0; 1)$	$O(D_4) \times O(A_2)$
$F_{4,6}A_{2,2}$	$\sqrt{6}D_4\sqrt{2}A_2$	$\sqrt{12}D_4^*\sqrt{2}A_2^*$	1	1	$O(D_4) \times O(A_2)$

Table 20: $K(V)$ and $\text{Out}(V)$ for the case $6G$

No.	$\mathfrak{g} = V_1$	$W(V_1)$	$\text{Out}(V_1)$	$\text{Out}(V)$	$K(V)$
3	$D_{4,12}A_{2,6}$	$W(D_4) \times W(A_2)$	$\mathfrak{S}_3 \times \mathbb{Z}_2$	\mathfrak{S}_3	1
14	$F_{4,6}A_{2,2}$	$W(F_4) \times W(A_2)$	\mathbb{Z}_2	1	\mathbb{Z}_3

5.9. **Conjugacy class $8E$ (Genus 1).** Assume that g belongs to the conjugacy class $8E$ of $O(\Lambda)$. Then $O(\text{Irr}(W), q_W) \cong 2^{12+9} \cdot \mathfrak{S}_6$. By Table 3, $\text{Aut}(W) (\cong \overline{\text{Aut}}(W))$ has the shape $2^{11+9} \cdot \mathfrak{S}_6$, which is an index 2 subgroup of $O(\text{Irr}(W), q_W)$. Note also that the Lie algebra structure of $\mathfrak{g} = V_1$ is $D_{5,8}A_{1,2}$ by Table 4.

Since the central charge of W is 18, the rank of L is 6. By (3.8) and Proposition 3.20, we have $Q_{\mathfrak{g}} \subset L \subset \sqrt{8}P_{\mathfrak{g}}^*$. It follows from Table 3 and $\mathcal{D}(L) \cong \text{Irr}(W)$ that $\mathcal{D}(L) \cong \mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_8^4$. Hence, we have $L_{\mathfrak{g}} = L \cong \sqrt{8}D_5^*\sqrt{2}A_1$ and $O(L_{\mathfrak{g}}) \cong O(D_5) \times W(A_1)$ (see Table 21).

By Proposition 3.17 and Lemma 3.18 (2), we have $\text{Out}(V) \cong O(L_{\mathfrak{g}})/\langle W(V_1), -1 \rangle$. The group $K(V)$ is determined by Proposition 3.12. See Table 22 for the structures.

Proposition 5.26. *Assume that g belongs to the conjugacy class $8E$. Then the shapes of the groups $K(V)$ and $\text{Out}(V)$ are given as in Table 22.*

Table 21: Even lattice of rank 6 for the case $8E$

$V_1 = \mathfrak{g}$	$Q_{\mathfrak{g}}$	$\sqrt{8}P_{\mathfrak{g}}^*$	$L_{\mathfrak{g}}/Q_{\mathfrak{g}}$	Glue	$O(L_{\mathfrak{g}})$
$D_{5,8}A_{1,2}$	$\sqrt{8}D_5\sqrt{2}A_1$	$\sqrt{8}D_5^*\sqrt{2}A_1^*$	\mathbb{Z}_4	(1; 0)	$O(D_5) \times W(A_1)$

Table 22: $K(V)$ and $\text{Out}(V)$ for the case $8E$

No.	$\mathfrak{g} = V_1$	$W(V_1)$	$\text{Out}(V_1)$	$\text{Out}(V)$	$K(V)$
10	$D_{5,8}A_{1,2}$	$W(A_5) \times W(A_1)$	\mathbb{Z}_2	1	\mathbb{Z}_2

5.10. **Conjugacy class $10F$ (Genus K).** Assume that g belongs to the conjugacy class $10F$ of $O(\Lambda)$. Then $O(\text{Irr}(W), q_W) \cong 2_+^{1+4} : (\mathfrak{S}_3 \times \mathfrak{S}_3) \times GO_4^+(5)$. By Table 3, $\text{Aut}(W) (\cong \overline{\text{Aut}}(W))$ has the shape $2_+^{1+4} : (2 \times \mathfrak{S}_3) \times GO_4^+(5)$, which is an index 3 subgroup of $O(\text{Irr}(W), q_W)$. By Table 4, the Lie algebra structure of $\mathfrak{g} = V_1$ is $C_{4,10}$.

