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Automorphisms and Quasi-Free States of the CAR Algebra

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Abstract. We study automorphisms of the CAR algebra which map the family of gauge-invariant, quasi-free states of the CAR algebra onto itself and show (Theorem 3.1) that they are one-particle automorphisms.

1. Introduction

The problem discussed in this paper arose from questions regarding equilibrium states of thermodynamical systems. Equilibrium states have been extensively discussed in the framework of C^*-algebras of observables (see, for example [6, 15]). Such states are labeled by a very small number of parameters. For example, in the case of a gas of identical particles, the equilibrium states are labeled by the temperature, the chemical potential, and the average velocity of the particles – quantities which relate directly with the conserved quantities – energy, particle number, and total momentum. Since conserved quantities are in one-one correspondence with one-parameter groups of transformations which leave the Hamiltonian invariant, we can describe the situation in a way which remains meaningful for infinite systems. Equilibrium states of thermodynamical systems are labeled by a very small number of parameters which relate directly with one-parameter, automorphism groups of the observable algebra that commute with the time-evolution automorphisms. The fact that there are so few parameters involved, which is related to the fact that there are only a small number of one-parameter groups of automorphisms that commute with the time-evolution, is an aspect of the ergodic nature of most large physical systems. A proof based on the dynamics of the system is still lacking, even though Sinai has obtained very interesting results in this direction for a classical system of N hard spheres.

Systems of particles without interaction do not behave ergodically in the above sense. This does not mean, however, that systems of noninteracting particles are uninteresting from the point of view of ergodicity. In [7, 8], the asymptotic time behavior of the free Fermi gas is discussed. It is found in [7] that, for increasing time, primary states of the CAR algebra are asymptotic to gauge-invariant, quasi-free states, provided these states satisfy a certain clustering property. In particular, primary, stationary (i.e. time-invariant) states with that clustering property are quasi-free. Quasi-free states [1–5, 9–14, 16, 17] are particularly simple states in the sense that they lack all except two-point correlations. Some basic facts about gauge-invariant, quasi-free states are given in Section 2. These states are in one-one correspondence with positive operators in the unit ball, the so-called one-particle operators, acting on the one-particle Hilbert space $\mathscr{H}$. Stationary, quasi-free states of the free fermions are not labeled by a finite number of parameters, as in the interacting case, but by one-particle operators that commute with the Hamiltonian

$$H(= -(\hbar^2/2m)(\partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2) = -(\hbar^2/2m)\Delta).$$

In particular, all operators of the form, $f(\hbar\partial/i\partial x, \hbar\partial/i\partial y, \hbar\partial/i\partial z)$ with f positive and having essential supremum 1, define translationally-invariant, stationary states of the free fermion system. In this case, $f(p_1, p_2, p_3)$ is the momentum distribution of the particles.

The free fermion system is of particular interest to us because, as noted before, the set of all primary states that satisfy a certain clustering property is known. It is a subset of the gauge-invariant, quasi-free states. This result is an important tool in determining the automorphisms that commute with the free-time evolution. Since a clustering property that is somewhat stronger than that which holds for all primary states may well be a condition one must impose upon a physically meaningful state, we shall restrict our considerations to those automorphisms of the CAR algebra whose transposes preserve this property. Let α be such an automorphism that, in addition, commutes with the free-time evolution. Its transpose maps the set of stationary, quasi-free states, that satisfy the clustering property, onto itself. What can one conclude about α?

Before attempting to solve this problem, one is faced with a more primitive question. What can be said about an automorphism α of the CAR algebra whose transpose maps the set of quasi-free states (or a certain subset of it) onto itself? Our main result (Theorem 3.1) states that, when the transpose leaves the set of gauge-invariant, quasi-free states stable, then, either the Fock state is mapped onto itself and there is a unitary operator U on $\mathscr{H}$, such that $\alpha(a(f)) = a(Uf)$, or the Fock state is mapped onto the anti-Fock state and there is a conjugate-unitary operator W on $\mathscr{H}$, such that $\alpha(a(f)) = a(Wf)^*$, where $a(f)$ is the annihilation operator on Fock space.

A related result is stated in Theorem 4.1. A unitary operator on one-particle space defines, in an obvious manner, a unitary operator on n-particle space. Theorem 4.1 characterizes such unitary operators on n-particle space as those which map anti-symmetrized products of one-particle wave functions *(product vectors)* onto product vectors.

Section 2 is devoted to notation and a number of preliminary results, which are used throughout the paper. The main theorem is proven in Section 3; and Section 4 contains some related results.

2. Some Preliminaries

An infinite system of identical Fermi particles can be represented, insofar as their algebraic interrelations are concerned, by a C^*-algebra, $\mathfrak{A}$, the so-called CAR algebra. The abstract algebra $\mathfrak{A}$ may be characterized as the norm closure (completion) of an algebra generated by a countably-infinite family of pairwise-commuting, self-adjoint algebras each isomorphic to the algebra of 2×2 complex matrices and each containing the unit, I, of $\mathfrak{A}$. Each $*$-representation of $\mathfrak{A}$ gives rise to a representation of the canonical anticommutation relations (CAR), and conversely.

For our purposes, it is more useful to describe $\mathfrak{A}$ in its *Fock representation*. With $\mathscr{H}$ a complex Hilbert space and $\mathscr{H}_n$ the n-fold tensor product, so that, for $x_1, \ldots, x_n, y_1, \ldots, y_n$ in $\mathscr{H}$, $\langle x_1 \otimes \cdots \otimes x_n | y_1 \otimes \cdots \otimes y_n \rangle = \langle x_1 | y_1 \rangle \cdots \langle x_n | y_n \rangle$, let S_n^- be the projection operator on $\mathscr{H}_n$ which assigns $\frac{1}{n!}\sum_\sigma \chi(\sigma) x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(n)}$ to $x_1 \otimes \cdots \otimes x_n$, where σ is a permutation of $\{1, \ldots, n\}$ and $\chi(\sigma)$ is $+1$ if σ is even, -1 if σ is odd. The range of S_n^- is the space $\mathscr{H}_n^{(a)}$ of antisymmetric tensors. We write $x_1 \wedge \cdots \wedge x_n$ for $(n!)^{1/2} S_n^-(x_1 \otimes \cdots \otimes x_n)$ (the "antisymmetrized, n-particle state with wave functions $x_1, \ldots, x_n$"). We have:

$$\begin{aligned}\langle x_1 \wedge \cdots \wedge x_n | y_1 \wedge \cdots \wedge y_n \rangle &= n! \langle x_1 \otimes \cdots \otimes x_n | S_n^-(y_1 \otimes \cdots \otimes y_n)\rangle \\ &= \sum_\sigma \chi(\sigma) \langle x_1 | y_{\sigma(1)} \rangle \cdots \langle x_n | y_{\sigma(n)} \rangle = \det(\langle x_i | y_j \rangle).\end{aligned}$$

Thus, assuming $x_1 \wedge \cdots \wedge x_n$ and $y_1 \wedge \cdots \wedge y_n$ are not 0, they are orthogonal if and only if there are scalars $c_1, \ldots, c_n$, not all 0, such that

$$0 = \sum_{i=1}^{n} \langle c_i x_i | y_j \rangle = \left\langle \sum_{i=1}^{n} c_i x_i | y_j \right\rangle ;$$

that is, if and only if the space, $[x_1, \ldots, x_n]$, generated by $x_1, \ldots, x_n$, contains a non-zero vector $(\Sigma c_i x_i)$ orthogonal to $[y_1, \ldots, y_n]$. If, in addition, the intersection, $[x_1, \ldots, x_n] \cap [y_1, \ldots, y_n]$, of the spaces $[x_1, \ldots, x_n]$ and $[y_1, \ldots, y_n]$ has dimension $n-1$ (in this case, we say that the spaces are "perpendicular"), the projections with ranges $[x_1, \ldots, x_n]$ and $[y_1, \ldots, y_n]$ commute. It follows that $\{e_{i_1} \wedge \cdots \wedge e_{i_n}\}$ is an orthonormal basis for $\mathscr{H}_n^{(a)}$ if $\{e_i\}$ is an orthonormal basis for $\mathscr{H}$. Moreover, $x_1 \wedge \cdots \wedge x_n = 0$ if and only if $x_1, \ldots, x_n$ are linearly dependent (if and only if $[x_1, \ldots, x_n]$ has dimension less than n). Thus $z \in [x_1, \ldots, x_n]$, if $z \wedge x_1 \wedge \cdots \wedge x_n = 0$ and $x_1 \wedge \cdots \wedge x_n \neq 0$. From this, if $x_1 \wedge \cdots \wedge x_n = y_1 \wedge \cdots \wedge y_n \neq 0$, then $[x_1, \ldots, x_n] = [y_1, \ldots, y_n]$. On the other hand, if $[x_1, \ldots, x_n] = [y_1, \ldots, y_n]$, then, expressing each y_j as a linear combination of $x_1, \ldots, x_n$, we see that $x_1 \wedge \cdots \wedge x_n$ and $y_1 \wedge \cdots \wedge y_n$ are scalar multiples of one another. We say that $x_1 \wedge \cdots \wedge x_n$ is a *product vector* – the *exterior* (or, *wedge*) *product* of $x_1, \ldots, x_n$.

The *antisymmetric Fock space*, $\mathscr{H}_{\mathscr{F}}^{(a)}$, is $\sum_{n=0}^{\infty} \oplus \mathscr{H}_n^{(a)}$. By definition $\mathscr{H}_0^{(a)}$ consists of complex scalar multiples of a single (unit) vector Φ_0, the *Fock vacuum*; and $\mathscr{H}_1^{(a)}$ is $\mathscr{H}$. If $\mathscr{H}$ were finite dimensional, $\mathscr{H}_{\mathscr{F}}^{(a)}$ would be the (finite-dimensional) "exterior" algebra over $\mathscr{H}$. The mapping, $\wedge$, from the n-fold Cartesian product $\mathscr{H} \times \cdots \times \mathscr{H}$ to $\mathscr{H}_n^{(a)}$ which assigns $x_1 \wedge \cdots \wedge x_n$ to $(x_1, \ldots, x_n)$ is an alternating, n-linear mapping. If $\tilde{a}$ is such a mapping of $\mathscr{H} \times \cdots \times \mathscr{H}$ into a space $\mathscr{K}$, there is a mapping $\hat{a}$ of $\mathscr{H}_n^{(a)}$ into $\mathscr{K}$ such that $\tilde{a} = \hat{a} \circ \wedge$. In particular if T is a linear mapping of $\mathscr{H}$ into $\mathscr{K}$ then $(x_1, \ldots, x_n) \to Tx_1 \wedge \cdots \wedge Tx_n$ is an alternating n-linear mapping of $\mathscr{H} \times \cdots \times \mathscr{H}$ into $\mathscr{K}_n^{(a)}$; so that there is a linear mapping $\hat{T}$ of $\mathscr{H}_{\mathscr{F}}^{(a)}$ into $\mathscr{K}_{\mathscr{F}}^{(a)}$ such that $\hat{T}(x_1 \wedge \cdots \wedge x_n) = Tx_1 \wedge \cdots \wedge Tx_n$. If T is a unitary transformation of $\mathscr{H}$ onto $\mathscr{K}$, metric considerations apply and $\hat{T}$ is a unitary transformation of $\mathscr{H}_{\mathscr{F}}^{(a)}$ onto $\mathscr{K}_{\mathscr{F}}^{(a)}$.

