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AUTOMORPHISM GROUPS, ISOMORPHIC TO $GL(3, \boldsymbol{F}_2)$, OF COMPACT RIEMANN SURFACES

AKIKAZU KURIBAYASHI AND HIDEYUKI KIMURA*

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Let X be a compact Riemann surface of genus $g \geq 2$. The automorphism group $\mathrm{Aut}(X)$ can be represented as a subgroup of $GL(g, \boldsymbol{C})$, since elements of $\mathrm{Aut}(X)$ act on the g-dimensional module of abelian differentials of X. We denote the representation by $\rho: \mathrm{Aut}(X) \to GL(g, \boldsymbol{C})$, and denote the image by $\rho(AG; X)$ for a subgroup AG of $\mathrm{Aut}(X)$. We have studied groups which are $GL(g, \boldsymbol{C})$-conjugate to $\rho(AG; X)$ for some X with fixed g and some AG. These groups are said to come from a Riemann surface X (see Definition 1). In this connection, we have introduced the CY-, RH- and EX-conditions (see Definitions 2, 3 and 5 in § 1). We saw in [6] that all groups which satisfy the CY- and RH-conditions come from Riemann surfaces except for two groups, i.e., the dihedral group $\mathscr{D}_8$ and the quaternion group $\mathscr{Q}_8$ in the case of $g=5$. Recently, on the other hand, Kimura [3], [4] studied which groups (isomorphic to $\mathscr{D}_8$, $\mathscr{Q}_8$ or $\mathfrak{U}_5$) come from Riemann surfaces for unspecified g (≥ 2).

In this paper, we consider for unspecified g (≥ 2) the CY- and RH-conditions for groups isomorphic to $GL(3, \boldsymbol{F}_2)$ of 3×3 invertible matrices with entries in the field $\boldsymbol{F}_2$ with two elements. We take the group $GL(3, \boldsymbol{F}_2)$ since it is the simple Hurwitz group of the smallest order. We apply the character theory of groups and see that if G $(\simeq GL(3, \boldsymbol{F}_2))$ satisfies the CY- and RH-conditions, then G comes from Riemann surfaces except in very few cases. This phenomenon seems to be rooted in some structure of groups although we cannot explicitly point out which.

1. Preliminaries.

DEFINITION 1 (cf. [5], [6]). A subgroup $G \subset GL(g, \boldsymbol{C})$ is said to come from a compact Riemann surface of genus g, if there exist a compact Riemann surface of genus g and a subgroup AG of $\mathrm{Aut}(X)$ such that $\rho(AG; X)$ is $GL(g, \boldsymbol{C})$-conjugate to G.

DEFINITION 2 (cf. [5], [6]). $G \subset GL(g, \boldsymbol{C})$ is said to satisfy the CY-condition if every element of $CY(G) = \{K \mid \text{nontrivial cyclic subgroup of } G\}$ comes from a compact Riemann surface of genus g.

DEFINITION 3 (cf. [8]). Assume that $G \subset GL(g, \boldsymbol{C})$ satisfies the E-condition, i.e.,

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$\{\mathrm{Tr}(\sigma)+\mathrm{Tr}(\sigma^{-1})\}$ is an integer for any $\sigma \in G$. For any $H \in CY(G)$ we set the following terminology:

$g_0(G)=(1/\#G)\sum_{\sigma\in G}\mathrm{Tr}(\sigma)$,
$r(H)=2-\{\mathrm{Tr}(\sigma)+\mathrm{Tr}(\sigma^{-1})\}$ for $H=\langle\sigma\rangle$,
$r_*(H:G)=r(H)-\sum_{K\in CY(G),\, H\subsetneqq K} r_*(K:G)$ (defined by descending condition)
$l(H:G)=r_*(H:G)/[N_G(H):H]$.

Here $N_G(H)$ means the normalizer of H in G. Then, we say that G satisfies the RH-condition if G satisfies the E-condition and if $l(H:G)$ is a non-negative integer for any $H\in CY(G)$.

DEFINITION 4 (cf. [8]). Assume that $G\subset GL(g, \boldsymbol{C})$ satisfies the RH-condition and let $\{H_1,\cdots, H_s\}$ be a complete set of representatives of the G-conjugacy classes of $CY(G)$. The RH-data $RH(G)$ of G is defined as

$$RH(G)=[g_0(G), \#G; \#H_1,\cdots,\#H_1,\cdots,\#H_s,\cdots,\#H_s]\,.$$

Here $\#H_i$ appears $l(H_i:G)$-times $(1\le i\le s)$.

REMARK 1 (cf. [8]). Assume that $G\subset GL(g, \boldsymbol{C})$ satisfies the E-condition. Then we have

$$2g-2=\#G[2g_0(G)-2+\sum_{i=1}^{s} l(H_i:G)(1-1/\#H_i)]\,.$$

DEFINITION 5 (cf. [9]). Assume that $G\subset GL(g, \boldsymbol{C})$ satisfies the RH-condition and let $[g_0, \#G; m_1,\cdots, m_r]$ be the RH-data of G. Then we can construct a Fuchsian group $\Gamma(G)$:

$$\Gamma(G)=\left\langle \alpha_1, \beta_1,\cdots,\alpha_{g_0},\beta_{g_0},\gamma_1,\cdots,\gamma_r;\ \prod_{j=1}^{r}\gamma_j\prod_{i=1}^{g_0}[\alpha_i,\beta_i]=1,\ \gamma_1^{m_1}=\cdots=\gamma_r^{m_r}=1\right\rangle\,.$$

We say that G satisfies the EX-condition if there exists a surjective homomorphism $\varphi:\Gamma(G)\to G$ with $\#\varphi(\gamma_j)=m_j$ $(j=1,\dots,r)$.

We remark here that $2g_0-2+\sum_{j=1}^{r}(1-1/m_j)>0$. See also [5], [11].