Since the central charge of W is 20, the rank of L is 4. By (3.8) and Proposition 3.20, we have $Q_{\mathfrak{g}} \subset L \subset \sqrt{20}P_{\mathfrak{g}}^*$. It follows from Table 3 and $\mathcal{D}(L) \cong \text{Irr}(W)$ that $\mathcal{D}(L) \cong \mathbb{Z}_2^2 \times \mathbb{Z}_4^2 \times \mathbb{Z}_5^4$. Then we have $L_{\mathfrak{g}} = L \cong \sqrt{10}D_4$ and $O(L_{\mathfrak{g}}) \cong O(D_4)$ (see Table 23).

Remark 5.27. For $U = \sqrt{20}L^*$, we have $\mathcal{D}(U) = \mathbb{Z}_2^2$ and $\text{rank}(U) = 4$. It is easy to show that $U \cong D_4$ and thus $L = \sqrt{20}U^* \cong \sqrt{20}D_4^* \cong \sqrt{10}D_4$.

Since $V_1 \cong C_{4,10}$, we have $\text{Out}(V_1) = 1$, and $\text{Out}(V) = 1$. The group $K(V)$ is trivial by Proposition 3.12. These group structures are summarized in Table 24.

Proposition 5.28. *Assume that g belongs to the conjugacy class $10F$. Then the shapes of the groups $K(V)$ and $\text{Out}(V)$ are given as in Table 24.*

Table 23: Even lattice of rank 4 for the case $10F$

$\mathfrak{g} = V_1$	$Q_{\mathfrak{g}}$	$\sqrt{20}P_{\mathfrak{g}}^*$	$L_{\mathfrak{g}}/Q_{\mathfrak{g}}$	Glue	$O(L_{\mathfrak{g}})$
$C_{4,10}$	$\sqrt{10}A_1^4$	$\sqrt{20}D_4^*$	\mathbb{Z}_2	(1111)	$O(D_4)$

Table 24: $K(V)$ and $\text{Out}(V)$ for the case $10F$

No.	V_1	$W(V_1)$	$\text{Out}(V_1)$	$\text{Out}(V)$	$K(V)$
4	$C_{4,10}$	$W(C_4)$	1	1	1

It is easy to see that μ_L is injective, that is, $\overline{O}(L) \cong O(L)$. Let φ be an isometry from $(\mathcal{D}(L), q_L)$ to $(\text{Irr}(W), -q_W)$. By Proposition 3.17 and $\text{Out}(V) = 1$, we have $\overline{O}(L) \cap \varphi^*(\overline{\text{Aut}}(W)) = W(V_1)$. Since $W(V_1) (\cong W(C_4))$ is an index 3 subgroup of $\overline{O}(L)$ (see Lemma 2.3), so is $\overline{O}(L) \cap \varphi^*(\overline{\text{Aut}}(W))$. Hence $\varphi^*(\overline{\text{Aut}}(W))$ and $\overline{O}(L)$ generate $O(\mathcal{D}(L), q_L)$. By Proposition 4.2, we obtain the following:

Proposition 5.29. *Assume that the conjugacy class of g is $10F$. Then, there exists exactly one holomorphic VOA of central charge 24 obtained as inequivalent simple current extensions of $V_L \otimes W$, up to isomorphism.*

As a consequence of our calculations, we have proved Theorem 1.4 and confirmed [Hö, Conjecture 4.8]. Combining with the characterization of Niemeier lattice VOAs in [DM04b], it provides another proof for the uniqueness of holomorphic vertex operator algebras of central charge 24 with non-trivial weight one Lie algebras.