If $\|T\| \leqq 1$, $\tilde{\mathscr{H}} = \mathscr{H} \oplus \mathscr{H}$, $\tilde{\mathscr{K}} = \mathscr{K} \oplus \mathscr{K}$, $P(x, y) = (x, 0)$, and $Q(u, v) = (u, 0)$, with x, y in $\mathscr{H}$ and u, v in $\mathscr{K}$, then there is a unitary transformation U of $\tilde{\mathscr{H}}$ onto $\tilde{\mathscr{K}}$ such that $QUP(x, y) = (Tx, 0)$ for all x, y in $\mathscr{H}$. Then $\hat{U}$ is a unitary transformation of $\tilde{\mathscr{H}}_n^{(a)}$ onto $\tilde{\mathscr{K}}_n^{(a)}$ and $\hat{Q}$ is a projection of $\tilde{\mathscr{K}}_n^{(a)}$ onto $\mathscr{K}_n^{(a)}$. Since $\hat{T}$ is the restriction of $\hat{Q}\hat{U}$ to $\mathscr{H}_n^{(a)}$, we see that $\|\hat{T}\| \leqq 1$. If T is a positive operator with pure point spectrum, computing norms with a basis of eigenvectors for T, we find that $\|\hat{T}|\mathscr{H}_n^{(a)}\| = \lambda_1 \cdots \lambda_n$, where $\lambda_1, \ldots, \lambda_n$ are the n largest eigenvalues of T (multiplicity included). An approximation argument provides the corresponding result for a general positive operator; and polar decomposition provides a norm formula for a general bounded operator. A simple check yields $(\hat{T})^* = \hat{T}^*$.

Since $(f_1, \ldots, f_n) \to f \wedge f_1 \wedge \cdots \wedge f_n$ is an alternating, n-linear mapping, there is a linear mapping, $a_n(f)^*$, of $\mathscr{H}_n^{(a)}$ into $\mathscr{H}_{n+1}^{(a)}$ with value $f \wedge f_1 \wedge \cdots \wedge f_n$ at $f_1 \wedge \cdots \wedge f_n$. The family $\{a_n(f)^*\}$ defines a mapping $a(f)^*$ on $\mathscr{H}_{\mathscr{F}}^{(a)}$. With $\{e_i\}$ an orthonormal basis for $\mathscr{H}$, $a(e_1)^*$ maps $\{e_{i_1} \wedge \cdots \wedge e_{i_n} : 1 \notin \{i_1, \ldots, i_n\}, n = 0, 1, 2, \ldots\}$ onto an orthonormal basis for the orthogonal complement, $\mathscr{H}_{\mathscr{F}}^{(a)} \ominus \mathscr{K}$; and $a(e_1)^*$ annihilates this complement. Thus $a(e_1)^*$ is a partial isometry with initial space $\mathscr{K}$ and final space $\mathscr{H}_{\mathscr{F}}^{(a)} \ominus \mathscr{K}$. It follows that $I = a(e_1)^* a(e_1) + a(e_1) a(e_1)^*$ $(= \{a(e_1), a(e_1)^*\}_+)$. More generally $a(f)a(f)^* + a(f)^* a(f) = \langle f | f \rangle I$. Polarization of this yields: $\{a(f), a(g)^*\}_+ = \langle f | g \rangle I$. We note that our inner product, $\langle f | g \rangle$, is linear in g and conjugate linear in f. We have $\{a(f)^*, a(g)^*\}_+ = 0$, as well. A conjugate-linear mapping $f \to a(f)$ of $\mathscr{H}$ onto operators $a(f)$ on a Hilbert space satisfying the relations (canonical anticommutation relations)

$$\{a(f), a(g)^*\}_+ = \langle f | g \rangle I, \ \{a(f), a(g)\}_+ = 0$$

is said to be a *representation* of the CAR. The particular representation we have exhibited on $\mathscr{H}_{\mathscr{F}}^{(a)}$ is called the *Fock representation.*

We can exhibit the *annihilator* $a(f)$ as explicitly as we described the *creator* $a(f)^*$ by expanding the determinant expression for $\langle f \wedge y_2 \wedge \cdots \wedge y_n | x_1 \wedge \cdots \wedge x_n \rangle$ in terms of its first row:

$$\begin{aligned} &\langle f \wedge y_2 \wedge \cdots \wedge y_n | x_1 \wedge \cdots \wedge x_n \rangle \\ &\quad = \langle y_2 \wedge \cdots \wedge y_n | a(f)(x_1 \wedge \cdots \wedge x_n) \rangle \\ &\quad = \sum_{j=1}^{n} (-1)^{j+1} \langle f | x_j \rangle \langle y_2 \wedge \cdots \wedge y_n | x_1 \wedge \cdots \wedge x_{j-1} \wedge x_{j+1} \wedge \cdots \wedge x_n \rangle ; \end{aligned}$$

so that

$$a(f)(x_1 \wedge \cdots \wedge x_n) = \sum_{j=1}^{n} (-1)^{j+1} \langle f | x_j \rangle x_1 \wedge \cdots \wedge x_{j-1} \wedge x_{j+1} \wedge \cdots \wedge x_n .$$

With E a projection on $\mathscr{H}$, we denote by $\mathfrak{A}_0(E)$ and $\mathfrak{A}(E)$ the $*$-algebra and C^*-algebra, respectively, on $\mathscr{H}_{\mathscr{F}}^{(a)}$ generated by $\{a(f) : Ef = f\}$. We write $\mathfrak{A}_0$ and $\mathfrak{A}$ in place of $\mathfrak{A}_0(I)$ and $\mathfrak{A}(I)$. The C^*-algebra $\mathfrak{A}$ is the CAR algebra and its action on $\mathscr{H}_{\mathscr{F}}^{(a)}$ is called its Fock representation. The state ϕ_0 of $\mathfrak{A}$ for which $\phi_0(A) = \langle \Phi_0 | A \Phi_0 \rangle$ is called the *Fock (vacuum) state* of $\mathfrak{A}$. Note that each $a(f)$ is in its left kernel $\mathscr{K}$ $(\phi_0(a(f)^* a(f)) = 0)$; so that each product of annihilators and creators *(monomial)* in which an annihilator appears to the right is in $\mathscr{K}$. Now each monomial is a sum of monomials in which all creators are to the left of all annihilators (we say that such a monomial is *Wick-ordered* – and *anti-Wick-ordered* if all creators are to the right of all annihilators); so that ϕ_0 annihilates all Wick-ordered monomials in $\mathfrak{A}_0$ other than I. These monomials span the null space of ϕ_0 on $\mathfrak{A}_0$. If ϱ is a state of $\mathfrak{A}$ and $\varrho \leqq 2\phi_0$, then $a(f)$ is in the left kernel of ϱ. Thus ϱ and ϕ_0 have the same null space in $\mathfrak{A}_0$ and agree at I. Hence $\varrho = \phi_0$; and ϕ_0 is a pure state of $\mathfrak{A}$. Exactly the same considerations apply to the restriction of ϕ_0 to $\mathfrak{A}(E)$, for each projection E on $\mathscr{H}$. Thus the restriction of ϕ_0 to $\mathfrak{A}(E)$ is pure.

The Hilbert space $\bar{\mathscr{H}}$, obtained from $\mathscr{H}$ by assigning an element $\bar{f}$ to each f in $\mathscr{H}$, defining $(\overline{cf+g})$ to be $c\bar{f}+\bar{g}$ and $\langle \bar{f}|\bar{g}\rangle$ to be $\langle g|f\rangle$, produces $\mathscr{H}_{\mathscr{F}}^{(a)}$, *anti-Fock space*, and $\bar{\Phi}_0$ is the *anti-Fock vacuum*. The mapping $f \to a(\bar{f})^*$ $(=\bar{a}(f))$ is a representation of the CAR (over $\mathscr{H}$), the *anti-Fock representation;* and the mapping $a(f) \to \bar{a}(f)$ extends to a $*$-isomorphism, $A \to \bar{A}$, of the CAR algebra $\mathfrak{A}$ over $\mathscr{H}$ onto the CAR algebra $\bar{\mathfrak{A}}$ over $\bar{\mathscr{H}}$. The state ϕ_I of $\mathfrak{A}$ defined by $A \to \langle \bar{\Phi}_0 | \bar{A}\bar{\Phi}_0\rangle$ is the *anti-Fock state* of $\mathfrak{A}$. Each $a(f)^*$ is in the left kernel of ϕ_I; so that, replacing $a(f)$ by $a(f)^*$ and using anti-Wick-ordered monomials instead of Wick-ordered monomials in the argument above, we have that the restriction of ϕ_I to each $\mathfrak{A}(E)$ is pure.

Since ϕ_0 is pure and Φ_0 is cyclic for $\mathfrak{A}$, the weak-operator closure, $\mathfrak{A}^-$, of $\mathfrak{A}$, is $\mathscr{B}(\mathscr{H}_{\mathscr{F}}^{(a)})$, the algebra of all bounded operators on $\mathscr{H}_{\mathscr{F}}^{(a)}$. Similarly $\mathfrak{A}(E)^-\tilde{E}_0 = \mathscr{B}([\mathfrak{A}(E)\Phi_0])$, where $\tilde{E}_0$ is the projection (in $\mathfrak{A}(E)'$) with range $[\mathfrak{A}(E)\Phi_0]$. If U_E is $(\widetilde{I-2E})$, then $U_E\Phi_0 = \Phi_0$, $a(g)U_E = U_E a(g)$, for each g in $(I-E)(\mathscr{H})$, and $a(f)U_E = -U_E a(f)$, for each f in $E(\mathscr{H})$. If A_0 is an even monomial in $\mathfrak{A}_0(I-E)$ (that is, A_0 is the product of an even total number of annihilators and creators) and A_1 is an odd monomial in $\mathfrak{A}_0(I-E)$, then A_0 and A_1U_E lie in $\mathfrak{A}(E)'$. Since $\mathfrak{A}_0(E)$ and $\mathfrak{A}_0(I-E)$ generate $\mathfrak{A}_0$ and Φ_0 is cyclic for $\mathfrak{A}_0$;

$$\mathscr{H}_{\mathscr{F}}^{(a)} = [\mathfrak{A}_0\Phi_0] = [\mathfrak{A}(E)\mathfrak{A}(I-E)\Phi_0] = [\mathfrak{A}(E)\mathfrak{A}(E)'\Phi_0].$$

Thus $\tilde{E}_0$ has central carrier I in $\mathfrak{A}(E)^-$; and the mapping ι_E of $\mathfrak{A}(E)^-\tilde{E}_0$ onto $\mathfrak{A}(E)^-$ which assigns A to $A\tilde{E}_0$ is a $*$-isomorphism.

