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Automorphisms and Homotopies of Groupoids and Crossed Modules

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Abstract

This paper[1] is concerned with the algebraic structure of groupoids and crossed modules of groupoids. We describe the group structure of the automorphism group of a finite connected groupoid C as a quotient of a semidirect product. We pay particular attention to the conjugation automorphisms of C, and use these to define a new notion of groupoid action. We then show that the automorphism group of a crossed module of groupoids $\mathcal{C}$, in the case when the range groupoid is connected and the source group totally disconnected, may be determined from that of the crossed module of groups $\mathcal{C}_u$ formed by restricting to a single object u. Finally, we show that the group of homotopies of $\mathcal{C}$ may be determined once the group of regular derivations of $\mathcal{C}_u$ is known.

[1]This is a slightly revised version of the paper published in Applied Categorical Structures, Volume 18 (October 2010), pp.473–504. This final publication is available at Springer via http://dx.doi.org/10.1007/s10485-008-9183-y. Hyperref links have been added (this was suggested by Ronnie Brown); the URLs in the list of references have been updated; and Murat Alp's new address and email are now shown.

1 Introduction

While the theory of groupoids has been extensively developed and found many applications and generalisations in areas such as algebraic topology, noncommutative geometry, Lie groupoids and theoretical physics, it appears that less attention has been paid to the strictly algebraic structure. Our aim in this paper is to make some progress towards remedying this omission.

We begin by investigating the automorphism group $\mathrm{Aut}\,\mathsf{C}$ of a finite groupoid C. Most of the detail is required in the case when C is connected. Three types of elementary automorphism are determined: by a permutation of the objects; by an automorphism of the vertex group C at an object u; and by a transversal for the set of stars $\{\mathsf{C}(u,v) \mid v \neq u\}$. We show that $\mathrm{Aut}\,\mathsf{C}$ is isomorphic to a quotient of $(S_n \times \mathrm{Aut}\,C) \ltimes C^n$ by a subgroup isomorphic to C, where n is the number of objects.

The traditional view of an action of a connected C on a set Y (see [5, § 10.4]) involves a partition of Y into subsets $\{Y_1, \ldots, Y_n\}$ and a partial function $Y \times \mathrm{Arr}(\mathsf{C}) \to Y,\ (y,\alpha) \mapsto y^{\alpha}$, defined when $(\alpha : u \to v)$ and $y \in Y_u$, and such that $y^{\alpha} \in Y_v$. Thus α acts as an isomorphism from Y_u to Y_v. Similarly, when $N \trianglelefteq C$ and N is the totally disconnected subgroupoid of C with N as the group at every object, we get an action of C on N with $(u,n,u)^{(u,c,v)} = (v,n^c,v)$. Here $\alpha = (u,c,v)$ acts as a conjugation isomorphism from $\mathsf{N}(u)$ to $\mathsf{N}(v)$.

We prefer an alternative view in which actions are no longer *partial* functions. Now α also determines an isomorphism from Y_v to Y_u (or $\mathsf{N}(v)$ to $\mathsf{N}(u)$) and fixes the remaining Y_w (or $\mathsf{N}(w)$). Indeed, C acts on itself, with α determining the *conjugation automorphism* $\wedge\alpha$ (read "to the α") which conjugates $\mathsf{C}(u)$ to $\mathsf{C}(v)$ and vice-versa; swaps $\mathsf{C}(u,v)$ with $\mathsf{C}(v,u)$; swaps the stars (costars) at u with those at v; and fixes the remaining arrows. We show in § 4.5 that these conjugations satisfy a set of *conjugation identities*. For example, when $\alpha' = (v,c',w)$ and $\{u,v,w\}$ are distinct objects, $\wedge(\alpha\alpha') = (\wedge\alpha) * (\wedge\alpha') * (\wedge\alpha) = (\wedge\alpha') * (\wedge\alpha) * (\wedge\alpha')$. This approach leads to a non-standard definition of normal subgroupoids.

Given objects U, V, W in a cartesian closed category $\mathbb{C}$, there is a product object $W \times U$, an internal morphism object V^U, and a natural bijection $\theta : \mathbb{C}(W \times U, V) \cong \mathbb{C}(W, V^U)$. So, when $W = V^U$, this θ gives a bijection $\mathbb{C}(V^U, V^U) \to \mathbb{C}(V^U \times U, V)$ which maps the identity on V^U to the evaluation $\varepsilon_{UV} : V^U \times U \to V$. From the map $\alpha : Z^Y \times Y^X \times X \xrightarrow{1 \times \varepsilon_{XY}} Z^Y \times Y \xrightarrow{\varepsilon_{YZ}} Z$ we obtain, by taking $W = Z^Y \times Y^X$, $U = X$, $V = Z$, the product of internal morphisms $* = \theta(\alpha) : Z^Y \times Y^X \to Z^X$. Then $\mathrm{END}(X) := (X^X, *)$ is a monoid object in $\mathbb{C}$, and $\mathrm{AUT}(X)$, its maximal subgroup, is a group object in $\mathbb{C}$. The objects of $\mathrm{END}(X)$ and $\mathrm{AUT}(X)$ are the arrows $\mathbb{C}(X,X)$ and the invertible ones, respectively. In particular, there is an identity object. See [9, Appendix B] for further details of this standard construction. As an example, see [12] where this theory is applied to a cartesian closed category $\mathbb{C} = \mathsf{Dgph}$ of digraphs, and then to undirected graphs.

The category Gpd of groupoids is also cartesian closed, so for each groupoid C there is a monoid-groupoid $\mathrm{END}\,\mathsf{C} = \mathsf{C}^{\mathsf{C}}$ and a group-groupoid $\mathrm{AUT}\,\mathsf{C}$, the full subcategory of $\mathrm{END}\,\mathsf{C}$ having the automorphisms in the group $\mathrm{Aut}\,\mathsf{C}$ as objects. In § 4 we investigate the combinatorics of this *automorphism groupoid* $\mathrm{AUT}\,\mathsf{C}$ of C. Then, in § 4.5, we give a new definition of an action of C on a groupoid B as a function $\mathrm{Arr}(\mathsf{C}) \to \mathrm{AUT}\,\mathsf{B}$ which satisfies the conjugation identities.

The category of crossed modules of groups and their morphisms may be viewed in many equivalent ways. We summarise a few of these here, giving further details in later sections, as required. A *crossed module* $\mathcal{X} = (\delta : B \to C)$ is a group homomorphism δ with a right action of C on B satisfying $\delta(b^c) = c^{-1}(\delta b)c$ and $(b')^{\delta b} = b^{-1}b'b$. The corresponding *cat*^{1}*-group* $\mathcal{C} = (\partial_0; \partial_1^-, \partial_1^+ : C \ltimes B \to C)$ has source and target surjections $\partial_1^-, \partial_1^+ : C \ltimes B \to C$, where $\partial_1^-(b,c) = c,\ \partial_1^+(b,c) = c(\delta b)$, and embedding $\partial_0 : C \to C \ltimes B,\ c \mapsto (c,1)$, satisfying $\partial_1^-\partial_0 c = \partial_1^+\partial_0 c = c$ and $[\ker\partial_1^-, \ker\partial_1^+] = 1$. The associated *group-groupoid* (or *categorical group*) $\mathcal{G}$ is the groupoid with objects C; arrows $C \ltimes B$; source and target given by $\partial_1^-, \partial_1^+$; and partial composition $(c_1,b_1) * (c_1(\delta b_1), b_3) = (c_1, b_1 b_3)$. The additional group structure $\otimes$ on the arrows is provided by the semidirect product $(c_1,b_1)\otimes(c_2,b_2) = (c_1c_2, b_1^{c_2} b_2)$. The arrows in B form the star at the identity. Another equivalent structure, which we shall not use here, is the notion of *strict* 2*-group*. From the discussion above, we see that the group-groupoid $\mathrm{AUT}\,\mathsf{C}$ may be considered as a crossed module of groups. A more extensive example, which includes crossed modules as the 2-dimensional case, is provided by the category of crossed complexes, which is shown in [8] to be cartesian closed, and is the main topic of [9].

Baez and Lauda [4] provide a review of all these equivalent structures, together with weaker versions such as coherent 2-groups, and many applications. The study of automorphisms of groupoids may be viewed as the *categorification* of permutation groups, and so forms part of the more general categorification process discussed in the Baez-Corfield-Schreiber *n-Category Cafè* online blog. As examples of recent papers discussed there, see Noohi [21] for a description of the groupoid of weak maps between two crossed modules, using a theory of *papillons*; and Roberts and Schreiber [23] on principal 2-bundles for 2-groups, with applications to 2-dimensional quantum field theory.

In § 5 we recall the definitions of a crossed module of groups $\mathcal{X}$; the Whitehead group of regular derivations $W(\mathcal{X})$; and the actor crossed square $\mathcal{S}(\mathcal{X})$. We then define a crossed module of groupoids $\mathcal{C} = (\partial : \mathsf{B} \to \mathsf{C})$, using our new notion of action. We are particular interested in the case when $\mathcal{C} = \mathcal{C}(\mathcal{X}, n) = (\partial : B_\bullet \times \mathsf{O}_n \to C_\bullet \times \mathsf{I}_n)$ is the (totally disconnected to connected) crossed module of groupoids with n objects and $\partial(q,b,q) = (q, \delta b, q)$, constructed from $\mathcal{X}$, and in § 5.3 we determine the automorphisms of $\mathcal{C}$ from those of $\mathcal{X}$.

Brown and Içen have investigated in [10] the homotopy group $H_{\mathsf{i}}^1(\mathcal{C})$, where i is the identity map on $\mathcal{C}$, and a homotopy is a pair of functions consisting of a section and a derivation. Our intuition was that it should be possible to determine $H_{\mathsf{i}}^1(\mathcal{C}(\mathcal{X}, n))$ given the Whitehead group $W(\mathcal{X})$. In Proposition 5.10 we show that this is indeed the case: $H_{\mathsf{i}}^1(\mathcal{C})$ is isomorphic to a quotient of $(S_n \ltimes C^n) \ltimes (W(\mathcal{X}) \ltimes B^n)$ by a subgroup isomorphic to B.

Many of the constructions described in this paper have been implemented in packages for the the computational discrete algebra system GAP4 [14]. The XMod package [1, 3] was introduced for GAP3 in 1996, implementing the actor crossed module for a crossed module of groups, as described in [2]. The Gpd package [20] with Moore in 2000 implemented finite groupoids, graphs of groups, graphs of groupoids, and provided normal forms for free products with amalgamation and HNN-extensions [11]. In the latest Gpd version 1.05 the basic structure of a groupoid has been completely rewritten: starting with a *magma with many objects*; associativity provides a *semigroup with many objects*; and when an identity at each object exists we obtain a *monoid with many objects*, which is just a category; finally, when every arrow is invertible, we obtain a *group with many objects* – a *groupoid*. New functions in Gpd 1.06 and XMod 2.13 will provide conjugation in groupoids; automorphism groups of groupoids; and crossed modules of groupoids[2].

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[2] At the time of revising this paper, in 2017, these new functions exist in Gpd 1.46 and XMod 2.58.

2 Groupoids

A *groupoid* is a small category in which every arrow is invertible. The books by Higgins [16] and Brown [5] are good references for the standard properties of groupoids[3]. In the notation used here, a finite groupoid $\mathsf{C} = (C_1, C_0)$ consists of the following:

- a set $\mathrm{Ob}(\mathsf{C}) = C_0$ of *objects*;
- a set $\mathrm{Arr}(\mathsf{C}) = C_1$ of *arrows*;
- *source* and *target* maps $\partial_1^-, \partial_1^+ : C_1 \to C_0$, so that we write $(\alpha : u \to v)$ whenever $\partial_1^- \alpha = u$ and $\partial_1^+ \alpha = v$, and denote by $\mathsf{C}(u, v)$ the *hom-set* of arrows with source u and target v;
- a function $\partial_0 : C_0 \to C_1,\ u \mapsto (1_u : u \to u)$, the *identity arrow* at u;
- an associative partial composition : $C_1 \times_0 C_1 \to C_1$, with $\alpha\beta$ defined whenever $\partial_1^+ \alpha = \partial_1^- \beta$, such that $\partial_1^-(\alpha\beta) = \partial_1^- \alpha$ and $\partial_1^+(\alpha\beta) = \partial_1^+ \beta$, so that $\mathsf{C}(u) := \mathsf{C}(u, u)$ is a group with identity 1_u, called the *object group* at u;
- for each arrow $(\alpha : u \to v)$ an inverse arrow $(\alpha^{-1} : v \to u)$ such that $\alpha\alpha^{-1} = 1_u$ and $\alpha^{-1}\alpha = 1_v$.

A morphism of groupoids, as for general categories, is called a functor. Thus a *functor* $\mathsf{g} = (g_1, g_0) : \mathsf{C} \to \mathsf{D}$ is a pair of maps $(g_1 : C_1 \to D_1,\ g_0 : C_0 \to D_0)$ such that $g_1 1_u = 1_{g_0 u}$ and $g_1(\alpha\beta) = (g_1\alpha)(g_1\beta)$ whenever the composite arrow is defined. It is often convenient to omit the subscripts $0, 1$ since it should be clear from the context whether an object or an arrow is being mapped. A morphism g is *injective* and/or *surjective* if both g_0, g_1 are.

The *underlying digraph* $\Gamma(\mathsf{C})$ of C is obtained by forgetting the composition, so the objects become vertices, the arrows become arcs, while the source and target maps have their usual digraph meaning. A groupoid is *connected* if its underlying digraph is connected, and then the digraph is regular and complete.

Example 2.1 (a) The categories of groups and groupoids, and their morphisms, are written **Gp**, **Gpd** respectively. There is a functor $\mathsf{Gpd} : \mathbf{Gp} \to \mathbf{Gpd},\ C \mapsto C_\bullet,\ c \mapsto (c : \bullet \to \bullet)$, where $C_\bullet$ is a groupoid with a single object $\bullet$.

(b) For X a set, the *trivial groupoid* $\mathsf{O}(X) = (O_1, O_0)$ *on* X has $O_0 = X$ and $O_1 = \{1_x \mid x \in X\}$. We denote $\mathsf{O}(\{1, \ldots, n\})$ by O_n.

(c) The *unit groupoid* I has objects $\{0, 1\}$ and four arrows. The two non-identity arrows are $(\iota : 0 \to 1)$ and its inverse $(\iota^{-1} : 1 \to 0)$.

(d) The connected *tree groupoid* I_n has objects $\{1, 2, \ldots, n\}$ and arrows $\{(p, q) \mid 1 \leqslant p, q \leqslant n\}$ where $\partial_1^-(p, q) = p$, $\partial_1^+(p, q) = q$, $(p, q)(q, r) = (p, r)$, and $(p, q)^{-1} = (q, p)$. Note that $\mathsf{I}_2 \cong \mathsf{I}$. We also write $\mathsf{I}(X)$ for the tree groupoid on a set of objects X. The name 'tree groupoid' comes from the fact that a subset of arrows which form a spanning tree in the underlying digraph generate the whole groupoid using composition and inversion. In particular, taking the subset $X_n = \{(1, p) \mid 2 \leqslant p \leqslant n\}$, we have $(q, r) = (1, q)^{-1}(1, r)$.

[3] See also Section III.1 of [18].

(e) The *product* $\mathsf{C} \times \mathsf{D}$ of groupoids C, D has objects $C_0 \times D_0$, arrows $C_1 \times D_1$, and composition $(\alpha_1, \beta_1)(\alpha_2, \beta_2) = (\alpha_1\alpha_2, \beta_1\beta_2)$, so that $(\alpha, \beta)^{-1} = (\alpha^{-1}, \beta^{-1})$. In particular, $\mathsf{C} = C_\bullet \times \mathsf{I}_n$ may be thought of as the groupoid with n objects $\{1, 2, \ldots, n\}$; $n^2|C|$ arrows $\{(p, c, q) \mid c \in C, 1 \leqslant p, q \leqslant n\}$; source $\partial_1^-(p, c, q) = p$; target $\partial_1^+(p, c, q) = q$; composition $(p, c, q)(q, c', r) = (p, cc', r)$; and inverses $(p, c, q)^{-1} = (q, c^{-1}, p)$. We shall sometimes find it convenient to write $c_{p,q}$ for (p, c, q). A generating set for C is given by $\{(1, c, 1) \mid c \in X_C\} \cup X_n$ where X_C is any generating set for C. Every finite, connected groupoid is isomorphic to a direct product of a group and a tree groupoid in this way, and we call such a representation a *standard connected groupoid*.

A *subgroupoid* $\mathsf{B} = (B_1, B_0)$ of $\mathsf{C} = (C_1, C_0)$ is a groupoid with $B_1 \subseteq C_1$, $B_0 \subseteq C_0$, having the same source, target and composition. A subgroupoid B is *full* if $\mathsf{B}(u, v) = \mathsf{C}(u, v)$ for all $u, v \in B_0$ and *wide* if $B_0 = C_0$. The *(connected) components* of C are its maximal connected subgroupoids, with one component C_i for each of the k connected components Γ_i of $\Gamma(\mathsf{C})$. We write $\mathsf{C} = \mathsf{C}_1 \cup \cdots \cup \mathsf{C}_k$. A groupoid, all of whose components have a single object, is a *union of groups*, and is said to be *totally disconnected*.

Given a wide subgroupoid $\mathsf{B} \subseteq \mathsf{C}$, there is an equivalence relation $\equiv_R$ on $\mathrm{Arr}(\mathsf{C})$ defined by $\alpha' \equiv_R \alpha \;\Leftrightarrow\; \alpha' = \beta\alpha$ for some $\beta \in \mathrm{Arr}(\mathsf{B})$. The equivalence classes $\mathsf{B}\alpha$ for this relation are called the *right cosets of* B *in* C. The *star* at u is $\mathrm{Star}(u) = \{\alpha \in C_1 \mid \partial_1^-\alpha = u\}$, the set of all arrows with source u. Similarly the *costar* at u is $\mathrm{Costar}(u) = \{\alpha \in C_1 \mid \partial_1^+\alpha = u\}$, the set of all arrows with target u. Note that each right coset of B in C is a subset of a costar. We may define a second equivalence relation $\equiv_L$ on $\mathrm{Arr}(\mathsf{C})$ by $\alpha' \equiv_L \alpha \Leftrightarrow \alpha' = \alpha\beta$ for some $\beta \in \mathrm{Arr}(\mathsf{B})$. The equivalence classes $\alpha\mathsf{B}$ for this relation are the *left cosets of* B *in* C, and this time each class is a subset of some star. The notion of *normal subgroupoid* will be considered in § 3.2.

