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# AUTOMORPHISMS OF A SURFACE OF GENERAL TYPE ACTING TRIVIALLY IN COHOMOLOGY

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**Abstract.** It is proved that, for a complex minimal smooth projective surface $S$ of general type, any automorphism group of $S$, inducing trivial actions on the second rational cohomology of $S$, is isomorphic to a cyclic group of order less than five or the product of two groups of order two, provided that the Euler characteristic of the structure sheaf of $S$ is larger than 188.

**Introduction.** It is well-known that, for a curve $C$ of genus $g \geq 2$, the automorphism group $\mathrm{Aut}\,C$ acts faithfully on $H^1(C, \boldsymbol{Q})$.

The case of surfaces has been studied by many authors. For K3 and Enriques surfaces $S$, $\mathrm{Aut}\,S$ acts faithfully on $H^2(S, \boldsymbol{Z})$ (cf. [BR], [Ue]); and there exists an Enriques surface $S$ for which $\mathrm{Aut}\,S$ does not act faithfully on $H^2(S, \boldsymbol{Q})$ (cf. [Pe]). For compact Kähler surfaces $S$ with $h^0(T_S) = 0$ and the canonical linear system $|K_S|$ base point free, Peters [Pe] proved that, if a non-trivial $\sigma \in \mathrm{Aut}\,S$ acts trivially on $H^2(S, \boldsymbol{Q})$, then either $K_S^2 = 8\chi(\mathcal{O}_S)$ and the order $\mathrm{o}(\sigma)$ of $\sigma$ is a power of 2 or $K_S^2 = 9\chi(\mathcal{O}_S)$ and $\mathrm{o}(\sigma)$ is a power of 3.

Taking the product of two hyperelliptic curves, one gets easily examples of surfaces of general type for which $\mathrm{Aut}\,S$ does not act faithfully on $H^2(S, \boldsymbol{Q})$. The aim of this paper is to prove the following

THEOREM A. *Let $S$ be a complex minimal smooth projective surface of general type, and $\chi(\mathcal{O}_S)$ the Euler characteristic of the structure sheaf of $S$. Let $G \subset \mathrm{Aut}\,S$ be a subgroup of automorphisms acting trivially on $H^2(S, \boldsymbol{Q})$. If $\chi(\mathcal{O}_S) > 188$, then $G$ is isomorphic to $C_n$ $(n \leq 4)$ or $C_2 \times C_2$, where $C_n$ is a cyclic group of order $n$.*

Theorem A is proved in Sections 2 through 4. Thanks to Beauville's theorem on the canonical map of $S$, the problem reduces to the analysis of the automorphisms of the canonical fiber surface $f : S \to B$, of genus $g \leq 5$. The main part of this paper is to treat the case $g = 3$ and $G$ nonabelian of order 8 or 6. The idea of the proof is to prove the existence of a $G$-invariant irreducible curve (in a singular fiber of $f$) on which $G$ acts faithfully and to analyze the action around it.

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We use standard notation as in [BPV] or [Ha]. In this paper we denote by $C_n$, $D_{2n}$ and $Q_8$ the cyclic group of order $n$, the dihedral group of order $2n$, and the quaternion group of order 8.

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**1. Preliminaries.** For the reader's convenience, in this section we recall several results from the literature.

(1.1) Let $S$ be a smooth complex projective surface of general type, with a fibration $f: S \to B$ of genus $g \geq 2$ over a smooth curve $B$. We assume that $f$ is relatively minimal, that is, $S$ has no $(-1)$-curves contained in a fiber of $f$. Denote by $F$ the general fiber of $f$. Let $K_S$ be the canonical divisor of $S$.

We say that $f$ is a *hyperelliptic* (resp. *nonhyperelliptic*) *fibration* if $F$ is a hyperelliptic (resp. nonhyperelliptic) curve. An irreducible curve $C$ on $S$ is *vertical* (with respect to $f$) if $f(C)$ is a point; otherwise, we say $C$ is *horizonal*.

(1.2) Let $f: S \to B$ be a relatively minimal fibration of genus $g \geq 2$, and $\sigma$ an involution of $S$ inducing the trivial action on $B$. Let $u : \tilde{S} \to S$ be the blowup of all isolated fixed points of $\sigma$, and $\tilde{\sigma}$ the induced involution on $\tilde{S}$. Let $P_\sigma = \tilde{S}/\tilde{\sigma}$. Then $f$ induces a fibration $h_\sigma: P_\sigma \to B$ of genus $g(F/\sigma)$ (not relatively minimal in general). We have a commutative diagram

$$\begin{array}{ccc} \tilde{S} & \xrightarrow{\pi} & P_\sigma \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle h_\sigma} \\ S & \xrightarrow[f]{} & B. \end{array}$$

(1.2.1) If $\Gamma < f^*b$ $(b \in B)$ is a $\sigma$-fixed curve, then from $(f \circ u)^*b = \pi^*(h_\sigma^* b)$, the coefficient of $\Gamma$ in $f^*b$ is divisible by 2. In particular, if $f^*b$ is reduced, then $\sigma$ acts nontrivially on any irreducible component of $f^*b$.

Let $F'$ be a semistable fiber of $f$ (i.e., $F'$ is reduced with only nodes as singularities), and $p \in F'$ a node. We say that $p$ is a *separating point* (resp. *nonseparating point*) of $F'$, if $F' \setminus \{p\}$ is disconnected (resp. connected) as a topological space.