APPENDIX A. ACTIONS OF AUTOMORPHISM GROUPS ON THE WEIGHT ONE SPACES

In this appendix, for holomorphic VOAs V of central charge 24 whose weight one Lie algebras are semisimple, we describe the subgroup $\text{Out}_1(V)$ of $\text{Out}(V)$ which preserves every simple ideal of V_1 and the quotient group $\text{Out}_2(V) = \text{Out}(V)/\text{Out}_1(V)$.

A.1. Simple current modules over $L_{\mathfrak{g}}(k, 0)$. Let \mathfrak{g} be a simple Lie algebra and let k be a positive integer. Let $L_{\mathfrak{g}}(k, 0)$ be the simple affine VOA associated with \mathfrak{g} at level k . Let $S_{\mathfrak{g}}$ be the set of isomorphism classes of simple current $L_{\mathfrak{g}}(k, 0)$ -modules. Then $S_{\mathfrak{g}}$ has an abelian group structure under the fusion product. The structures of $S_{\mathfrak{g}}$ are well-known (see [Li01, Remark 2.21] and reference therein), which are summarized in Table 25. Here $\Gamma(\mathfrak{g})$ is the diagram automorphism group of \mathfrak{g} and $[\Lambda]$ is the irreducible $L_{\mathfrak{g}}(k, 0)$ -module $L_{\mathfrak{g}}(k, \Lambda)$. Note that the notations $[i](= i[1])$, $[s]$ and $[c]$ are used in [Sc93].

Table 25: Simple current $L_{\mathfrak{g}}(k, 0)$ -modules

Type	level	$S_{\mathfrak{g}}$	$\Gamma(\mathfrak{g})$	generators of $S_{\mathfrak{g}}$
A_1	k	\mathbb{Z}_2	1	$[1] = [k\Lambda_1]$
A_n ($n \geq 2$)	k	\mathbb{Z}_{n+1}	\mathbb{Z}_2	$[1] = [k\Lambda_1]$
B_n ($n \geq 2$)	k	\mathbb{Z}_2	1	$[1] = [k\Lambda_1]$
C_n ($n \geq 2$)	k	\mathbb{Z}_2	1	$[1] = [k\Lambda_n]$
D_4	k	$\mathbb{Z}_2 \times \mathbb{Z}_2$	\mathfrak{S}_3	$[s] = [k\Lambda_{n-1}], [c] = [k\Lambda_n]$
D_{2n} ($n \geq 3$)	k	$\mathbb{Z}_2 \times \mathbb{Z}_2$	\mathbb{Z}_2	$[s] = [k\Lambda_{n-1}], [c] = [k\Lambda_n]$
D_{2n+1} ($n \geq 2$)	k	\mathbb{Z}_4	\mathbb{Z}_2	$[s] = [k\Lambda_{n-1}]$
E_6	k	\mathbb{Z}_3	\mathbb{Z}_2	$[1] = [k\Lambda_1]$
E_7	k	\mathbb{Z}_2	1	$[1] = [k\Lambda_6]$
E_8	2	\mathbb{Z}_2	1	$[1] = [\Lambda_7]$
E_8	$k \neq 2$	1	1	
F_4	k	1	1	
G_2	k	1	1	

A.2. Glue codes of holomorphic VOAs of central charge 24. Let V be a holomorphic VOA of central charge 24 with $0 < \text{rank } V_1 < 24$. Let $\text{Out}_1(V)$ be the subgroup of $\text{Out}(V)$ which preserves every simple ideal of V_1 and set $\text{Out}_2(V) = \text{Out}(V)/\text{Out}_1(V)$. Then $\text{Out}_2(V)$ is the permutation group on the set of simple ideals of V_1 induced from $\text{Out}(V)$.