REMARK 2 (cf. [9], [10]). If $G\subset GL(g, \boldsymbol{C})$ satisfies the EX-condition, there exist a compact Riemann surface X of genus g and an injective homomorphism : $G\to\mathrm{Aut}(X)$. Further, we have

$$\mathrm{Tr}\,\rho(\sigma; X)=1+\sum_{(u,m)=1}\sum_{m\mid m_j}(1/m_j)\#\{\alpha\in G\,|\,\sigma=\alpha\varphi(\gamma_j)^{um_j/m}\alpha^{-1}\}\zeta_m^u/(1-\zeta_m^u)$$

for σ $(\#\sigma=m\ge 2)$. Here m_j is the order of γ_j and $\zeta_m=\exp(2\pi i/m)$. If there exists a surjective homomorphism $\varphi:\Gamma(G)\to G$ such that

$$\mathrm{Tr}\,\sigma=\mathrm{Tr}\,\rho(\sigma; X)\qquad\text{for every}\quad \sigma\in G\,,$$

then we see that G comes from the compact Riemann surface X.

We shall give a necessary and sufficient condition for a finite cyclic group $H \subset GL(g, \boldsymbol{C})$ to come from a compact Riemann surface of genus g. Under the notation of [10], we have:

RESULT 1 (cf. [10]). Let χ_1 be a character of a representation of H with $\chi_1(1) \geq 2$. If there exists a normal rotation datum λ of H such that $\chi_1 = 1 + \chi(\lambda)_1$, then there exist a compact Riemann surface X and an injective homomorphism $H \to \mathrm{Aut}(X)$ such that for every $\sigma \in H$

$$\chi_1(\sigma) = \mathrm{Tr}\, \rho(\sigma; X)\,.$$

We also have:

RESULT 2 (cf. [7]). Let A be an element of prime order n of $GL(g, \boldsymbol{C})$. The following two conditions are equivalent:

(1) There is a compact Riemann surface X of genus g and an automorphism σ of X such that $\rho(\sigma; X)$ is conjugate to A.

(2) There are s (≥ 0) integers $\nu_1, \cdots, \nu_s$ which are prime to n such that

$$\mathrm{Tr}\, A = 1 + \sum_{i=1}^{s} \zeta_n^{\nu_i}/(1 - \zeta_n^{\nu_i}) \qquad \text{(Trace formula of Eichler)}\,.$$

2. The fundamental properties of $GL(3, \boldsymbol{F}_2)$. We consider the group $GL(3, \boldsymbol{F}_2)$, which is known to be the unique simple group of order 168. The following facts are well known.

FACT 2.1 (cf. [13]). $GL(3, \boldsymbol{F}_2)$ has no subgroups of order 28, 42, 56 or 84.

FACT 2.2 (cf. [13]). Maximal subgroups of $GL(3, \boldsymbol{F}_2)$ are of orders 21 and 24.

Throughout this section, we denote by a, b and c elements of order 2, 3 and 4, respectively, and we denote by d_1 and d_2 elements of order 7 whose traces are 0 and 1, respectively.

FACT 2.3 (cf. [1]). $GL(3, \boldsymbol{F}_2)$ has the following generators and relations:

(1) $a^2 = b^3 = d^7 = abd = (dba)^4 = e$.

For example,

$$a = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad b = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad d = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

(2) $b_1^3 = b_2^3 = (b_1 b_2)^4 = (b_1^{-1} b_2)^4 = e$.

For example,

$$b_1=\begin{pmatrix}1&0&0\\0&0&1\\0&1&1\end{pmatrix},\qquad b_2=\begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix}.$$

FACT 2.4 (cf. [12]). $GL(3, \boldsymbol{F}_2)$ has the following character table:

$\lvert C\rvert$	168	8	3	4	7	7
	e	a	b	c	d_1	d_2
χ_1	1	1	1	1	1	1
χ_2	3	-1	0	1	α	$\bar{\alpha}$
χ_3	3	-1	0	1	$\bar{\alpha}$	α
χ_4	6	2	0	0	-1	-1
χ_5	7	-1	1	-1	0	0
χ_6	8	0	-1	0	1	1

Here the second row shows the representatives of conjugacy classes of $GL(3, \boldsymbol{F}_2)$. We know that if d belongs to the class d_1, then d^3, d^5 and d^6 belong to the class d_2. Further,

$$\bar{\alpha}=(-1-\sqrt{-7})/2,\quad \alpha=(-1+\sqrt{-7})/2 \quad\text{and}\quad \alpha=\zeta_7+\zeta_7^2+\zeta_7^4\,.$$

We shall show some simple properties of $GL(3, \boldsymbol{F}_2)$.

PROPOSITION 2.1. $GL(3, \boldsymbol{F}_2)=\{[x, y]\mid x, y\in GL(3, \boldsymbol{F}_2)\}$.

PROOF. It is sufficient to consider representatives of conjugacy classes of $GL(3, \boldsymbol{F}_2)$ since we have the relation:

$$g^{-1}[x, y]g=[g^{-1}xg, g^{-1}yg]$$

for $g\in GL(3, \boldsymbol{F}_2)$.

(1) order 2. Put

$$a=\begin{pmatrix}1&0&1\\0&1&0\\0&0&1\end{pmatrix},\qquad b=\begin{pmatrix}1&0&0\\0&0&1\\0&1&1\end{pmatrix}.$$

Then we have

$$[a, b]=a^{-1}b^{-1}ab=\begin{pmatrix}1&1&0\\0&1&0\\0&0&1\end{pmatrix}.$$

Hence $[a, b]$ is of order 2.

(2) order 3. Put

$$c=\begin{pmatrix}1&0&1\\0&0&1\\0&1&0\end{pmatrix},\qquad d_2=\begin{pmatrix}1&0&1\\1&1&1\\0&1&1\end{pmatrix}.$$

Then we have

$$[c, d_2]=c^{-1}d_2^{-1}cd_2=\begin{pmatrix}0&0&1\\0&1&0\\1&0&1\end{pmatrix}.$$

Hence $[c, d_2]$ is of order 3.