(1.3) (cf. [Ca, Lemma 2.4]) Let $f: S \to B$ and $\sigma$ be as above, and $F'$ a semistable singular fiber of $f$. If $p \in F'$ is an isolated fixed point of $\sigma$, then $p$ is a node of $F'$, and moreover if $\sigma$ is a hyperelliptic involution of $S$, then $p$ is a separating point of $F'$.

(1.4) Notation as in (1.2). If $f$ is a relatively minimal hyperelliptic fibration, gluing the hyperelliptic involution of $F$ gives an everywhere defined involution $\sigma$ on $S$. Then $h_\sigma: P_\sigma \to B$ is a ruled surface. Let $(\tilde{R}, \tilde{\delta})$ be the double cover data corresponding to $\pi: \tilde{S} \to P_\sigma$. One has a minimal ruled surface $P$, and a (possibly singular) double cover data $(R, \delta)$ on $P$, satisfying the following conditions:

(i) There is a birational morphism $\phi: P_\sigma \to P$ such that $\tilde{R}$ is the reduced inverse image of $R$;

(ii) Let $R_h$ be the sum of the nonvertical irreducible components of $R$. Then the singularities of $R_h$ are at most of order $g+1$, and $R^2$ is the smallest among all such choices (cf. [X1, Lemma 6]). $(P, R, \delta)$ is called the *genus $g$ data* corresponding to $f$.

(1.4.1) ([X1, Definition 5]) Let $f: S \to B$ be a hyperelliptic fibration corresponding to genus $g$ data $(P, R, \delta)$. For any fiber $F$ of $f$ and $i = 3, \dots, g+2$, we define the $i$-singularity $s_i(F)$ of $F$ as follows:

If $i$ is odd, $s_i(F)$ equals the number of singularities of type $i \to i$ (that is, infinitely near points of multiplicity $i$) of $R$ on the image of $F$.

If $i$ is even, $s_i(F)$ equals the number of singularities of order $i$ of $R$ on the image of $F$, not belonging to a singularity of type $i-1 \to i-1$ or $i+1 \to i+1$.

The singularities $s_i(F)$ do not depend on the choice of the contraction map $\phi\colon\ P_\sigma \to P$ (cf. [X1, Lemma 8]). Clearly there are only a finite number of fibers $F$ with $s_i(F) \neq 0$ for each $i$. A fiber $F$ is *essential*, if $s_i(F) \neq 0$ for some $i$.

(1.4.2) (Xiao [X1, Theorem 1]) Let $f: S \to B$ be the hyperelliptic fibration corresponding to genus $g$ data $(P, R, \delta)$. If $f$ has no essential fibers, then

$$K_S^2 = \frac{4g-4}{g}\chi(\mathcal{O}_S) - \frac{4(g^2-1)(g(B)-1)}{g}\,.$$

(1.5) (Reid [Re]) Let $f: S \to B$ be a nonhyperelliptic fibration of genus $g = 3$. Then the natural morphism of sheaves

$$r\colon S^2(f_*\omega_{S/B}) \to f_*\omega_{S/B}^2$$

is generically surjective. Let $\mathcal{M} = \operatorname{Coker} r$. Then $\mathcal{M} = \bigoplus_{b\in B} \mathcal{M}_b$, where $\mathcal{M}_b$ is the stalk of $\mathcal{M}$ at $b \in B$, which is an $\mathcal{O}_{B,b}$-module of finite length. Let $H(S/B, b) = \operatorname{length} \mathcal{M}_b$. For any $b \in B$, if $f^*b$ is a smooth nonhyperelliptic curve or an irreducible nonhyperelliptic curve with one node whose normalization is a curve of genus 2, then $H(S/B, b) = 0$. Using the Riemann-Roch theorem on $S$ and the Leray spectral sequence, we have

$$K_S^2 = 3\chi(\mathcal{O}_S) + 10(g(B)-1) + \sum_{b\in B} H(S/B, b)\,.$$

For any normal surface $X$, we denote by $p_g(X)$ the geometric genus of a nonsingular model of $X$.

(1.6) (Beauville [Be]) Let $S$ be a projective minimal nonsingular surface of general type with $\chi(\mathcal{O}_S) \geq 21$, and $\phi_S\colon S - - \to \boldsymbol{P}^{p_g(S)-1}$ the canonical map. There are two cases:

(1.6.1) $\phi_S$ is composed with a pencil. Then the moving part of $|K_S|$ is base point free. Let $f: S \to B$ be the fibration associated with $\phi_S$, and $g$ the genus of the general fiber of $f$. Then $2 \leq g \leq 5$ and $K_S^2 \geq (2g-2)(\chi(\mathcal{O}_S) - 2)$.

(1.6.2) $\dim \operatorname{Im} \phi_S = 2$. If $\chi(\mathcal{O}_S) \geq 31$, then either (i) $p_g(\operatorname{Im}\phi_S) = 0$ and $\deg \phi_S \leq 9$ or (ii) $p_g(\operatorname{Im}\phi_S) = p_g(S)$ and $\deg\phi_S \leq 3$.

(1.7) (Xiao [X2]) Let $f: S \to B$ be as in (1.6.1). Then $g(B) \leq 1$.