Now, $a(f)\Phi_0 = 0$ and, when $Ef = 0$, $a(f)(\mathfrak{A}_0(E)\Phi_0) = (0)$. Thus $a(f)\tilde{E}_0 = 0$ and $\tilde{E}_0 a(f)^* = 0$, when $Ef = 0$; so that $\tilde{E}_0 A\tilde{E}_0 = \lambda\tilde{E}_0$ when A is in $\mathfrak{A}_0(I-E)$. It follows that $B \to \tilde{E}_0 B\tilde{E}_0$ is a (completely-) positive, linear mapping of $\mathscr{B}(\mathscr{H}_{\mathscr{F}}^{(a)})$ onto $\mathfrak{A}(E)^-\tilde{E}_0$. The composition of this mapping with ι_E is a completely-positive, linear mapping, ψ_E, of $\mathscr{B}(\mathscr{H}_{\mathscr{F}}^{(a)})$ onto $\mathfrak{A}(E)^-$. By construction of ψ_E,

$$\psi_E(a(x_n)^* \ldots a(x_1)^* a(y_1)\ldots a(y_m)) = a(Ex_n)^* \ldots a(Ex_1)^* a(Ey_1)\ldots a(Ey_m).$$

More generally:

Proposition 2.1. *If T is a linear transformation of one Hilbert space, $\mathscr{H}$, into another, $\mathscr{K}$, and $\|T\| \leqq 1$, then the mapping*

$$a(x_n)^* \ldots a(x_1)^* a(y_1)\ldots a(y_m) \to a(Tx_n)^* \ldots a(Tx_1)^* a(Ty_1)\ldots a(Ty_m)$$

extends (uniquely) to a completely-positive, linear mapping ψ_T of the CAR *algebra, $\mathfrak{A}_{\mathscr{H}}$, over $\mathscr{H}$ into the* CAR *algebra, $\mathfrak{A}_{\mathscr{K}}$, over $\mathscr{K}$.*

Proof. If $\tilde{\mathscr{H}} = \mathscr{H} \oplus \mathscr{H}$, $\tilde{\mathscr{K}} = \mathscr{K} \oplus \mathscr{K}$, $P(h,h') = (h,0)$ for h, h' in $\mathscr{H}$, $Q(k,k') = (k,0)$ for k, k' in $\mathscr{K}$, and $\tilde{T}(h,h') = (Th, 0)$, then there is a unitary transformation U of $\tilde{\mathscr{H}}$ onto $\tilde{\mathscr{K}}$ such that $QUP = \tilde{T}$. The mapping $a(f) \to a(Uf)$ extends, uniquely, to a $*$-isomorphism of $\mathfrak{A}_{\tilde{\mathscr{H}}}$ onto $\mathfrak{A}_{\tilde{\mathscr{K}}}$. The composition of the restriction of this isomorphism to $\mathfrak{A}_{\tilde{\mathscr{H}}}(P)$ and ψ_Q is ψ_T.

We note that the characterization of ψ_T as the result of distributing T throughout a Wick-ordered monomial is independent of the ordering *only if* T is an isometry; for $\psi_T(a(f)a(f)^*)=\psi_T(I-a(f)^*a(f))=I-a(Tf)^*a(Tf)\neq a(Tf)a(Tf)^*$, when $\|f\|=1$, unless $\langle Tf|Tf\rangle=1$.

If $A\in\mathscr{B}(\mathscr{H})$ and $0\leqq A\leqq I$, we call $\phi_I\circ\psi_{A^{\frac{1}{2}}}$ the *gauge-invariant, quasi-free state* of $\mathfrak{A}$ with *one-particle operator* A. We write ϕ_A for this state and note that there is no conflict between this notation and the designation of the Fock and anti-Fock states of by ϕ_0 and ϕ_I (i.e. these states are quasi-free with one-particle operators 0 and I, respectively). Note that

$$\begin{aligned}
&\phi_A(a(f_n)^*\dots a(f_1)^*a(g_1)\dots a(g_n))\\
&=\phi_I(a(A^{\frac{1}{2}}f_n)^*\dots a(A^{\frac{1}{2}}f_1)^*a(A^{\frac{1}{2}}g_1)\dots a(A^{\frac{1}{2}}g_n))\\
&=\langle\bar{\Phi}_0|a(\overline{A^{\frac{1}{2}}f_n})\dots a(\overline{A^{\frac{1}{2}}f_1})a(\overline{A^{\frac{1}{2}}g_1})^*\dots a(\overline{A^{\frac{1}{2}}g_n})^*\bar{\Phi}_0\rangle\\
&=\langle\overline{A^{\frac{1}{2}}f_1}\wedge\dots\wedge\overline{A^{\frac{1}{2}}f_n}|\overline{A^{\frac{1}{2}}g_1}\wedge\dots\wedge\overline{A^{\frac{1}{2}}g_n}\rangle\\
&=\det(\langle\overline{A^{\frac{1}{2}}f_i}|\overline{A^{\frac{1}{2}}g_j}\rangle)=\det(\langle g_j|Af_i\rangle)\\
&=\det(\langle g_i|Af_j\rangle)=\langle g_1\wedge\dots\wedge g_n|Af_1\wedge\dots\wedge Af_n\rangle\,.
\end{aligned}$$

Proposition 2.2. *If E is a finite-dimensional projection on $\mathscr{H}$ with $\{e_1,\dots,e_n\}$ an orthonormal basis for $E(\mathscr{H})$, then $\phi_E(T)=\langle e_1\wedge\dots\wedge e_n|T(e_1\wedge\dots\wedge e_n)\rangle$.*

Proof. Let $\{e_j\}$ be an orthonormal basis for $\mathscr{H}$, and T be a Wick-ordered monomial in annihilators and creators corresponding to basis elements. Then $\langle e_1\wedge\dots\wedge e_n|T(e_1\wedge\dots\wedge e_n)\rangle$ is 0 unless T has the form $a(e_{i_{\sigma(m)}})^*\dots a(e_{i_{\sigma(1)}})^*$ $\cdot a(e_{i_1})\dots a(e_{i_m})$, with $\{i_1\dots,i_m\}$ an m-element subset of $\{1,\dots,n\}$, in which case its value and that of $\phi_E(T)$ is $\chi(\sigma)$. If T does not have this form $\psi_E(T)=0$, so $\phi_E(T)=0$. Thus our equality holds.

If follows that ϕ_E is pure when E is a finite-dimensional projection on $\mathscr{H}$. More generally, if E is any orthogonal projection on $\mathscr{H}$ and ϱ is a state of $\mathfrak{A}$ such that $\varrho\leqq 2\phi_E$ then the restrictions of ϱ to $\mathfrak{A}(E)$ and $\mathfrak{A}(I-E)$ coincide with those of ϕ_I and ϕ_0, respectively. Using the fact that monomials A and A' in $\mathfrak{A}_0(E)$ and $\mathfrak{A}_0(I-E)$, respectively, commute or anti-commute and that Wick-ordered monomials are in the left or right kernels of ϕ_0 while anti-Wick ordered monomials are in the left or right kernels of ϕ_I, (other than cI, $c\neq 0$), we conclude that $\varrho(AA')=\varrho(A)\varrho(A')$. The same is true for A in $\mathfrak{A}(E)$ and A' in $\mathfrak{A}(I-E)$. Thus $\varrho=\phi_E$ and ϕ_E is pure.

If $0\leqq A_0\leqq I$ with $A_0(\neq A_0^2)$ in $\mathscr{B}(\mathscr{H})$, using the Spectral Theorem, there is a one-dimensional projection E_1 on $\mathscr{H}$ and a positive number t such that $0\leqq A_1\leqq I$ and $0\leqq A_2\leqq I$, where $A_1=A_0+tE_1$ and $A_2=A_0-tE_1$. Computing with an orthonormal basis $\{e_j\}$ for $\mathscr{H}$ such that $E_1e_1=e_1$, we have that $\phi_{A_0}=\frac{1}{2}(\phi_{A_1}+\phi_{A_2})$. To see this, note that

$$\begin{aligned}
&\phi_{A_k}(a(e_{i_n})^*\dots a(e_{i_1})^*a(e_{j_1})\dots a(e_{j_n}))\\
&\qquad=\langle e_{j_1}\wedge\dots\wedge e_{j_n}|A_ke_{i_1}\wedge\dots\wedge A_ke_{i_n}\rangle\,,
\end{aligned}$$

where $k=0,1,2$; and that $A_0e_j=A_1e_j=A_2e_j$, when $j\neq 1$. Thus ϕ_A is pure if and only if A is a projection.

From the foregoing, if E is a finite-dimensional projection, ϕ_E is a pure, gauge-invariant, quasi-free state equivalent to the Fock state. Conversely, if E is a projection on one-particle space $\mathscr{H}$ and ϕ_E is equivalent to the Fock state, then E is a finite-dimensional projection. This follows as a special case of [12; Theorem 2.8]. A direct proof is not difficult. If $\phi_E = \omega_x | \mathfrak{A}$, for some unit vector x of $\mathscr{H}_{\mathscr{F}}^{(a)}$, then $1 = \phi_E(a(e_j)^* a(e_j)) = \omega_x(a(e_j)^* a(e_j))$, where $\{e_j\}$ is an orthonormal basis for $E(\mathscr{H})$. Thus $a(e_j)^* x = 0$, for each j. If

$$x = \sum_{i_1 < \cdots < i_n; j_1 < \cdots < j_m} c_{i_1 \ldots i_n; j_1 \ldots j_m} e_{i_1} \wedge \cdots \wedge e_{i_n} \wedge e'_{j_1} \wedge \cdots \wedge e'_{j_m},$$

where $\{e'_j\}$ is an orthonormal basis for $(I-E)(\mathscr{H})$, then

$$0 = a(e_j)^* x = \Sigma c_{i_1 \ldots i_n; j_1 \ldots j_m} e_j \wedge e_{i_1} \wedge \cdots \wedge e_{i_n} \wedge e'_{j_1} \wedge \cdots \wedge e'_{j_m};$$

so that $j \in \{i_1, \ldots, i_n\}$ unless $c_{i_1 \ldots i_n; j_1 \ldots j_m} = 0$. If $E(\mathscr{H})$ is infinite-dimensional, we can choose j not in $\{i_1, \ldots, i_n\}$; and $x = 0$, contradicting the assumption that x is a unit vector. Thus E is a finite-dimensional projection.

3. Automorphisms Preserving Gauge-Invariant, Quasi-Free States

Our main result is contained in the following theorem.