(3) order 4. Put

$$d_1=\begin{pmatrix}0&0&1\\1&0&1\\0&1&0\end{pmatrix},\qquad a=\begin{pmatrix}1&0&1\\0&1&0\\0&0&1\end{pmatrix}.$$

Then we have

$$[d_1, a]=d_1^{-1}a^{-1}d_1a=\begin{pmatrix}1&1&1\\0&1&0\\0&1&1\end{pmatrix}.$$

Hence $[d_1, a]$ is of order 4.

(4) order 7 of trace 0. Put

$$c_1=\begin{pmatrix}1&1&0\\0&0&1\\0&1&0\end{pmatrix},\qquad c_2=\begin{pmatrix}1&0&0\\1&0&1\\0&1&0\end{pmatrix}.$$

Then we have

$$[c_1, c_2]=c_1^{-1}c_2^{-1}c_1c_2=\begin{pmatrix}0&1&0\\0&1&1\\1&0&1\end{pmatrix}.$$

Hence $[c_1, c_2]$ is of order 7 and $\mathrm{Tr}[c_1, c_2]=0$.

(5) order 7 of trace 1. Put

$$c_1=\begin{pmatrix}1&0&0\\1&0&1\\0&1&0\end{pmatrix},\qquad c_2=\begin{pmatrix}1&1&0\\0&0&1\\0&1&0\end{pmatrix}.$$

Then we have

$$[c_1, c_2]=c_1^{-1}c_2^{-1}c_1c_2=\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}.$$

Hence $[c_1, c_2]$ is of order 7 and $\mathrm{Tr}[c_1, c_2]=1$. q.e.d.

We get easily the following:

PROPOSITION 2.2.

(1) *Neither* $d_1 \cdot d_2$ *nor* $d_2 \cdot d_1$ *can be of order* 2.

(2) *Neither* $d_1 \cdot d_2$ *nor* $d_2 \cdot d_1$ *can be of order* 7, *except when* d_2 *is a power of* d_1.

(3) *Neither* $d_1 \cdot d_1'$ *nor* $d_2 \cdot d_2'$ *can be of order* 4. *Here*, d_i *and* d_i' $(i=1, 2)$ *belong to the same conjugacy class.*

(4) *If* $c \cdot c'$ *is of order* 2, *then* $c=c'$.

PROPOSITION 2.3. *There exist* x *and* y *in* $GL(3, \boldsymbol{F}_2)$ *such that*

$$GL(3, \boldsymbol{F}_2)=\langle x, y\rangle$$

with $[x, y]$ *having order* 3, 4 *or* 7. *However there do not exist* x, y *such that* $\langle x, y\rangle=GL(3, \boldsymbol{F}_2)$ *with* $[x, y]$ *having order* 2.

PROOF. In the cases of order 3, order 4 and order 7, we see that the x and y in Proposition 2.1 are generators of $GL(3, \boldsymbol{F}_2)$ by considering Fact 2.1 and Fact 2.2. We shall prove the case $\sharp[x, y]=2$, where we denote by $\sharp$ the order of an element.

1. $[d, x]=d^{-1}x^{-1}dx$. Since $x^{-1}dx$ and d^{-1} belong to different conjugacy classes, $[d, x]$ cannot be of order 2 by Proposition 2.2, (1).

2. Assume that $\sharp[c, a]=2$. By Proposition 2.2, (4), we see that $a^{-1}ca=c^{-1}$. Put $S=c$ and $T=a$. Then $S^4=T^2=(ST)^2=E$, which define the group $\mathscr{D}_8$ of order 8 (cf. [2]).

3. Assume that $\sharp[c, b]=2$. By Proposition 2.2, (4), we see that $b^{-1}cb=c^{-1}$, and $(cb)^2=b^2$. Hence $\sharp(cb)=3$ or $\sharp(cb)=6$. This is absurd since $GL(3, \boldsymbol{F}_2)$ has no element of order 6 and $c\neq e$.

4. Assume that $\sharp[c, c']=2$. By Proposition 2.2, (4), we see that $c'^{-1}cc'=c^{-1}$. Put $S=c$, $T=c'$. Then $S^4=T^4=E$, $T^{-1}ST=S^{-1}$, which define the group $\langle 2, 2|4; 2\rangle$ of order 16 (cf. [2]).

5. Assume that $\sharp[b_1, b_2]=2$. Assume that $\sharp b_1b_2=2$ or $\sharp b_1^{-1}b_2=2$. Put $S=b_1$, $T=b_2$. Then $S^3=T^3=(ST)^2=E$, which define the group $\mathfrak{U}_4$. Assume that $\sharp b_1b_2=\sharp b_1^{-1}b_2=3$. Put $S=b_1$, $T=b_2$. Then $S^3=T^3=(ST)^3=(S^{-1}T)^3=E$, which define the group $(3, 3|3, 3)$ of order 27. Assume that $\sharp b_1b_2=4$. We have $\sharp b_2b_1=\sharp(b_2b_1)^{-1}=4$ since $b_1b_2\sim b_2b_1$. From $\sharp[b_1, b_2]=2$, we see by Proposition 2.2, (4) that $(b_2b_1)^{-1}=b_1b_2$. Hence $b_1=b_2^2$ and so $\langle b_1, b_2\rangle$ cannot be $GL(3, \boldsymbol{F}_2)$. Assume that $\sharp b_1^{-1}b_2=4$. Then we see that $\sharp b_1b_2^{-1}=4$. From $\sharp[b_1, b_2]=2$, we see that $\sharp[b_1^{-1}, b_2]=2$. Hence by Proposition 2.2, (4) we have $b_1=b_2$. This is absurd. Assume that $\sharp b_1b_2=7$. Similarly as above, we see that $b_2b_1\sim b_1b_2$ and in this case $(b_2b_1)^{-1}$ is not conjugate to b_1b_2. So $[b_1, b_2]$ cannot

be of order 2 by Proposition 2.2, (1). In the case $\#b_1^{-1}b_2=7$, we have the same result. Thus, all cases have been checked.