(1.8) (A special case of the logarithmic Miyaoka-Yau inequality. cf. [Sa]) Let $S$ be a projective nonsingular complex surface of general type and $C \subset S$ a nonsingular curve. Then $K_S^2 \le 9\chi(\mathcal{O}_S) + (g(C)-1) - K_S C/4$.

(1.9) (Accola [Ac]) Let $C$ be a curve of genus $g$, and $G \subset \mathrm{Aut}\, C$ a finite group. If $G$ admits a partition, i.e., $G = \bigcup_{i=1}^s G_i$, where $G_i$ are subgroups of $G$ satisfying $G_i \cap G_j = \langle 1_G \rangle$ for all $i \neq j$, then

$$(s-1)g + |G|g(C/G) = \sum_{i=1}^{s} |G_i| g(C/G_i)\,.$$

For example, assume that $G = D_{2n}$ is a dihedral group of order $2n$. Let $\alpha \in G$ generate the cyclic subgroup of order $n$, and let $\beta \in G$ be an element of order 2 not in $\langle\alpha\rangle$. Then $\beta_i = \alpha^i\beta$ $(i = 1, 2, \dots, n)$ are elements in $G$ not in $\langle\alpha\rangle$. So $G$ admits a partition and we have

$$g + 2g(C/G) = g(C/\langle\alpha\rangle) + g(C/\langle\beta_1\rangle) + g(C/\langle\beta_2\rangle)\,.$$

(1.10) Let $S$ be a smooth surface, $\sigma \in \mathrm{Aut}\, S$, and $p \in S$ a fixed point of $\sigma$. Then $\sigma$ induces a linear action on the tangent space $T_pS$ of $S$ at $p$. If this action is trivial, then $\sigma$ is trivial.

A curve $C \subset S$ is *$\sigma$-invariant* (resp. *$\sigma$-fixed*), if $\sigma(C) = C$ (resp. $\sigma(p) = p$ for any $p \in C$).

(1.11) If a reduced $\sigma$-fixed curve $C$ is singular, then $\sigma$ is trivial. This follows from (1.10), since the induced action of $\sigma$ on the tangent space at the singular point of $C$ is trivial.

(1.12) Let $C$ be a curve of genus $g$, and $G \subset \mathrm{Aut}\, C$ a finite group. If $G$ has a fixed point, then $G$ is cyclic.

(1.13) Let $C$ be a curve of genus $g \ge 2$, and $G \subset \mathrm{Aut}\, C$ an abelian group. Assume that $g(C/G) = 1$. Let $\pi : C \to C/G$ be the quotient map. Let $q_i$ $(i = 1, \dots, k)$ be the points over which $\pi$ is ramified and $r_i$ the ramification number of $\pi$ over $q_i$. Then $k \ge 2$, and if $k = 2$ then $r_1 = r_2$. Indeed, $G$ is an abelian quotient of $\pi_1(C/G \setminus \{q_1, \dots, q_k\})$, which is generated by $\alpha, \beta, \gamma_1, \dots, \gamma_k$ with one relation $\alpha\beta\alpha^{-1}\beta^{-1}\gamma_1 \cdots \gamma_k = 1$, where $\alpha$ and $\beta$ are generators of $\pi_1(C/G)$ and $\gamma_i$ is a small loop around $q_i$. Let $\bar{\gamma}_i$ be the image of $\gamma_i$ in $G$. Then $\bar{\gamma}_i$ is of order $r_i$ and $\bar{\gamma}_1 \cdots \bar{\gamma}_k = 1$.

(1.14) Let $S$ be a smooth projective surface, and $G \subset \mathrm{Aut}\, S$ a finite subgroup such that $G$ acts trivially on $H^2(S, \boldsymbol{Q})$. By the argument of [Pe, Lemma 2], we have that, if $p \in S$ a $\sigma$-fixed point for some $\mathrm{id} \neq \sigma \in G$, then either $p \in \mathrm{Bs}|K_S|$ (the base locus of $|K_S|$) or $p$ is an isolated $\sigma$-fixed point. This implies:

(1.14.1) If $C \subset S$ is a $\sigma$-fixed curve for some $\mathrm{id} \neq \sigma \in G$, then $C \subset \mathrm{Bs}|K_S|$.

(1.14.2) If $C \subset S$ is a $G$-invariant curve, and $C \not\subset \mathrm{Bs}|K_S|$, then $G$ acts faithfully on $C$, i.e., $G \hookrightarrow \mathrm{Aut}\, C$.

(1.15) Let $S$ and $G$ be as in (1.14). Assume that $S$ has a fibration $f : S \to B$ and $G$ induces the trivial action on $B$. If $p_g(S) > 0$ then $g(F/G) > 0$, where $F$ is a general fiber of $f$. Indeed, We have $p_g(S/G) = \dim H^0(S, \omega_S)^G$ (cf. [Fr, p. 99]). By Hodge theory,

$H^0(S, \omega_S)^G = H^0(S, \omega_S)$. So $p_g(S/G) = p_g(S)$ and thus the general fiber of $S/G \to B$ is not rational if $p_g(S) > 0$.

**2. First reductions.** To prove Theorem A, let me start by fixing notation.

(2.1) Let $S$ be a complex minimal nonsingular projective surface of general type with $\chi(\mathcal{O}_S) \geq 21$. Assume that the canonical map $\phi_S$ of $S$ is composed with a pencil.