Let $V_1 = \bigoplus_{i=1}^s \mathfrak{g}_i$ be the direct sum of simple ideals. Let $S_i (= S_{\mathfrak{g}_i})$ be the set of (the isomorphism classes of) simple current $L_{\mathfrak{g}_i}(k_i, 0)$ -modules, where k_i is the level of \mathfrak{g}_i in V . Then S_i has an abelian group structure under the fusion product.

Let $S_{\mathfrak{g}} = \prod_{i=1}^s S_i$ be the direct product of the groups S_i . We often view a simple current $\langle V_1 \rangle$ -module as an element of $S_{\mathfrak{g}}$ via the map $\bigotimes_{i=1}^s M^i \mapsto (M^1, \dots, M^s)$. Let $\{1, 2, \dots, s\} = \bigcup_{b \in B} I_b$ be the partition such that $\mathfrak{g}_i \cong \mathfrak{g}_j$ if and only if $i, j \in I_b$ for some $b \in B$, where B is an index set. The automorphism group $\text{Aut}(S_{\mathfrak{g}})$ of $S_{\mathfrak{g}}$ is defined to be $(\prod_{i=1}^s \Gamma(\mathfrak{g}_i)) : (\prod_{b \in B} \mathfrak{S}_{|I_b|})$, where the symmetric group $\mathfrak{S}_{|I_b|}$ acts naturally on $\prod_{i \in I_b} S_i$.

Let G_V be the subgroup of $S_{\mathfrak{g}}$ consisting of all (isomorphism classes of) simple current $\langle V_1 \rangle$ -submodules of V , which we call the *Glue code* of V . The automorphism group $\text{Aut}(G_V)$ of G_V is defined to be the subgroup of $\text{Aut}(S_{\mathfrak{g}})$ stabilizing G_V . Let $\text{Aut}_1(G_V) = (\prod_{i=1}^s \Gamma(\mathfrak{g}_i)) \cap \text{Aut}(G_V)$ and $\text{Aut}_2(G_V) = \text{Aut}(G_V)/\text{Aut}_1(G_V)$. Then $\text{Aut}_1(G_V)$ is the subgroup of $\text{Aut}(G_V)$ stabilizing every S_i , and $\text{Aut}_2(G_V)$ acts faithfully on $\{S_i \mid 1 \leq i \leq s\}$, or $\{\mathfrak{g}_i \mid 1 \leq i \leq s\}$, as a permutation group. Clearly $\text{Aut}(V)$ preserves $S_{\mathfrak{g}}$. Hence $\text{Out}_i(V) \subset \text{Aut}_i(G_V)$ for $i = 1, 2$.

By using the generators of the glue codes G_V in [Sc93], we can easily determine $\text{Aut}_1(G_V)$ and $\text{Aut}_2(G_V)$ explicitly. We also determine the shapes of $\text{Out}_1(V)$ and $\text{Out}_2(V)$; see Tables 26.

Table 26: $\text{Aut}_i(G_V)$ and $\text{Out}_i(V)$

No.	Genus	$\mathfrak{g} = V_1$	$\text{Aut}_1(G_V)$	$\text{Out}_1(V)$	$\text{Aut}_2(G_V)$	$\text{Out}_2(V)$
15	A	$A_{1,1}^{24}$	1	1	M_{24}	M_{24}
24		$A_{2,1}^{12}$	\mathbb{Z}_2	\mathbb{Z}_2	M_{12}	M_{12}
30		$A_{3,1}^8$	\mathbb{Z}_2	\mathbb{Z}_2	$AGL_3(2)$	$AGL_3(2)$
37		$A_{4,1}^6$	\mathbb{Z}_2	\mathbb{Z}_2	\mathfrak{S}_5	\mathfrak{S}_5
42		$D_{4,1}^6$	\mathbb{Z}_3	\mathbb{Z}_3	\mathfrak{S}_6	\mathfrak{S}_6
43		$A_{5,1}^4 D_{4,1}$	\mathbb{Z}_2	\mathbb{Z}_2	\mathfrak{S}_4	\mathfrak{S}_4
46		$A_{6,1}^4$	\mathbb{Z}_2	\mathbb{Z}_2	\mathfrak{A}_4	\mathfrak{A}_4
49		$A_{7,1}^2 D_{5,1}^2$	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}_2^2	\mathbb{Z}_2^2
51		$A_{8,1}^3$	\mathbb{Z}_2	\mathbb{Z}_2	\mathfrak{S}_3	\mathfrak{S}_3
54		$D_{6,1}^4$	1	1	\mathfrak{S}_4	\mathfrak{S}_4
55		$A_{9,1}^2 D_{6,1}$	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}_2