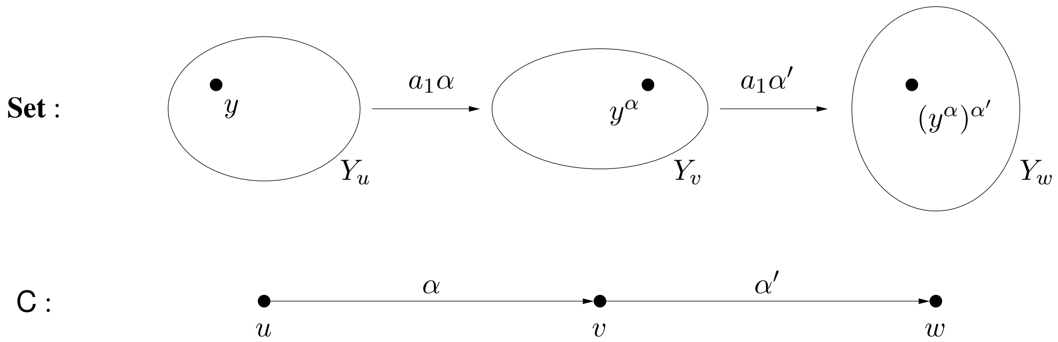


Figure 1: Groupoid action on sets.

We now consider the traditional notion of an action of a groupoid C. We restrict to the case when C is connected since there is a clear extension to the general case. For C a groupoid, a C-set-system (or, by abuse of language, a C-set) is a functor $\mathsf{a} = (a_1, a_0)$ from C to **Set**, mapping arrows to bijections. So, for $(\alpha : u \to v) \in \mathrm{Arr}(\mathsf{C})$, there are sets $a_0u = Y_u$, $a_0v = Y_v$ and a bijection $a_1\alpha : Y_u \to Y_v$. We also call a an *action of* C *on* $\bigsqcup_{u \in \mathrm{Ob}(\mathsf{C})} Y_u$. If $(\alpha' : v \to w)$ is a second arrow in C and $a_0w = Y_w$ then, since a preserves composition, we have

$$a_1(\alpha\alpha') \;=\; (a_1\alpha) * (a_1\alpha') \;=\; (a_1\alpha') \circ (a_1\alpha) \;:\; Y_u \to Y_w.$$

For $y \in Y_u$ we denote, in the usual way, $(a_1\alpha)(y)$ by y^α, and then the condition becomes $(y^\alpha)^{\alpha'} = y^{\alpha\alpha'}$. Figure 1 illustrates the situation.

A similar notion applies to sets with structure. For example, C-*graphs* are functors from C to the groupoid of (combinatorial) graphs and their isomorphisms.

A C-*group-system* (or C-*group*) provides, for each object u a group B_u with identity e_u and, for each $(\alpha : u \to v)$, an isomorphism of groups $a_1\alpha : B_u \to B_v$. We write b^α for $(a_1\alpha)(b)$ when $b \in B_u$. Since the group structure has to be preserved, as well as $(b^\alpha)^{\alpha'} = b^{(\alpha\alpha')}$, we require ${e_u}^\alpha = e_v$ and $(b_1b_2)^\alpha = ({b_1}^\alpha)({b_2}^\alpha)$. A C-*module* is a C-group in which all the B_u are abelian.

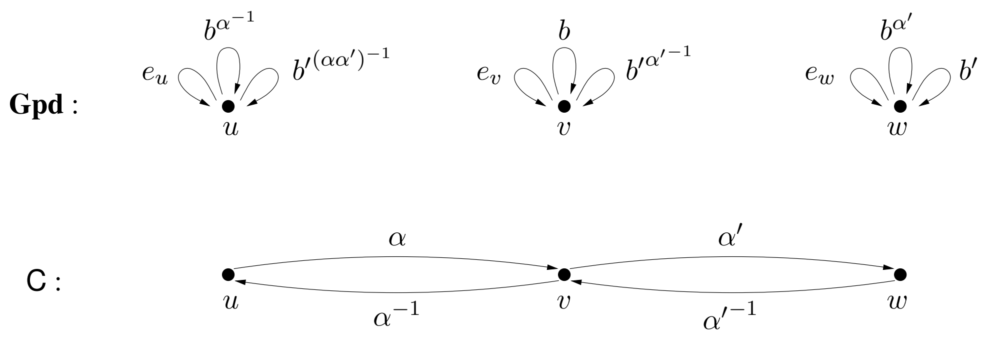


Figure 2: Groupoid action on groupoids

A C-*groupoid-system* is a functor $\mathsf{a} = (a_1, a_0)$ from C to **Gpd**, where now there are groupoids $a_0u = \mathsf{B}_u,\ a_0v = \mathsf{B}_v$ and an invertible functor $a_1\alpha : \mathsf{B}_u \to \mathsf{B}_v$. As a simple case, note that a C-group determines a C-groupoid on replacing each B_u by $\mathsf{B}_u = \mathsf{Gpd}(B_u)$, taking u as the single object. Thus a C-module may be consided as an abelian C-groupoid. Figure 2 shows part of the structure in such a case.

A particular example, when $\mathsf{C} = C_\bullet \times \mathsf{I}_n$ and $N \trianglelefteq C$, is given by taking $B_u \cong N$ for all $u \in \mathrm{Ob}(\mathsf{C})$ and the action to be conjugation:

$$(p,n,p)^{(p,c,q)} \;=\; (q,c^{-1},p)(p,n,p)(p,c,q) \;=\; (q,n^c,q)\,.$$

This will provide one of our first examples of a crossed module of groupoids in Example 5.1(c). We shall have more to say about groupoid actions in § 4.5.

3 Automorphisms of Groupoids

An automorphism of a category C is a functor $\mathsf{a} : \mathsf{C} \to \mathsf{C}$ which is an isomorphism. Let C be the connected groupoid with object set $U = \{u_1, \ldots, u_n\}$ and let $\{(\alpha_p : u_1 \to u_p) \mid 2 \leqslant p \leqslant n\}$ be a spanning tree in the underlying digraph. If C is the object group at u_1, an automorphism of C is obtained on choosing

- $\kappa \in \mathrm{Aut}\,C$,
- $\{(\beta_p : u_1 \to u_p) \mid 2 \leqslant p \leqslant n\}$, replacing the α_p in the tree,
- $\pi \in \mathrm{Symm}(U)$, permuting the objects in U.

Thus there are in total $n! \times |\mathrm{Aut}\,C| \times |C|^{n-1}$ automorphisms of C.

3.1 Automorphisms of standard connected groupoids

We now analyse the standard case where $\mathsf{C} = C_\bullet \times \mathsf{I}_n$ is the groupoid constructed in Example 2.1(e). If C has generating set $X_C = \{c_1, \ldots, c_\ell\}$ then, for each object p, C is generated by the set

$$X_p \;=\; \{(p, c_k, p) \mid c_k \in X_C\} \cup \{(p, e, q) \mid q \neq p\},$$

where the right-hand set forms a spanning tree T_p in $\Gamma(\mathsf{C})$. The remaining arrows are given as the composites:

$$\begin{array}{rcll} (p,c,p) &=& (p, c_{k_1}, p)(p, c_{k_2}, p)\ldots(p, c_{k_j}, p) & \text{when } c = c_{k_1}c_{k_2}\ldots c_{k_j} \in C,\ c_{k_i} \in X_C, \\ (q,c,r) &=& (p,e,q)^{-1}(p,c,p)(p,e,r). & \end{array}$$

An automorphism of C will be specified by giving the images of the arrows in one of the X_p.

There are three sets of automorphisms which generate the group $A = \mathrm{Aut}(\mathsf{C})$.

(1) For π a permutation in the symmetric group S_n we define an automorphism a_π by

$$\mathsf{a}_\pi(q, c, r) \;=\; (\pi q, c, \pi r).$$

(2) We may apply an automorphism κ of C to the loops at object p, giving an automorphism a_κ of C fixing each object, and defined on generators by

$$\mathsf{a}_\kappa(p, c, p) \;=\; (p, \kappa c, p)\,, \qquad \mathsf{a}_\kappa(p, e, q) \;=\; (p, e, q)\,.$$

It follows that $\mathsf{a}_\kappa(q, c, r) = (q, \kappa c, r)$, so a_κ applies κ to all the hom-sets simultaneously.

(3) The hom-set $\mathsf{C}(q, r)$ provides a regular representation of C with action $(q, c, r)^{c'} = (q, cc', r)$. For each $1 \leqslant p \leqslant n$ choose $c_p \in C$. The n-tuple $\boldsymbol{c} = (c_1, \ldots, c_n)$ determines an automorphism $\mathsf{a}_{\boldsymbol{c}}$ of C, fixing the objects, where

$$\mathsf{a}_{\boldsymbol{c}}(q, c, r) \;=\; (q, c_q^{-1} c c_r, r)\,.$$

At the vertex groups this gives conjugates $\mathsf{a}_{\boldsymbol{c}}(\mathsf{C}(q)) = (\mathsf{C}(q))^{c_q}$.

We now investigate composites of the set

$$X_A \;=\; \{\mathsf{a}_\pi \mid \pi \in S_n\} \cup \{\mathsf{a}_\kappa \mid \kappa \in \mathrm{Aut}\, C\} \cup \{\mathsf{a}_{\boldsymbol{c}} \mid \boldsymbol{c} \in C^n\}\,,$$

and obtain an explicit form for $\mathrm{Aut}\,\mathsf{C}$ as the quotient of a semidirect product. In keeping with the use of right actions, we write $\mathsf{a} * \mathsf{b}$ for the composite mapping $\mathsf{b} \circ \mathsf{a}$.

There are actions of both S_n and $\mathrm{Aut}\, C$ on C^n, where

$$\boldsymbol{c}^\pi \;=\; \pi\boldsymbol{c} \;=\; (c_{\pi^{-1}1}, \ldots, c_{\pi^{-1}n}), \qquad \boldsymbol{c}^\kappa \;=\; \kappa\boldsymbol{c} \;=\; (\kappa c_1, \ldots, \kappa c_n),$$

and these actions commute, giving an action of $S_n \times \mathrm{Aut}\, C$ on C^n. We denote by C^n_p the subset $\{\boldsymbol{c} \in C^n \mid c_p = e\}$, and note that C^n_p is closed under multiplication in C^n, and that

$$\mathsf{a}_{\boldsymbol{c}} \;=\; \mathsf{a}_{\wedge c_p} * \mathsf{a}_{c_p^{-1}\boldsymbol{c}} \qquad \text{where} \qquad c_p^{-1}\boldsymbol{c} \;=\; (c_p^{-1}c_1, \ldots, c_p^{-1}c_n) \in C^n_p \tag{1}$$

and where $\wedge c_p$ (read "to the c_p") denotes conjugation of C by c_p.

Proposition 3.1 *The automorphism group of* $\mathsf{C} = C_\bullet \times \mathsf{I}_n$ *is given by*

$$\mathrm{Aut}\,\mathsf{C} \;\cong\; \left((S_n \times \mathrm{Aut}\,C) \ltimes C^n\right)/K_1(C)$$

where $K_1(C) = \{(((\,),\wedge c),(c^{-1},\ldots,c^{-1})) \mid c \in C\} \cong C$*, and* $(\,)$ *is the identity permutation.*

Proof: We define a map

$$\theta_{\mathsf{C}} \;:\; (S_n \times \mathrm{Aut}\,C) \ltimes C^n \to \mathrm{Aut}\,\mathsf{C},\;\; ((\pi,\kappa),\boldsymbol{c}) \mapsto \mathsf{a}_\pi * \mathsf{a}_\kappa * \mathsf{a}_{\boldsymbol{c}}.$$

It is straightforward to verify that pairs of automorphisms in X_A compose as follows, where $\pi,\xi \in S_n,\ \kappa,\lambda \in \mathrm{Aut}\,C$, and $\boldsymbol{c},\boldsymbol{d} \in C^n$:

$$\begin{array}{rcccl}
(\mathsf{a}_\pi * \mathsf{a}_\xi)(q,c,r) &=& \mathsf{a}_{\pi*\xi}(q,c,r) &=& ((\pi*\xi)q, c, (\pi*\xi)r)\,,\\
(\mathsf{a}_\kappa * \mathsf{a}_\lambda)(q,c,r) &=& \mathsf{a}_{\kappa*\lambda}(q,c,r) &=& (q, (\kappa*\lambda)c, r),\\
(\mathsf{a}_{\boldsymbol{c}} * \mathsf{a}_{\boldsymbol{d}})(q,c,r) &=& \mathsf{a}_{\boldsymbol{cd}}(q,c,r) &=& (q, (c_q d_q)^{-1} c (c_r d_r), r),\\
(\mathsf{a}_\kappa * \mathsf{a}_\pi)(q,c,r) &=& (\mathsf{a}_\pi * \mathsf{a}_\kappa)(q,c,r) &=& (\pi q, \kappa c, \pi r),\\
(\mathsf{a}_{\boldsymbol{c}} * \mathsf{a}_\pi)(q,c,r) &=& (\mathsf{a}_\pi * \mathsf{a}_{\pi\boldsymbol{c}})(q,c,r) &=& (\pi q, c_q^{-1} c c_r, \pi r),\\
(\mathsf{a}_{\boldsymbol{c}} * \mathsf{a}_\kappa)(q,c,r) &=& (\mathsf{a}_\kappa * \mathsf{a}_{\kappa\boldsymbol{c}})(q,c,r) &=& (q, \kappa(c_q^{-1} c c_r), r).
\end{array}$$

These formulae show that θ_{C} is surjective, and that

$$(\mathsf{a}_\pi * \mathsf{a}_\kappa * \mathsf{a}_{\boldsymbol{c}}) * (\mathsf{a}_\xi * \mathsf{a}_\lambda * \mathsf{a}_{\boldsymbol{d}}) \;=\; \mathsf{a}_{\pi*\xi} * \mathsf{a}_{\kappa*\lambda} * \mathsf{a}_{(\xi\lambda\boldsymbol{c})\boldsymbol{d}}\,. \tag{2}$$

The semidirect product rule gives $((\pi,\kappa),\boldsymbol{c})((\xi,\lambda),\boldsymbol{d}) = ((\pi\xi,\kappa\lambda),\boldsymbol{c}^{(\xi,\lambda)}\boldsymbol{d})$, which shows that θ_{C} is a homomorphism. Since

$$\mathsf{a}_\pi * \mathsf{a}_\kappa * \mathsf{a}_{\boldsymbol{c}} \quad : \quad \left\{\begin{array}{lcl} (1,c,1) &\mapsto& (\pi 1, {c_{\pi 1}}^{-1}(\kappa c) c_{\pi 1}, \pi 1),\\ (1,e,j) &\mapsto& (\pi 1, {c_{\pi 1}}^{-1} c_{\pi j}, \pi j), \end{array}\right.$$

it follows that $\theta_{\mathsf{C}}((\pi,\kappa),\boldsymbol{c})$ is the identity automorphism provided

- π is the identity permutation,
- $c_j = c_1$ for all $2 \leqslant j \leqslant n$, so $\boldsymbol{c} = (c_1, c_1, \ldots, c_1)$,
- $\kappa c = c_1 c c_1^{-1}$ for all $c \in C$, so $\kappa = \wedge(c_1^{-1})$.

Hence $\ker\theta_{\mathsf{C}}$ is the specified group $K_1(C)$. □

It is clear that the group A_1 generated by the a_π is isomorphic to S_n; that the group A_2 generated by the a_κ is isomorphic to $\mathrm{Aut}\,C$; and that the group A_3 generated by the $\mathsf{a}_{\boldsymbol{c}}$ is isomorphic to C^n. We denote by $A_{1,3}, A_{2,3}$ the subgroups of $\mathrm{Aut}\,\mathsf{C}$ generated by $A_1 \cup A_3$ and $A_2 \cup A_3$ respectively. The join $A_{1,2}$ of A_1 and A_2 is isomorphic to $A_1 \times A_2$. The proof of Proposition 3.1 may be adjusted to show that

$$A_{1,3} \;\cong\; (S_n \ltimes C^n)/\hat{Z}(C) \quad \text{and} \quad A_{2,3} \;\cong\; (\mathrm{Aut}\,C \ltimes C^n)/\{(\wedge c, (c^{-1},\ldots,c^{-1})) \mid c \in C\}\,,$$

where $Z = Z(C)$ is the centre of C and $\hat{Z}(C) = \{((\,),(z,\ldots,z)) \mid z \in Z\}$.

Since the elements $((\pi,\kappa),(e,c_2,\ldots,c_n))$ form a transversal for the cosets of $K_1(C)$, we observe that an automorphism $\mathsf{f} = (f_1,f_0) : \mathsf{C} \to \mathsf{C}$ is specified by giving

- the permutation f_0 of the objects;
- an automorphism κ_{f} of the object group C, so that $f_1(1,c,1) = (f_0 1, \kappa_{\mathsf{f}} c, f_0 1)$;
- images $f_1(1,e,q) = (f_0 1, c_{\mathsf{f},q}, f_0 q)$ for the tree T_1, determining $\boldsymbol{c}_{\mathsf{f}} := (e, c_{\mathsf{f},2}, \ldots, c_{\mathsf{f},n})$.

A convenient *standard form* for f is therefore $\mathsf{f} = \mathsf{a}_{\kappa_{\mathsf{f}}} * \mathsf{a}_{\boldsymbol{c}_{\mathsf{f}}} * \mathsf{a}_{f_0}$, where

$$f_1(q,c,r) \;=\; (f_0 q,\; c_{\mathsf{f},q}^{-1}(\kappa_{\mathsf{f}} c) c_{\mathsf{f},r},\; f_0 r).$$

It is clear how to replace object 1 by an arbitrary object p to obtain an alternative standard form. The formulae in Proposition 3.1 and equation (1) enable us to write down the composite of two standard forms in standard form as:

$$\left(\mathsf{a}_{\kappa_{\mathsf{f}}} * \mathsf{a}_{\boldsymbol{c}_{\mathsf{f}}} * \mathsf{a}_{f_0}\right) * \left(\mathsf{a}_{\kappa_{\mathsf{g}}} * \mathsf{a}_{\boldsymbol{c}_{\mathsf{g}}} * \mathsf{a}_{g_0}\right) \;=\; \mathsf{a}_{\kappa_{\mathsf{f}} * \kappa_{\mathsf{g}} * (\wedge c_{\mathsf{g}, f_0 1})} * \mathsf{a}_{c_{\mathsf{g}, f_0 1}^{-1}(\kappa_{\mathsf{g}} \boldsymbol{c}_{\mathsf{f}})(f_0^{-1} \boldsymbol{c}_{\mathsf{g}})} * \mathsf{a}_{f_0 * g_0}.$$

The next type of groupoid to consider is the disjoint union D of m copies of a connected groupoid C. An automorphism of D which does not interchange the components is obtained by choosing an automorphism for each component, and these form a group isomorphic to $(\mathrm{Aut}\ \mathsf{C})^m$. The automorphism group of D is the wreath product $S_m \wr \mathrm{Aut}\ \mathsf{C}$ with action

$$(\mathsf{f}_1, \ldots, \mathsf{f}_m)^{\pi} \;=\; (\mathsf{f}_{(\pi^{-1}1)}, \ldots, \mathsf{f}_{(\pi^{-1}m)})\,.$$

In particular, groupoids of the form $\mathsf{B} = B_{\bullet} \times \mathsf{O}_m$, a disjoint union of isomorphic groups, will be used in Section 5. Clearly $\mathrm{Aut}\ \mathsf{B} \cong S_m \wr \mathrm{Aut}\ B$.