6. Assume that $\#[b,a]=2$. If $\#ba=3$, put $S=(ba)^{-1}$, $T=b$. Then, $S^3=T^3=(ST)^2=E$, which define the group $\mathfrak{U}_4$. If $\#ba=2$, put $S=b$, $T=a$. Then, $S^3=T^2=(ST)^2=E$, which define the group $\mathfrak{S}_3$. If $\#ba=4$, put $S=ba$, $T=a$. Then, $S^4=T^2=(ST)^3=E$, which define the group $\mathfrak{S}_4$. If $\#ba=7$, we see that $b^{-1}a^{-1}\sim(ba)^{-1}$. On the other hand $(ba)^{-1}$ is not conjugate to ba. Hence $[b,a]$ cannot be of order 2 by Proposition 2.2, (1).

7. Assume that $\#[a_1,a_2]=2$. Then $(a_1a_2)^4=e$. Put $S=a_1a_2$, $T=a_1$. Then, $S^4=T^2=(ST)^2=E$, which define the group $\mathscr{D}_8$. q.e.d.

3. Automorphism groups of Riemann surfaces. Let G be a finite subgroup of $GL(g, \boldsymbol{C})$ with a fixed isomorphism $\iota: GL(3, \boldsymbol{F}_2)\to G$. We denote the images of a, b, c, d_1 and d_2 under ι by A, B, C, D_1 and D_2, respectively. Throughout this section we keep this notation.

PROPOSITION 3.1. *Let χ_G be the character of the natural representation $G\to GL(g, \boldsymbol{C})$ and let $\chi_G=n_1\chi_1+\dots+n_6\chi_6$ ($n_i\in\boldsymbol{Z}_{\geq 0}$, $i=1,\dots,6$) be the decomposition into irreducible characters of χ_G. Then G satisfies the CY- and RH-conditions if and only if n_i's satisfy the following five relations*:

(1) $n_2+n_3-n_4\geq 0$,

(2) $1-(n_1+n_5-n_6)\geq 0$,

(3) $1-(n_1+n_2+n_3-n_5)\geq 0$,

(4) $p=(1-n_1+2n_2-n_3+n_4-n_6)/3$ is a non-negative integer,

(5) $q=(1-n_1-n_2+2n_3+n_4-n_6)/3$ is a non-negative integer.

PROOF. The "only if" part is trivial. We shall prove the "if" part. First, we shall show that G satisfies the RH-condition. To see this, it is sufficient to consider $\langle A\rangle$, $\langle B\rangle$, $\langle C\rangle$ and $\langle D\rangle$. Here $\langle D\rangle$ is a cyclic subgroup of order 7. We see that

$$\begin{aligned} r_*(\langle C^2\rangle : G) &= r(\langle C^2\rangle)-r_*(\langle C\rangle : G)\\ &=\{2-2(n_1-n_2-n_3+2n_4-n_5)\}-\{2-2(n_1+n_2+n_3-n_5)\}\\ &=4(n_2+n_3-n_4)\,. \end{aligned}$$

Hence by considering $[N_G(\langle A\rangle):\langle A\rangle]=4$, we see that

$$l(\langle A\rangle : G)=n_2+n_3-n_4\,.$$

Similarly,

$$l(\langle B\rangle : G)=1-(n_1+n_5-n_6)\,,$$

$$l(\langle C\rangle : G)=1-(n_1+n_2+n_3-n_5)\,.$$

For $l(\langle D\rangle : G)$ we have

$$r_*(\langle D\rangle:G)=2-(\chi_G(D_1)+\chi_G(D_1^{-1}))$$
$$=2-2\left(n_1-\frac{n_2}{2}-\frac{n_3}{2}-n_4+n_6\right)$$

and $[N_G(\langle D\rangle):\langle D\rangle]=3$. Hence we see that

$$l(\langle D\rangle:G)=\frac{1}{3}(2-2n_1+n_2+n_3+2n_4-2n_6)=p+q\,.$$

By assumption, all l's are non-negative integers. Thus G satisfies the RH-condition. We denote $l(\langle A\rangle:G)$ by $l(A)$ and so on.

Second, we shall show that G satisfies the CY-condition. To see this it is sufficient to consider $\langle B\rangle$, $\langle C\rangle$ and $\langle D\rangle$.

As for $\langle B\rangle$, we see that $\chi_G(B)=n_1+n_5-n_6$. Put $s=2-2\chi_G(B)$. Put $v_1=\cdots=v_{s/2}=1$, $v_{s/2+1}=\cdots=v_s=2$ in the trace formula. Then we have

$$\mathrm{Tr}(B)=1+\{1-(n_1+n_5-n_6)\}\cdot\left(\frac{\zeta_3}{1-\zeta_3}+\frac{\zeta_3^2}{1-\zeta_3^2}\right).$$

Hence $\langle B\rangle$ comes from a compact Riemann surface of genus g by Result 2.

As for $\langle C\rangle$, since its order is not prime, we use Result 1. (See [9] for notation and terminology.) Result 1 works well for the case $\langle B\rangle$ proved above. Define a rotation datum λ_C of $\langle C\rangle$ as follows:

$$\begin{cases} C,\ C^3\to(1-(n_1+n_2+n_3-n_5))(\lambda_{41}+\lambda_{43}) \\ C^2\to(2-2(n_1-n_2-n_3+2n_4-n_5))\lambda_{21} \\ E\to(2-2g)\lambda_{11}\,. \end{cases}$$

Then Red λ_C is given by

$$\begin{cases} C,\ C^3\to\lambda_C(C) \\ C^2\to4(n_2+n_3-n_4)\lambda_{21}\,, \end{cases}$$

and

$$l(C,\lambda_C)=2-2(n_1+n_2+n_3-n_5)\,,\qquad l(C^2,\lambda_C)=2(n_2+n_3-n_4)\,.$$