Theorem 3.1. *If $\mathfrak{A}$ is the* CAR *algebra in its Fock representation on the complex Hilbert space $\mathscr{H}_{\mathscr{F}}^{(a)}$ of antisymmetric tensors over $\mathscr{H}$ and α is an automorphism of $\mathfrak{A}$ whose transpose $\hat{\alpha}$ carries the set of gauge-invariant, quasi-free states onto itself, then, either the Fock state is mapped onto itself by $\hat{\alpha}$ and there is a unitary operator U on $\mathscr{H}$ such that $\alpha(a(f)) = a(Uf)$, or the Fock state is mapped onto the anti-Fock state by $\hat{\alpha}$ and there is a conjugate-linear, unitary operator W on $\mathscr{H}$ such that $\alpha(a(f)) = a(Wf)^*$.*

The proof of Theorem 3.1 will be effected with the aid of the following results (3.2–3.13). During its course, we will note, in the case where $\hat{\alpha}(\phi_0) = \phi_0$, that $\hat{U}$ implements α (see Section 2).

Lemma 3.2. *The image of ϕ_0 under $\hat{\alpha}$ is either ϕ_0 or ϕ_I.*

Proof. Since $\hat{\alpha}(\phi_0)$ is pure and, by assumption, a quasi-free, gauge-invariant state of $\mathfrak{A}$; $\hat{\alpha}(\phi_0) = \phi_{E_0}$, for some projection E_0 on $\mathscr{H}$ (see Section 2). If π is the representation of $\mathfrak{A}$ on $\mathscr{H}_\pi$ determined by ϕ_{E_0}, then the mapping $A\Phi_0 \to \pi(\alpha^{-1}(A))x_{E_0}$, where x_{E_0} is a unit vector in $\mathscr{H}_\pi$ such that $\phi_{E_0}(A) = \langle x_{E_0} | \pi(A) x_{E_0} \rangle$ for all A in $\mathfrak{A}$, extends to a unitary transformation U of $\mathscr{H}_{\mathscr{F}}^{(a)}$ onto $\mathscr{H}_\pi$. If E_0 is neither 0 nor I, there is a unit vector f in $E_0(\mathscr{H})$ and a unit vector g orthogonal to $E_0(\mathscr{H})$. We shall arrive at a contradiction from this assumption, so that E_0 is either 0 or I and $\hat{\alpha}(\phi_0)$ is either the Fock or anti-Fock state.

If E_1 is the projection on $\mathscr{H}$ with $\{x : \langle x | f \rangle = 0, E_0 x = x\}$ as range then $\phi_{E_0}(A) = \phi_{E_1}(a(f) A a(f)^*)$, for $\phi_{E_1}(a(f) a(h) a(h)^* a(f)^*) = 0$, when $E_1 h = h$ or when $h = f$; and $\phi_{E_1}(a(f) a(k)^* a(k) a(f)^*) = \phi_{E_1}(a(f) a(f)^* a(k)^* a(k)) = 0$, when $E_0 k = 0$. Thus ϕ_{E_1} is equivalent to ϕ_{E_0}; and there is a vector x_{E_1} in $\mathscr{H}_\pi$ such that $\phi_{E_1}(A) = \langle x_{E_1} | \pi(A) x_{E_1} \rangle$. Similarly, if E_2 is the projection on $\mathscr{H}$ with range generated by $E_1(\mathscr{H})$ and g, then $\phi_{E_2}(A) = \phi_{E_1}(a(g) A a(g)^*)$, for all A in $\mathfrak{A}$; and there is a vector x_{E_2} in $\mathscr{H}_\pi$ such that $\phi_{E_2}(A) = \langle x_{E_2} | \pi(A) x_{E_2} \rangle = \langle \pi(a(g)^*) x_{E_1} | \pi(A) \pi(a(g)^*) x_{E_1} \rangle$.

Since π is irreducible, $x_{E_2}=c_2\pi(a(g)^*)x_{E_1}$, and, similarly, $x_{E_0}=c_0\pi(a(f)^*)x_{E_1}$, where c_0 and c_2 are scalars of modulus 1. Now,

$$2^{-\frac{1}{2}}(x_{E_0}+x_{E_2})=\pi(a(2^{-\frac{1}{2}}(c_0 f+c_2 g))^*)x_{E_1}(=x_F)\;;$$

so that $\phi_F(A)=\langle x_F|\pi(A)x_F\rangle$, for all A in $\mathfrak{A}$, where F is the projection on $\mathscr{H}$ with range generated by $E_1(\mathscr{H})$ and $c_0 f+c_2 g$. If $Ux_1=x_F$ and $Ux_2=x_{E_2}$ then, since $U\Phi_0=x_{E_0}$, $\Phi_0+x_2=2^{\frac{1}{2}}x_1$. Noting that $UAU^{-1}=\pi(\alpha^{-1}(A))$, for each A in $\mathfrak{A}$, we have $\langle x_2|Ax_2\rangle=\langle x_{E_2}|UAU^{-1}x_{E_2}\rangle=\langle x_{E_2}|\pi(\alpha^{-1}(A))x_{E_2}\rangle=\phi_{E_2}(\alpha^{-1}(A))$. Similarly $\langle x_1|Ax_1\rangle=\phi_F(\alpha^{-1}(A))$ and $\langle\Phi_0|A\Phi_0\rangle=\phi_{E_0}(\alpha^{-1}(A))$. Thus $\omega_{x_1}|\mathfrak{A}$ and $\omega_{x_2}|\mathfrak{A}$ are pure, gauge-invariant, quasi-free states which transform under $\hat{\alpha}$ into ϕ_F and ϕ_{E_2}. Since ϕ_F and ϕ_{E_2} are equivalent to ϕ_{E_0}, $\omega_{x_1}|\mathfrak{A}$ and $\omega_{x_2}|\mathfrak{A}$ are equivalent to ϕ_0. From Proposition 2.2, x_1 and x_2 are product vectors in $\mathscr{H}_{\mathscr{F}}^{(a)}$. But $\Phi_0+x_2=2^{\frac{1}{2}}x_1$, and each product vector lies in an n-particle space. Thus x_1 and x_2 are multiples of Φ_0; and $\phi_{E_2}=\phi_0$, contrary to the choice of E_2 different from 0.

In case $\hat{\alpha}(\phi_0)=\phi_I$, $\hat{\alpha}'(\phi_0)=\phi_0$, where $\alpha'=\alpha\circ\sigma$ and $\sigma(a(f))=a(W_0 f)^*$ with W_0 a conjugate-unitary operator on $\mathscr{H}$ (obtained, for example, by transforming each linear combination of elements in an orthonormal basis for $\mathscr{H}$ into the linear combination resulting from replacing each coefficient by its complex-conjugate). Since σ determines an automorphism of $\mathfrak{A}$ which interchanges the Fock and anti-Fock states and which maps the set of gauge-invariant, quasi-free states onto itself; $\hat{\alpha}'$ maps the set of gauge-invariant, quasi-free states onto itself. If we prove that there is a unitary operator U_0 on $\mathscr{H}$ such that $\alpha'(a(f))=a(U_0 f)$, then $\alpha(a(W_0 f)^*)=a(U_0 f)$, and $\alpha(a(f))=a(U_0 W_0^* f)^*$, with W the conjugate-unitary operator $U_0 W_0^*$ on $\mathscr{H}$.

We assume, henceforth, that $\hat{\alpha}(\phi_0)=\phi_0$ and use the notation of Theorem 3.1 throughout the remainder of this section. With this assumption, U, constructed in Lemma 3.2, is a unitary operator on $\mathscr{H}_{\mathscr{F}}^{(a)}$ which carries product vectors onto product vectors. Although the components of the argument proving that are to be found in the proof of Lemma 3.2, we make the statement and proof explicit in Lemma 3.3. Note that the automorphism σ, above, is a special case of the larger class of Bogoliubov transformations. In Section 2, we introduced the notation $\hat{T}$ to denote a certain transformation on $\mathscr{H}_{\mathscr{F}}^{(a)}$ arising from T defined on $\mathscr{H}$. In Lemma 3.3 and the results following, we construct a unitary operator $\hat{U}$ on $\mathscr{H}_{\mathscr{F}}^{(a)}$. We will eventually locate a unitary operator U on $\mathscr{H}$ for which $\hat{U}$ is the transformation on $\mathscr{H}_{\mathscr{F}}^{(a)}$ arising from it – justifying this notation.

Lemma 3.3. *There is a unitary operator $\hat{U}$ on $\mathscr{H}_{\mathscr{F}}^{(a)}$ which implements α and maps product vectors onto product vectors.*

Proof. Since $\hat{\alpha}(\phi_0)=\phi_0$, the mapping $A\Phi_0\to\alpha(A)\Phi_0$ extends to a unitary operator $\hat{U}$ on $\mathscr{H}_{\mathscr{F}}^{(a)}$ such that $\hat{U}A\hat{U}^*=\alpha(A)$. We show that $\hat{U}(x_1\wedge\cdots\wedge x_n)$ is a product vector, for all $x_1,\ldots,x_n$ in $\mathscr{H}$. Since $x_1\wedge\cdots\wedge x_n$ is a scalar multiple of the wedge-product of an orthonormal set of vectors (a basis for $[x_1,\ldots,x_n]$, when $x_1\wedge\cdots\wedge x_n\neq 0$), we may assume that $\{x_1,\ldots,x_n\}$ is an orthonormal set. If E_0 is the projection with range $[x_1,\ldots,x_n]$, $\phi_{E_0}=\omega_{x_1\wedge\cdots\wedge x_n}$, from Proposition 2.2. As α is implemented by a unitary operator on $\mathscr{H}_{\mathscr{F}}^{(a)}$, $\hat{\alpha}(\phi_{E_0})$ is a vector state of $\mathfrak{A}$. By assumption, $\hat{\alpha}(\phi_{E_0})$ is a gauge-invariant quasi-free state of $\mathfrak{A}$ (equivalent to ϕ_0, from the preceding remark). Thus $\hat{\alpha}(\phi_{E_0})=\omega_{y_1\wedge\cdots\wedge y_m}|\mathfrak{A}$ where $y_1\wedge\cdots\wedge y_m=\hat{U}^*(x_1\wedge\cdots\wedge x_n)$.

A linear transformation (such as $\hat{U}$, above) which is defined on a subspace of $\mathscr{H}_{\mathscr{F}}^{(a)}$ and maps product vectors onto product vectors will be said to be a *product* linear transformation (or *product* unitary, etc.).