The final case to consider is that of an arbitrary groupoid G, whose connected components form isomorphism classes $[\mathsf{G}_i]$ with m_i components in $[\mathsf{G}_i]$. The automorphism group A_i of $[\mathsf{G}_i]$ is $S_{m_i} \wr \mathrm{Aut}\ \mathsf{G}_i$, and the automorphism group $\mathrm{Aut}\ \mathsf{G}$ is the direct product of these A_i.

3.2 Conjugation in groupoids

Each element c of a group C determines the inner automorphism $\wedge c : C \to C,\ c' \mapsto c^{-1}c'c$, where the orbits are the conjugacy classes, and $\wedge c = \wedge c'$ whenever $c' = zc$ for some $z \in Z(C)$. A similar notion holds for a connected groupoid but, when there is more than one object, the automorphisms $\wedge c_{p,q}$ with p, q fixed are all distinct.

Definition 3.2 *For $c_{p,q} = (p,c,q)$ an arrow in a connected groupoid $\mathsf{C} = C_{\bullet} \times \mathsf{I}_n$ (with $p \neq q$), conjugation of C by $c_{p,q}$ is the automorphism $\wedge c_{p,q} := \mathsf{a}_{(p,q)} * \mathsf{a}_{\boldsymbol{c}}$ where $\boldsymbol{c}$ has components $c_p = c^{-1}$, $c_q = c$, $c_r = e$ otherwise. This automorphism interchanges:*

- *p with q, and fixes the remaining objects;*
- *the loops at p and q : $(p,b,p) \mapsto (q, c^{-1}bc, q),\ (q,b,q) \mapsto (p, cbc^{-1}, p)$;*
- *the hom-sets $\mathsf{C}(p,q),\ \mathsf{C}(q,p)$: $(p,b,q) \mapsto (q, c^{-1}bc^{-1}, p),\ (q,b,p) \mapsto (p, cbc, q)$;*
- *the rest of the costars at p, q : $(r,b,p) \mapsto (r, bc, q),\ (r,b,q) \mapsto (r, bc^{-1}, p)$;*

- *the rest of the stars at* $p, q\;:\;(p,b,r) \mapsto (q, c^{-1}b, r),\;(q,b,r) \mapsto (p, cb, r)\,$*;*

where $r \notin \{p,q\}$*. The remaining arrows are unchanged.*

Conjugation by $c_{p,p}$ *is* $\wedge c_{p,p} = \mathsf{a}_{\mathbf{c}}$*, where* $\mathbf{c}$ *has components* $c_p = c$*,* $c_r = e$ *otherwise. All the objects are fixed; loops at* p *are conjugated by* c*,* $(p,b,p) \mapsto (p, c^{-1}bc, p)$*; and the rest of the star and costar at* c *are permuted:* $(p,b,r) \mapsto (p, c^{-1}b, r),\;(r,b,p) \mapsto (r, bc, p)$ *for* $r \neq p$*.*

These constructions may be remembered as: "for p or q as source, multiply b on the left by c^{-1} or c respectively; and for p or q as target, multiply b on the right by c or c^{-1}".

It is *not* the case that the map $\wedge : \mathsf{C} \to (\mathrm{Aut}\,\mathsf{C})_{\bullet}$ is a groupoid morphism. Indeed, it is straightforward to verify that, when $\alpha_1 = (p, c_1, q),\; \alpha_2 = (q, c_2, r)$ with p, q, r distinct,

$$\wedge(\alpha_1\alpha_2) \;=\; (\wedge\alpha_1) * (\wedge\alpha_2) * (\wedge\alpha_1) \;=\; (\wedge\alpha_2) * (\wedge\alpha_1) * (\wedge\alpha_2)\,. \tag{3}$$

The image of this identity, under the map $\wedge(\mathsf{C}) \to S_n\,,\;\wedge(p,c,q) \mapsto (p,q)$, is the permutation identity $(p,r) = (p,q)(q,r)(p,q) = (q,r)(p,q)(q,r)$. There are other identities satisfied by these $\wedge c_{p,q}$ and $\wedge c_{p,p}$ which we shall use when defining the notion of groupoid action in § 4.5. If $\beta_1 = (p, d_1, p),\; \beta_2 = (p, d_2, p)$ and $\beta_3 = (q, d_3, q)$, then

$$\begin{array}{rcl} \wedge(\beta_1\beta_2) & = & (\wedge\beta_1) * (\wedge\beta_2)\,, \\ \wedge(\beta_1\alpha_1) & = & (\wedge\beta_1) * (\wedge\alpha_1) * (\wedge\beta_1)^{-1}\,, \\ \wedge(\alpha_1\beta_3) & = & (\wedge\beta_3)^{-1} * (\wedge\alpha_1) * (\wedge\beta_3)\,. \end{array} \tag{4}$$

Thirdly, if $\alpha_3 = (q, c_3, p),\; \alpha_4 = (u, c_4, v)$ and $\beta_4 = (q, c_3c_1, q)$, with p, q, u, v all distinct, then

$$\begin{array}{rcl} \wedge(\alpha_1\alpha_3) & = & (\wedge\alpha_1) * (\wedge\alpha_3) * (\wedge\beta_4)\,, \\ (\wedge\alpha_1) * (\wedge\alpha_4) & = & (\wedge\alpha_4) * (\wedge\alpha_1)\,. \end{array} \tag{5}$$

Proposition 3.3 *When* $\mathsf{C} = C_{\bullet} \times \mathsf{I}_n$ *and* $|C| = k$*, the number of distinct conjugation automorphisms is* $\omega_{\mathsf{C}} = 1 + n(k-1) + \frac{1}{2}n(n-1)k$*.*

Proof: First note that $e_{p,p}$ is the identity automorphism for every object p. Since $\wedge c_{p,p}$, with $c \neq e$, acts with c or c^{-1} on the star and costar at p, it is only the case that $\wedge c_{p,p} = \wedge c'_{p',p'}$ when $p = p'$ and $c = c'$. This gives the term $n(k-1)$. Thirdly, considering $\wedge c_{p,q} = \wedge c'_{p',q'}$, we see that $\{p,q\} = \{p',q'\}$. It is easy to check that $\wedge c_{p,q} = \wedge c^{-1}_{q,p}$, but that this is the only possible equality. □

In Subsection 4.2 we shall investigate the full subgroupoid of the automorphism groupoid of C whose objects are the conjugation automorphisms of C.

When C is a connected component of a groupoid B, and $\alpha \in C_1$, we define $\wedge\alpha : \mathsf{B} \to \mathsf{B}$ to be the automorphism of B which acts as $\wedge\alpha$ on C and fixes all the other components.

We are now in a position to give a *non-standard definition* of normality for groupoids.

Definition 3.4 *A subgroupoid* $\mathsf{N} = (N_1, N_0)$ *of* C *is* normal in C*, written* $\mathsf{N} \trianglelefteq \mathsf{C}$*, if* $\beta^{\alpha} \in N_1$ *for all* $\beta \in N_1,\; \alpha \in C_1$*.*

The usual definition of normality (see [5, § 8.3]) requires that N is wide in C, and that $\alpha^{-1}\mathsf{N}(u)\alpha = \mathsf{N}(v)$ for all $(\alpha : u \to v) \in \mathsf{C}$. This allows both $\mathsf{N}_{\bullet} \times \mathsf{I}_n$ to be normal in $\mathsf{C}_{\bullet} \times \mathsf{I}_n$ when $N \lhd C$, and also $(C_{\bullet} \times \mathsf{I}_{\{u,v\}}) \cup (C_{\bullet} \times \mathsf{I}_{\{w\}})$ to be normal in $C_{\bullet} \times \mathsf{I}_{\{u,v,w\}}$. Our definition is more restrictive, and excludes these examples.

Proposition 3.5 *The normal subgroupoids of* $\mathsf{C} = C_\bullet \times \mathsf{I}_n$ *are* C *itself, and the totally disconnected subgroupoids* $N_\bullet \times \mathsf{O}_n$ *for all* $N \trianglelefteq C$.

Proof: If N_0 contains the object p then, for each object q, conjugation by (p, e, q) maps p to q, so N must be a wide subgroupoid of C. If N is totally disconnected, and if the component at p has vertex group N, then conjugation by (p, e, q) shows that the vertex group at q is also N, so that $\mathsf{N} \cong N_\bullet \times \mathsf{O}_n$, and conjugation by (p, c, p) shows that $N \trianglelefteq C$.

If $(q, n, r) \in N_1$ with $q \neq r$ then, for all $c \in C$, conjugation by (p, c, q) maps (q, n, r) to (p, cn, r), so $N = C$ and hence $\mathsf{N} = \mathsf{C}$. □

Example 3.6 The normal subgroupoids of $\mathsf{C} \cup \mathsf{D}$ where $\mathsf{C} = C_\bullet \times \mathsf{I}_n$ and $\mathsf{D} = D_\diamond \times \mathsf{I}_m$, are as follows:

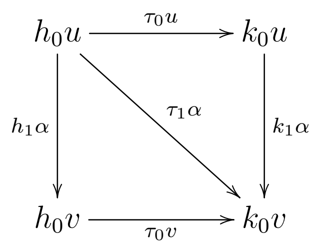
- $\mathsf{C} \cup \mathsf{D}$ itself;
- $\mathsf{C} \cup (M_\diamond \times \mathsf{O}_m)$ for each $M \trianglelefteq D$;
- $(N_\bullet \times \mathsf{O}_n) \cup \mathsf{D}$ for each $N \trianglelefteq C$;
- $(N_\bullet \times \mathsf{O}_n) \cup (M_\diamond \times \mathsf{O}_m)$ for each $N \trianglelefteq C,\ M \trianglelefteq D$.

It is clear how to generalise this example to a groupoid with more than two components.

4 Automorphism Groupoids and Sections

4.1 Natural Transformations

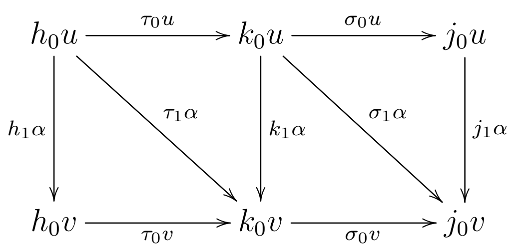
Functors are related by *natural transformations*. If $\mathsf{h}, \mathsf{k} : \mathsf{C} \to \mathsf{D}$ are functors, then a natural transformation $\tau : \mathsf{h} \to \mathsf{k}$ is determined by a function $\tau_0 : \mathrm{Ob}(\mathsf{C}) \to \mathrm{Arr}(\mathsf{D})$, such that for every arrow $(\alpha : u \to v) \in \mathsf{C}$ the following diagram commutes.



(6)

Commutativity of the diagram enables us to define a function $\tau_1 : \mathrm{Arr}(\mathsf{C}) \to \mathrm{Arr}(\mathsf{D})$, where $\tau_1\alpha$ is this diagonal arrow and $\tau_1 1_u = \tau_0 u$ for each object u. This function τ_1 is also known as the *evaluation morphism* $\varepsilon_{\mathsf{CD}}$, and will be discussed further in Subsection 4.3.

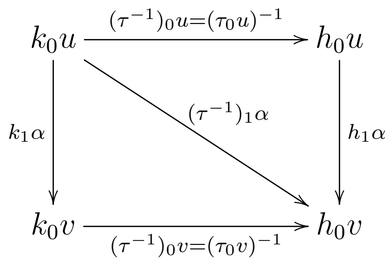
Natural transformations compose in the obvious way. If j is a third functor from C to D, and if $\sigma : \mathsf{k} \to \mathsf{j}$ is a second natural transformation, then we obtain the diagram:



The composite natural transformation $\tau * \sigma \ : \ \mathsf{h} \to \mathsf{j}$ is defined by:

$$\begin{aligned}(\tau * \sigma)_0 u &= (\tau_0 u)(\sigma_0 u), \\ (\tau * \sigma)_1 \alpha &= (\tau_1 \alpha)(\sigma_0 v) \;=\; (\tau_0 u)(k_1 \alpha)(\sigma_0 v) \;=\; (\tau_0 u)(\sigma_1 \alpha).\end{aligned}$$

Restricting to groupoids, so that arrows are invertible, we have $\tau_0 v = (h_1\alpha)^{-1}(\tau_0 u)(k_1\alpha)$, so τ is defined if we are given, for each component of C, the image of one object. Furthermore, the transformation τ has inverse $\tau^{-1} : \mathsf{k} \to \mathsf{h}$ where $(\tau^{-1})_0 u = (\tau_0 u)^{-1}$ and $(\tau^{-1})_1 \alpha = (k_1\alpha)(\tau_0 v)^{-1} = (\tau_0 u)^{-1}(h_1\alpha)$,



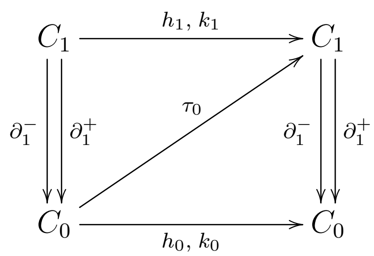
so τ is a *natural equivalence*. In this way we obtain the groupoid $\mathrm{HOM}(\mathsf{C}, \mathsf{D})$ with functors as objects and natural transformations as arrows. The identity equivalence ι_{h} at h is given by $\iota_{\mathsf{h},0} u = 1_{h_0 u}, \ \iota_{\mathsf{h},1}\alpha = h_1\alpha$. It is this construction which makes $\mathbf{Gpd}$ cartesian closed, as discussed in the Introduction, and which we consider further in § 4.3.

4.2 Automorphism groupoid of a groupoid

When $\mathsf{C} = \mathsf{D}$ and h, k are isomorphisms, we obtain our first example of a *homotopy*, with $\tau : \mathsf{C} \times \mathsf{I} \to \mathsf{D}$ being considered as a groupoid (h, k)-homotopy (see [5, § 6.5]) with

$$\tau(u, 0) = h_0 u, \quad \tau(u, 1) = k_0 u, \quad \tau(\alpha, 0) = h_1 \alpha, \quad \tau(\alpha, 1) = k_1 \alpha.$$

The significant feature of τ is that it lifts from one level to the next, as in the following diagram:



We thus obtain the *automorphism groupoid* $\mathrm{AUT}\,\mathsf{C}$ of C whose objects are the automorphisms of C and whose arrows are the natural equivalences between these automorphisms.

Given $\tau : \mathsf{h} \to \mathsf{k}$ and a third isomorphism j, we may define a $(\mathsf{j} * \mathsf{h}, \mathsf{j} * \mathsf{k})$-homotopy ρ by

$$\rho_0 u \;=\; \tau_0(j_0 u), \qquad \rho_1 \alpha \;=\; \tau_1(j_1 \alpha) \;=\; (\rho_0 u)(k_1 j_1 \alpha) \;=\; (h_1 j_1 \alpha)(\rho_0 v). \tag{7}$$

Using this construction we may obtain all the (h, k)-homotopies from the $(\mathsf{k}^{-1} * \mathsf{h}, \mathsf{i})$-homotopies, where i is the identity functor on C.

Proposition 4.1 *The combinatorial structures of the automorphism groupoids of* $\mathsf{C} = C_\bullet \times \mathsf{I}_n$ *and* $\mathsf{B} = B_\bullet \times \mathsf{O}_n$ *are given in the following table.*

	$\mathsf{C} = C_\bullet \times \mathsf{I}_n$	$\mathsf{B} = B_\bullet \times \mathsf{O}_n$
number of objects (automorphisms)	$n!\,\lvert\mathrm{Aut}\,C\rvert\,\lvert C\rvert^{n-1}$	$n!\,\lvert\mathrm{Aut}\,B\rvert^n$
number of arrows (natural equivalences)	$(n!)^2\,\lvert\mathrm{Aut}\,C\rvert\,\lvert C\rvert^{2n-1}$	$n!\,\lvert\mathrm{Aut}\,B\rvert^n\,\lvert B\rvert^n$
vertex groups	$Z(C) \cong C/\mathrm{Inn}\,C$	$(Z(B))^n$
number of connected components	$\lvert\mathrm{Out}\,C\rvert$	$n!\,\lvert\mathrm{Out}\,B\rvert^n$
number of objects in each component	$n!\,\lvert\mathrm{Inn}\,C\rvert\,\lvert C\rvert^{n-1}$	$\lvert\mathrm{Inn}\,B\rvert^n$

Proof: We have already determined the number of automorphisms in these two cases.

When specifying a natural equivalence in $\mathrm{AUT}\,\mathsf{C}$, first choose $\mathsf{h} \in \mathrm{Aut}\,\mathsf{C}$ and set $\pi = \mathsf{h}_0$. Then choose $\xi \in S_n$ and $c_p \in C,\ 1 \leqslant p \leqslant n$ so as to specify τ with $\tau_0 p = (\pi p, c_p, \xi p)$ and $\tau_1(p,c,q) = h_1(p,c,q)\,(\tau_0 q) : \pi p \to \xi q$. This τ is an equivalence $\mathsf{h} \to \mathsf{k}$ where $k_0 = \xi$ and $k_1(p,c,q) = (\tau_0 p)^{-1}\,h_1(p,c,q)\,(\tau_0 q)$. By Proposition 3.1 the total number of equivalences is $n!\,\lvert\mathrm{Aut}\,C\rvert\,\lvert C\rvert^{n-1}.n!\,\lvert C\rvert^n$.