Let $G \subset \operatorname{Aut} S$ be a subgroup of automorphisms of $S$, inducing trivial actions on $H^2(S, \boldsymbol{Q})$.

Let $M$ and $Z$ be the moving part and the fixed part of $|K_S|$, respectively. By (1.6.1), $|M|$ has no base points. Let

$$\phi_S = \varphi \circ f : S \to B \to \operatorname{Im} \phi_S \subset \boldsymbol{P}^{p_g(S)-1}$$

be the Stein factorization of $\phi_S$. We call $f : S \to B$ the *canonical fibration* associated with $\phi_S$. Let $F$ be a general fiber of $f$, and $g$ the genus of $F$.

Let $d$ and $L$ be the degree and the hyperplane section of $\operatorname{Im} \phi_S$ in $\boldsymbol{P}^{p_g(S)-1}$ respectively. We have $\mathcal{O}_S(M) = f^*\varphi^*L$ and $M \sim_{\text{num}} \deg \varphi d F$. Note that $h^1(B, \varphi^*L) = 0$, since $g(B) \leq 1$ by (1.7), and $d \geq \operatorname{codim} \operatorname{Im} \phi_S + 1$ (cf. [Mu]). From

$$p_g(S) = h^0(S, \varphi^*L) = \deg(\varphi^*L) + 1 - g(B) + h^1(B, \varphi^*L) = \deg \varphi d + 1 - g(B),$$

we get

$$\deg \varphi = 1 \quad \text{and} \tag{2.1.1}$$

$$d = \begin{cases} \chi(\mathcal{O}_S) & \text{if } g(B) = 1, \\ \chi(\mathcal{O}_S) - 2 + q(S) & \text{if } g(B) = 0. \end{cases} \tag{2.1.2}$$

(2.2) Since $H^0(S, \omega_S)$ is a direct factor of $H^2(S, \boldsymbol{C})$ by Hodge theory, $G$ acts trivially on $H^0(S, \omega_S)$. This implies that $G$ acts trivially on $\operatorname{Im} \phi_S$ and there is a homomorphism $h$ of $G$ into $\operatorname{Aut} B$. Since $\deg \varphi = 1$ (2.1.1), we have that $\operatorname{Ker} h = G$, i.e., $G$ induces the trivial action on $B$, and $G \hookrightarrow \operatorname{Aut} F$ for a general fiber $F$ of $f$.

NOTATION 2.3. Let $f : S \to B$ and $G$ be as above.

(i) We write $Z = H + V$ and $H = n_1\Gamma_1 + n_2\Gamma_2 + \cdots$ with $n_1 \geq n_2 \geq \cdots$, where $H$ (resp. $V$) is the horizontal part (resp. the vertical part) of $Z$, and $\Gamma_i$ ($i = 1, 2, \ldots$) are the irreducible components of $H$, with $n_i$ the multiplicity of $\Gamma_i$ in $H$.

(ii) For a general fiber $F$ of $f$, let $R_F$ be the set of ramified points of the quotient map $F \to F/G$. For any two curves $C$ and $D$ on $S$, we denote by $C \cap D$ the set-theoretic intersection $\operatorname{supp} C \cap \operatorname{supp} D$.

LEMMA 2.4. *Let $f : S \to B$, $H$, $\Gamma_i$ and $G$ be as in* (2.1). *Let $F$ be a general fiber of $f$. Then*

(2.4.1) $R_F \subset H \cap F$.

(2.4.2) *If $R_F = H \cap F$, then $\Gamma_i$ is smooth for every $i$.*

PROOF. (i) Suppose that there is a point $p \in F$ such that $p \in R_F$ and $p \notin H \cap F$. Then there exists an element id $\neq \sigma \in G$ such that $p$ is $\sigma$-fixed. Since $F$ is a general fiber, $p$ is not an isolated fixed point of $\sigma$. So there exists a $\sigma$-fixed curve $C$ passing through $p$. By (1.14.1), $C \subset \mathrm{Bs}|K_S|$. $C$ is not vertical since $F$ is a general fiber. So $C < H$, which contradicts the assumption that $p \notin H \cap F$.

(ii) For a general point $p \in \Gamma_i$, $p \in H \cap f^*(f(p)) = R_{f^*(f(p))}$. This implies there exists id $\neq \sigma_p \in G$ such that $p$ is $\sigma_p$-fixed. Since $G$ is finite, there is a id $\neq \sigma \in G$ such that $\Gamma_i$ is $\sigma$-fixed. So $\Gamma_i$ is smooth by (1.11). □

LEMMA 2.5. *Let $f : S \to B$, $H$, $g$ and $G$ be as in* (2.1). *Let $F$ be a general fiber of $f$. If $2 \leq g \leq 4$, then either $|G| \leq 2g - 2$, or $G$ is nonabelian, $G$ acts transitively on $H \cap F'$ for any fiber $F'$ of $f$, and the only possibilities for the triple $(g, |G|, \#(H \cap F))$ are as follows*:

$$(3,\ 8,\ 4)\,,\quad (3,\ 6,\ 2)\,,\quad (4,\ 12,\ 6)\,,\quad (4,\ 8,\ 2)\,.$$

*Moreover, if $(g, |G|, \#(H \cap F)) = (3,\ 8,\ 4)$ or $(4,\ 12,\ 6)$, then $H$ is reduced and each irreducible component of $H$ is smooth.*