We see that λ_C is a normal rotation datum, since we have

$$\frac{\mathrm{Red}\,\lambda_C(C)}{[\langle C\rangle:\langle C\rangle]}=\frac{l(C,\lambda_C)}{2}(\lambda_{41}+\lambda_{43})$$

and

$$\frac{\operatorname{Red}\lambda_C(C^2)}{[\langle C\rangle:\langle C^2\rangle]}=\frac{4(n_2+n_3-n_4)}{2}\lambda_{21}=l(C^2,\lambda_C)\lambda_{21}\,.$$

Hence we see that

$$\chi_G|_{\langle C\rangle}=1+\chi(\lambda_C)_1 \quad\text{and}\quad \chi_G|_{\langle A\rangle}=1+\chi(\lambda_A)_1\,.$$

Thus $\langle C\rangle$ and $\langle A\rangle$ come from a compact Riemann surface of genus g by Result 1.

As for $\langle D\rangle$ $(=\langle D_1\rangle)$, if we put

$$\begin{aligned}&\nu_1=\cdots=\nu_p=1\,,\quad \nu_{p+1}=\cdots=\nu_{2p}=2\,,\quad \nu_{2p+1}=\cdots=\nu_{3p}=4\,,\\ &\nu_{3p+1}=\cdots=\nu_{3p+q}=3\,,\quad \nu_{3p+q+1}=\cdots=\nu_{3p+2q}=5\,,\\ &\nu_{3p+2q+1}=\cdots=\nu_{3p+3q}=6\,,\end{aligned}$$

then we have, with $\zeta=\zeta_7$,

$$\operatorname{Tr}D_1=1+p\left(\frac{\zeta}{1-\zeta}+\frac{\zeta^2}{1-\zeta^2}+\frac{\zeta^4}{1-\zeta^4}\right)+q\left(\frac{\zeta^3}{1-\zeta^3}+\frac{\zeta^5}{1-\zeta^5}+\frac{\zeta^6}{1-\zeta^6}\right).$$

Hence $\langle D\rangle$ comes from a compact Riemann surface of genus g by Result 2.

Thus we see that G satisfies the CY-condition. q.e.d.

The conditions in Proposition 3.1 are rewritten as follows:

$$(*)\begin{cases}n_2=l(A)+l(B)+l(C)+2p+q-3+3n_1\,,\\ n_3=l(A)+l(B)+l(C)+p+2q-3+3n_1\,,\\ n_4=l(A)+2l(B)+2l(C)+3p+3q-6+6n_1\,,\\ n_5=2l(A)+2l(B)+3l(C)+3p+3q-7+7n_1\,,\\ n_6=2l(A)+3l(B)+3l(C)+3p+3q-8+8n_1\,.\end{cases}$$

Here we note that

(i) $g_0(G)=n_1$,

(ii) $g=n_1+3n_2+3n_3+6n_4+7n_5+8n_6\geq 2$.

Now we recall the Fuchsian group $\Gamma(G)$, considered in §1. We rewrite $\Gamma(G)$ as follows:

$$\Gamma(G)=\langle\alpha_1,\beta_1,\ldots,\alpha_{n_1},\beta_{n_1},\gamma_1,\ldots,\gamma_{l(A)},\delta_1,\ldots,\delta_{l(B)},\varepsilon_1,\ldots,\varepsilon_{l(C)},\eta_1,\ldots,\eta_p,\xi_1,\ldots,\xi_q;$$
$$\prod\gamma_j\prod\delta_k\prod\varepsilon_l\prod\eta_m\prod\xi_n\prod[\alpha_i,\beta_i]=1,\ \gamma_j^2=\delta_k^3=\varepsilon_l^4=\eta_m^7=\xi_n^7=1\rangle\,.$$

We wish to find out which groups come from Riemann surfaces by means of the 5-tuple $(l(A), l(B), l(C), p, q)$. For this purpose we use the following lemma:

REDUCTION LEMMA. *For an arbitrary element Ξ of $GL(3,\boldsymbol{F}_2)$ and any positive integer N, there exist $\Xi_1,\ldots,\Xi_N$ which are $GL(3,\boldsymbol{F}_2)$-conjugate to Ξ such that $\Xi=\Xi_1\cdots\Xi_N$.*

PROOF. If $\#\Xi=1$, there is nothing to prove. Assume that $\#\Xi=2$. We have

$$\Xi=\begin{pmatrix}1&0&1\\0&1&0\\0&0&1\end{pmatrix}=\begin{pmatrix}1&1&1\\0&1&0\\0&0&1\end{pmatrix}\cdot\begin{pmatrix}1&1&0\\0&1&0\\0&0&1\end{pmatrix}.$$

If $N\equiv 0 \pmod 2$, put Ξ_1, Ξ_2 to be the first, the second matrices on the right hand side and put $\Xi_3=\cdots=\Xi_N=\Xi_2$. Then we have

$$\Xi=\Xi_1\Xi_2\Xi_3\cdots\Xi_N .$$

If $N\equiv 1 \pmod 2$, it is sufficient to set $\Xi=\Xi_1=\cdots=\Xi_N$. In the cases $\#\Xi=3$, 4 and 7, the proofs are similar. q.e.d

Let $P=(r_1,\dots,r_5)$ and $Q=(s_1,\dots,s_5)$. $P\searrow Q$ means that $r_i\geq s_i$ $(i=1,\dots,5)$ and that if $r_j>s_j$ for some j, then $s_j>0$. We say Q to be minimal if Q is one of the last ones through operations of the reduction.

REMARK. By the Reduction Lemma, we see that, for example,

(1) $(3,2,0,0,0)\searrow(2,2,0,0,0)$,
(2) $(2,3,0,0,0)\searrow(2,2,0,0,0)$,
(3) $(1,0,4,0,0)\searrow(1,0,3,0,0)$,
(4) $(3,0,0,0,2)\searrow(3,0,0,0,1)$.