We now seek the size of the vertex group at the identity automorphism i. If τ is such a loop and $\tau_0 1 = (1,z,1)$, then $\alpha = (1,c,1)$ in (6) gives $zc = cz$ for all $c \in C$, so $z \in Z(C)$. Taking $\alpha = (1,e,q)$ we find that $\tau_0 q = (q,z,q)$, so τ is completely determined by z. Hence the number of objects in the component containing i is $(n!\,\lvert C^n\rvert)/\lvert Z\rvert = n!\,\lvert\mathrm{Inn}\,C\rvert\,\lvert C\rvert^{n-1}$.

The automorphism group acts on the objects of the automorphism groupoid by right multiplication, permuting the components, so the components are isomorphic and their number is the obvious quotient $\lvert\mathrm{Out}\,C\rvert$.

For B, which is totally disconnected, an equivalence from $(\pi, (\eta_1,\ldots,\eta_n))$ to $(\xi, (\zeta_1,\ldots,\zeta_n))$ can only exist when $\xi = \pi$. Taking $\alpha = (p,b,p)$ in (6), we find that $\zeta_p b = (\eta_p b)^{\tau_0 p}$ for each object p, where $\tau_0 p$ is any loop at πp, so for each automorphism h the outdegree is $\lvert B\rvert^n$. The choices for π, η_p and $\tau_0 p$ determine a total of $n!\,\lvert\mathrm{Aut}\,B\rvert^n\,\lvert B\rvert^n$ equivalences.

When $\zeta_p = \eta_p$ we see that $\tau_0 p = z_p \in Z = Z(B)$. Since there is no interaction between $\tau_0 p$ and $\tau_0 q$, there are $\lvert Z\rvert^n$ choices, and commutativity of the center ensures that the vertex groups are isomorphic to Z^n. So the number of objects in the component containing i is $\lvert B\rvert^n/\lvert Z\rvert^n = \lvert\mathrm{Inn}\,B\rvert^n$. Again, the components of $\mathrm{AUT}\,\mathsf{B}$ are all isomorphic, and there are $(n!\,\lvert\mathrm{Aut}\,B\rvert^n)/\lvert\mathrm{Inn}\,B\rvert^n = n!\,\lvert\mathrm{Out}\,B\rvert^n$ of them. □

When $\mathsf{C} = C_\bullet$ is a group C considered as a one-object groupoid, the automorphism groupoid has $\lvert\mathrm{Aut}\,C\rvert$ objects; $\lvert\mathrm{Aut}\,C\rvert.\lvert C\rvert$ natural equivalences; $\lvert\mathrm{Out}\,C\rvert$ components; $\lvert\mathrm{Inn}\,C\rvert$ objects in each component; and degree $\lvert Z(C)\rvert$. The automorphisms connected to the identity automorphism are the conjugations $\wedge c$, known as inner automorphisms, with equivalences $\tau : \mathsf{i} \to \wedge c$ given by $\tau_0 = zc,\ z \in Z(C)$. For groupoids, we say that h is an *inner automorphism* if there is an equivalence $\mathsf{i} \to \mathsf{h}$ in the automorphism groupoid.

Corollary 4.2 *The inner automorphisms in* $\mathrm{AUT}\,\mathsf{C}$ *are generated by the conjugations* $\wedge c_{p,q}$ *for* $1 \leqslant p,q \leqslant n,\ c \in C$. *The inner automorphisms in* $\mathrm{AUT}\,\mathsf{B}$ *are the conjugations* $\wedge(b_1,\ldots,b_n) := (\wedge b_1,\ldots,\wedge b_n)$.

Proof: The $\wedge e_{p,q}$ generate the symmetric group S_n, permuting the objects, and for $\boldsymbol{c} = (c_1,\ldots,c_n)$ it follows from Definition 3.2 that $\mathsf{a}_{\boldsymbol{c}} = (\wedge(c_1)_{1,1}) * \cdots * (\wedge(c_n)_{n,n})$. Thus the conjugations generate the group $((S_n \times \mathrm{Inn}C) \ltimes C^n)/K_1(C)$, which has the required order (where $K_1(C)$ was defined in Proposition 3.1), and the outer automorphisms of C form a transversal for the cosets, each coset forming a connected component of $\mathrm{AUT}\,\mathsf{C}$.

We have seen that the equivalences $\mathsf{i} \to \mathsf{h}$ in $\mathrm{AUT}\,\mathsf{B}$ have $\mathsf{h} = ((\ ), (\zeta_1, \ldots, \zeta_n))$ with $\zeta_p b = b^{\tau_0 p}$ for each $\tau_0 p \in B$, so the equivalences are just products of conjugations. □

Example 4.3 If we define the *conjugation groupoid* $\mathrm{CONJ}\,\mathsf{C}$ of $\mathsf{C} = C_\bullet \times \mathsf{I}_n$ to be the full subgroupoid of the identity component of $\mathrm{AUT}\,\mathsf{C}$ whose objects are the conjugation automorphisms then, by Proposition 3.3, $\mathrm{CONJ}\,\mathsf{C}$ has the form $Z_\bullet \times \mathsf{I}_{\omega_\mathsf{C}}$, where $Z = Z(C)$. To show that the vertex group *is* $Z = Z(C)$ we obtain formulae for the natural equivalences $\mathsf{i} \to \wedge c_{p,q}$. Considering, in turn, as α in equation (6), the arrows $(p,b,p), (q,b,q), (p,b,q), (r,b,p)$, where $r \notin \{p,q\},\ b \in C$, we find a τ for each $z \in Z(C)$ where

$$\tau_0\, p = (p, zc, q), \quad \tau_0\, q = (q, zc^{-1}, p), \quad \tau_0\, r = (r, z, r).$$

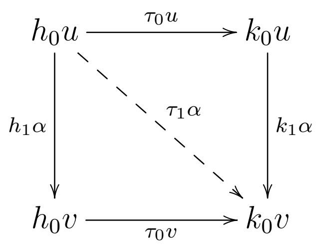
For $\tau' : \wedge c_{p,q} \to \mathsf{i}$ the corresponding formulae are

$$\tau'_0\, p = (q, c^{-1}z, p), \quad \tau'_0\, q = (p, cz, q), \quad \tau'_0\, r = (r, z, r).$$

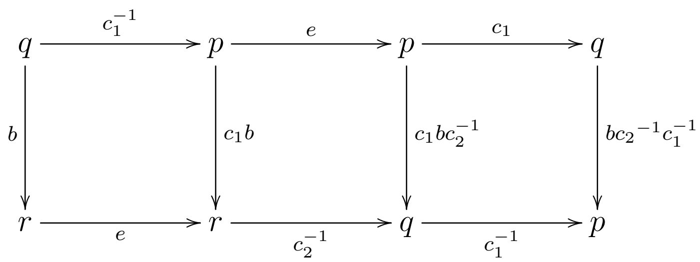
Similarly, there is an equivalence $\tau : \mathsf{i} \to \wedge c_{p,p}$ with $\tau_0\, p = (p, zc, p),\ \tau_0\, r = (r, z, r)$, and a τ' : $\wedge c_{p,p} \to \mathsf{i}$ with $\tau'_0\, p = (p, c^{-1}z, p),\ \tau'_0\, r = (r, z, r)$.

It is well-known that the group of natural equivalences $(\mathrm{AUT}\,\mathsf{C})_1$ is isomorphic to the group of functors $\mathsf{C} \to \Box\mathsf{C}$ where $\Box\mathsf{C}$ is the groupoid of *commutative squares in* C [9, § 6.1]. This isomorphism maps τ to f_τ where

$\mathsf{f}_\tau \alpha \quad =$



We may use these squares for the τ above, with $z = e$, to illustrate the proof of the identity (3) for conjugations. The image $\wedge(\alpha_1\alpha_2)(q,b,r) = ((\wedge\alpha_1) * (\wedge\alpha_2) * (\wedge\alpha_1))(q,b,r)$, for example, is $(q, b(c_1c_2)^{-1}, p)$:



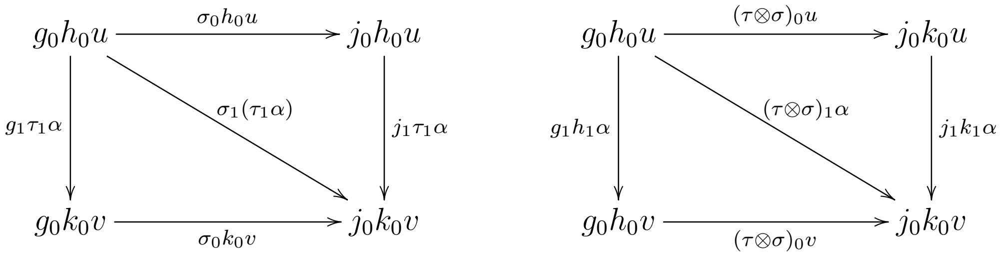
4.3 The group structure $\otimes$ on $\mathrm{AUT}\,\mathsf{C}$

As we explained in the Introduction, the automorphism groupoid $\mathrm{AUT}\,\mathsf{C}$ also has a group structure making it a group-groupoid, or crossed module.

The group multiplication $\otimes$ is essentially composition of natural equivalences. If $(\tau : \mathsf{h} \to \mathsf{k})$ and $(\sigma : \mathsf{g} \to \mathsf{j})$, then if $(\alpha : u \to v)$, $(\beta : x \to y) \in \mathsf{C}$ we have $\tau_1\alpha = (h_1\alpha)(\tau_0 v) = (\tau_0 u)(k_1\alpha)$ and $\sigma_1\beta = (g_1\beta)(\sigma_0 y) = (\sigma_0 x)(j_1\beta)$. We define

$$(\tau : \mathsf{h} \to \mathsf{k}) \otimes (\sigma : \mathsf{g} \to \mathsf{j}) \;=\; (\tau \otimes \sigma : \mathsf{h} * \mathsf{g} \to \mathsf{k} * \mathsf{j}) \quad \text{where} \quad (\tau \otimes \sigma)_1\alpha = \sigma_1(\tau_1\alpha)\,.$$

The relevant commutative diagrams are as follows:



The condition for a natural transformation is easily checked, giving:

$$\begin{array}{rcl} (\tau \otimes \sigma)_0\, u &=& (g_1\tau_0 u)(\sigma_0 k_0 u) \;=\; (\sigma_0 h_0 u)(j_1\tau_0 u) \\ (\tau \otimes \sigma)_1\, \alpha &=& (g_1\tau_1\alpha)(\sigma_0 k_0 v) \;=\; (g_1\tau_0 u)(g_1 k_1\alpha)(\sigma_0 k_0 v) \;=\; (g_1\tau_0 u)(\sigma_0 k_0 u)(j_1 k_1\alpha) \\ &=& (\sigma_0 h_0 u)(j_1\tau_0 u)(j_1 k_1\alpha) \;=\; (\sigma_0 h_0 u)(j_1\tau_1\alpha)\,. \end{array} \tag{8}$$

As expected, $\iota_{\mathsf{h}} * \iota_{\mathsf{k}} = \iota_{\mathsf{h}*\mathsf{k}}$.

We find the following straightforward construction particularly useful, since it enables us to transfer calculations at the identity object in a group-groupoid to an arbitrary object. For k an invertible element in a monoid M, the monoid $(M, *_k)$ has multiplication $*_k$ defined in terms of the usual multiplication by

$$m *_k n \;:=\; mk^{-1}n \tag{9}$$

and has identity k. If $m \in M$ is invertible in M then m has $*_k$-inverse $\overline{m} := km^{-1}k$. When M is a group, the $*_k$-conjugation automorphism is:

$$\wedge_k\, m : M \to M, \quad n \mapsto \overline{m} *_k n *_k m \;=\; km^{-1}nk^{-1}m\,. \tag{10}$$

If $\gamma : X \times M \to X,\ (x, m) \mapsto x^m$ is an action of M on a set X, then the corresponding $*_k$-*action* $\gamma_k : X \times M \to X$ is given by $(x, m) \mapsto x^{k^{-1}m}$, so that

$$\gamma_k(\gamma_k(x, m), n) \;=\; \gamma_k(x^{k^{-1}m}, n) \;=\; x^{k^{-1}mk^{-1}n} \;=\; x^{k^{-1}(m *_k n)} \;=\; \gamma_k(x, m *_k n)\,. \tag{11}$$

This generalises to a category $\mathbb{C}$ if we choose an invertible element (u, k_u, u) at each object u and define multiplication $*_{\boldsymbol{k}}$ by

$$(u, a, v) *_{\boldsymbol{k}} (v, b, w) \;:=\; (u, ak_v^{-1}b, w).$$

The resulting category $(\mathbb{C}, *_{\boldsymbol{k}})$ has identities (u, k_u, u), and if (u, a, v) has inverse (v, a^{-1}, u) in $\mathbb{C}$ then the $*_{\boldsymbol{k}}$-inverse is $\overline{(u, a, v)} = (v, k_v a^{-1} k_u, u)$.

One application we shall require later is to the monoid of endomorphisms of a crossed module, so that $\mathrm{End}_\kappa(\mathcal{X})$, where $\kappa = (\kappa_2, \kappa_1)$ is an automorphism of $\mathcal{X}$, has multiplication $\eta *_\kappa \zeta \;:=\; (\eta_2 *_{\kappa_2}$

$\zeta_2,\ \eta_1 *_{\kappa_1} \zeta_1)$. More immediately, if we fix a permutation $k_0 \in \mathrm{Symm}(C_0)$, we may define a multiplication $*_{k_0}$ on $\mathrm{Symm}(C_0)$ by

$$h_0 *_{k_0} g_0 \ := \ h_0 * k_0^{-1} * g_0,$$

such that k_0 is the $*_{k_0}$-identity and h_0 has $*_{k_0}$-inverse $\overline{h_0} := k_0 * h_0^{-1} * k_0$.

Returning to $\mathrm{AUT}\,\mathsf{C}$, it is easy to see that when $\mathsf{h} = \mathsf{g} = \mathsf{i}$, the identity automorphism, then $(\tau \otimes \sigma : \mathsf{i} \to \mathsf{k} * \mathsf{j})$, so that the star of equivalences at i form a group. This is also true for the costar at i. More generally, setting $\mathsf{g} = \mathsf{h}$, we obtain a group structure $\otimes_{\mathsf{h}}$ on the star at h by defining

$$\tau \otimes_{\mathsf{h}} \sigma \ := \ \tau \otimes \iota_{\mathsf{h}}^{-1} \otimes \sigma \quad \text{for} \quad \tau, \sigma \in \mathrm{Star}(\mathsf{h})\,. \tag{12}$$

Restricting the formulae in (8) to this special case, we obtain

$$\begin{aligned}
(\tau \otimes_{\mathsf{h}} \sigma)_0\, u \ &= \ (\tau_0 u)(\sigma_0 h_0^{-1} k_0 u) \ = \ (\sigma_0 u)(j_1 h_1^{-1} \tau_0 u) \ : \ h_0 u \to (k_0 *_{h_0} j_0) u\,,\\
(\tau \otimes_{\mathsf{h}} \sigma)_1\, \alpha \ &= \ (h_1 \alpha)(\tau_0 v)(\sigma_0 h_0^{-1} k_0 v) \ = \ (\sigma_0 u)(j_1 \alpha)(j_1 h_1^{-1} \tau_0 v) \ : \ h_0 u \to (k_0 *_{h_0} j_0) v\,.
\end{aligned}$$

Similarly, setting $\mathsf{j} = \mathsf{k}$, we obtain a group structure on the costar at k,

$$\tau \otimes_{\mathsf{k}} \sigma \ := \ \tau \otimes \iota_{\mathsf{k}}^{-1} \otimes \sigma \quad \text{for} \quad \tau, \sigma \in \mathrm{Costar}(\mathsf{k})\,. \tag{13}$$

The product formulae in this case are as follows:

$$\begin{aligned}
(\tau \otimes_{\mathsf{k}} \sigma)_0\, u \ &= \ (\sigma_0 k_0^{-1} h_0 u)(\tau_0 u) \ = \ (g_1 k_1^{-1} \tau_0 u)(\sigma_0 u) \ : \ (h_0 *_{k_0} g_0) u \to k_0 u\,,\\
(\tau \otimes_{\mathsf{k}} \sigma)_1\, \alpha \ &= \ (\sigma_0 k_0^{-1} h_0 u)(\tau_0 u)(k_1 \alpha) \ = \ (g_1 k_1^{-1} \tau_0 u)(g_1 \alpha)(\sigma_0 v) \ : \ (h_0 *_{k_0} g_0) u \to k_0 v\,.
\end{aligned}$$

4.4 Admissible and coadmissible sections

For h_0, k_0 a pair of permutations of the objects of a groupoid C, an (h_0, k_0)-*section* $t_0 : C_0 \to C_1$ is a map which composes with the source and target maps to give h_0 and k_0 respectively:

$$h_0 \ = \ t_0 * \partial_1^-\,, \qquad k_0 \ = \ t_0 * \partial_1^+.$$

Note that if $\tau : \mathsf{h} \to \mathsf{k}$ is a natural equivalence between automorphisms of C, then τ_0 is such a section. An (h_0, k_0)-section is also called an *admissible* h_0*-section* and a *coadmissible* k_0*-section*.

The *groupoid of sections* $\mathrm{Sect}(\mathsf{C})$ of C has the permutations of C_0 as objects and the (h_0, k_0)-sections as elements of the hom-set from h_0 to k_0. Composition in $\mathrm{Sect}(\mathsf{C})$ is defined by

$$((t_0 : h_0 \to k_0) * (r_0 : k_0 \to j_0) \ : \ h_0 \to j_0) \ \ : \ u \mapsto (t_0 u)(r_0 u)\,.$$

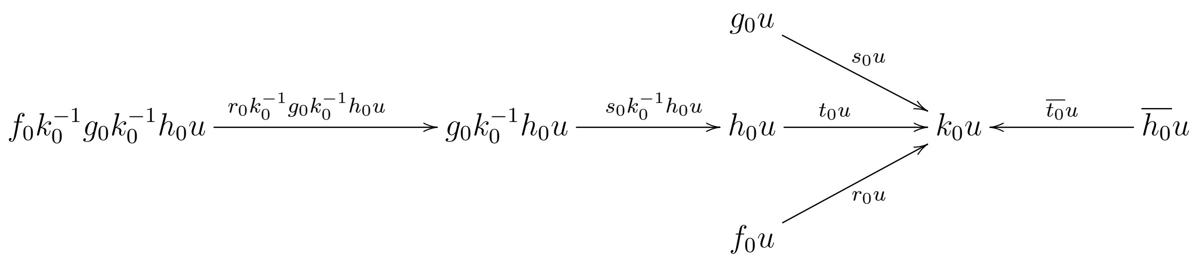
We do *not* require commutativity of a diagram, as for natural transformations in (6), so there is no $t_1 \alpha$ for $\alpha \in C_1$, and so no multiplication $\otimes$. However, we *can* define group structures on the stars and costars, as in (12) and (13). The group $M^1_{k_0}(\mathsf{C})$ of coadmissible k_0-sections of C has a product, written $\star_{k_0}$, such that the composite k_0-section $t_0 \star_{k_0} s_0$ is defined at u by

$$((t_0 : h_0 \to k_0) \star_{k_0} (s_0 : g_0 \to k_0)) u \ := \ \big((s_0 k_0^{-1} h_0 u)(t_0 u) \ : \ (h_0 *_{k_0} g_0) u \to k_0 u\big)\ . \tag{14}$$

It is straightforward to verify that this product is associative, and that

$$(t_0 \star_{k_0} s_0 \star_{k_0} r_0) u \ = \ \big((r_0 k_0^{-1} g_0 k_0^{-1} h_0 u)(s_0 k_0^{-1} h_0 u)(t_0 u) : (h_0 *_{k_0} g_0 *_{k_0} f_0) u \to k_0 u\big)\,.$$

Here is a sketch illustrating the situation.