58		$E_{6,1}^4$	\mathbb{Z}_2	\mathbb{Z}_2	\mathfrak{S}_4	\mathfrak{S}_4
59		$A_{11,1}D_{7,1}E_{6,1}$	\mathbb{Z}_2	\mathbb{Z}_2	1	1
60		$A_{12,1}^2$	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}_2
61		$D_{8,1}^3$	1	1	\mathfrak{S}_3	\mathfrak{S}_3
63		$A_{15,1}A_{9,1}$	\mathbb{Z}_2	\mathbb{Z}_2	1	1
64		$D_{10,1}E_{7,1}^2$	1	1	\mathbb{Z}_2	\mathbb{Z}_2
65		$A_{17,1}E_{7,1}$	1	1	1	1
66		$D_{12,1}^2$	1	1	\mathbb{Z}_2	\mathbb{Z}_2
67		$A_{24,1}$	\mathbb{Z}_2	\mathbb{Z}_2	1	1
68		$E_{8,1}^3$	1	1	\mathfrak{S}_3	\mathfrak{S}_3
69		$D_{16,1}E_{8,1}$	1	1	1	1
70		$D_{24,1}$	1	1	1	1
5	B	$A_{1,2}^{16}$	1	1	$\text{AGL}_4(2)$	$\text{AGL}_4(2)$
16		$A_{3,2}^4A_{1,1}^4$	\mathbb{Z}_2	\mathbb{Z}_2	$\mathbb{Z}_2^4 : \mathfrak{S}_3$	$\mathbb{Z}_2^4 : \mathfrak{S}_3$
25		$D_{4,2}^2C_{2,1}^4$	1	1	$\mathfrak{S}_2 \times \mathfrak{S}_4$	$\mathfrak{S}_2 \times \mathfrak{S}_4$
26		$A_{5,2}^2C_{2,1}A_{2,1}^2$	\mathbb{Z}_2	\mathbb{Z}_2	$\mathfrak{S}_2 \times \mathfrak{S}_2$	$\mathfrak{S}_2 \times \mathfrak{S}_2$
31		$D_{5,2}^2A_{3,1}^2$	\mathbb{Z}_2	\mathbb{Z}_2	$\mathfrak{S}_2 \times \mathfrak{S}_2$	$\mathfrak{S}_2 \times \mathfrak{S}_2$
33		$A_{7,2}C_{3,1}^2A_{3,1}$	\mathbb{Z}_2	\mathbb{Z}_2	\mathfrak{S}_2	\mathfrak{S}_2
38		$C_{4,1}^4$	1	1	\mathfrak{S}_4	\mathfrak{S}_4
39		$D_{6,2}C_{4,1}B_{3,1}^2$	1	1	\mathfrak{S}_2	\mathfrak{S}_2
40		$A_{9,2}A_{4,1}B_{3,1}$	\mathbb{Z}_2	\mathbb{Z}_2	1	1
44		$E_{6,2}C_{5,1}A_{5,1}$	\mathbb{Z}_2	\mathbb{Z}_2	1	1
47		$D_{8,2}B_{4,1}^2$	1	1	\mathfrak{S}_2	\mathfrak{S}_2
48		$C_{6,1}^2B_{4,1}$	1	1	\mathfrak{S}_2	\mathfrak{S}_2
50		$D_{9,2}A_{7,1}$	\mathbb{Z}_2	\mathbb{Z}_2	1	1
52		$C_{8,1}F_{4,1}^2$	1	1	\mathfrak{S}_2	\mathfrak{S}_2
53		$E_{7,2}B_{5,1}F_{4,1}$	1	1	1	1
56		$C_{10,1}B_{6,1}$	1	1	1	1
62		$B_{8,1}E_{8,2}$	1	1	1	1
6	C	$A_{2,3}^6$	\mathbb{Z}_2	1	\mathfrak{S}_6	\mathfrak{S}_6
17		$A_{5,3}D_{4,3}A_{1,1}^3$	\mathbb{Z}_2	1	\mathfrak{S}_3	\mathfrak{S}_3
27		$A_{8,3}A_{2,1}^2$	\mathbb{Z}_2	1	\mathfrak{S}_2	\mathfrak{S}_2
32		$E_{6,3}G_{2,1}^3$	\mathbb{Z}_2	1	\mathfrak{S}_3	\mathfrak{S}_3
34		$D_{7,3}A_{3,1}G_{2,1}$	\mathbb{Z}_2	1	1	1
45		$E_{7,3}A_{5,1}$	\mathbb{Z}_2	1	1	1
2	D	$A_{1,4}^{12}$	1	1	\mathfrak{S}_{12}	M_{12}
12		$B_{2,2}^6$	1	1	\mathfrak{S}_6	\mathfrak{S}_5