Lemma 3.4. *The intersection of an infinite, commuting family of m-dimensional projections each pair of which has m-1 dimensional intersection is m-1 dimensional.*

Proof. Since each projection is m-dimensional, each is the sum of m one-dimensional projections. Since the family is commutative, we can find an orthogonal set of one-dimensional projections such that each projection of our commuting family is the sum of m of them. In this way, our problem reduces to showing that if $S = \bigcup_a S_a$ where $\{S_a\}$ is an infinite family of sets each of which has m elements and such that each pair has intersection with m-1 elements, then $\bigcap_a S_a$ has m-1 elements (in other words, each S_c contains $S_a \cap S_b$, when $a \neq b$). Suppose S_c does not contain $S_a \cap S_b$. Then $S_a \cap S_c$ and $S_b \cap S_c$ are distinct m-1 element sets; so that their union has at least m elements. As this union is contained in S_c it coincides with S_c; and $S_c \subseteqq S_a \cup S_b$. By the same token, with $c \neq d$, S_d does not contain both $S_a \cap S_c$ and $S_b \cap S_c$; so that S_d is contained in either $S_a \cup S_c$ or $S_b \cup S_c$. But this contradicts the assumption that $\{S_a\}$, and, hence, $\cup S_a$ are infinite.

Proposition 3.5. *If V is an isometric, linear mapping of an infinite-dimensional subspace $\mathscr{K}$ of $\mathscr{H}$ onto a set of product vectors in $\mathscr{H}_m^{(a)}$, then $\bigcap [Vx]$ has dimension m-1. ($[Vx]$ is the subspace of H determined by Vx.)*

Proof. With $\{e_j\}$ an orthonormal basis for $\mathscr{K}$, $Ve_1 = x_1 \wedge \cdots \wedge x_m$, $Ve_2 = y_1 \wedge \cdots \wedge y_m$, and $V(e_1 + e_2) = z_1 \wedge \cdots \wedge z_m$, we have

$$x_1 \wedge \cdots \wedge x_m + y_1 \wedge \cdots \wedge y_m = z_1 \wedge \cdots \wedge z_m .$$

Some z_j, say z_1, is not in $[x_1, \ldots, x_m]$. Thus $z_1 \wedge x_1 \wedge \cdots \wedge x_m \wedge y_j = 0$; and $[y_1, \ldots, y_m] \subseteqq [z_1, x_1, \ldots, x_m]$. Since Ve_1 and Ve_2 are not 0, $[x_1, \ldots, x_m]$ and $[y_1, \ldots, y_m]$ are m-dimensional subspaces of $[z_1, x_1, \ldots, x_m]$, a space of dimension $m+1$. Thus $[x_1, \ldots, x_m] \cap [y_1, \ldots, y_m]$ has dimension at least $m-1$; and

$$Ve_1 = x \wedge v_1 \wedge \cdots \wedge v_{m-1}, \; Ve_2 = y \wedge v_1 \wedge \cdots \wedge v_{m-1},$$

with $\{x, v_1, \ldots, v_{m-1}\}$ and $\{y, v_1, \ldots, v_{m-1}\}$ orthonormal sets. In addition, $0 = \langle Ve_1 | Ve_2 \rangle = \langle x | y \rangle \det(\langle v_i | v_j \rangle) = \langle x | y \rangle$. Hence, E_1 and E_2 commute, where E_j is the orthogonal projection on $\mathscr{H}$ with range $[Ve_j]$. Thus $\{E_j\}$ is an infinite, commuting family of projections on $\mathscr{H}$ such that $E_j E_k(\mathscr{H})$ has dimension $m-1$ when $j \neq k$. From Lemma 3.4, $\bigcap_j E_j(\mathscr{H})$ has dimension $m-1$.

Lemma 3.6. *If V is a product isometry of $\mathscr{K}_n^{(a)}$ into $\mathscr{H}_{\mathscr{F}}^{(a)}$, with $\mathscr{K}$ a subspace of $\mathscr{H}$, then V has range in some $\mathscr{H}_m^{(a)}$. If $n \leqq m$ and $\mathscr{K}$ is infinite dimensional then $[V(x_1 \wedge \cdots \wedge x_n)] \cap [V(y_1 \wedge \cdots \wedge y_n)]$ has dimension at least $m-n$.*

Proof. If $\{e_j\}$ is an orthonormal basis for $\mathscr{K}$ and $\{i_1, \ldots, i_t\}$, $\{j_1, \ldots, j_t\}$ are disjoint sets of indices,

$$\begin{aligned} 1 &= \langle (e_{i_1} + e_{j_1}) \wedge e_{i_2} \wedge \cdots \wedge e_{i_t} \wedge e_{k_{t+1}} \wedge \cdots \wedge e_{k_n} | e_{i_1} \wedge \cdots \wedge e_{i_t} \wedge e_{k_{t+1}} \wedge \cdots \wedge e_{k_n} \rangle \\ &= \langle V((e_{i_1} + e_{j_1}) \wedge e_{i_2} \wedge \cdots \wedge e_{i_t} \wedge e_{k_{t+1}} \wedge \cdots \wedge e_{k_n}) | V(e_{i_1} \wedge \cdots \wedge e_{i_t} \wedge e_{k_{t+1}} \wedge \cdots \wedge e_{k_n}) \rangle . \end{aligned}$$

Replacing $V(e_{i_1} \wedge \cdots \wedge e_{i_t} \wedge e_{k_{t+1}} \wedge \cdots \wedge e_{k_n})$ by $V(e_{j_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_t} \wedge e_{k_{t+1}} \wedge \cdots \wedge e_{k_n})$ in the preceding computation, it follows that both (product) vectors lie in the same space $\mathscr{H}_m^{(a)}$ as $V((e_{i_1}+e_{j_1}) \wedge e_{i_2} \wedge \cdots \wedge e_{i_t} \wedge e_{k_{t+1}} \wedge \cdots \wedge e_{k_n})$ does. Applying this result to successive replacements of $i_2, i_3, \ldots, i_t$ by $j_2, j_3, \ldots, j_t$, one concludes that $V(e_{j_1} \wedge \cdots \wedge e_{j_t} \wedge e_{k_{t+1}} \wedge \cdots \wedge e_{k_n})$ lies in $\mathscr{H}_m^{(a)}$. Thus V maps $\mathscr{K}_n^{(a)}$ into $\mathscr{H}_m^{(a)}$, since $\{e_{i_1} \wedge \cdots \wedge e_{i_n}\}$ is an orthonormal basis for $\mathscr{K}_n^{(a)}$.

Since $[x_2, \ldots, x_n]$ has dimension $n-1$, its orthogonal complement and $[y_1, \ldots, y_n]$ are not disjoint. Let z_1 be a unit vector in their intersection. Then $[V(z_1 \wedge x_2 \wedge \cdots \wedge x_n)] \cap [V(x_1 \wedge \cdots \wedge x_n)]$ has dimension at least $m-1$, from Proposition 3.5; for $x \to V(x \wedge x_2 \wedge \cdots \wedge x_n)$ is an isometric linear mapping of $\mathscr{K} \ominus [x_2, \ldots, x_n]$ onto a set of product vectors in $\mathscr{H}_m^{(a)}$.[1] Let z_2 be a unit vector in $[y_1, \ldots, y_n]$ orthogonal to $[z_1, x_3, \ldots, x_n]$. Then

$$[V(z_1 \wedge x_2 \wedge \cdots \wedge x_n)] \cap [V(z_1 \wedge z_2 \wedge x_3 \wedge \cdots \wedge x_n)]$$

has dimension at least $m-1$; so that its intersection with

$$[V(z_1 \wedge x_2 \wedge \cdots \wedge x_n)] \cap [V(x_1 \wedge \cdots \wedge x_n)]$$

has dimension at least $m-2$ (both are subspaces of the m-dimensional space $[V(z_1 \wedge x_2 \wedge \cdots \wedge x_n)]$). Thus $[V(z_1 \wedge z_2 \wedge x_3 \wedge \cdots \wedge x_n)] \cap [V(x_1 \wedge \cdots \wedge x_n)]$ has dimension at least $m-2$. If we have found mutually orthogonal unit vectors $z_1, \ldots, z_{k-1}$ in $[y_1, \ldots, y_n]$ such that $[V(z_1 \wedge \cdots \wedge z_{k-1} \wedge x_k \wedge \cdots \wedge x_n)]$ and $[V(x_1 \wedge \cdots \wedge x_n)]$ have an intersection of dimension at least $m-k+1$, choose a unit vector z_k in $[y_1, \ldots, y_n]$ orthogonal to $[z_1, \ldots, z_{k-1}, x_{k+1}, \ldots, x_n]$. Then $[V(z_1 \wedge \cdots \wedge z_{k-1} \wedge x_k \wedge \cdots \wedge x_n)]$ and $[V(z_1 \wedge \cdots \wedge z_k \wedge x_{k+1} \wedge \cdots \wedge x_n)]$ have intersection of dimension at least $m-1$. Thus $[V(z_1 \wedge \cdots \wedge z_k \wedge x_{k+1} \wedge \cdots \wedge x_n)]$ and $[V(x_1 \wedge \cdots \wedge x_n)]$ have intersection of dimension at least $m-k$. Finally, $z_1 \wedge \cdots \wedge z_n = c y_1 \wedge \cdots \wedge y_n$ and $[V(x_1 \wedge \cdots \wedge x_n)] \cap [V(y_1 \wedge \cdots \wedge y_n)]$ has dimension at least $m-n$.

Lemma 3.7. *For each n, $\hat{U}$ maps $\mathscr{H}_n^{(a)}$ onto $\mathscr{H}_n^{(a)}$.*

Proof. From Lemma 3.6, $\hat{U}$ maps $\mathscr{H}_n^{(a)}$ into some $\mathscr{H}_m^{(a)}$. Since $\hat{U}^*$ satisfies the same hypotheses as $\hat{U}$, $\hat{U}^*$ maps $\mathscr{H}_m^{(a)}$ into $\mathscr{H}_n^{(a)}$ (again, from Lemma 3.6). Thus $\hat{U}$ maps $\mathscr{H}_n^{(a)}$ onto $\mathscr{H}_m^{(a)}$. For some orthonormal sets $\{x_1, \ldots, x_n\}$ and $\{y_1, \ldots, y_n\}$ in $\mathscr{H}$, $\hat{U}(x_1 \wedge \cdots \wedge x_n) = e_1 \wedge \cdots \wedge e_m$ and $\hat{U}(y_1 \wedge \cdots \wedge y_n) = e_{m+1} \wedge \cdots \wedge e_{2m}$, where $\{e_1, \ldots, e_{2m}\}$ is an orthonormal set in $\mathscr{H}$. If $n \leqq m$, from Lemma 3.6,

$$[e_1, \ldots, e_m] \cap [e_{m+1}, \ldots, e_{2m}] \; (=(0))$$

has dimension at least $m-n$. Thus $m \leqq n$. Applying this conclusion to $\hat{U}^*, n \leqq m$; so that $m=n$.

We denote the dimension of a (finite-dimensional) subspace E of $\mathscr{H}$ by $d(E)$.