The identity coadmissible (k_0, k_0)-section in $M^1_{k_0}(\mathsf{C})$ is i_0 where $i_0u = 1_{k_0u}$ for all $u \in C_0$. The $\star_{k_0}$-inverse of t_0 is the coadmissible section $\overline{t_0}$ where

$$\overline{t_0}u = (t_0h_0^{-1}k_0u)^{-1}, \quad \text{so} \quad \overline{t_0}k_0^{-1}h_0u = (t_0u)^{-1} \quad \text{and} \quad \overline{t_0} * \partial_1^- = k_0 * h_0^{-1} * k_0 = \overline{h_0}.$$

Note that the map from $(M^1_{k_0}(\mathsf{C}), \star_{k_0})$ to $(\mathrm{Symm}(C_0), *_{k_0})$, mapping t_0 to $t_0 * \partial_1^-$, is a group homomorphism.

The definition for the group of admissible h_0-sections, corresponding to (14), is

$$((t_0 : h_0 \to k_0) \star_{h_0} (s_0 : h_0 \to j_0))u \ := \ \big((t_0u)(s_0h_0^{-1}k_0u) \ : \ h_0u \to (k_0 *_{h_0} j_0)u\big) \ ,$$

but we shall not use this construction in this paper.

Example 4.4 Let $\mathsf{C} = C_\bullet \times \mathsf{I}_n$. For $\pi \in S_n$ and $\boldsymbol{d} \in C^n$ define $t_{\pi,0}, s_{\boldsymbol{d},0} \in M^1_{k_0}(\mathsf{C})$ by $t_{\pi,0}p = (\pi p, e, k_0p)$ and $s_{\boldsymbol{d},0}p = (k_0p, d_p^{-1}, k_0p)$. Since $(s_{\boldsymbol{d},0} \star_{k_0} t_{\pi,0})\,p = (\pi p, d_p^{-1}, k_0p)$ it is clear that every section in $M^1_{k_0}(\mathsf{C})$ can be expressed as a product in this way. Also $(t_{\pi,0} \star_{k_0} s_{\boldsymbol{d},0})\,p = (\pi p, d^{-1}_{(k_0^{-1}\pi p)}, k_0p)$, so $s_{\boldsymbol{d},0} \star_{k_0} t_{\pi,0} = t_{\pi,0} \star_{k_0} s_{(k_0^{-1}\pi)^{-1}\boldsymbol{d},0}$. Hence $M^1_{k_0}(\mathsf{C}) \cong (S_n, *_{k_0}) \ltimes C^n$ using the $*_{k_0}$-action defined in (11).

Forgetting the $\star$-products, and just considering $\mathrm{Sect}\,\mathsf{C}$ as a groupoid, it is clear that $\mathrm{Sect}\,\mathsf{C} \cong (C^n)_\bullet \times \mathsf{I}_{n!}$. Proposition 3.1 then gives $\mathrm{Aut}\,\mathrm{Sect}\,\mathsf{C} \cong ((S_{n!} \times (S_n \wr \mathrm{Aut}C)) \ltimes (C^n)^{n!})/K_1(C^n)$ where $K_1(C^n) = \{(((\,), \wedge\boldsymbol{d}), (\boldsymbol{d}^{-1}, \ldots\ldots, \boldsymbol{d}^{-1})) \mid \boldsymbol{d} \in C^n\}$.

4.5 Groupoid Actions

For groups the *inner homomorphism* $G \to \mathrm{Aut}\,G,\ g \mapsto \wedge g$, expresses the conjugation action of G on itself. Converting to single-object groupoids, the homomorphism becomes a functor $G_\bullet \to (\mathrm{Aut}\,G)_\bullet$, but there is no functor $G_\bullet \to \mathrm{AUT}(G_\bullet)$ since g is an arrow while $\wedge g$ is now an object.

An action of a groupoid C on a groupoid B is usually defined in the case where B is a union of groups having the same objects as C, and for each object u the group $\mathsf{C}(u)$ acts on the group $\mathsf{B}(u)$. These actions are required to be compatible, so there is essentially one action per component. Thus, when $(\alpha : u \to v) \in \mathsf{C}$ and $\beta \in \mathsf{B}(u)$, we have $\beta^\alpha \in \mathsf{B}(v)$. A particular case of this situation is when B is a normal, totally disconnected subgroupoid of C and the action is conjugation, $\beta^\alpha = \alpha^{-1}\beta\alpha$. It *appears* from this description that α does not act by permuting the arrows of B, but by providing an isomorphism from $\mathsf{B}(u)$ to $\mathsf{B}(v)$. However, if we mimic the rules for $\wedge c_{p,q}$ in Definition 3.2 and define, for $\beta' \in \mathsf{B}(v)$, ${\beta'}^\alpha := \beta \in \mathsf{B}(u)$ whenever $\beta^\alpha = \beta'$, the action by α also provides an isomorphism from $\mathsf{B}(v)$ to $\mathsf{B}(u)$. The hom-sets $\mathsf{B}(u, v)$ and $\mathsf{B}(v, u)$ are empty, so the arrows of B *are* permuted.

A more general definition of an action of C on B would be a functor $\Phi : \mathsf{C} \to \mathrm{AUT}\,\mathsf{B}$, which does not require B to be totally disconnected, and which *does* provide a permutation of the arrows. One possible approach is to convert $\mathrm{AUT}\,\mathsf{C}$ to a 2-*groupoid* — a special type of 2-category (see, for example, Kamps and Porter [17]), but this is beyond the scope of this paper.

As a result of these considerations, we propose a new definition of groupoid action.

Definition 4.5 *An* action *of a groupoid* C *on a groupoid* B *is a function* $\omega : C_1 \to \mathrm{Aut}\,\mathsf{B}$ *which satisfies the conjugation relations of (3), (4), and (5). Specifically, when*

$$\begin{array}{llll} \alpha_1 = (p, c_1, q), & \alpha_2 = (q, c_2, r), & \alpha_3 = (q, c_3, p), & \alpha_4 = (u, c_4, v), \\ \beta_1 = (p, c_1, p), & \beta_2 = (p, d_2, p), & \beta_3 = (q, d_3, q), & \beta_4 = (q, c_3 c_1, q), \end{array}$$

with p, q, r, u, v *all distinct, then the following identities hold:*

$$\begin{array}{rcl} \omega(\alpha_1\alpha_2) & = & (\omega\alpha_1) * (\omega\alpha_2) * (\omega\alpha_1) \;=\; (\omega\alpha_2) * (\omega\alpha_1) * (\omega\alpha_2)\,, \\ \omega(\beta_1\alpha_1) & = & (\omega\beta_1) * (\omega\alpha_1) * (\omega\beta_1)^{-1}\,, \\ \omega(\alpha\beta_3) & = & (\omega\beta_3)^{-1} * (\omega\alpha_1) * (\omega\beta_3)\,, \\ \omega(\beta_1\beta_2) & = & (\omega\beta_1) * (\omega\beta_2)\,, \\ \omega(\alpha_1\alpha_3) & = & (\omega\alpha_1) * (\omega\alpha_3) * (\omega\beta_4), \\ (\omega\alpha_1) * (\omega\alpha_4) & = & (\omega\alpha_4) * (\omega\alpha_1)\,. \end{array}$$

Various particular cases of this definition should be noted:

(a) a *totally disconnected groupoid action* is an action of C on B when B is totally disconnected;

(b) when C has one object we obtain the usual notion of a group action;

(c) C acts on itself by $\omega(\alpha) = \wedge\alpha$.

As we shall see in the following section, a more general notion of groupoid action allows for a more general notion of crossed module of groupoids. These ideas will be developed in future papers.

5 Crossed modules of groupoids

5.1 Basic definitions for groups

We first recall the standard constructions for groups – see, for example, [2, 6] and [9, chapters 2,3]. A *crossed module of groups* $\mathcal{X} = (\delta : B \to C)$ comprises a group homomorphism δ and a right action of C on B such that

$$\delta(b^c) = c^{-1}(\delta b)c \quad \text{and} \quad (b')^{\delta b} = b^{-1}b'b \quad \text{for all} \quad c \in C,\ b, b' \in B\,.$$

We denote by $\mathrm{Triv}(C)$ the subgroup of $Z(C)$ consisting of those $c \in C$ which act trivially on both B and C. An endomorphism η of $\mathcal{X}$ is a pair of homomorphisms

$$\eta = (\eta_2 : B \to B,\ \eta_1 : C \to C) \quad \text{such that} \quad \eta_2 * \delta = \delta * \eta_1 \quad \text{and} \quad \eta_2(b^c) = (\eta_2 b)^{\eta_1 c}\,.$$

Following the alternative multiplication in (9), when κ is an automorphism of $\mathcal{X}$ we define a product $*_\kappa$ on $\mathrm{End}\,\mathcal{X}$ by $\eta *_\kappa \zeta := \eta * \kappa^{-1} * \zeta$, having identity κ, and denote the resulting monoid by $\mathrm{End}_\kappa\mathcal{X}$, and its maximal subgroup by $\mathrm{Aut}_\kappa\mathcal{X}$.

The automorphism structure of $\mathcal{X}$ has been developed by Whitehead [25], Lue [19] and Norrie [22], and forms an *actor crossed square* $\mathcal{S}(\mathcal{X})$. For further details of crossed squares see, for example, Ellis and Steiner [13]. One of the four groups in $\mathcal{S}(\mathcal{X})$ is the group $W(\mathcal{X})$ of regular derivations, introduced by Whitehead. For κ an automorphism of $\mathcal{X}$, Gilbert [15] has extended these derivations to κ-derivations, which are used extensively by Brown and Içen in [10]. We have not seen this idea extended to a κ-version $\mathcal{S}_\kappa\mathcal{X}$ of the actor crossed square, so include the details of this structure here. The proofs are straightforward generalisations of the identity case, and may be found in the online notes [24].

$$\mathcal{S}_\kappa\mathcal{X} \quad = \quad \begin{array}{ccc} B & \xrightarrow{\nu_{\kappa,2}} & W_\kappa \\ {\scriptstyle\delta}\downarrow & \searrow & \downarrow{\scriptstyle\Delta_\kappa} \\ C & \xrightarrow[\nu_{\kappa,1}]{} & A_\kappa \end{array} \tag{15}$$

A κ-derivation of $\mathcal{X}$ is a map

$$\phi : C \to B \quad \text{such that} \quad \phi(cc') = (\phi c)^{\kappa_1 c'}(\phi c')\,,$$

from which it follows that $\phi e = e$ and $(\phi c)^{-1} = (\phi(c^{-1}))^{\kappa_1 c}$. Such a ϕ determines a second, source endomorphism ζ_ϕ where

$$\zeta_\phi : \mathcal{X} \to \mathcal{X}, \quad \zeta_{\phi,2}\, b = (\kappa_2 b)(\phi\delta b), \quad \zeta_{\phi,1}\, c = (\kappa_1 c)(\delta\phi c), \tag{16}$$

and ϕ is called a (ζ_ϕ, κ)-*derivation* (see, for example, [6]). For $a \in B$ the *principal κ-derivation* ϕ_a is the map

$$\phi_a : C \to B,\ c \mapsto (a^{-1})^{\kappa_1 c} a, \quad \text{so that} \quad \phi_a\delta b = [\kappa_2 b, a],\ \delta\phi_a c = [\kappa_1 c, \delta a]\,. \tag{17}$$

This ϕ_a is a (ζ_a, κ)-derivation where $\zeta_{a,2}\, b = (\kappa_2 b)^a$ and $\zeta_{a,1}\, c = (\kappa_1 c)^{\delta a}$.

The κ-derivations form a monoid $\mathrm{Der}_\kappa(\mathcal{X})$ with *Whitehead product* $\star_\kappa$ given by

$$(\phi \star_\kappa \psi)c \;=\; (\psi c)(\phi c)(\psi\kappa_1^{-1}\delta\phi c) \;=\; (\phi c)(\psi\kappa_1^{-1}\zeta_{\phi,1}c) \;=\; (\psi c)(\zeta_{\psi,2}\kappa_2^{-1}\phi c)\,, \tag{18}$$

and $\phi \star_\kappa \psi$ is a $(\zeta_\phi *_\kappa \zeta_\psi, \kappa)$-derivation. The zero derivation $0 : c \mapsto e$ is an identity for this product. Now ϕ is invertible with respect to $\star_\kappa$ precisely when ζ_ϕ is an automorphism of $\mathcal{X}$, and the invertible derivations form the *Whitehead group* $W_\kappa(\mathcal{X})$ of the monoid. Note that $W_\kappa(\mathcal{X}) \cong W(\mathcal{X}) := W_\iota(\mathcal{X})$, where ι is the identity automorphism of $\mathcal{X}$.

If λ is another automorphism of $\mathcal{X}$, there is an isomorphism $\theta_{\kappa,\lambda} : \mathrm{Der}_\kappa(\mathcal{X}) \to \mathrm{Der}_\lambda(\mathcal{X})$ given by:

$$\theta_{\kappa,\lambda}(\phi) = \chi_\phi, \quad \chi_\phi c = \phi\kappa_1^{-1}\lambda_1 c, \qquad \text{and} \qquad \theta_{\kappa,\lambda}^{-1}(\chi) = \phi_\chi, \quad \phi_\chi c = \chi\lambda_1^{-1}\kappa_1 c.$$

A useful, special case of this result is when $\lambda = \iota$. If μ is yet another automorphism of $\mathcal{X}$, we may define a $(\mu * \kappa)$-derivation ψ by $\psi c = \phi\mu_1 c$. The source automorphism for ψ satisfies:

$$\zeta_{\psi,2}\, b \;=\; ((\mu * \kappa)_2 b)(\psi\delta b) \;=\; (\kappa_2\mu_2 b)(\phi\mu_1\delta b) \;=\; (\kappa_2\mu_2 b)(\phi\delta\mu_2 b) \;=\; \zeta_{\phi,2}\,\mu_2\, b\,. \tag{19}$$

This result, together with a similar calculation for $\zeta_{\psi,1}\, c$, shows that ψ is a $(\mu * \zeta_\phi, \mu * \kappa)$-derivation. Hence, as for homotopies (see (7)), the (ζ, κ)-derivations may be obtained from the $(\kappa^{-1} * \zeta, \iota)$-derivations.

The group of automorphisms $A_\kappa = \mathrm{Aut}_\kappa \mathcal{X}$ has right actions on B and C given by

$$b^\eta \;:=\; \eta_2\kappa_2^{-1} b\,, \qquad c^\eta \;:=\; \eta_1\kappa_1^{-1} c\,. \tag{20}$$

The Norrie crossed module $\mathcal{N}_\kappa(\mathcal{X})$ forms the bottom of the square:

$$\mathcal{N}_\kappa(\mathcal{X}) = (\nu_{\kappa,1} : C \to A_\kappa) \;\text{ where }\; \nu_{\kappa,1} c = \zeta_c,\; \zeta_{c,1} d = (\kappa_1 d)^c,\; \zeta_{c,2} a = (\kappa_2 a)^c,$$

and the action is given by (20).
Similarly, the Lue crossed module $\mathcal{L}_\kappa(\mathcal{X})$ forms the diagonal of the square:

$$\mathcal{L}_\kappa(\mathcal{X}) = (\delta * \nu_{\kappa,1} : B \to A_\kappa) \;\text{ where }\; \nu_{\kappa,1}\delta b = \zeta_{\delta b},\; \zeta_{\delta b,1} d = (\kappa_1 d)^{\delta b},\; \zeta_{\delta b,2} a = (\kappa_2 a)^b,$$

and the action is also given by (20).

The *actor crossed module* of $\mathcal{X}$ is $\mathcal{A}_\kappa(\mathcal{X}) = (\Delta_\kappa : W_\kappa \to A_\kappa)$ where $\Delta_\kappa\phi$ is the ζ_ϕ of (16), and the action of A_κ on W_κ is given by

$$\phi^\eta \;:=\; \kappa_1 * \eta_1^{-1} * \phi * \kappa_2^{-1} * \eta_2 \;:\; C \to B\,. \tag{21}$$

This ϕ^η is a $((\wedge_\kappa\eta)\zeta_\phi, \kappa)$-derivation using the κ-conjugation formula (10), so $\wedge_\kappa\eta : A_\kappa \to A_\kappa$, $\zeta \mapsto \zeta^\eta = \kappa * \eta^{-1} * \zeta * \kappa^{-1} * \eta\,.$

The *Whitehead crossed module* of $\mathcal{X}$ for κ is $\mathcal{W}_\kappa(\mathcal{X}) = (\nu_{\kappa,2} : B \to W_\kappa)$ where $\nu_{\kappa,2} a$ is the principal derivation ϕ_a given by (17). The action of W_κ on B is given by

$$a^\phi \;:=\; a^{\zeta_\phi} \;=\; \zeta_{\phi,2}\kappa_2^{-1} a \;=\; a(\phi\kappa_1^{-1}\delta a)\,.$$

Note that, for $\eta \in \mathrm{Aut}_\kappa(\mathcal{X})$, $(\phi_a)^\eta = \phi_{a^\eta}$.

The boundary maps of $\mathcal{W}_\kappa$ and $\mathcal{N}_\kappa$ form an *inner morphism* of crossed modules $\nu_\kappa : \mathcal{X} \to \mathcal{A}_\kappa(\mathcal{X})$, as shown in (15). The *inner actor crossed module* of $\mathcal{X}$ for κ is the image $\nu_\kappa\mathcal{X}$, where the source group consists of the principal κ-derivations, and the range group consists of the κ-conjugation automorphisms. For further details see [22] and the XMod manual [3].