PROOF. For any point $p \in S$, let $\mathrm{stab}(p) = \{\tau \in G | \tau(p) = p\}$. If $r_p := |\mathrm{stab}(p)| = 1$ for some $p \in H \cap F$, then $|G| \leq \#(H \cap F) \leq 2g - 2$. So we can assume that $|\mathrm{stab}(p)| \geq 2$ for each $p \in H \cap F$. Let $m$ be the number of orbits of $H \cap F$ under the action of $G$. Then by (2.4.1), the quotient map $\pi : F \to F/G$ has exactly $m$ branch points. Using the Hurwitz formula for $\pi$, we get $|G| \leq 2g - 2$ if either $g(F/G) \geq 2$, or $g(F/G) = 1$ and either $G$ is abelian or $m \geq 2$. Hence we can assume that $g(F/G) = 1$ ($g(F/G) \neq 0$ by (1.15)), $G$ is nonabelian and $m = 1$. Then $G$ acts transitively on $H \cap F$ and hence on $H \cap F'$ for any fiber $F'$ of $f$. In this case, for any point $p \in H \cap F$, we have $|G|/r_p = \#(H \cap F)$ and $\#(H \cap F) \mid 2g - 2$. Using the Hurwitz formula for $\pi$ again, we have $|G| = \#(H \cap F) + 2g - 2$. Note that $G$ is nonabelian in this case, and we get that $(g, |G|, \#(H \cap F))$ equals one of the triples listed in the lemma. The last statement follows by (2.4.2). □

REMARK 2.6. If $(g, |G|, \#(H \cap F)) = (3,\ 6,\ 2)$, then either $H = 2\Gamma_1 + 2\Gamma_2$ or $H = 2\Gamma$, where $\Gamma_i$ are sections of $f$ and $\Gamma$ is an irreducible smooth curve with $\Gamma F = 2$.

PROPOSITION 2.7. *Let $S$ be a complex minimal nonsingular projective surface of general type with $\chi(\mathcal{O}_S) \geq 21$, and let $G \subset \mathrm{Aut}\, S$ be a subgroup of automorphisms of $S$ inducing trivial actions on $H^2(S, \boldsymbol{Q})$. Assume that the canonical map $\phi_S$ is composed with a pencil. Let $f : S \to B$ be the canonical fibration associated with $\phi_S$, and $g$ the genus of a general fiber of $f$. Furthermore, assume $\chi(\mathcal{O}_S) > 188$ if $g = 4$, and $\chi(\mathcal{O}_S) > 60$ if $g = 5$. Then $|G| \leq 4$.*

PROOF. By (1.6.1), $2 \leq g \leq 5$. If $g = 2$, we have $|G| \leq 2$ by Lemma 2.5. The proof of the case $3 \leq g \leq 5$ is longer and is postponed till the next two sections. □

PROOF OF THEOREM A. By Proposition 2.7, we can assume that $\phi_S$ is generically finite. Since $H^0(S, \omega_S)$ is a direct factor of $H^2(S, \boldsymbol{Q})$ by Hodge theory, $G$ acts trivially on $H^0(S, \omega_S)$. This implies that $G$ induces trivial actions on $\mathrm{Im}\,\phi_S$. So $\phi_S$ factors through the quotient map

$$\phi_S = \alpha \circ q : S \xrightarrow{q} S/G \xrightarrow{\alpha} \operatorname{Im} \phi_S .$$

Thus $|G| = \deg \phi_S / \deg \alpha$. Now if $S$ is as in case (ii) of (1.6.2), then $|G| \leq 3$. If $S$ is as in case (i) of (1.6.2), then $\deg \alpha \geq 2$, since $p_g(S/G) = p_g(S) \neq 0 = p_g(\operatorname{Im} \phi_S)$. So $|G| \leq \deg \phi_S/2 \leq 9/2$. □

### 3. Proof of Proposition 2.7, the case $g = 3$.

LEMMA 3.1. *Let $S$ be a complex nonsingular projective surface, and $G \subset \operatorname{Aut} S$ a subgroup of automorphisms of $S$ inducing trivial actions on $H^2(S, \boldsymbol{Q})$. Let $C \subset S$ be an irreducible curve. If $C^2 < 0$, then $C$ is $G$-invariant.*

PROOF. Indeed, if $C$ is not $\sigma$-invariant for some $\mathrm{id} \neq \sigma \in G$, then $(\sigma^* C)C \geq 0$. On the other hand, $\sigma^* C$ is numerically equivalent to $C$, since $G$ acts trivially on $\mathrm{NS}(S) \otimes \boldsymbol{Q} \hookrightarrow H^2(S, \boldsymbol{Q})$. So $(\sigma^* C)C = C^2 < 0$, a contradiction. □

LEMMA 3.2. *Let $f : S \to B$, $H$, $g$ and $G$ be as in* (2.1). *Assume that $g = 3$ and $G$ is a nonabelian group of order* 8.