Proposition 3.8. *If V is a product isometry of $\mathscr{K}_n^{(a)}$ into $\mathscr{H}_m^{(a)}$, where $n \leqq m$ and $\mathscr{K}$ is an infinite-dimensional subspace of $\mathscr{H}$, and*

$$d([V(e_{i_1} \wedge \cdots \wedge e_{i_n})] \cap [V(e_{j_1} \wedge \cdots \wedge e_{j_n})]) = m-n$$

for some $e_{i_1}, \ldots, e_{i_n}, e_{j_1}, \ldots, e_{j_n}$, where $\{e_j\}$ is an orthonormal basis for $\mathscr{K}$, then $d\left(\bigcap_{x_1, \ldots, x_n} [V(x_1 \wedge \cdots \wedge x_n)]\right) = m-n$.

[1] Without loss of generality we assume that $\{x_1, \ldots, x_n\}$ and $\{y_1, \ldots, y_n\}$ are orthonormal sets.

Proof. We write E_k for $[V(e_{k_1}\wedge\cdots\wedge e_{k_n})]$, $E_{\bar{k}t}$ for

$$[V(e_{k_1}\wedge\cdots\wedge e_{k_{r-1}}\wedge e_t\wedge e_{k_{r+1}}\wedge\cdots\wedge e_{k_n})],$$

and F_t for $E_i\cap E_{\bar{j}t}$. If z lies in $\bigcap_k E_k$, then, with $x_1,\ldots,x_n$ in $\mathscr{K}$, $z\wedge V(x_1\wedge\cdots\wedge x_n)=0$, since $z\wedge V(e_{k_1}\wedge\cdots\wedge e_{k_n})=0$ for all $k_1,\ldots,k_n$. Hence $z\in[V(x_1\wedge\cdots\wedge x_n)]$. It will suffice to show that $d\left(\bigcap_k E_k\right)=m-n$ (equivalently, that $E_i\cap E_j\subseteqq E_k$ for each k). Proposition 3.5 establishes our result when $n=1$. Suppose we have proved it for values less than n.

Since $x_1\wedge\cdots\wedge x_{r-1}\wedge x_{r+1}\wedge\cdots\wedge x_n\to V(x_1\wedge\cdots\wedge x_{r-1}\wedge e_{i_r}\wedge x_{r+1}\wedge\cdots\wedge x_n)$ is an isometric, linear mapping of $(\mathscr{K}\ominus[e_{i_r}])^{(a)}_{n-1}$ into $\mathscr{H}^{(a)}_m$; $d(F_{i_r})\geqq m-n+1$, from Lemma 3.6. If $d(F_{i_r})>m-n+1$, then, since $d(E_j\cap E_{\bar{j}i_r})=m-1$ (from Proposition 3.5 – as argued in Lemma 3.6); $d(E_i\cap E_j)>m-n$, contrary to assumption. Thus $d(F_{i_r})=m-n+1$. Our inductive hypothesis applies, and $d\left(\bigcap_{k_1,\ldots,k_n} E_{\bar{k}i_r}\right)=m-n+1$. Since $\bigcap_{k_1,\ldots,k_n} E_{\bar{k}i_r}\subseteqq E_{\bar{j}i_r}$;

$$d\left(\left(\bigcap_{k_1,\ldots,k_n} E_{\bar{k}i_r}\right)\cap E_j\right)=d\left(\left(\bigcap_{k_1,\ldots,k_n} E_{\bar{k}i_r}\right)\cap E_j\cap E_{\bar{j}i_r}\right)=m-n=d(E_i\cap E_j).$$

But $\left(\bigcap_{k_1,\ldots,k_n}\right)E_{\bar{k}i_r}\cap E_j\subseteqq E_i\cap E_j$. Thus

$$E_i\cap E_j\subseteqq\bigcap_{k_1,\ldots,k_n} E_{\bar{k}i_r}\subseteqq F_{i_r}.$$

In particular, we have established (under the induction hypothesis) that if $d(E_i\cap E_j)=m-n$ then $E_i\cap E_j\subseteqq E_k$ provided one of the $k_1,\ldots,k_n$ is in $\{i_1,\ldots,i_n,j_1,\ldots,j_n\}$. (In our argument $k_r=i_r$.)

Having proved that $d(F_{i_r})=m-n+1$ and $E_i\cap E_j\subseteqq F_{i_r}$, it follows that F_{i_r} is generated by $E_i\cap E_j$ and a unit vector f_r orthogonal to it. Moreover $\{f_1,\ldots,f_n\}$ are linearly independent. To see this, note that $\bigcap_t E_{\bar{j}t}$ is $m-1$ dimensional, from Proposition 3.5, so that $E_{\bar{j}t}$ is generated by $\bigcap_t E_{\bar{j}t}$ and a unit vector g_t orthogonal to $\bigcap_t E_{\bar{j}t}$. In addition,

$$0=\langle V(e_{j_1}\wedge\cdots\wedge e_{j_{r-1}}\wedge e_t\wedge e_{j_{r+1}}\wedge\cdots\wedge e_{j_n})|V(e_{j_1}\wedge\cdots\wedge e_{j_{r-1}}\wedge e_{t'}\wedge e_{j_{r+1}}\wedge\cdots\wedge e_{j_n})\rangle$$
$$=\langle g_t|g_{t'}\rangle.$$

Thus no $E_{\bar{j}t}$ is contained in the union of the others (for g_t is orthogonal to that union). Now f_r is not in $\bigcap_t E_{\bar{j}t}$; for, otherwise f_r is in E_j, hence, in $E_i\cap E_j$, contrary to the choice of f_r. Thus f_r and $\bigcap_t E_{\bar{j}t}$ generate $E_{\bar{j}i_r}$; so that a linear relation among $\{f_1,\ldots,f_n\}$ would entail that some $E_{\bar{j}t}$ is contained in the union of the others. By the same token, if F_t has dimension $m-n+1$ or greater and t is not in $\{i_1,\ldots,i_n\}$, F_t contains a unit vector f_0 orthogonal to $E_i\cap E_j$ and linearly independent of $\{f_1,\ldots,f_n\}$. Recalling that $E_i\cap E_j\subseteqq F_t$ (as established before), and $F_t\subseteqq E_i$ (by definition of F_t); we see that $d(E_i)$ would exceed m. Thus $d(F_t)=m-n$ when $t \notin \{i_1, \ldots, i_n\}$; so that $E_i \cap E_j \subseteqq F_{k_r} \subseteqq E_k$. (We now have that

$$d([V(e_{i_1} \wedge \cdots \wedge e_{i_n})] \cap [V(e_{j_1} \wedge \cdots \wedge e_{j_{r-1}} \wedge e_{k_r} \wedge e_{j_{r+1}} \wedge \cdots \wedge e_{j_n})]) = m-n$$

when $k_r \notin \{i_1, \ldots, i_n\}$; and, from the first part of this proof,

$$F_{k_r} \subseteqq E_k = [V(e_{k_1} \wedge \cdots \wedge e_{k_n})]$$

since

$$k_r \in \{i_1, \ldots, i_n, j_1, \ldots, j_{r-1}, k_r, j_{r+1}, \ldots, j_n\}.)$$

Lemma 3.9. *The equality,*

$$\begin{aligned} &d([x_1, \ldots, x_n] \cap \cdots \cap [z_1, \ldots, z_n]) \\ &\quad = d([\hat{U}(x_1 \wedge \cdots \wedge x_n)] \cap \cdots \cap [\hat{U}(z_1 \wedge \cdots \wedge z_n)]) \end{aligned}$$

is valid for finite and infinite intersections.

Proof. We establish the assertion of the lemma, first, for the intersection $[x_1, \ldots, x_n] \cap [z_1, \ldots, z_n]$ of two n-dimensional subspaces of $\mathscr{H}$. If $\{v_1, \ldots, v_k\}$ is an orthonormal basis for this intersection, changing notation, we can write $x_1 \wedge \cdots \wedge x_{n-k} \wedge v_1 \wedge \cdots \wedge v_k$ and $z_1 \wedge \cdots \wedge z_{n-k} \wedge v_1 \wedge \cdots \wedge v_k$ in place of $x_1 \wedge \cdots \wedge x_n$ and $z_1 \wedge \cdots \wedge z_n$. In Lemma 3.7 we noted that $\hat{U}$ maps $\mathscr{H}_n^{(a)}$ into $\mathscr{H}_n^{(a)}$; so that $y_1 \wedge \cdots \wedge y_{n-k} \to \hat{U}(y_1 \wedge \cdots \wedge y_{n-k} \wedge v_1 \wedge \cdots \wedge v_k)$ is a product isometry of $(\mathscr{H} \ominus [v_1, \ldots, v_k])_{n-k}^{(a)}$ into $\mathscr{H}_n^{(a)}$. From Lemma 3.6,

$$[\hat{U}(x_1 \wedge \cdots \wedge x_n)] \cap [\hat{U}(z_1 \wedge \cdots \wedge z_n)]$$

has dimension at least k. Applying this to $\hat{U}^{-1}$, we see that

$$d([\hat{U}(x_1 \wedge \cdots \wedge x_n)] \cap [\hat{U}(z_1 \wedge \cdots \wedge z_n)]) = k = d([x_1, \ldots, x_n] \cap [z_1, \ldots, z_n]).$$

Let $w_1, \ldots, w_r$ be an orthonormal basis for $[x_1, \ldots, x_n] \cap \cdots \cap [z_1, \ldots, z_n]$; and let $u_1, \ldots, u_{n-r}, u'_1, \ldots, u'_{n-r}$ be an orthonormal set of vectors in $\mathscr{H} \ominus [w_1, \ldots, w_r]$. Then, from the preceding,

$$d([\hat{U}(u_1 \wedge \cdots \wedge u_{n-r} \wedge w_1 \wedge \cdots \wedge w_r)] \cap [\hat{U}(u'_1 \wedge \cdots \wedge u'_{n-r} \wedge w_1 \wedge \cdots \wedge w_r)]) = r.$$

Moreover $y_1 \wedge \cdots \wedge y_{n-r} \to \hat{U}(y_1 \wedge \cdots \wedge y_{n-r} \wedge w_1 \wedge \cdots \wedge w_r)$ is a product isometry of $(\mathscr{H} \ominus [w_1, \ldots, w_r])_{n-r}^{(a)}$ into $\mathscr{H}_n^{(a)}$. From Proposition 3.8,

$$d\Big(\bigcap_{y_1, \ldots, y_{n-r}} [\hat{U}(y_1 \wedge \cdots \wedge y_{n-r} \wedge w_1 \wedge \cdots \wedge w_r)]\Big) = r.$$

Thus $d([\hat{U}(x_1 \wedge \cdots \wedge x_n)] \cap \cdots \cap [\hat{U}(z_1 \wedge \cdots \wedge z_n)]) \geqq r$. Applying this to $\hat{U}^{-1}$, we have

$$\begin{aligned} r &= d([x_1 \wedge \cdots \wedge x_n] \cap \cdots \cap [z_1 \wedge \cdots \wedge z_n]) \\ &\geqq d([\hat{U}(x_1 \wedge \cdots \wedge x_n)] \cap \cdots \cap [\hat{U}(z_1 \wedge \cdots \wedge z_n)]), \end{aligned}$$

from which our lemma follows.