Indeed, we can find Ξ'_1, Ξ'_2, Ξ'_3 so that $\Xi'_1=\Xi_1$, $\Xi'_2\Xi'_3=\Xi_2$, in (1). The proofs of (2), (3) and (4) are similar to (1).

PROPOSITION 3.2. *Assume that $n_1\geq 2$. If G satisfies the CY- and RH-conditions, then G comes from a Riemann surface.*

PROOF. We shall construct a sujective homomorphism $\varphi\colon\Gamma(G)\to G$ as follows:

$$\alpha_1\to X\,,\quad \beta_1\to Y\,,\quad \alpha_2\to U\,,\quad \beta_2\to V\,,\quad \alpha_i,\beta_i\to E\ (i=3,\dots,n_1)\,,$$
$$\gamma_j\to A\ (j=1,\dots,l(A))\,,\quad \delta_k\to B\ (j=1,\dots,l(B))\,,\quad \varepsilon_l\to C\ (j=1,\dots,l(C))\,,$$
$$\eta_m\to D_1\ (m=1,\dots,p)\,,\quad \xi_n\to D_2\ (n=1,\dots,q)\,.$$

Here X, Y are elements of $GL(3,\boldsymbol{F}_2)$ so that $GL(3,\boldsymbol{F}_2)=\langle X, Y\rangle$ and U, V are elements of $GL(3,\boldsymbol{F}_2)$ so that $[U,V]=[X,Y]^{-1}\{\prod\gamma_j\prod\delta_k\prod\varepsilon_l\prod\eta_m\prod\xi_n\}^{-1}$. These are possible by Fact 2.3 and by Proposition 2.1.

Now, we shall prove the $GL(g,\boldsymbol{C})$-conjugacy of the group. It is sufficient to check for A, B, C and D_1, D_2.

$$\begin{aligned}\operatorname{Tr}\rho(A;X)=1&+\sum_j^{l(A)}\frac{1}{2}\#\{\alpha\in G\,|\,A=\alpha\varphi(\gamma_j)\alpha^{-1}\}\frac{-1}{1-(-1)}\\&+\sum_j^{l(C)}\frac{1}{4}\#\{\alpha\in G\,|\,A=\alpha\varphi(\gamma_j)^2\alpha^{-1}\}\frac{-1}{1-(-1)}\end{aligned}$$

$$=1-2l(A)-l(C)=n_1-n_2-n_3+2n_4-n_5=\chi_G(A)\,.$$

Indeed, we see that both $\#\{\cdots\}$ are 8 from the character table in §2. Hence $\mathrm{Tr}(A)=\mathrm{Tr}\,\rho(A;\,X)$.

$$\mathrm{Tr}\,\rho(B;X)=1+\sum_{j}^{l(B)}\frac{1}{3}\#\{\alpha\in G\,|\,B=\alpha\varphi(\gamma_j)\alpha^{-1}\}\frac{\zeta_3}{1-\zeta_3}$$
$$+\sum_{j}^{l(B)}\frac{1}{3}\#\{\alpha\in G\,|\,B=\alpha\varphi(\gamma_j)^2\alpha^{-1}\}\frac{\zeta_3^2}{1-\zeta_3^2}$$
$$=1-l(B)=n_1+n_5-n_6=\chi_G(B)\,.$$

We see that both $\#\{\cdots\}$ are 3 from the character table in §2. Hence $\mathrm{Tr}(B)=\mathrm{Tr}\,\rho(B;\,X)$.

For the $\mathrm{Tr}\,\rho(C;\,X)$, $\mathrm{Tr}\,\rho(D_1;\,X)$ and $\mathrm{Tr}\,\rho(D_2;\,X)$, the proofs are similar. So we can omit them. q.e.d.

PROPOSITION 3.3. *Assume that $n_1=1$. If G satisfies the CY- and RH-conditions, then G comes from a Riemann surface except in the case of the* 5-*tuple* (1, 0, 0, 0, 0).

PROOF. It is sufficient to consider the following 5-tuples.

	$l(A)$	$l(B)$	$l(C)$	p	q
(1)	1	0	0	0	0
(2)	0	1	0	0	0
(3)	0	0	1	0	0
(4)	0	0	0	1	0
(5)	0	0	0	0	1
(6)	1	0	1	0	0
(7)	2	0	0	0	0
(8)	0	0	0	1	1
(9)	0	0	0	2	0
(10)	0	0	0	0	2

Except in (1), we can define homomorphisms φ easily. For the $GL(g,\,\boldsymbol{C})$-conjugacy of the group, the proof is similar to Proposition 3.2. q.e.d.

REMARK. We have a Riemann surface for (2, 0, 0, 0, 0).

PROPOSITION 3.4. *Assume that $n_1=0$. Considering* $(*)$ *we get a table of minimal* 5 *tuples*:

		$l(A)$	$l(B)$	$l(C)$	p	q
[I]	(1)	1	1	1	1	1
[II]	(1)	1	1	1	1	0

	(2)	1	1	1	0	1
	(3)	1	1	0	1	1
	(4)	1	0	1	1	1
	(5)	0	1	1	1	1
[III]	(1-a)	2	1	1	0	0
	(1-b)	1	2	1	0	0
	(1-c)	1	1	2	0	0
	(2)	1	1	0	1	0
	(3)	1	1	0	0	1
	(4)	1	0	1	1	0
	(5)	1	0	1	0	1
	(6)	1	0	0	1	1
	(7)	0	1	1	1	0
	(8)	0	1	1	0	1
	(9)	0	1	0	1	1
	(10)	0	0	1	1	1
[IV]	(1-a)	4	1	0	0	0
	(1-b)	2	2	0	0	0
	(1-c)	1	3	0	0	0
	(2-a)	4	0	1	0	0
	(2-b)	2	0	2	0	0
	(2-c)	1	0	3	0	0
	(3-a)	3	0	0	1	0
	(3-b)	3	0	0	0	1
	(4-a)	1	0	0	2	0
	(4-b)	1	0	0	0	2
	(5-a)	0	2	1	0	0
	(5-b)	0	1	2	0	0
	(6-a)	0	2	0	1	0
	(6-b)	0	2	0	0	1
	(7-a)	0	1	0	2	0
	(7-b)	0	1	0	0	2
	(8-a)	0	0	2	1	0
	(8-b)	0	0	2	0	1
	(9-a)	0	0	1	2	0
	(9-b)	0	0	1	0	2
	(10-a)	0	0	0	2	1
	(10-b)	0	0	0	1	2
[V]	(1)	6	0	0	0	0
	(2)	0	4	0	0	0
	(3)	0	0	3	0	0