5.2 Basic definitions for groupoids

We now consider the corresponding constructions for groupoids (a standard reference is Brown & Higgins [7]). Let $\mathsf{C}_1 = (C_1, C_0)$ be a connected groupoid and $\mathsf{C}_2 = (C_2, C_0)$ a groupoid with the same object set, and let $\mathsf{C}_2, \mathsf{C}_1$ act upon themselves by conjugation as in Definition 3.2 and § 4.5: $\beta \mapsto \wedge\beta : \mathsf{C}_2 \to \mathsf{C}_2$ and $\alpha \mapsto \wedge\alpha : \mathsf{C}_1 \to \mathsf{C}_1$. As noted previously, the usual definition requires C_2 to be totally disconnected, whereas our version will allow more general actions.

A *pre-crossed module of groupoids* $\mathcal{C} = (\partial : \mathsf{C}_2 \to \mathsf{C}_1)$ consists, firstly, of a morphism of groupoids $\partial = (\partial_2, \mathrm{id})$, which is the identity on objects, called the *boundary*, and pictured as:

$$\begin{array}{ccc} C_2 & \xrightarrow{\partial_2} & C_1 \\ \partial_1^- \big\downarrow\big\downarrow \partial_1^+ & & \partial_1^- \big\downarrow\big\downarrow \partial_1^+ \\ C_0 & \xrightarrow[\mathrm{id}]{} & C_0 \end{array} \qquad \text{or, more simply,} \qquad C_2 \xrightarrow{\partial_2} C_1 \overset{\partial_1^-}{\underset{\partial_1^+}{\rightrightarrows}} C_0 \,.$$

Secondly, there is an action of C_1 on C_2 such that ∂ is a C_1-morphism, so $\partial_2 * \partial_1^- = \partial_1^-$, $\partial_2 * \partial_1^+ = \partial_1^+$ and, for all $\beta \in C_2$ and $\alpha \in C_1$,

$$\textbf{X1:} \qquad \partial_2(\beta^\alpha) \;=\; (\partial_2\beta)^\alpha \,.$$

The pre-crossed module $\mathcal{C}$ is a *crossed module of groupoids* if for all $\beta, \beta_1 \in \mathsf{C}_2$,

$$\textbf{X2:} \qquad {\beta_1}^{\partial_2\beta} \;=\; {\beta_1}^{\beta} \,.$$

Note that, when both axioms are satisfied, the restriction $(\partial_u : \mathsf{C}_2(u) \to \mathsf{C}_1(u))$ is a crossed module of groups for all $u \in C_0$.

A *morphism of pre-crossed modules* $\mathsf{f} : \mathcal{C} \to \mathcal{C}'$, where $\mathcal{C}' = (\partial' : \mathsf{C}'_2 \to \mathsf{C}'_1)$, is a triple $\mathsf{f} = (f_2, f_1, f_0)$, where $\mathsf{f}_2 = (f_2, f_0) : \mathsf{C}_2 \to \mathsf{C}'_2$ and $\mathsf{f}_1 = (f_1, f_0) : \mathsf{C}_1 \to \mathsf{C}'_1$ are morphisms of groupoids satisfying

$$f_2 * \partial'_2 \;=\; \partial_2 * f_1, \qquad f_2(\beta^\alpha) \;=\; (f_2\beta)^{f_1\alpha}, \tag{22}$$

making the following diagram commute:

$$\begin{array}{ccccc} C_2 & \xrightarrow{\partial_2} & C_1 & \overset{\partial_1^-}{\underset{\partial_1^+}{\rightrightarrows}} & C_0 \\ f_2 \big\downarrow & & f_1 \big\downarrow & & \big\downarrow f_0 \\ C'_2 & \xrightarrow[\partial'_2]{} & C'_1 & \overset{\partial_1'^-}{\underset{\partial_1'^+}{\rightrightarrows}} & C'_0 \end{array}$$

When $\mathcal{C}, \mathcal{C}'$ are crossed modules, f is a *morphism of crossed modules*.

We now list some examples of crossed modules of groupoids – those in (a), (c) and (d) have totally disconnected source.

Example 5.1 (a) The crossed module of groupoids corresponding to $\mathcal{X} = (\delta : B \to C)$ is $\mathcal{X}_\bullet = (\delta_\bullet : B_\bullet \to C_\bullet)$ where $\delta_\bullet(\bullet, b, \bullet) = (\bullet, \delta b, \bullet)$ and $(\bullet, b, \bullet)^{(\bullet, c, \bullet)} = (\bullet, b^c, \bullet)$.

(b) Since $\mathsf{C} = C_\bullet \times \mathsf{I}_n$ acts on itself by $\alpha \mapsto \wedge\alpha$, we obtain the identity crossed module $(\mathrm{id} : \mathsf{C} \to \mathsf{C})$.

(c) Given $N \trianglelefteq C$, let $\mathsf{C}_1 = C_\bullet \times \mathsf{I}_n$ and let $\mathsf{C}_2 = N_\bullet \times \mathsf{O}_n$ be the totally disconnected subgroupoid consisting of n copies of $N_\bullet$. Then C_1 acts on C_2 by conjugation, giving the *normal subgroupoid crossed module* $\mathcal{C} = (\mathrm{inc} : \mathsf{C}_2 \to \mathsf{C}_1)$ where inc is the inclusion functor. When C is the trivial group we obtain $\mathcal{I}_n := (\iota_n : \mathsf{O}_n \to \mathsf{I}_n)$.

(d) More generally, if $\mathcal{X} = (\delta : B \to C)$ is a crossed module of groups, let $\mathsf{B} = B_\bullet \times \mathsf{O}_n$ and $\mathsf{C} = C_\bullet \times \mathsf{I}_n$. Then B is a C-groupoid-system, and $\mathcal{C} = (\partial : \mathsf{B} \to \mathsf{C}) \cong \mathcal{X}_\bullet \times \mathcal{I}_n$ is a crossed module with $\partial(p,b,p) = (p,\delta b,p)$.

(e) If $\mathcal{X} = (\delta : B \to C)$ as before, let $\mathsf{B} = B_\bullet \times \mathsf{I}_m$. Then $(\partial : \mathsf{B} \to C_\bullet)$ is a crossed module with $\partial(u,b,v) = \delta b$ and $(u,b,v)^c = (u,b^c,v)$.

(f) Combining the previous two cases, let $\mathsf{B} = (B_\bullet \times \mathsf{I}_m) \times \mathsf{O}_n$, another C-groupoid-system. Again $(\partial : \mathsf{B} \to \mathsf{C})$ is a crossed module with $\partial(p,(u,b,v),q) = (p,\delta b,q)$ and action $(p,c,q) \mapsto \wedge c_{p,q}$ where, for example, $\wedge c_{p,q}\,(p,(u,b,v),p) = (q,(u,b^c,v),q)$.

(g) The *direct product* $\mathcal{C} \times \mathcal{C}'$ is $(\partial \times \partial' : \mathsf{C}_2 \times \mathsf{C}_2' \to \mathsf{C}_1 \times \mathsf{C}_1')$, with $(\beta,\beta')^{(\alpha,\alpha')} := (\beta^\alpha, {\beta'}^{\alpha'})$.

5.3 Automorphism group of $\mathcal{X}_\bullet \times \mathcal{I}_n$

The expectation here is that when $\mathcal{X} = (\delta : B \to C)$ is a crossed module of groups and $\mathcal{C} = (\partial : B_\bullet \times \mathsf{O}_n \to C_\bullet \times \mathsf{I}_n) \cong \mathcal{X}_\bullet \times \mathcal{I}_n$ is the corresponding (totally disconnected to connected) crossed module of groupoids with n objects and $\partial(q,b,q) = (q,\delta b,q)$, then it should be possible to determine the automorphisms of $\mathcal{C}$ from those of $\mathcal{X}$. We have seen earlier that the automorphism group of $B_\bullet \times \mathsf{O}_n$ is $S_n \ltimes (\mathrm{Aut}\,B)^n$, and that of $C_\bullet \times \mathsf{I}_n$ is $((S_n \times \mathrm{Aut}\,C) \ltimes C^n)/K_1(C)$.

There are three types of automorphism, $\mathsf{f}_\pi, \mathsf{f}_\kappa, \mathsf{f}_{\boldsymbol{c}}$, of $\mathcal{C}$, corresponding to the three types of automorphism $\mathsf{a}_\pi, \mathsf{a}_\kappa, \mathsf{a}_{\boldsymbol{c}}$ of a connected groupoid, as discussed in Section 3.1.

(1) For $\pi \in S_n$ we define $\mathsf{a}'_\pi \in \mathrm{Aut}(B_\bullet \times \mathsf{O}_n)$ by $(p,b,p) \mapsto (\pi p, b, \pi p)$. Then $\mathsf{f}_\pi = (\mathsf{a}'_\pi, \mathsf{a}_\pi, \pi)$ is an automorphism of $\mathcal{C}$.

(2) For $\kappa = (\kappa_2,\kappa_1) \in \mathrm{Aut}\,\mathcal{X}_\bullet$ we define $\mathsf{a}'_{\kappa_2} \in \mathrm{Aut}(B_\bullet \times \mathsf{O}_n)$ by $(p,b,p) \mapsto (p,\kappa_2 b,p)$. Then $\mathsf{f}_\kappa = (\mathsf{a}'_{\kappa_2}, \mathsf{a}_{\kappa_1}, (\,)) \in \mathrm{Aut}\,\mathcal{C}$.

(3) For $\boldsymbol{c} = (c_1,\ldots,c_n) \in C^n$, we define $\mathsf{a}'_{\boldsymbol{c}} \in \mathrm{Aut}(B_\bullet \times \mathsf{O}_n)$ by $(p,b,p) \mapsto (p,b^{c_p},p)$. Then $\mathsf{f}_{\boldsymbol{c}} = (\mathsf{a}'_{\boldsymbol{c}}, \mathsf{a}_{\boldsymbol{c}}, (\,)) \in \mathrm{Aut}\,\mathcal{C}$.

Proposition 5.2 *Composition rules for these automorphisms of $\mathcal{C}$ are as follows,*

$$\begin{array}{ccc} \mathsf{f}_\pi * \mathsf{f}_\xi = \mathsf{f}_{\pi*\xi}, & \mathsf{f}_\kappa * \mathsf{f}_\lambda = \mathsf{f}_{\kappa_2*\lambda_2,\kappa_1*\lambda_1}, & \mathsf{f}_{\boldsymbol{c}} * \mathsf{f}_{\boldsymbol{d}} = \mathsf{f}_{\boldsymbol{cd}}, \\ \mathsf{f}_\kappa * \mathsf{f}_\pi = \mathsf{f}_\pi * \mathsf{f}_\kappa, & \mathsf{f}_{\boldsymbol{c}} * \mathsf{f}_\pi = \mathsf{f}_\pi * \mathsf{f}_{\pi\boldsymbol{c}}, & \mathsf{f}_{\boldsymbol{c}} * \mathsf{f}_\kappa = \mathsf{f}_\kappa * \mathsf{f}_{\kappa_1\boldsymbol{c}}. \end{array}$$

Proof: Replacing $\mathsf{a}_\pi, \mathsf{a}_\kappa, \mathsf{a}_{\boldsymbol{c}}$ by $\mathsf{a}'_\pi, \mathsf{a}'_{\kappa_2}, \mathsf{a}'_{\boldsymbol{c}}$ in the formulae of Proposition 3.1 gives composition identities for these automorphisms of $B_\bullet \times \mathsf{O}_n$ with one exception. Since

$$\mathsf{a}'_{\boldsymbol{c}} * \mathsf{a}'_{\kappa_2}\,(p,b,p) \;=\; (p,\kappa_2(b^{c_p}),p) \;=\; (p,(\kappa_2 b)^{\kappa_1 c_p},p) \;=\; \mathsf{a}'_{\kappa_2} * \mathsf{a}'_{\kappa_1\boldsymbol{c}}\,(p,b,p),$$

the sixth identity is $\mathsf{a}'_{\boldsymbol{c}} * \mathsf{a}'_{\kappa_2} = \mathsf{a}'_{\kappa_2} * \mathsf{a}'_{\kappa_1\boldsymbol{c}}$.

Combining the identities for the a's with those for the a''s gives the required identities for the f's. For example, in the exceptional, sixth case (omitting the identity maps on the objects),

$$\begin{array}{rcl} \mathsf{f}_{\boldsymbol{c}} * \mathsf{f}_\kappa & = & (\mathsf{a}'_{\boldsymbol{c}}, \mathsf{a}_{\boldsymbol{c}}) * (\mathsf{a}'_{\kappa_2}, \mathsf{a}_{\kappa_1}) \;=\; (\mathsf{a}'_{\boldsymbol{c}} * \mathsf{a}'_{\kappa_2}, \mathsf{a}_{\boldsymbol{c}} * \mathsf{a}_{\kappa_1}) \\ & = & (\mathsf{a}'_{\kappa_2} * \mathsf{a}'_{\kappa_1\boldsymbol{c}}, \mathsf{a}_{\kappa_1} * \mathsf{a}_{\kappa_1\boldsymbol{c}}) \;=\; (\mathsf{a}'_{\kappa_2}, \mathsf{a}_{\kappa_1}) * (\mathsf{a}'_{\kappa_1\boldsymbol{c}}, \mathsf{a}_{\kappa_1\boldsymbol{c}}) \;=\; \mathsf{f}_\kappa * \mathsf{f}_{\kappa_1\boldsymbol{c}}\,. \end{array}$$

□

For each $c \in C$ there is an automorphism $\wedge c = (\wedge c, \wedge c)$ of $\mathcal{X}_\bullet$ where $\wedge c(\bullet, b, \bullet) = (\bullet, b^c, \bullet)$ and $\wedge c(\bullet, c', \bullet) = (\bullet, c^{-1}c'c, \bullet)$. The automorphism $\mathsf{a}'_{\wedge c}$ of $B_\bullet \times \mathsf{O}_n$, defined in case (2) above, maps (p, b, p) to (p, b^c, p). We may check that $\mathsf{f}_{\wedge c} := (\mathsf{a}'_{\wedge c}, \mathsf{a}_{\wedge c}, (\,))$ is an automorphism of $\mathcal{C}$ as follows,

$$\mathsf{a}'_{\wedge c}(p, b^{c'}, p) \;=\; (p, (b^{c'})^c, p) \;=\; (p, (b^c)^{c^{-1}c'c}, p) \;=\; (p, b^c, p)^{(p, c'^c, p)} \;=\; (\mathsf{a}'_{\wedge c}(p, b, p))^{\mathsf{a}_{\wedge c}(p, c', p)}.$$

Since $\mathsf{f}_{\wedge c}$ is the identity automorphism only when c acts trivially on both b and c', the set $\{c \in C \mid \mathsf{f}_{\wedge c} = \mathsf{i}\}$ is the subgroup $\mathrm{Triv}(C)$ of $Z(C)$. It is easy to check that, for any $\mu \in \mathrm{Aut}\,\mathcal{C}$, the conjugate automorphism $(\mathsf{f}_{\wedge c})^\mu$ is equal to $\mathsf{f}_{\wedge(\mu_1 c)}$.

The following result generalises Proposition 3.1.

Proposition 5.3 *The automorphism group of* $\mathcal{C} = (\partial : (B_\bullet \times \mathsf{O}_n) \to (C_\bullet \times \mathsf{I}_n))$ *is given by*

$$\mathrm{Aut}\,\mathcal{C} \;\cong\; ((S_n \times \mathrm{Aut}\,\mathcal{X}) \ltimes C^n)\,/K_2(C),$$

where $\mathcal{X} = (\delta : B \to C)$ *and* $K_2(C) = \{(((\,), (\wedge c, \wedge c)), (c^{-1}, \ldots, c^{-1})) \mid c \in C\} \cong C.$

Proof: Let $\mathsf{g} = (g_2, g_1, g_0)$ be an automorphism of $\mathcal{C}$. Since g_0 is the permutation of the objects achieved by g, the automorphism $\mathsf{g}' = \mathsf{g} * \mathsf{f}_{g_0}^{-1}$ fixes the objects. If, for each $q \neq 1$, g'_1 maps $(1, e, q)$ to $(1, c_q, q)$, and if $\boldsymbol{c} = (e, c_2, \ldots, c_n)$, then $\mathsf{g}'' = \mathsf{g}' * \mathsf{f}_{\boldsymbol{c}}^{-1}$ fixes all the tree generators $(1, e, q)$. It follows that $g''_1 = \kappa_1 \in \mathrm{Aut}\,C$, and $g''_2(q, b, q) = (q, \overline{\kappa}_q b, q)$ where the $\overline{\kappa}_q$ are automorphisms of B. Applying the second axiom in (22), we obtain:

$$(q, \overline{\kappa}_q b, q) \;=\; g''_2(q, b, q) \;=\; (g''_2(1, b, 1))^{g''_1(1, e, q)} \;=\; (1, \overline{\kappa}_1 b, 1)^{(1, e, q)} \;=\; (q, \overline{\kappa}_1 b, q),$$

so the $\overline{\kappa}_q$ are all equal. Thus $g''_2 = \kappa_2 \in \mathrm{Aut}\,B$ and $\kappa = (\kappa_2, \kappa_1) \in \mathrm{Aut}\,\mathcal{X}$. Hence $\mathsf{g} = \mathsf{f}_\kappa * \mathsf{f}_{\boldsymbol{c}} * \mathsf{f}_{g_0}$, which we take as the standard form for g.