(i) *Let $\sigma$ be the generator of the center of $G$, which is clearly a cyclic subgroup of order* 2. *Then $H$ is $\sigma$-fixed* (*and hence smooth*), *and $G/\langle\sigma\rangle \hookrightarrow \operatorname{Aut} H$.*

(ii) *Let $F'$ be a singular fiber of $f$ and $C < F'$ an irreducible component with $CH \neq 0$. Then $G \hookrightarrow \operatorname{Aut} C$.*

PROOF. (i) Let $F$ be a general fiber of $f$. Let $\bar{F} = F/\langle\sigma\rangle$ and $\bar{G} = G/\langle\sigma\rangle$. Since $g(F/G) = 1$ and $|\bar{G}| = 4$, using the Hurwitz formula for $\bar{F} \to \bar{F}/\bar{G} \simeq F/G$, we get $g(\bar{F}) = 1$. So $\sigma$ has four fixed points on $F$. Since $\#(H \cap F) = 4$ in Lemma 2.5, by (2.4.1), $H$ is $\sigma$-fixed and hence smooth by (1.11). Since $G$ acts transitively on $H \cap F$ and $\#(H \cap F) = 4$, we have $G/\langle\sigma\rangle \hookrightarrow \operatorname{Aut} H$.

(ii) If $F'$ is reducible, $C$ is $G$-invariant by (3.1); if the reduced scheme $F'_{\mathrm{red}}$ of $F'$ is irreducible, then $C = F'_{\mathrm{red}}$ is clearly $G$-invariant. So there is a homomorphism $h : G \to \operatorname{Aut} C$.

Let $\sigma$ be as in (i). If $\sigma \in \operatorname{Ker} h$, then $C + H$ is $\sigma$-fixed. So $\sigma$ is trivial by (1.11). This is impossible. Hence the lemma follows by showing that $\sigma \in \operatorname{Ker} h$ if $\operatorname{Ker} h$ is not trivial.

Suppose that $\operatorname{Ker} h$ is not trivial. If $G \simeq Q_8$, we have that $\sigma \in \operatorname{Ker} h$ since there is only one element of order 2 in $Q_8$. Now assume that $G \simeq D_8$. If $|\operatorname{Ker} h| = 2$, we get $\operatorname{Ker} h = \langle\sigma\rangle$ since a normal subgroup of order 2 must be contained in the center of $G$; If $|\operatorname{Ker} h| = 4$, let $\alpha \in G$ be an element of order 4. Then $\sigma = \alpha^2$ and $h(\alpha^2) = h(\alpha)^2 = \mathrm{id}$. So $\sigma \in \operatorname{Ker} h$. □

LEMMA 3.3. (i) *Let $G$ be a nonabelian group of order* 8. *Assume that $G \hookrightarrow \operatorname{Aut} C$ for some smooth curve $C$ of genus $\leq 1$. Then $G \simeq D_8$. Moreover, if $g(C) = 1$, the elements of order* 4 *of $G$ act freely on $C$.*

(ii) *If $G \simeq D_6 \hookrightarrow \operatorname{Aut} C$ for some smooth elliptic curve $C$, then the elements of order* 3 *of $G$ act freely on $C$.*

PROOF. (i) If $C \simeq \boldsymbol{P}^1$, the lemma follows by the well known fact that a finite subgroup of Aut $\boldsymbol{P}^1$ is isomorphic to one of the following groups: $C_n$, $D_{2n}$, $T_{12}$, $O_{24}$ and $I_{60}$, where $T_{12}$, $O_{24}$ and $I_{60}$ are the polyhedral groups of indicated orders.

If $C$ is an elliptic curve, then $G = T \rtimes A$ (a semi-direct product), where $T$ is a group of translations and $A \subset \operatorname{Aut} C$ is a subgroup preserving the group structure. If $T \simeq C_2$, then $G$ must be abelian, which contradicts the assumption. Now assume that $|T| = 4$. Let $\alpha \in G$ be an element of order 4. Then it is easy to see that $\alpha^2 \in T$ since $|A| = 2$ in this case. So $\alpha^2$ and hence $\alpha$ has no fixed points. This implies $\alpha \in T$. Hence $T \simeq C_4$, and the result follows.

(ii) follows by an argument similar to that in (i). □

LEMMA 3.4. *Let $f: S \to B$, $g$ and $G$ be as in* (2.1). *Assume that $g = 3$.*

(i) *If $G \simeq Q_8$, $f$ is nonhyperelliptic.*

(ii) *If $G \simeq D_8$ or $D_6$, $f$ is hyperelliptic.*

PROOF. (i) Otherwise, let $\tau$ be the hyperelliptic involution of a general fiber $F$ of $f$. Since $g(F/G) = 1$ by (1.15), we get $\tau \notin G$. This implies $G \hookrightarrow \operatorname{Aut} \boldsymbol{P}^1$, since Aut $F$ is a $\langle\tau\rangle$-extension of a subgroup of Aut $\boldsymbol{P}^1$. This is impossible by Lemma 3.3.