(4)	0	0	0	3	0
(5)	0	0	0	0	3

In the above table, some of them are easily induced from the other. Let us denote $P \mapsto\!\!\!\!\rightarrow Q$ when Q is induced from P. That is;

[I](1) $\mapsto\!\!\!\!\rightarrow$ [II](1) by considering $D_1 \cdot D_1^3 = D_1^4$.

[II](2) $\mapsto\!\!\!\!\rightarrow$ [II](1) by considering the inverse.

[II](3), (4), (5) $\mapsto\!\!\!\!\rightarrow$ [III](3), (4), (7) respectively, by $D_1 \cdot D_1^3 = D_1^4$.

[III](3), (5), (8) $\mapsto\!\!\!\!\rightarrow$ [III](2), (4), (7) respectively, by considering the inverse of each member.

[IV](3-b), (4-b), (6-b), (7-b), (8-b), (9-b), (10-b) $\mapsto\!\!\!\!\rightarrow$ [IV](3-a), (4-a), (6-a), (7-a), (8-a), (9-a), (10-a) respectively, by considering the inverse of each member.

[V](5) $\mapsto\!\!\!\!\rightarrow$ [V](4), by considering the inverse.

We see that [III](6), [IV](9-a) and (10-a) cannot come from Riemann surfaces by Proposition 2.2, (1), (3) and (2). We see also that [IV](9-b) and [IV](10-b) cannot come from Riemann surfaces. Indeed, we see that [IV](9-b) $\mapsto\!\!\!\!\rightarrow$ [IV](9-a) and [IV](10-b) $\mapsto\!\!\!\!\rightarrow$ [IV](10-a). However, the following five cases come from Riemann surfaces as we can see in the proof of the next Theorem:

(1) (2, 0, 0, 1, 1), (1, 0, 0, 2, 1).

(2) (0, 0, 2, 2, 0), (0, 0, 1, 3, 0).

(3) (0, 0, 0, 2, 2), (0, 0, 0, 3, 1).

Thus we have determined all cases which do not come from Riemann surfaces. We shall express this fact in terms of $(n_1, \ldots, n_6)$ in $(*)$, because it is more directly related to the group G.

THEOREM. *Assume that G satisfies the CY- and RH-conditions. If G does not come from a Riemann surface of genus g then* $(n_1, \ldots, n_6)$ *is equal to one of the following*:

(1) (1, 1, 1, 1, 2, 2) $(g=43)$,

(2) (0, 1, 1, 1, 1, 0) $(g=19)$,

(3) (0, 2, 0, 2, 2, 1) $(g=40)$,

(4) (0, 0, 2, 2, 2, 1) $(g=40)$,

(5) (0, 2, 1, 3, 2, 1) $(g=49)$,

(6) (0, 1, 2, 3, 2, 1) $(g=49)$.

PROOF. By Propositions 3.2, 3.3 and 3.4 it is sufficient to prove that the reduced members in Proposition 3.4 come from Riemann surfaces except in the five cases (2)–(6). Further, the surjectivity is easy to show and so we shall show only the relation. Here, for example, in [II](1) we mean $A \cdot B \cdot C \cdot D = E$ and $\langle A, B, C, D\rangle = GL(3, \boldsymbol{F}_2)$.

[II] (1) $\quad A=\begin{pmatrix}1&1&1\\0&1&0\\0&0&1\end{pmatrix}, \quad B=\begin{pmatrix}0&1&1\\0&1&0\\1&1&1\end{pmatrix}, \quad C=\begin{pmatrix}0&1&1\\1&0&0\\0&0&1\end{pmatrix},$

$$D=\begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}.$$

[III](1-a) $$A=\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad A'=\begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix}, \quad B=\begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix},$$

$$C=\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

(1-b) $$A=\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad B=B'=\begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \quad C=\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix}.$$

(1-c) $$A=\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad B=\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad C=\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix},$$

$$C'=\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

(2) $$A=\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad B=\begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \quad D=\begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}.$$

(4) $$A=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \quad C=\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \quad D=\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

(7) $$B=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \quad C=\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \quad D=\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

(9) $$B=\begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad D=\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad D'=\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

(10) $$C=\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \quad D=\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad D'=\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}.$$

[IV](1-a) $A_1=\begin{pmatrix}1&1&1\\0&1&0\\0&0&1\end{pmatrix}$, $A_2=\begin{pmatrix}1&0&1\\0&1&1\\0&0&1\end{pmatrix}$, $A_3=\begin{pmatrix}0&1&1\\0&1&0\\1&1&0\end{pmatrix}$,

$A_4=\begin{pmatrix}1&1&1\\0&0&1\\0&1&0\end{pmatrix}$, $B=\begin{pmatrix}1&0&0\\1&0&1\\0&1&1\end{pmatrix}$.

(1-b) $A_1=A_2=\begin{pmatrix}1&0&1\\0&1&0\\0&0&1\end{pmatrix}$, $B_1=\begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix}$, $B_2=\begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}$.