Corollary 3.10. *With e_0 a unit vector in $\mathscr{H}$,*

$$d\Big(\bigcap_{x_1, \ldots, x_{n-1}} [\hat{U}(x_1 \wedge \cdots \wedge x_{n-1} \wedge e_0)]\Big) = 1.$$

Lemma 3.11. *For each unit vector e_0 in $\mathscr{H}$, $\hat{U}e_0$ lies in*

$$\bigcap_{x_1,\ldots,x_{n-1}} [\hat{U}(x_1 \wedge \cdots \wedge x_{n-1} \wedge e_0)]\,.$$

Proof. From Corollary 3.10, there is unit vector f_1 which generates $\bigcap_{x_1,\ldots,x_{n-1}} [\hat{U}(x_1 \wedge \cdots \wedge x_{n-1} \wedge e_0)]$. Thus $\hat{U}(x_1 \wedge \cdots \wedge x_{n-1} \wedge e_0) = x'_1 \wedge \cdots \wedge x'_{n-1} \wedge f_1$. Now $d\Big(\bigcap_{y_1,\ldots,y_{n-1}} [\hat{U}^*(y_1 \wedge \cdots \wedge y_{n-1} \wedge f_1)]\Big) = 1$, from Corollary 3.10 and

$$\begin{aligned}1 &= d\Big(\bigcap_{x_1,\ldots,x_{n-1}} [\hat{U}^*\hat{U}(x_1 \wedge \cdots \wedge x_{n-1} \wedge e_0)]\Big)\\ &= d\Big(\bigcap_{x_1,\ldots,x_{n-1}} [\hat{U}^*(x'_1 \wedge \cdots \wedge x'_{n-1} \wedge f_1)]\Big)\\ &\geqq d\Big(\bigcap_{y_1,\ldots,y_{n-1}} [\hat{U}^*(y_1 \wedge \cdots \wedge y_{n-1} \wedge f_1)]\Big) = 1\,;\end{aligned}$$

so that each $[\hat{U}^*(y_1 \wedge \cdots \wedge y_{n-1} \wedge f_1)]$ contains e_0. Thus

$$\hat{U}^*(y_1 \wedge \cdots \wedge y_{n-1} \wedge f_1) = y'_1 \wedge \cdots \wedge y'_{n-1} \wedge e_0\,.$$

Suppose f_1 is not a scalar multiple of $\hat{U}e_0 (= e_1)$. Then $\langle f_1 | e_2 \rangle \neq 0$, where e_2 is a unit vector in $[e_1, f_1]$ orthogonal to e_1. Let $\{e_j\}_{j=1,2,\ldots}$ be an orthonormal basis for $\mathscr{H}$; and let A be $\hat{U}a(e_0)^* a(e_0)\hat{U}^*$. Then

$$\begin{aligned}A(f_1 \wedge e_{j_2} \wedge \cdots \wedge e_{j_n}) &= \hat{U}a(e_0)^* a(e_0)\,(e_0 \wedge e'_{j_2} \wedge \cdots \wedge e'_{j_n})\\ &= f_1 \wedge e_{j_2} \wedge \cdots \wedge e_{j_n}\,.\end{aligned}$$

Finite sums, $\Sigma c_{i_1\ldots i_p;j_1\ldots j_q} a(e_{i_1})^* \ldots a(e_{i_p})^* a(e_{j_1}) \ldots a(e_{j_q})\ (= B)$, form a norm-dense subset of $\mathfrak{A}$. Let ε be $|\langle f_1 | e_2 \rangle|/5$, and choose B such that $\|A - B\| < \varepsilon$. Then

$$\varepsilon^2 > \|(A-B)e_1\|^2 \geqq |c_{2;1}|^2\,,$$

(examining the coefficient of e_2),

$$\varepsilon^2 > \|(A-B)e_m\|^2 \geqq |c_0|^2\,,$$

where c_0 is the coefficient of I in the sum representation of B and all e_i, e_j appearing in this sum are among $e_1, \ldots, e_{m-1}$ (so that A and all terms of B other than $c_0 I$ map e_m to 0 or a multiple of a basis element other than e_m), and

$$\varepsilon^2 > \|(A-B)e_2\|^2 \geqq |c_0 + c_{2;2}|^2$$

(examining the coefficient of e_2).

$$\text{Thus} \quad |c_0| < \varepsilon, \quad |c_{2;1}| < \varepsilon \quad \text{and} \quad |c_{2;2}| < 2\varepsilon\,.$$

The only terms in the sum for B that yield multiples of $e_2 \wedge e_{m+2} \wedge \cdots \wedge e_{m+n}$ when B acts on

$$\begin{aligned}&f_1 \wedge e_{m+2} \wedge \cdots \wedge e_{m+n}\\ &\quad (= \langle e_1 | f_1 \rangle e_1 \wedge e_{m+2} \wedge \cdots \wedge e_{m+n} + \langle e_2 | f_1 \rangle e_2 \wedge e_{m+2} \wedge \cdots \wedge e_{m+n})\end{aligned}$$

are $c_0 I$, $c_{2;1}a^*(e_2)a(e_1)$, and $c_{2;2}a^*(e_2)a(e_2)$. Thus

$$\begin{aligned}|\langle f_1|e_2\rangle|^2/25 &= \varepsilon^2\\ &\geqq \|(A-B)(f_1\wedge e_{m+2}\wedge\cdots\wedge e_{m+n})\|^2\\ &\geqq |\langle e_2|f_1\rangle - c_0\langle e_2|f_1\rangle - c_{2;1}\langle e_1|f_1\rangle - c_{2;2}\langle e_2|f_1\rangle|^2\end{aligned}$$

so that

$$\begin{aligned}&|\langle f_1|e_2\rangle|/5\\ &\geqq |\langle f_1|e_2\rangle| - |c_0|\,|\langle f_1|e_2\rangle| - |c_{2;1}|\,|\langle f_1|e_1\rangle| - |c_{2;2}|\,|\langle f_1|e_2\rangle|\\ &> |\langle f_1|e_2\rangle| - 4\varepsilon = |\langle f_1|e_2\rangle|/5\end{aligned}$$

a contradiction. Thus $f_1 = c' e_1$ for some scalar c'.

Lemma 3.12. *There is a sequence $\{c_n\}$ of complex scalars of modulus 1 such that $\hat{U}(x_1\wedge\cdots\wedge x_n) = c_n(Ux_1\wedge\cdots\wedge Ux_n)$, where U is the restriction of $\hat{U}$ to one-particle space.*

Proof. Let $\{e_j\}$ be an orthonormal basis for $\mathscr{H}$. Then

$$Ue_j \in \bigcap_{x_1,\ldots,x_{n-1}} [\hat{U}(x_1\wedge\cdots\wedge x_{n-1}\wedge e_j)]$$

for all j and n. Thus $\hat{U}(e_{j_1}\wedge\cdots\wedge e_{j_n})$ and $Ue_{j_1}\wedge\cdots\wedge Ue_{j_n}$ differ by a phase factor. Say

$$\hat{U}(e_{j_1}\wedge\cdots\wedge e_{j_n}) = cUe_{j_1}\wedge\cdots\wedge Ue_{j_n}$$

and

$$\hat{U}(e_{i_1}\wedge e_{j_2}\wedge\cdots\wedge e_{j_n}) = c'Ue_{i_1}\wedge Ue_{j_2}\wedge\cdots\wedge Ue_{j_n}.$$

Writing f for $(e_{j_1}+e_{i_1})/\sqrt{2}$ and f' for $(ce_{j_1}+c'e_{i_1})/\sqrt{2}$, $\hat{U}(f\wedge e_{j_2}\wedge\cdots\wedge e_{j_n}) = Uf'\wedge Ue_{j_2}\wedge\cdots\wedge Ue_{j_n}$. Since $\hat{U}(f\wedge e_{j_2}\wedge\cdots\wedge e_{j_n}) = c''Uf\wedge Ue_{j_2}\wedge\cdots\wedge Ue_{j_n}$ and $Uf' - c''Uf$ is orthogonal to $Ue_{j_2},\ldots,Ue_{j_n}$; we have $Uf' = Uc''f$. Thus $ce_{j_1}+c'e_{i_1} = c''e_{j_1}+c''e_{i_1}$; and $c = c'' = c'$. Changing basis elements successively, we conclude that $\hat{U}(e_{j_1}\wedge\cdots\wedge e_{j_n}) = cUe_{j_1}\wedge\cdots\wedge Ue_{j_n}$, for all $j_1,\ldots,j_n$. Writing c_n for c, we have $\hat{U}(x_1\wedge\cdots\wedge x_n) = c_n(Ux_1\wedge\cdots\wedge Ux_n)$. Of course $c_1 = 1$; and $\hat{U}\Phi_0 = \Phi_0$.

Lemma 3.13. *For all $x_1,\ldots,x_n$ in $\mathscr{H}$ and $n = 0,1,2,\ldots$, $\hat{U}(x_1\wedge\cdots\wedge x_n) = Ux_1\wedge\cdots\wedge Ux_n$.*

Proof. From Lemma 3.12, there is a sequence $\{c_n\}$ of scalars of modulus 1 such that $\hat{U}(x_1\wedge\cdots\wedge x_n) = c_n(Ux_1\wedge\cdots\wedge Ux_n)$. If $\{e_j\}$ is an orthonormal basis for $\mathscr{H}$, so is $\{Ue_j\}$. Write f_j for Ue_j. Since $\hat{U}a(e_j)\hat{U}^*(f_j\wedge f_{j_2}\wedge\cdots\wedge f_{j_n}) = \bar{c}_n\hat{U}(e_{j_2}\wedge\cdots\wedge e_{j_n}) = c_{n-1}\bar{c}_n(f_{j_2}\wedge\cdots\wedge f_{j_n})$ and $\hat{U}a(e_j)\hat{U}^*(f_{j_1}\wedge\cdots\wedge f_{j_n}) = 0$ if $j\notin\{j_1,\ldots,j_n\}$; we have that $\hat{U}a(e_j)\hat{U}^*|\mathscr{H}_n^{(a)} = c_{n-1}\bar{c}_n a(f_j)|\mathscr{H}_n^{(a)}$.