(ii) Let $F$ be a general fiber of $f$. If $G \simeq D_{2n}$ for $n = 3$ or 4, then by (1.9) we have $g(F) + 2g(F/D_{2n}) = g(F/\langle\alpha\rangle) + g(F/\langle\beta_1\rangle) + g(F/\langle\beta_2\rangle)$, where $\alpha$ and $\beta_i$ are as in (1.9). Since $g(F/D_{2n}) = 1$ and $g(F/\langle\alpha\rangle) = 1$, we get $g(F/\langle\beta_i\rangle) = 2$. So $F$ is étale over a curve of genus 2. This implies $F$ is hyperelliptic by [Ac]. □

LEMMA 3.5. *Let $f: S \to B$ be a nonhyperelliptic fibration of genus* 3, *and $G \subset \operatorname{Aut} S$ a subgroup inducing the trivial action on $B$. Let $F'$ be a fiber of $f$. Assume that $G \simeq Q_8$, and that $F'$ is either a smooth hyperelliptic curve or a multiple fiber $2C$ with $C$ smooth of genus* 2. *Then the kernel of the homomorphism $h: G \to \operatorname{Aut} F'_{\mathrm{red}}$ is not trivial.*

PROOF. Suppose that $\ker h$ is trivial. Denote by $\sigma$ the unique element of order 2 in $G$.

First we assume that $F' = 2C$, where $C$ is a smooth curve of genus 2. Let $p' = f(F')$ and fix a point $p \in B$ such that $f^*p$ is smooth. Let $\tilde{B} \to B$ be a double cover ramified exactly at $p$ and $p'$, and let $\pi': \tilde{S} \to \tilde{B} \times_B S$ be the normalization. Then $\pi := p_2 \circ \pi': \tilde{S} \to S$ is ramified along $f^*p$, and $\tilde{f} := p_1 \circ \pi': \tilde{S} \to \tilde{B}$ is a fibration of genus 3, where $p_1$ and $p_2$ are the projections of $\tilde{B} \times_B S$ onto its factors. Let $\tilde{p}'$ be the inverse image of $p'$. Then $\tilde{F}' := \tilde{f}^*\tilde{p}'$ is a smooth hyperelliptic curve. Since $G$ induces the trivial action on $B$, $\tilde{B} \times_B S \subset \tilde{B} \times S$ is $G$-invariant. So $G$ acts on $\tilde{S}$, inducing the trivial action on $\tilde{B}$. We have $G \hookrightarrow \operatorname{Aut} \tilde{F}'$ if $\operatorname{Ker} h$ is trivial. Hence the lemma is reduced to the case when $F'$ is a smooth hyperelliptic curve.

Now assume that $F'$ is a smooth hyperelliptic curve. Let $\tau$ be the hyperelliptic involution of $F'$. If $\tau \notin G$, then $G \hookrightarrow \operatorname{Aut} \boldsymbol{P}^1$ since Aut $F'$ is a $\langle\tau\rangle$-extension of a subgroup of Aut $\boldsymbol{P}^1$. This is impossible by Lemma 3.3. So we can assume that $\sigma$ is the hyperelliptic involution of $F'$. Then there are eight $\sigma$-fixed points on $F'$. By (1.3), there exists a $\sigma$-fixed curve $D$ passing through these points. Since $G \simeq Q_8 \hookrightarrow \operatorname{Aut} F'$ by assumption, we get $F' \neq D$. Now for a general fiber $F$, there are at least $\#(D \cap F) = DF = DF' \geq 8$ $\sigma$-fixed points.

This implies that $\sigma$ is the hyperelliptic involution of $F$, contradicting the assumption that $f$ is nonhyperelliptic. □

LEMMA 3.6. *Let $f: S \to B$, $H$, $g$ and $G$ be as in* (2.1). *Assume that $g = 3$ and $G$ is a nonabelian group of order* 8. *Let $F'$ be a singular fiber of $f$ and $C < F'$ an irreducible component. Denote by $\tilde{C}$ the normalization of $C$. If $g(\tilde{C}) \geq 2$, then $F'$ belongs to one of the following possible types.*

(i) *$F' = 2C$, and $C$ is smooth*;

(ii) *$F' = C$ is an irreducible curve with one node, and the normalization of $F'$ is a curve of genus* 2;

(iii) *$F' = C + D$, where $C$ and $D$ are irreducible smooth curves meeting transversally at two points, and $g(C) = 2$ and $g(D) = 0$.*

PROOF. We have either $p_a(C) = 3$ or $C = \tilde{C}$. In the former case, $F' = C$ is an irreducible curve with one singularity, say $q$, and its normalization is a curve of genus 2. If $q \in F'$ is a cusp, the inverse image $\tilde{q}$ of the cusp $q \in F'$ under the normalization map is $G$-fixed. This implies $G$ is cyclic by (1.12), a contradiction. So $F'$ is of type (ii). In the latter case, since $K_S C = 2 - C^2 \geq 2$ and $K_S F' = 4$, we get either $C^2 = 0$ ($F'$ is of type (i)) or $\mathrm{mult}_C\, F' = 1$. Now assume that $\mathrm{mult}_C\, F' = 1$. Then $F'$ is 1-connected. Let $D < F'$ be an irreducible curve such that $DC > 0$. If $\#(D \cap C) = 1$, $G$ is cyclic, a contradiction. So $\#(D \cap C) \geq 2$ and hence $DC \geq 2$. Note that $p_a(D + C) \leq 3$, hence we have $DC = 2$ and $F'$ is of type (iii). □

PROOF OF PROPOSITION 2.7, THE CASE $g = 3$. Let $f: S \to B$ be the canonical fibration associated with $\phi_S$. By (2.2), $G$ induces the trivial action on $B$, and $G \hookrightarrow \mathrm{Aut}\, F$, where $F$ is a general fiber of $f$. Assume $g = 3$. By Lemma 2.5, if $|G| > 4$, then $G$ is isomorphic to $Q_8$, $D_8$ or $D_6$. Now the result follows by the next claims. □

CLAIM 3.7. *$G \simeq Q_8$ does not occur.*

PROOF. Suppose $G \simeq Q_8$. Then by Lemma 3.4, $f$ is nonhyperelliptic. Since $g(B) \leq 1$ by (1.7), $f$ has singular fibers.