(1-c) $A_1=\begin{pmatrix}1&1&1\\0&0&1\\0&1&0\end{pmatrix}$, $B_1=\begin{pmatrix}1&0&0\\1&0&1\\0&1&1\end{pmatrix}$, $B_2=\begin{pmatrix}1&0&0\\0&0&1\\0&1&1\end{pmatrix}$,

$B_3=\begin{pmatrix}1&1&1\\0&1&0\\1&0&0\end{pmatrix}$.

(2-a) $A_1=\begin{pmatrix}1&1&1\\0&1&0\\0&0&1\end{pmatrix}$, $A_2=\begin{pmatrix}1&0&1\\0&1&1\\0&0&1\end{pmatrix}$, $A_3=A_4=\begin{pmatrix}0&1&1\\0&1&0\\1&1&0\end{pmatrix}$,

$C=\begin{pmatrix}1&1&0\\0&1&1\\0&0&1\end{pmatrix}$.

(2-b) $A_1=A_2=\begin{pmatrix}0&1&1\\0&1&0\\1&1&0\end{pmatrix}$, $C=\begin{pmatrix}1&1&0\\0&1&1\\0&0&1\end{pmatrix}$, $C'=\begin{pmatrix}1&1&1\\0&1&1\\0&0&1\end{pmatrix}$.

(2-c) $A_1=\begin{pmatrix}1&1&1\\0&1&0\\0&0&1\end{pmatrix}$, $C_1=\begin{pmatrix}1&1&0\\0&0&1\\0&1&0\end{pmatrix}$, $C_2=\begin{pmatrix}0&1&0\\1&0&1\\0&0&1\end{pmatrix}$,

$C_3=\begin{pmatrix}0&1&1\\1&1&0\\0&1&0\end{pmatrix}$.

(3-a) Put A_1, A_2, A_3 as in (1-a) and put $D_1=\begin{pmatrix}0&1&0\\0&1&1\\1&0&1\end{pmatrix}$.

(4-a) $A=\begin{pmatrix}1&0&1\\0&1&1\\0&0&1\end{pmatrix},\quad D_1=\begin{pmatrix}0&1&0\\1&0&1\\1&0&0\end{pmatrix},\quad D_1'=\begin{pmatrix}0&0&1\\1&0&1\\0&1&0\end{pmatrix}.$

(5-a) $B_1=\begin{pmatrix}1&0&0\\0&0&1\\0&1&1\end{pmatrix},\quad B_2=\begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix},\quad C=\begin{pmatrix}0&1&1\\0&1&0\\1&0&0\end{pmatrix}.$

(5-b) In (5-a), put B_1, $C_1=B_1B_2$ and $C_2=B_2^{-1}B_1$.

(6-a) $B_1=\begin{pmatrix}1&0&0\\0&0&1\\0&1&1\end{pmatrix},\quad B_2=\begin{pmatrix}0&0&1\\0&1&0\\1&0&1\end{pmatrix},\quad D_1=\begin{pmatrix}1&1&0\\0&1&1\\1&0&0\end{pmatrix}.$

(7-a) $B=\begin{pmatrix}0&1&0\\0&1&1\\1&1&1\end{pmatrix},\quad D_1=\begin{pmatrix}0&1&0\\0&1&1\\1&0&1\end{pmatrix},\quad D_1'=\begin{pmatrix}0&0&1\\0&1&1\\1&1&1\end{pmatrix}.$

(8-a) $C_1=\begin{pmatrix}1&1&0\\1&0&1\\1&0&0\end{pmatrix},\quad C_2=\begin{pmatrix}1&1&0\\0&0&1\\0&1&0\end{pmatrix},\quad D_1=\begin{pmatrix}0&1&0\\0&1&1\\1&0&1\end{pmatrix}.$

[V] (1) $A_1=A_6=\begin{pmatrix}1&1&1\\0&1&0\\0&0&1\end{pmatrix},\quad A_2=A_5=\begin{pmatrix}1&0&1\\0&1&1\\0&0&1\end{pmatrix},\quad A_3=A_4=\begin{pmatrix}0&1&1\\0&1&0\\1&1&0\end{pmatrix}.$

(2) $B_1=\begin{pmatrix}1&0&0\\0&0&1\\0&1&1\end{pmatrix},\quad B_2=B_1^{-1},\quad B_3=\begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix},\quad B_4=B_3^{-1}.$

(3) $C_1=\begin{pmatrix}1&0&1\\0&0&1\\0&1&0\end{pmatrix},\quad C_2=\begin{pmatrix}0&1&0\\1&0&1\\0&0&1\end{pmatrix},\quad C_3=\begin{pmatrix}0&1&1\\1&1&0\\0&1&0\end{pmatrix}.$

(4) $D_1=\begin{pmatrix}0&0&1\\1&0&1\\0&1&0\end{pmatrix},\quad D_1'=\begin{pmatrix}1&0&1\\1&0&0\\0&1&1\end{pmatrix},\quad D_1''=\begin{pmatrix}0&0&1\\0&1&1\\1&1&1\end{pmatrix}.$

We see easily what we stated just before the Theorem.

(1) $(2,0,0,1,1)\mapstochar\twoheadrightarrow(0,0,1,1,1)$; $(1,0,0,2,1)\mapstochar\twoheadrightarrow(1,1,0,0,1)$.

(2) $(0,0,2,2,0)\mapstochar\twoheadrightarrow(0,1,0,2,0)$; $(0,0,1,3,0)\mapstochar\twoheadrightarrow(0,1,1,1,0)$.

(3) $(0,0,0,2,2)\mapstochar\twoheadrightarrow(0,1,0,2,0)$; $(0,0,0,3,1)\mapstochar\twoheadrightarrow(0,1,0,1,1)$.

Finally, for the $GL(g, \boldsymbol{C})$-conjugacy of the groups the proof is the same as that of Proposition 3.2. q.e.d.

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DEPARTMENT OF MATHEMATICS
CHUO UNIVERSITY
TOKYO 112
JAPAN

TOKYO INSTITUTE OF TECHNOLOGY
DEPARTMENT OF MATHEMATICS
TOKYO 152
JAPAN