As in the proof of Proposition 3.1, there is an action of $(S_n \times \mathrm{Aut}\,\mathcal{X})$ on C^n, where

$$\boldsymbol{c}^\pi \;=\; \pi\boldsymbol{c} \;=\; (c_{\pi^{-1}1}, \ldots, c_{\pi^{-1}n}), \qquad \boldsymbol{c}^\kappa \;=\; \kappa_1\boldsymbol{c} \;=\; (\kappa_1 c_1, \ldots, \kappa_1 c_n).$$

We define a map

$$\theta_{\mathcal{C}} \;:\; (S_n \times \mathrm{Aut}\,\mathcal{X}) \ltimes C^n \to \mathrm{Aut}\,\mathcal{C}, \qquad ((\pi, \kappa), \boldsymbol{c}) \mapsto \mathsf{f}_\pi * \mathsf{f}_\kappa * \mathsf{f}_{\boldsymbol{c}}.$$

It is then straightforward to show that (compare this with equation (2))

$$(\mathsf{f}_\pi * \mathsf{f}_\kappa * \mathsf{f}_{\boldsymbol{c}}) * (\mathsf{f}_\xi * \mathsf{f}_\lambda * \mathsf{f}_{\boldsymbol{d}}) \;=\; \mathsf{f}_{\pi * \xi} * \mathsf{f}_{\kappa_2 * \lambda_2, \kappa_1 * \lambda_1} * \mathsf{f}_{(\xi\lambda_1\boldsymbol{c})\boldsymbol{d}},$$

so that $\theta_{\mathcal{C}}$ is a group homomorphism. Since

$$(\mathsf{f}_\pi * \mathsf{f}_\kappa * \mathsf{f}_{\boldsymbol{c}})_1 \;:\; \begin{cases} (1, c, 1) \;\mapsto\; (\pi 1, c_{\pi 1}^{-1}(\kappa_1 c)c_{\pi 1}, \pi 1), \\ (1, e, q) \;\mapsto\; (\pi 1, c_{\pi 1}^{-1}c_{\pi q}, \pi q), \end{cases}$$

$$(\mathsf{f}_\pi * \mathsf{f}_\kappa * \mathsf{f}_{\boldsymbol{c}})_2 \;:\; (q, b, q) \;\mapsto\; (\pi q, (\kappa_2 b)^{c_{\pi q}}, \pi q),$$

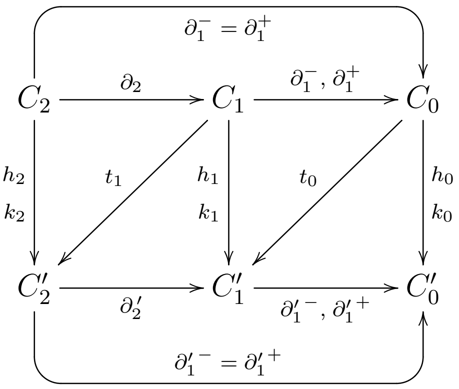
it follows that $\theta_{\mathcal{C}}((\pi, \kappa), \boldsymbol{c})$ is the identity automorphism provided

- π is the identity permutation,
- $c_q = c_1$ for all $2 \leqslant q \leqslant n$, so $\boldsymbol{c} = (c_1, c_1, \ldots, c_1)$,
- $\kappa_1 c = c_1 c c_1^{-1}$ for all $c \in C$, so $(\kappa_1, (\,)) = \mathsf{a}_{\wedge(c_1^{-1})}$,
- $\kappa_2 b = b^{c_1^{-1}}$ for all $b \in B$, so $(\kappa_2, (\,)) = \mathsf{a}'_{\wedge(c_1^{-1})}$.

Hence $\ker\theta_{\mathcal{C}}$ is the specified group $K_2(C)$. □

5.4 Homotopies between morphisms

In this Subsection we first review (with different notation) the definition and properties of (h,k)-homotopies for crossed modules of groupoids, given in Brown and İçen [10, § 2]. Again our expectation is that it should be possible to determine the group of homotopies of $\mathcal{X}_\bullet \times \mathcal{I}_n$ given the Whitehead group $W(\mathcal{X})$, and we show in Proposition 5.10 that this is the case.



Definition 5.4 [10, Definition 2.1 and Proposition 2.2] *Let* $\mathsf{h},\mathsf{k} : \mathcal{C} = (\partial : \mathsf{C}_2 \to \mathsf{C}_1) \to \mathcal{C}' = (\partial' : \mathsf{C}'_2 \to \mathsf{C}'_1)$ *be morphisms of crossed modules. A* crossed module (h,k)-homotopy $\mathsf{t} : \mathsf{h} \simeq \mathsf{k}$ *is a pair of functions* (t_1, t_0) *such that the following three conditions hold.*

(a) $t_0 : C_0 \to C'_1$ *is an* (h_0, k_0)*-section, so* $t_0 * \partial_1^- = h_0,\ \ t_0 * \partial_1^+ = k_0$.

(b) $t_1 : C_1 \to C'_2$ *is a* k*-derivation, by which we mean that* $t_1\alpha$ *is a loop at* $\partial_1^+ k_1\alpha$*, and that* $t_1(\alpha\alpha') = (t_1\alpha)^{k_1\alpha'}(t_1\alpha')$ *when the composite is defined.*

(c) For all $(\alpha : u \to v) \in C_1$ *and* $(\beta : v \to v) \in C_2$ *the loops* $\partial'_2 t_1\alpha$ *and* $t_1\partial_2\beta$ *measure the divergence from commutativity of the following squares (in the second square dashed lines denote arrows in* C'_1*) generalising (16),*

$$\text{(i)}\quad \partial'_2 t_1\alpha = (k_1\alpha)^{-1}(t_0u)^{-1}(h_1\alpha)(t_0v), \qquad \text{(ii)}\quad t_1\partial_2\beta = (k_2\beta)^{-1}(h_2\beta)^{t_0v}. \tag{23}$$

We then call t_1 *an* (h,k)*-derivation.*

Again our definition is not the standard one, since we do not require B to be totally disconnected. Note that we *do* require $t_1\alpha$ to be a *loop* in C'_2, so that condition (c) can be satisfied.

From now on we consider the case when $\mathcal{C} = \mathcal{C}'$ and h,k are automorphisms. In the special case that $\mathsf{k} = \mathsf{i}$ it is usual to call t a *free homotopy* and t_1 a *free derivation*. In another special case, when

$t_0 u = 1_u$ for all $u \in C_0$ we call t a *homotopy over the identity section* and t_1 a *derivation over the identity section*. A free derivation over the identity is simply called a *derivation*. We denote by $W^1_{\mathsf{k}}(\mathcal{C})$ the set of k-derivations over the identity section.

The following result generalises the construction in (7).

Proposition 5.5 *If* $\mathsf{t} = (t_1, t_0)$ *is an* (h, k)*-homotopy and* j *is an automorphism of* $\mathcal{C}$*, then a* $(\mathsf{j}*\mathsf{h}, \mathsf{j}*\mathsf{k})$*-homotopy* $\mathsf{r} = (r_1, r_0)$ *is defined by*

$$r_0 u \;:=\; t_0(j_0 u), \qquad r_1\alpha \;:=\; t_1(j_1\alpha).$$

Proof: Using equations (7), (19) and Definition 5.4 we check that:

$$\begin{array}{rcl} r_1(\alpha\alpha') & = & t_1((j_1\alpha)(j_1\alpha')) \;=\; (t_1 j_1\alpha)^{k_1 j_1\alpha'}(t_1 j_1\alpha') \;=\; (r_1\alpha)^{(\mathsf{j}*\mathsf{k})_1\alpha'}(r_1\alpha'), \\ \partial_2 r_1\alpha & = & \partial_2 t_1 j_1\alpha \;=\; (k_1 j_1\alpha)^{-1}(t_0 j_0 u)^{-1}(h_1 j_1\alpha)(t_0 j_0 v) \;=\; (k_1 j_1\alpha)^{-1}(r_0 u)^{-1}(h_1 j_1\alpha)(r_0 v), \\ r_1\partial_2\beta & = & t_1 j_1 \partial_2\beta \;=\; t_1\partial_2 j_2\beta \;=\; (k_2 j_2\beta)^{-1}(h_2 j_2\beta)^{t_0 j_0 v} \;=\; ((\mathsf{j}*\mathsf{k})_2\beta)^{-1}((\mathsf{j}*\mathsf{h})_2\beta)^{r_0 v}. \end{array}$$

It follows that the (h, k)-homotopies may be obtained from the $(\mathsf{k}^{-1} * \mathsf{h}, \mathsf{i})$-homotopies. □

On fixing an automorphism k of $\mathcal{C}$, let $*_{\mathsf{k}}$ be the multiplication on $\mathrm{Aut}\,\mathcal{C}$ given in terms of the standard composition by $\mathsf{h} *_{\mathsf{k}} \mathsf{g} \;:=\; \mathsf{h} * \mathsf{k}^{-1} * \mathsf{g}$, such that k is the $*_{\mathsf{k}}$-identity and h has $*_{\mathsf{k}}$-inverse $\overline{\mathsf{h}} := \mathsf{k} * \mathsf{h}^{-1} * \mathsf{k}$. The next result combines the product in (14) for sections with the product in (18) for derivations to give a product for homotopies.

Proposition 5.6 [10, Proposition 2.4] *The set* $H^1_{\mathsf{k}}(\mathcal{C})$ *of* (h, k)*-homotopies of* $\mathcal{C}$ *with fixed* k *form a monoid with product* $\star_{\mathsf{k}}$*, where the composite* $(\mathsf{h} *_{\mathsf{k}} \mathsf{g}, \mathsf{k})$*-homotopy* $\mathsf{t} \star_{\mathsf{k}} \mathsf{s}$ *is defined by:*

$$(\mathsf{t} \star_{\mathsf{k}} \mathsf{s})_0\, u \;:=\; (t_0 \star_{k_0} s_0)u \;=\; (s_0 k_0^{-1} h_0 u)(t_0 u), \quad (\mathsf{t} \star_{\mathsf{k}} \mathsf{s})_1\, \alpha \;=\; (t_1\alpha)(s_1 k_1^{-1} h_1 \alpha)^{t_0 v}. \tag{24}$$

Proof: A proof when $\mathsf{k} = \mathsf{i}$ is given in [10], and is easily adapted to the general case. □

The $(\mathsf{h} *_{\mathsf{k}} \mathsf{g} *_{\mathsf{k}} \mathsf{f}, \mathsf{k})$-homotopy $\mathsf{t} \star_{\mathsf{k}} \mathsf{s} \star_{\mathsf{k}} \mathsf{r}$ is given by:

$$\begin{array}{rcl} (\mathsf{t} \star_{\mathsf{k}} \mathsf{s} \star_{\mathsf{k}} \mathsf{r})_0\, u & = & (r_0 k_0^{-1} g_0 k_0^{-1} h_0 u)(s_0 k_0^{-1} h_0 u)(t_0 u), \\ (\mathsf{t} \star_{\mathsf{k}} \mathsf{s} \star_{\mathsf{k}} \mathsf{r})_1\, \alpha & = & (t_1\alpha)(s_1 k_1^{-1} h_1\alpha)^{t_0 v}(r_1 k_1^{-1} g_1 k_1^{-1} h_1 \alpha)^{(\mathsf{t}\star_{\mathsf{k}}\mathsf{s})_0 v}. \end{array} \tag{25}$$

The arrows in these composite formulae are shown in Figure 3, which extends the sketch in § 4.4.

Following § 4.4, we call an invertible (h, k)-homotopy both an *admissible* h*-homotopy* and a *coadmissible* k*-homotopy*.

As an introduction to the methods used in the n-object case in § 5.5, we investigate the (h, k)-homotopies $\mathsf{t} = (t_1, t_0)$ of $\mathcal{X}_\bullet = (\delta_\bullet : B_\bullet \to C_\bullet)$. Since, in this single object case, $h_0 = k_0 = (\ )$, the identity map on $\{\bullet\}$, condition 5.4(a) is trivial. Any map $\{\bullet\} \to C$ is an (h_0, k_0)-section, so we may choose $t_0(\bullet) = (\bullet, d, \bullet)$. Condition 5.4(b) is just the requirement that t_1 is a (k_2, k_1)-derivation of $\mathcal{X} = (\delta : B \to C)$. The table in (26) compares the conditions in 5.4(c) for t_1 to be an (h, k)-derivation of $\mathcal{X}_\bullet$, where $\alpha = (\bullet, c, \bullet) \in C_\bullet$, and $\beta = (\bullet, b, \bullet) \in B_\bullet$, with the corresponding requirements in (16) for an (η, κ)-derivation ϕ of $\mathcal{X}$.

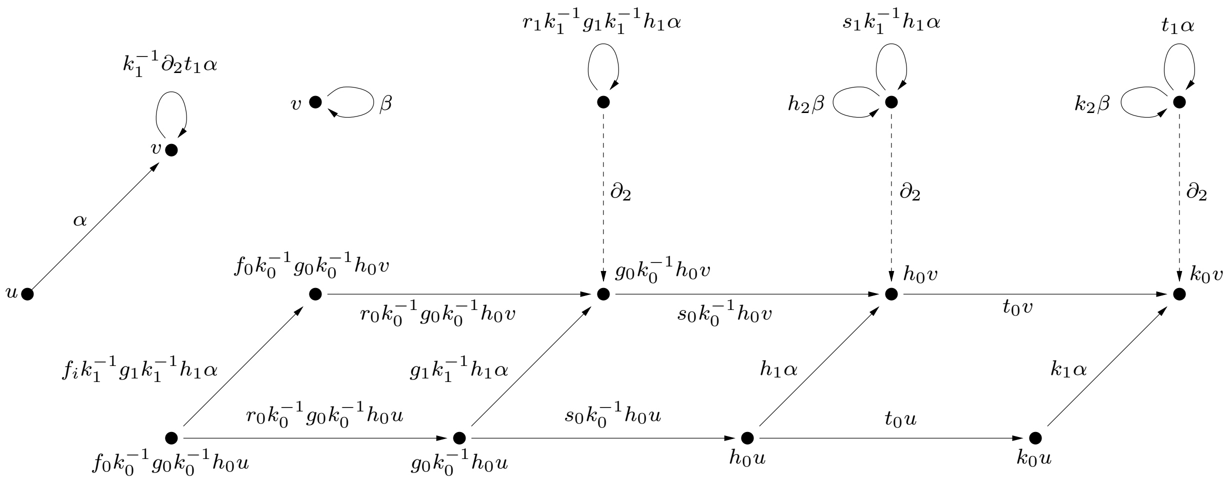


Figure 3: Composite homotopy

condition	$\mathcal{X}_\bullet$	$\mathcal{X}$
5.4(c)(i)	$(k_1\alpha)(\delta_\bullet t_1\alpha) = (h_1\alpha)^{(\bullet,d,\bullet)}$	$(\kappa_1 c)(\partial\phi c) = \eta_1 c$
5.4(c)(ii)	$(k_2\beta)(t_1\delta_\bullet\beta) = (h_2\beta)^{(\bullet,d,\bullet)}$	$(\kappa_2 b)(\phi\partial b) = \eta_2 b$

(26)

Since h_0, k_0 are trivial we may consider h, k as automorphisms of $\mathcal{X}$, so t_1 is an $(\mathsf{h} * \wedge d, \mathsf{k})$-derivation of $\mathcal{X}$.

Proposition 5.7 *The group $H^1_{\mathsf{i}}(\mathcal{X}_\bullet)$ of coadmissible* i*-homotopies of $\mathcal{X}_\bullet$ is isomorphic to the semidirect product $C \ltimes W(\mathcal{X})$, where the action is that in* $\mathrm{Act}(\mathcal{X})$.

Proof: Each $d \in C$ acts via $\nu_{\iota,1}$ on $\phi \in W(\mathcal{X})$ as $\phi^d := \phi^{\wedge d} = (\wedge d^{-1}) * \phi * (\wedge d)$, so

$$\phi^d c \;=\; \left(\phi(dcd^{-1})\right)^d \;=\; \left((\phi d)^{cd^{-1}}(\phi c)^{d^{-1}}(\phi(d^{-1}))\right)^d \;=\; (\phi d)^c(\phi c)(\phi d)^{-1}\,. \tag{27}$$

For $\phi \in W(\mathcal{X})$ the (ζ_ϕ, i)-homotopy t_ϕ is defined by $\mathsf{t}_{\phi,0}(\bullet) = (\bullet, e, \bullet)$, $\mathsf{t}_{\phi,1}(\bullet, c, \bullet) = (\bullet, \phi c, \bullet)$. Similarly, we may define s_d for $d \in C$ by $\mathsf{s}_{d,0}(\bullet) = (\bullet, d^{-1}, \bullet)$, $\mathsf{s}_{d,1}(\bullet, c, \bullet) = (\bullet, e, \bullet)$. Using equations (23) to calculate $g_1\alpha$ and $g_2\beta$, we find that s is a (g, i)-homotopy where $g_1(\bullet, c, \bullet) = (\bullet, d^{-1}cd, \bullet)$ and $g_2(\bullet, b, \bullet) = (\bullet, b^d, \bullet)$, so $\mathsf{g} = \wedge d$. We have seen that if $\{d_1, \ldots, d_\ell\}$ and $\{\phi_1, \ldots, \phi_m\}$ are generating sets for C and $W(\mathcal{X})$ respectively, then $\{\mathsf{s}_{d_1}, \ldots, \mathsf{s}_{d_\ell}, \mathsf{t}_{\phi_1}, \ldots, \mathsf{t}_{\phi_m}\}$ is a generating set for $H^1_{\mathsf{i}}(\mathcal{X})$. We claim that there is an isomorphism $M^1_{\mathsf{i}}(\mathcal{X}_\bullet) \to C \ltimes W(\mathcal{X})$ mapping s_{d_i} to $(d_i, 0)$ and t_{ϕ_j} to (e, ϕ_j) so that $\mathsf{s}_{d_i} \star \mathsf{t}_{\phi_j} \mapsto (d_i, \phi_j)$ and $\mathsf{t}_{\phi_j} \star \mathsf{s}_{d_i} \mapsto (d_i, {\phi_j}^{d_i})$. The product equations (24) give:

$$\begin{array}{rclrcl} (\mathsf{t}_\phi \star \mathsf{s}_d)_0(\bullet) &=& (\bullet, d^{-1}, \bullet)\,, & (\mathsf{t}_\phi \star \mathsf{s}_d)_1(\bullet, c, \bullet) &=& (\bullet, \phi c, \bullet)\,, \\ (\mathsf{s}_d \star \mathsf{t}_\phi)_0(\bullet) &=& (\bullet, d^{-1}, \bullet)\,, & (\mathsf{s}_d \star \mathsf{t}_\phi)_1(\bullet, c, \bullet) &=& (\bullet, \phi^{d^{-1}} c, \bullet)\,. \end{array}$$

Hence $\mathsf{t}_\phi \star \mathsf{s}_d \;=\; \mathsf{s}_d * \mathsf{t}_{\phi^d}$ and, since the generators satisfy the rules for a semidirect product, an isomorphism is obtained. □

5.5 Homotopy group of $\mathcal{X}_\bullet \times \mathcal{I}_n$

Consider the case, as in Subsection 5.3, when $\mathcal{C} = \mathcal{X}_\bullet \times \mathcal{I}_n$ is connected, and $\mathcal{X} = (\delta : B \to C)$. Again the expectation is that it should be possible to determine the homotopies of $\mathcal{C}$ given the regular derivations of $\mathcal{X}$. We have seen in Proposition 5.5 that, once we know the (h,i)-homotopies $\mathsf{t}=(t_1,t_0)$ of $\mathcal{C}$, then the rest are easily obtained. So we attempt to enumerate the former, assuming known the (η,ι)-derivations of $\mathcal{X}$. Because of the multiplication rule for t_1, we may define an (h,i)-derivation by specifying the images of a generating set (just as we did for automorphisms).

