BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 77, Number 6, November 1971

# AUTOMORPHISMS OF A FREE ASSOCIATIVE ALGEBRA OF RANK 2

BY ANASTASIA CZERNIAKIEWICZ

Communicated by Hyman Bass, May 28, 1971

We announce here that the answer to the following conjecture [3, p. 197] is in the affirmative:

If $R$ is a Generalized Euclidean Domain then every automorphism of the free associative algebra of rank 2 over $R$ is tame, i.e. a product of elementary automorphisms.

We state here the necessary steps to prove the conjecture; detailed proofs will appear in [4] and [5].

*Notation.* $R$ stands for a commutative domain with 1;

$R\langle x, y\rangle$ is the free associative algebra of rank 2 over $R$, on the *free* generators $x$ and $y$;

$R(\tilde{x}, \tilde{y})$ is the polynomial algebra over $R$ on the *commuting* indeterminates $\tilde{x}$ and $\tilde{y}$.

We write $R\langle x, y\rangle$ as a bigraded algebra

$$R\langle x, y\rangle = \bigoplus_{r \geqq s \geqq 0} \mathfrak{A}_r^s$$

where the subindex denotes the homogeneous degree and the upper index stands for the degree in $x$. We will write $P = \sum P_r^s$ where $P_r^s \in \mathfrak{A}_r^s$ for every $P \in R\langle x, y\rangle$.

The elementary automorphisms of $R\langle x, y\rangle$ are by definition the following:

(i) $\sigma \in \operatorname{Aut}_R(R\langle x, y\rangle)$; $\sigma(x) = y$; $\sigma(y) = x$.

(ii) $\varphi_{\alpha,\beta} \in \operatorname{Aut}_R(R\langle x, y\rangle)$, $\alpha$, $\beta$ units of $R$;

$$\varphi_{\alpha,\beta}(x) = \alpha x; \qquad \varphi_{\alpha,\beta}(y) = \beta y.$$

(iii) $\tau_{f(y)} \in \operatorname{Aut}_R(R\langle x, y\rangle)$, where $f(y)$ is any polynomial that does not depend on $x$;

$$\tau_{f(y)}(x) = x + f(y); \qquad \tau_{f(y)}(y) = y.$$

In a completely parallel way one defines the elementary automorphisms of $R(\tilde{x}, \tilde{y})$.

---

*AMS* 1970 *subject classifications.* Primary 16A06, 16A72; Secondary 20F55, 16A02.

*Key words and phrases.* Free associative algebra, endomorphisms, automorphisms, elementary automorphisms, tame automorphisms, wild automorphisms, polynomial rings, euclidean domains.

THEOREM 1. *The map*

$$\mathrm{Aut}_R(R\langle x, y\rangle) \to \mathrm{Aut}_R(R(\tilde{x}, \tilde{y}))$$

*induced by the abelianization functor is a monomorphism.*

The proof of Theorem 1 is an immediate corollary of the more technical result:

THEOREM 2. *Let* $P, Q, E \in R\langle x, y\rangle$ *satisfy the following requirements:*
(i) $P_0^0 = Q_0^0 = 0,\ E_0 = E_1 = 0.$
(ii) $P_n^0 = 0$ *for all* $n \geqq 1$; $Q_m^0 = 0$ *for all* $m \geqq 2$; $E_r^0 = 0$ *for all* $r \geqq 2$.
(iii) $E(P,Q) = xy - yx$.
*Then we conclude*

$$P = P_1^1 = \alpha x; \qquad Q = Q_1^0 + \sum_n Q_n^n = \beta y + f(x);$$

$$E = (\alpha\beta)^{-1}(xy - yx), \qquad \alpha, \beta \text{ are units of } R.$$

The proof of Theorem 2 is obtained with slight modifications from the proof of the main theorem in [4].

In fact, for every rational number $\lambda \geqq 0$ we define

$$\chi_\lambda = \left\{P_a^\alpha;\ a > 1,\ \alpha \geqq 1,\ \frac{\alpha - 1}{a - 1} = \lambda\right\}$$

$$\cup \left\{Q_b^\beta;\ b > 1,\ \beta \geqq 0,\ \frac{\beta}{b - 1} = \lambda\right\}$$

$$\cup \left\{E_m^\mu;\ m > 2,\ \mu \geqq 1,\ \frac{\mu - 1}{m - 2} = \lambda\right\}.$$

As we have only a finite set of rational numbers $\lambda$ for which $\chi_\lambda \neq \{0\}$ we use the ordering of the rational numbers to prove inductively that if $\chi_\lambda = \{0\}$ for all $\lambda < \lambda_0$ then $\chi_{\lambda_0} = \{0\}$.

To achieve this purpose we exhibit a relation of algebraic dependence between two elements of $\chi_{\lambda_0}$ and using a result of P. M. Cohn [2] about homogeneous elements of $R\langle x, y\rangle$ we conclude $\chi_{\lambda_0} = \{0\}$.

COROLLARY. *If $R$ is a generalized euclidean domain then every automorphism of $R\langle x, y\rangle$ is tame.*

In fact, let $\phi$ be an automorphism of $R\langle x, y\rangle$. Using a theorem of Jung [6] that says that every automorphism of $R(\tilde{x}, \tilde{y})$ is tame, we can assume that, modulo a tame automorphism of $R\langle x, y\rangle$, the map $\mathrm{Aut}_R(R\langle x, y\rangle) \to \mathrm{Aut}_R(R(\tilde{x}, \tilde{y}))$ carries $\phi$ into the identity. Hence using

Theorem 1 it follows that $\phi$ must be the identity of $\mathrm{Aut}_R(R\langle x, y\rangle)$.

## REFERENCES

1. P. M. Cohn, *Free associative algebras*, Bull. London Math. Soc. **1** (1969), 1–39; correction, ibid. **1** (1969), 218. MR **39** #2800; #5614.

2. ———, *On subsemigroups of free semigroups*, Proc. Amer. Math. Soc. **13** (1962), 347–351. MR **25** #3111.

3. ———, *Lecture notes on free rings*, Yale University, New Haven, Conn., 1962.

4. A. Czerniakiewicz, *Automorphisms of a free associative algebra of rank* 2 Thesis, New York University, 1970.

5. ———, *Automorphisms of a free associative algebra of rank* 2, Trans. Amer. Math. Soc. **160** (1971), 393–401.

6. H. W. E. Jung, *Über ganze birationale Transformationen der Ebene*, J. Reine Angew. Math. **184** (1942), 161–174. MR **5**, 74.

COLUMBIA UNIVERSITY, NEW YORK, NEW YORK 10027