23		$B_{3,2}^4$	1	1	\mathfrak{S}_4	\mathfrak{A}_4
29		$B_{4,2}^3$	1	1	\mathfrak{S}_3	\mathfrak{S}_3
41		$B_{6,2}^2$	1	1	\mathfrak{S}_2	\mathfrak{S}_2
57		$B_{12,2}$	1	1	1	1
13		$D_{4,4}A_{2,2}^4$	$\mathfrak{S}_3 \times \mathbb{Z}_2$	\mathbb{Z}_2	\mathfrak{S}_4	\mathfrak{S}_4
22		$C_{4,2}A_{4,2}^2$	\mathbb{Z}_2	\mathbb{Z}_2	\mathfrak{S}_2	\mathfrak{S}_2
36		$A_{8,2}F_{4,2}$	\mathbb{Z}_2	\mathbb{Z}_2	1	1
7	E	$A_{3,4}^3A_{1,2}$	\mathbb{Z}_2^3	\mathbb{Z}_2^2	\mathfrak{S}_3	\mathfrak{S}_3
18		$A_{7,4}A_{1,1}^3$	\mathbb{Z}_2	1	\mathbb{Z}_2	\mathbb{Z}_2
19		$D_{5,4}C_{3,2}A_{1,1}^2$	\mathbb{Z}_2	1	\mathbb{Z}_2	\mathbb{Z}_2
28		$E_{6,4}A_{2,1}B_{2,1}$	\mathbb{Z}_2	1	1	1
35		$C_{7,2}A_{3,1}$	\mathbb{Z}_2	1	1	1
9	F	$A_{4,5}^2$	\mathbb{Z}_2^2	\mathbb{Z}_2	\mathfrak{S}_2	\mathfrak{S}_2
20		$D_{6,5}A_{1,1}^2$	1	1	\mathfrak{S}_2	1
8	G	$A_{5,6}B_{2,3}A_{1,2}$	\mathbb{Z}_2	\mathbb{Z}_2	1	1
21		$C_{5,3}G_{2,2}A_{1,1}$	1	1	1	1
11	H	$A_{6,7}$	\mathbb{Z}_2	1	1	1
10	I	$D_{5,8}A_{1,2}$	\mathbb{Z}_2	1	1	1
3	J	$D_{4,12}A_{2,6}$	$\mathfrak{S}_3 \times \mathbb{Z}_2$	\mathfrak{S}_3	1	1
14		$F_{4,6}A_{2,2}$	\mathbb{Z}_2	1	1	1
4	K	$C_{4,10}$	1	1	1	1

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