Proposition 5.8 *An (h,i)-derivation t_1 is defined by specifying*

- $t_1(1,c,1)=(1,\phi c,1)$*, where $\phi : C \to B$ is a chosen ι-derivation for $\mathcal{X}$,*
- *a choice of images $t_1(1,e,q)=(q,b_q,q),\ 2 \leqslant q \leqslant n$, for arrows in the tree T_1.*

The resulting $(\zeta_\phi * (\wedge \boldsymbol{b}), \mathsf{i})$-derivation, where $\boldsymbol{b}=(e,b_2,\ldots,b_n)$, is given by:

$$t_1(p,c,q) \;=\; (q,(b_p^{-1})^c(\phi c)b_q,q) \quad \textit{for all} \quad 1 \leqslant p,q \leqslant n \quad \textit{and} \quad c \in C. \tag{28}$$

Proof: Applying the multiplication rule in Definition 5.4(b) we find, for $p,q \geqslant 2$, that

$$\begin{aligned}
t_1(p,e,1) &= (p,b_p^{-1},p)^{(p,e,1)}t_1(1,e,1) \;=\; (1,b_p^{-1},1),\\
t_1(1,c,p) &= (1,\phi c,1)^{(1,e,p)}(p,b_p,p) \;=\; (p,(\phi c)b_p,p),\\
t_1(p,c,p) &= (1,b_p^{-1},1)^{(1,c,p)}(p,(\phi c)b_p,p) \;=\; (p,(b_p^{-1})^c(\phi c)b_p,p)\\
t_1(p,c,1) &= (1,b_p^{-1},1)^{(1,c,1)}(1,\phi c,1) \;=\; (1,(b_p^{-1})^c(\phi c),1),\\
t_1(p,c,q) &= (1,(b_p^{-1})^c(\phi c),1)^{(1,e,q)}(q,b_q,q) \;=\; (q,(b_p^{-1})^c(\phi c)b_q,q).
\end{aligned}$$

It is then easy to check that the final formula holds for all $1 \leqslant p,q \leqslant n$. We then verify the multiplication rule of Definition 5.4 (b) as follows:

$$\begin{aligned}
(t_1(p,c,q))^{(q,c',r)}\,t_1(q,c',r) &= (q,(b_p^{-1})^c(\phi c)b_q,q)^{(q,c',r)}\,(r,(b_q^{-1})^{c'}(\phi c')b_r,r)\\
&= (r,(b_p^{-1})^{cc'}(\phi c)^{c'}(\phi c')b_r,r)\\
&= t_1(p,cc',r).
\end{aligned}$$

To determine the source automorphism of t_1 we note that equations (23) reduce, in the case of derivations over the identity section, to $h_1\alpha=(k_1\alpha)(\partial t_1\alpha)$ and $h_2\beta=(k_2\beta)(t_1\partial\beta)$, agreeing with the definition of ζ_ϕ in (16). Here $\mathsf{k}=\mathsf{i}$, and we find that

$$h_1(p,c,q) \;=\; (p,(\delta b_p)^{-1}(\zeta_{\phi,1}c)(\delta b_q),q) \qquad \text{and} \qquad h_2(q,b,q) \;=\; (q,b_q^{-1}(\zeta_{\phi,2}b)b_q,q)\,.$$

Hence $\mathsf{h}=\zeta_\phi * (\wedge \boldsymbol{b})$. □

Proposition 5.9 *The group $W_{\mathsf{i}}^1(\mathcal{C})$ is isomorphic to $(W(\mathcal{X}) \ltimes B^n)/K_3(B)$ where $K_3(B)=\{(\phi_a,(a^{-1},\ldots,a^{-1}) \mid a \in B\}$.*

Proof: Two sets of derivations generate the group $W_{\mathsf{i}}^1(\mathcal{C})$.

- For each $\phi \in W(\mathcal{X})$ let $t_{\phi,1}$ be the derivation obtained by taking $b_q=e$ for all $q \geqslant 2$. Then $t_{\phi,1}(p,c,q)=(q,\phi c,q)$ for all p,q and c.
- For $\boldsymbol{a} \in B^n$ let $t_{\boldsymbol{a},1}$ be the derivation obtained by taking $b_q=a_1^{-1}a_q$, and let ϕ be the principal derivation ϕ_{a_1} of (17). Then $t_{\boldsymbol{a},1}(p,c,q)=(q,(a_p^{-1})^c a_q,q)$ for all p,q and c.

The product rule $(t_1 \star s_1)\alpha = (t_1\alpha)(s_1 h_1 \alpha)$ gives $(t_{\phi,1} \star t_{\boldsymbol{a},1})(p,c,q) = (q, (a_p^{-1})^c(\phi c)a_q, q)$, so this product is the general t_1 in Proposition 5.8. Also,

$$\begin{aligned}(t_{\boldsymbol{a},1} \star t_{\phi,1})(p,c,q) &= (q, (a_p^{-1})^c a_q, q) t_{\phi,1}(p, (\delta a_p)^{-1} c (\delta a_q), q) \\ &= (q, (a_p^{-1})^c a_q (\phi((\delta a_p)^{-1}))^{c(\delta a_q)} (\phi c)^{\delta a_q} (\phi \delta a_q), q) \\ &= (q, (a_p^{-1})^c (\phi(\delta a_p^{-1}))^c (\phi c) a_q (\phi \delta a_q), q) \\ &= (q, (\zeta_{\phi,2} a_p^{-1})^c (\phi c)(\zeta_{\phi,2} a_q), q)\,,\end{aligned}$$

so $t_{\boldsymbol{a},1} \star t_{\phi,1} = t_{\phi,1} \star t_{\zeta_{\phi,2}\boldsymbol{a},1} = t_{\phi,1} \star t_{\boldsymbol{a}^\phi,1}$. This gives the required isomorphism. □

In conclusion, we have obtained four sets of homotopies generating the group $H^1_{\mathsf{i}}(\mathcal{C})$.

(1) For each $\pi \in S_n$ the $(\mathsf{h}_\pi, \mathsf{i})$-homotopy which simply permutes the objects is t_π where

$$t_{\pi,0}\, p = (\pi p, e, p), \qquad t_{\pi,1}(p,c,q) = (q,e,q), \qquad \mathsf{h}_\pi = (\mathrm{id}, \mathrm{id}, \pi)\,.$$

(2) For each $\boldsymbol{d} \in C^n$ there is an $(\mathsf{f}_{\boldsymbol{d}}, \mathsf{i})$-homotopy $\mathsf{s}_{\boldsymbol{d}}$ where

$$s_{\boldsymbol{d},0}\, p = (p, d_p^{-1}, p), \qquad s_{\boldsymbol{d},1}(p,c,q) = (q,e,q)\,.$$

(3) For each (ζ_ϕ, ι)-derivation $\phi \in W(\mathcal{X})$ there is an $(\mathsf{h}_\phi, \mathsf{i})$-homotopy t_ϕ where

$$t_{\phi,0}\, p = (p,e,p), \qquad t_{\phi,1}(p,c,q) = (q, \phi c, q), \qquad \mathsf{h}_\phi = (\zeta_{\phi,2}, \zeta_{\phi,1}, (\,))\,.$$

(4) For each $\boldsymbol{a} \in B^n$ there is an $(\mathsf{f}_{\delta_2 \boldsymbol{a}}, \mathsf{i})$-homotopy $\mathsf{t}_{\boldsymbol{a}}$ where $\delta_2 \boldsymbol{a} = (\delta_2 a_1, \ldots, \delta_2 a_n)$ and

$$t_{\boldsymbol{a},0}\, p = (p,e,p), \qquad t_{\boldsymbol{a},1}(p,c,q) = (q, (a_p^{-1})^c a_q, q)\,.$$

The (i,i)-homotopy e, where $e_0 p = (p,e,p)$ and $e_1(p,c,q) = (q,e,q)$ for all $1 \leqslant p,q \leqslant n,\ c \in C$, is the identity element in the group.

We now investigate composites of the set

$$X_M = \{\mathsf{t}_\pi \mid \pi \in S_n\} \cup \{\mathsf{s}_{\boldsymbol{d}} \mid \boldsymbol{d} \in C^n\} \cup \{\mathsf{t}_\phi \mid \phi \in W(\mathcal{X})\} \cup \{\mathsf{t}_{\boldsymbol{a}} \mid \boldsymbol{a} \in B^n\}\,.$$

Brown and İçen in [10, Theorem 2.6] have shown, generalising (21), that $\mathrm{Aut}\,\mathcal{C}$ acts on $H^1_{\mathsf{i}}(\mathcal{C})$. (We have used the action on sections in § 4.5.) The more general k-action is given by

$$(\mathsf{t}^{\mathsf{f}})_0 := k_0 * f_0^{-1} * t_0 * k_1^{-1} * f_1\,, \qquad (\mathsf{t}^{\mathsf{f}})_1 := k_1 * f_1^{-1} * t_1 * k_2^{-1} * f_2\,.$$

When $\mathsf{k} = \mathsf{i}$, automorphism f_π acts trivially on t_ϕ, while other particular cases are given by:

$$(\mathsf{t}_{\boldsymbol{a}})^{\mathsf{f}_\pi} = \mathsf{t}_{\pi\boldsymbol{a}}\,, \quad (\mathsf{s}_{\boldsymbol{d}})^{\mathsf{f}_\pi} = \mathsf{s}_{\pi\boldsymbol{d}}\,, \quad (\mathsf{t}_{\boldsymbol{a}})^{\mathsf{f}_\phi} = \mathsf{t}_{\zeta_{\phi,2}\boldsymbol{a}}\,, \quad (\mathsf{t}_{\boldsymbol{a}})^{\mathsf{f}_{\boldsymbol{d}}} = \mathsf{t}_{\boldsymbol{a}^{\boldsymbol{d}}}\,, \tag{29}$$

where $\boldsymbol{a}^{\boldsymbol{d}} = (a_1^{d_1}, \ldots, a_n^{d_n})$. Furthermore, a generalised version of (27) is given by $(\mathsf{t}_\phi^{\mathsf{f}_{\boldsymbol{d}}})_1\,(p,c,q) = (q, (\phi d_p)^c(\phi c)(\phi d_q)^{-1}, q)$. Hence, by (28) and Proposition 5.9,

$$\mathsf{t}_\phi^{\mathsf{f}_{\boldsymbol{d}}} = \mathsf{t}_\phi \star \mathsf{t}_{(\phi\boldsymbol{d})^{-1}} \qquad \text{where} \quad \phi\boldsymbol{d} := (\phi d_1, \ldots, \phi d_n) \in B^n \tag{30}$$

and, using (25), we may check that $\mathsf{t}_\phi \star \mathsf{s}_{\boldsymbol{d}} = \mathsf{s}_{\boldsymbol{d}} \star \mathsf{t}_\phi \star \mathsf{t}_{(\phi\boldsymbol{d})^{-1}}$.

Proposition 5.10 *The homotopy group $H^1_{\mathsf{i}}(\mathcal{C})$ for $\mathcal{C} = \mathcal{X}_\bullet \times \mathcal{I}_n$ is given by*

$$H^1_{\mathsf{i}}(\mathcal{C}) \;\cong\; ((S_n \ltimes C^n) \ltimes (W(\mathcal{X}) \ltimes B^n))\,/K_4(B)$$

where $K_4(B) \;=\; \{(((\,), (e,\ldots,e)), (\phi_a, (a^{-1},\ldots,a^{-1}))) \mid a \in B\} \;\cong\; B.$

Proof: We have already seen that $M^1_{\mathsf{i}}(\mathcal{C}) \cong S_n \ltimes C^n$ and that $W^1_{\mathsf{i}}(\mathcal{C}) \cong (W(\mathcal{X}) \ltimes B^n)/K_3(B)$. It is suggested in [10, Theorem 2.7] that an isomorphism $H^1_{\mathsf{i}}(\mathcal{C}) \cong M^1_{\mathsf{i}}(\mathcal{C}) \ltimes W^1_{\mathsf{i}}(\mathcal{C})$ is immediate from the definition of homotopy multiplication in (24), so there is no more to do. However, since the calculations are quite intricate, we prefer to outline the details. Define a map

$$\theta_H \;:\; (S_n \ltimes C^n) \ltimes (W(\mathcal{X}) \ltimes B^n) \to H^1_{\mathsf{i}}(\mathcal{C}), \qquad ((\pi, \boldsymbol{d}), (\phi, \boldsymbol{a})) \mapsto \mathsf{t}_\pi * \mathsf{s}_{\boldsymbol{d}} * \mathsf{t}_\phi * \mathsf{t}_{\boldsymbol{a}}\,.$$

Pairs of homotopies in X_M compose as follows, where $\pi, \xi \in S_n,\; \boldsymbol{c}, \boldsymbol{d} \in C^n,\; \phi, \psi \in W(\mathcal{X})$ and $\boldsymbol{a}, \boldsymbol{b} \in B^n$:

$$\begin{array}{lll}
\mathsf{t}_\pi \star \mathsf{t}_\xi = \mathsf{t}_{\pi\star\xi}, & \mathsf{s}_{\boldsymbol{d}} \star \mathsf{t}_\pi = \mathsf{t}_\pi \star \mathsf{s}_{\pi\boldsymbol{d}}, & \mathsf{t}_\phi \star \mathsf{t}_\pi = \mathsf{t}_\pi \star \mathsf{t}_\phi, \qquad \mathsf{t}_{\boldsymbol{a}} \star \mathsf{t}_\pi = \mathsf{t}_\pi \star \mathsf{t}_{\pi\boldsymbol{a}}, \\
& \mathsf{s}_{\boldsymbol{d}} \star \mathsf{s}_{\boldsymbol{c}} = \mathsf{s}_{\boldsymbol{dc}}, & \mathsf{t}_\phi \star \mathsf{s}_{\boldsymbol{d}} = \mathsf{s}_{\boldsymbol{d}} \star \mathsf{t}_\phi \star \mathsf{t}_{(\phi\boldsymbol{d})^{-1}}, \qquad \mathsf{t}_{\boldsymbol{a}} \star \mathsf{s}_{\boldsymbol{d}} = \mathsf{s}_{\boldsymbol{d}} \star \mathsf{t}_{\boldsymbol{a}^{\boldsymbol{d}}}, \\
& & \mathsf{t}_\phi \star \mathsf{t}_\psi = \mathsf{t}_{\phi\star\psi}, \qquad \mathsf{t}_{\boldsymbol{a}} \star \mathsf{t}_\phi = \mathsf{t}_\phi \star \mathsf{t}_{\zeta_{\phi,2}\,\boldsymbol{a}}, \\
& & \qquad \mathsf{t}_{\boldsymbol{a}} \star \mathsf{t}_{\boldsymbol{b}} = \mathsf{t}_{\boldsymbol{ab}}.
\end{array}$$

These formulae, together with the actions in (29) and (30) show that θ_H is surjective, preserves the multiplication, and that

$$(\mathsf{t}_\pi \star \mathsf{s}_{\boldsymbol{d}} \star \mathsf{t}_\phi \star \mathsf{t}_{\boldsymbol{a}}) \star (\mathsf{t}_\xi \star \mathsf{s}_{\boldsymbol{c}} \star \mathsf{t}_\psi \star \mathsf{t}_{\boldsymbol{b}}) \;=\; \mathsf{t}_{\pi*\xi} \star \mathsf{s}_{(\xi\boldsymbol{d})\boldsymbol{c}} \star \mathsf{t}_{\phi\star\psi} \star \mathsf{t}_{(\zeta_{\psi,2}((\phi\boldsymbol{c})^{-1}(\xi\boldsymbol{a})^{\boldsymbol{c}}))\boldsymbol{b}}\,.$$

It is clear that $\mathsf{t}_\pi \star \mathsf{s}_{\boldsymbol{d}} \star \mathsf{t}_\phi \star \mathsf{t}_{\boldsymbol{a}}$ is the identity homotopy e provided

- π is the identity permutation,
- $\boldsymbol{d}$ is the identity vector $(e,\ldots,e)$,
- $(\mathsf{t}_{\boldsymbol{a}} \star \mathsf{t}_\phi)_1(p,c,q) \;=\; (q, (a_p^{-1})^c(\phi c)a_q, q) = (q,e,q)$ for all p, q and c.

From the third of these conditions we first deduce that $\boldsymbol{a}$ is a constant vector $(a,\ldots,a)$, and then that $\phi c = a^c a^{-1}$ for all $c \in C$. Hence ϕ is the principal derivation $\phi_{a^{-1}}$ and $\ker\theta_H$ is the specified subgroup $K_4(C)$. □

Note that, when $n = 1$, this result reduces to $H^1_{\mathsf{i}}(\mathcal{X}_\bullet) \cong (C \ltimes (W(\mathcal{X}) \ltimes B))/B$, which simplifies to $C \ltimes W(\mathcal{X})$ as shown in Proposition 5.7.

Finally, let us compare the homotopy group of $\mathcal{C}_1 = (\mathrm{id} : \mathsf{C} \to \mathsf{C})$ with that of $\mathcal{C}_2 = (\partial : C_\bullet \times \mathsf{O}_n \to \mathsf{C})$, where $\mathsf{C} = C_\bullet \times \mathsf{I}_n$. Clearly the (h_0, k_0)-sections are the same in both cases. Conditions (23) for derivations become, for both $\mathcal{C}_1$ and $\mathcal{C}_2$,

$$t_1\alpha \;=\; (k_1\alpha)^{-1}(t_0u)^{-1}(h_1\alpha)(t_0v)\,, \qquad t_1\beta \;=\; (k_2\beta)^{-1}(t_0v)^{-1}(h_2\beta)(t_0v)\,.$$

Since $h_1 = h_2$ and $k_1 = k_2$, the second equation specialises the first, so the β in $\mathcal{C}_1$ which are not loops provide no extra requirements. Hence $H^1_{\mathsf{i}}(\mathcal{C}_1) \cong H^1_{\mathsf{i}}(\mathcal{C}_2)$.

The work in [10] goes on to explore a 2-crossed module $M^2(\mathcal{C}) \to H^1(\mathcal{C}) \to \mathrm{Aut}\,\mathcal{C}$ where $M^2(\mathcal{C})$ is the group of 2-sections $T_0 : C_0 \to C_2$. We hope to extend our algebraic investigations to this situation in a future paper.

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