Writing c'_n for $c_{n-1}\bar{c}_n$ (so that $c'_1 = 1$), we show that $c'_n = c'_{n-1} = \cdots = 1$. Let $\Sigma\, c_{i_1\ldots i_p;j_1\ldots j_2}a(f_{i_1})^*\ldots a(f_{i_p})^*a(f_{j_1})\ldots a(f_{j_q})$ $(=B)$ be chosen such that $\|A-B\| < \varepsilon$, where $A = \hat{U}a(e_1)\hat{U}^*$. Suppose $i_1,\ldots,i_p;j_1,\ldots,j_q$ are less than m. Then, examining the coefficient of $f_{m+2}\wedge\cdots\wedge f_{m+n}$,

$$\varepsilon^2 > \|(A-B)(f_1\wedge f_{m+2}\wedge\cdots\wedge f_{m+n})\|^2 \geqq |c'_n - c_{0;1}|^2.$$

This inequality holds for $n = 1,2,\ldots$, so that $|c'_n - c'_{n-1}| < 2\varepsilon$ for all positive ε. Thus $1 = c'_1 = c'_2 = \cdots$, and $c_n = c_{n-1} = \cdots = c_0 = 1$.

Proof of Theorem 3.1. We know that $\hat{U}$ implements α (Lemma 3.3) and is the product unitary operator on $\mathscr{H}_{\mathscr{F}}^{(a)}$ corresponding to U on $\mathscr{H}$, when $\hat{\alpha}(\phi_0)=\phi_0$ (Lemma 3.13). The case in which $\hat{\alpha}(\phi_0)=\phi_I$ is reduced to this case following the proof of Lemma 3.2. To conclude the proof, we note that $\alpha(a(f))=a(Uf)$, or, equivalently, that $\alpha(a(f)^*)=a(Uf)^*$. For this, observe that

$$\begin{aligned}
&\alpha(a(f)^*)(f_1 \wedge \cdots \wedge f_n)\\
&\quad=\hat{U}a(f)^*\hat{U}^*(f_1 \wedge \cdots \wedge f_n)\\
&\quad=\hat{U}a(f)^*(U^*f_1 \wedge \cdots \wedge U^*f_n)\\
&\quad=\hat{U}(f \wedge U^*f_1 \wedge \cdots \wedge U^*f_n)\\
&\quad=(Uf)\wedge f_1 \wedge \cdots \wedge f_n\\
&\quad=a(Uf)^*(f_1 \wedge \cdots \wedge f_n),
\end{aligned}$$

for all $f_1, \ldots, f_n$ in $\mathscr{H}$. Thus $\alpha(a(f)^*)=a(Uf)^*$.

4. Product Unitary Operators on n-Particle Space

In the preceding section, after showing (Lemma 3.3) that $\hat{U}$ is a product unitary operator on Fock space, we make no further use of the hypothesis that $\hat{U}$ induces an automorphism of $\mathfrak{A}$ until Lemma 3.11. With this hypothesis, it is proved (Lemma 3.13) that $\hat{U}$ is induced by a unitary operator on one-particle space. The fact that this same result is valid for a product unitary operator defined only on n-particle space is proved in the theorem that follows.

Theorem 4.1. *If $\hat{U}$ is a product unitary operator on n-particle space $\mathscr{H}_n^{(a)}$, there is a unitary operator U on one-particle space $\mathscr{H}$ such that $\hat{U}(x_1 \wedge \cdots \wedge x_n) = Ux_1 \wedge \cdots \wedge Ux_n$.*

We prove this theorem with the aid of the following lemmas (notation as in Theorem 4.1). Again, the argument will justify the use of the notation $\hat{U}$.

Lemma 4.2. *If $\{e_j\}$ is an orthonormal basis for $\mathscr{H}$, there is a unit vector f_j in $\bigcap_{x_1,\ldots,x_{n-1}} [\hat{U}(x_1 \wedge \cdots \wedge x_{n-1} \wedge e_j)]$ such that $\{f_j\}$ is an orthonormal basis for $\mathscr{H}$.*

Proof. To show that $\langle f_j | f_k\rangle = 0$, when $j \neq k$, we may assume that j and k are in $\{1, \ldots, n+1\}$. Let E_j be the n-dimensional space,

$$[\hat{U}(e_1 \wedge \cdots \wedge e_{j-1} \wedge e_{j+1} \wedge \cdots \wedge e_{n+1})],\ j=1, \ldots, n+1,$$

and E be the space $E_1 \vee E_2$. From Lemma 3.9, $E_1 \cap E_2$, $E_1 \cap E_j$, and $E_2 \cap E_j$ are $n-1$ dimensional; and $E_1 \cap E_2 \cap E_j$ is $n-2$ dimensional, when $j \neq 1, 2$. Thus E is $n+1$ dimensional; and E_1, E_2 contain unit vectors v_1, v_2, respectively, that lie in E_j but not in $E_1 \cap E_2 \cap E_j$. It follows that $E_1 \cap E_2 \cap E_j$ and v_1 generate an $n-1$ dimensional subspace of $E_1 \cap E_j$ (which is, therefore, $E_1 \cap E_j$). As v_2 is in E_2 but not in $E_1 \cap E_2 \cap E_j$, v_2 is not in $E_1 \cap E_j$; so that $E_1 \cap E_2 \cap E_j$, v_1 and v_2 generate an n-dimensional subspace of E_j, which is therefore, E_j. Thus E_j is a subspace of E. Let f_j be a unit vector in E (unique up to a phase factor) orthogonal to E_j. From Section 2 and Lemma 3.9, E_j and E_k are perpendicular, when $j \neq k$; and both are subspaces of E. Hence f_j is in E_k and f_k is in E_j, when $j \neq k$. Since f_j is orthogonal to E_j; $\langle f_j | f_k \rangle = 0$, when $j \neq k$. Thus $\{f_1, \ldots, f_{n+1}\}$ is an orthonormal basis for E, and $E_j = [f_1, \ldots, f_{j-1}, f_{j+1}, \ldots, f_{n+1}]$. Hence f_j generates $\bigcap_{k \neq j} E_k$. From Corollary 3.10, $\bigcap_{x_1, \ldots, x_{n-1}} [\hat{U}(x_1 \wedge \cdots \wedge x_{n-1} \wedge e_j)]$ is a one-dimensional space $\left(\text{contained in } \bigcap_{k \neq j} E_k\right)$; so that f_j generates it.

Lemma 4.3. *If $\hat{U}$ is a product unitary operator on $\mathscr{H}_n^{(a)}$ and $\{e_j\}$ is an orthonormal basis for $\mathscr{H}$, there is an orthonormal basis $\{f_j\}$ for $\mathscr{H}$ such that $\hat{U}(e_{i_1} \wedge \cdots \wedge e_{i_n}) = f_{i_1} \wedge \cdots \wedge f_{i_n}$ for all $i_1, \ldots, i_n$.*

Proof. From Lemma 4.2, we can find an orthonormal basis $\{f'_j\}$ such that $f'_j \in \bigcap_{x_1, \ldots, x_{n-1}} [\hat{U}(x_1 \wedge \cdots \wedge x_{n-1} \wedge e_j)]$ for all j. Thus $\hat{U}(e_{i_1} \wedge \cdots \wedge e_{i_n}) = c_{i_1 \ldots i_n} f'_{i_1} \wedge \cdots \wedge f'_{i_n}$, where $|c_{i_1 \ldots i_n}| = 1$. Let c'_j be $c_{1 \ldots j-1\, j+1 \ldots n+1}$, and let c_j be $\bar{c}'_j \left(\prod_{k=1}^{n+1} c'_k \right)^{1/n}$. With f_j as $c_j f'_j$,

$$\begin{aligned}
&\hat{U}(e_1 \wedge \cdots \wedge e_{j-1} \wedge e_{j+1} \wedge \cdots \wedge e_{n+1}) \\
&\quad = c'_j f'_1 \wedge \cdots \wedge f'_{j-1} \wedge f'_{j+1} \wedge \cdots \wedge f'_{n+1} \\
&\quad = c'_j \overline{(c_1 \ldots c_{j-1} c_{j+1} \ldots c_{n+1})} f_1 \wedge \cdots \wedge f_{j-1} \wedge f_{j+1} \wedge \cdots \wedge f_{n+1} \\
&\quad = f_1 \wedge \cdots \wedge f_{j-1} \wedge f_{j+1} \wedge \cdots \wedge f_{n+1} .
\end{aligned}$$

Suppose, now, that we have chosen $f_1, \ldots, f_m$, $m > n$, so that f_j is a multiple of f'_j and $\hat{U}(e_{i_1} \wedge \cdots \wedge e_{i_n}) = f_{i_1} \wedge \cdots \wedge f_{i_n}$ when $1 \leqq i_1 < i_2 < \cdots < i_n \leqq m$. Suppose $\hat{U}(e_1 \wedge \cdots \wedge e_{n-1} \wedge e_{m+1}) = c' f_1 \wedge \cdots \wedge f_{n-1} \wedge f'_{m+1}$ and $\hat{U}(e_2 \wedge \cdots \wedge e_n \wedge e_{m+1}) = c'' f_2 \wedge \cdots \wedge f_n \wedge f'_{m+1}$. From Lemma 4.2, there is a vector f such that

$$\hat{U}(e_1 \wedge \cdots \wedge e_{n-1} \wedge (e_{m+1} + e_{n+1})) = f_1 \wedge \cdots \wedge f_{n-1} \wedge f$$

and

$$\hat{U}(e_2 \wedge \cdots \wedge e_n \wedge (e_{m+1} + e_{n+1})) = c f_2 \wedge \cdots \wedge f_n \wedge f .$$

Since

$$\hat{U}(e_1 \wedge \cdots \wedge e_{n-1} \wedge (e_{m+1} + e_{n+1})) = f_1 \wedge \cdots \wedge f_{n-1} \wedge (c' f'_{m+1} + f_{n+1})$$

and

$$\hat{U}(e_2 \wedge \cdots \wedge e_n \wedge (e_{m+1} + e_{n+1})) = f_2 \wedge \cdots \wedge f_n \wedge (c'' f'_{m+1} + f_{n+1}) ;$$

$f = c' f'_{m+1} + f_{n+1}$ and $c f = c'' f'_{m+1} + f_{n+1}$. Thus $(cc' - c'') f'_{m+1} + (c-1) f_{n+1} = 0$; and $c = 1$, $c' = c''$.

Applying this conclusion to step-by-step replacements, we have that $\hat{U}(e_{i_1} \wedge \cdots \wedge e_{i_n} \wedge e_{m+1}) = c f_{i_1} \wedge \cdots \wedge f_{i_n} \wedge f'_{m+1}$ for one phase factor c and all $i_1, \ldots, i_n$ less than $m+1$. Defining f_{m+1} to be $c f'_{m+1}$, our induction yields the basis $\{f_j\}$.

Proof of Theorem 4.1. With $\{e_j\}$ and $\{f_j\}$ as in Lemma 4.3, let U be the unitary operator on $\mathscr{H}$ for which $U e_j = f_j$, $j = 1, 2, \ldots$. Then $\hat{U}(e_{i_1} \wedge \cdots \wedge e_{i_n}) = U e_{i_1} \wedge \cdots \wedge U e_{i_n}$; so that $\hat{U}(x_1 \wedge \cdots \wedge x_n) = U x_1 \wedge \cdots \wedge U x_n$ for all $x_1, \ldots, x_n$ in $\mathscr{H}$.

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