Let $F'$ be a singular fiber of $f$, and let $C < F'$ be an irreducible component such that $CH \neq 0$. By Lemma 3.2 (ii), we have $G \hookrightarrow \mathrm{Aut}\, C$. By Lemmas 3.3 (i), 3.5 and 3.6, we have that $F'$ is of type (ii) or (iii) of (3.6).

If $F'$ is of type (iii) of (3.6), we have that either $D \not< V$ or $HD > 0$, where $V$ is as in (2.1). Indeed, if both $D < V$ and $HD = 0$ hold, then $C \not< V$ and thus $VD < 0$. But from $0 = K_S D = (M + H + V)D$, we get $VD = 0$, a contradiction. Now by Lemma 3.2 (ii) and (1.14.2), $Q_8 \hookrightarrow \mathrm{Aut}\, D$. This is impossible by Lemma 3.3 (i).

Now if $F'$ is of type (ii) of Lemma 3.6, we show that $F'$ is nonhyperelliptic.

Let $\sigma$ be the generator of the center of $G$. We have $G/\langle\sigma\rangle \simeq C_2 \times C_2$. First we claim that the node $q \in F'$ is an isolated $\sigma$-fixed point. Otherwise, there is a $\sigma$-fixed curve $D$ passing through $q$. By (1.14.1), $D < H$. Since $q \in F'$ is $G$-fixed, $q \in H$ is $G/\langle\sigma\rangle$-fixed. By Lemma 3.2 (i) and (1.12), $G/\langle\sigma\rangle$ is cyclic, a contradiction.

Second, we claim that $\sigma$ preserves the local two branches at $q$. Indeed, let $G' \subset G$ be the subgroup preserving the local two branches at $q$. Clearly $G'$ is cyclic of order 4. Let $\alpha$ be a generator of $G'$. If $\sigma \notin G'$, then $\sigma$ and $\alpha$ generate $G$, and it is easy to see that $\sigma\alpha\sigma = \alpha^{-1}$. This implies that $G \simeq D_8$, a contradiction.

Now we have that $q$ is an isolated $\sigma$-fixed point and that $\sigma$ preserves the local two branches at $q$. So $h_\sigma^*(f(q))$ consists of two irreducible smooth curves meeting transversally at two points, where $h_\sigma\colon P_\sigma \to B$ is as in (1.2). Since $h_\sigma$ is of genus 1, $\sigma$ is a hyperelliptic involution of the normalization $\tilde{F}'$ of $F'$. This implies that $F'$ is a nonhyperelliptic fiber. Indeed, if there exists an involution $\tau$ on $F'$ such that $F'/\langle\tau\rangle \simeq \boldsymbol{P}^1$, then $\tau$ exchanges the local two branches at $q$, and $\tau$ is a hyperelliptic involution of $\tilde{F}'$. This implies that $\sigma = \tau$ on $F'$, which is absurd since one preserves the local two branches at $q$ while the other not.

By the above argument, we have that any singular fiber $F'$ of $f$ is a nonhyperelliptic irreducible curve with one node. By Lemma 3.5 and (1.14.2), $f$ has no smooth hyperelliptic fibers. Thus $f$ has no fibers $F'$ with non-vanishing $H(S/B, f(F'))$ (see (1.5) for the notation). By (1.5), we have that

$$K_S^2 = 3\chi(\mathcal{O}_S) + 10(g(B) - 1)\,.$$

We get a contradiction by (1.6.1). □

CLAIM 3.8. *$G \simeq D_8$ or $D_6$ does not occur.*

PROOF. Suppose $G \simeq D_6$. The proof of the case $G \simeq D_8$ is similar and is left to the reader. By Lemma 3.4, $f$ is hyperelliptic. We will show that

(3.8.1) any singular fiber of $f$ belongs to one of the following types:

(i) $F' = C$ is an irreducible curve with one node, and the normalization of $F'$ is a curve of genus 2;

(ii) $F' = C$ is an irreducible curve with three nodes, and the normalization of $F'$ is isomorphic to $\boldsymbol{P}^1$;

(iii) $F' = C + D$, where $C$ and $D$ are irreducible smooth curves meeting transversally at two points, and $g(C) = 2$ and $g(D) = 0$.

We note that, if $F'$ belongs to one of the types (i)–(iii), the singularities $s_i(F') = 0$ for $i \geq 3$ (see (1.4.1) for the definition). (We check it when $F'$ is of the type (iii); the other cases are similar. Let $q_1$ and $q_2$ be nodes of $F' = C + D$. Let $\tau$ be the hyperelliptic involution of $f$. Let the notation be as in (1.2) and (1.4). Since the dual graph of $h_\tau(f(F'))$ is a tree, we have $\tau q_1 = q_2$. Let $\bar{C}$ and $\bar{D}$ be the images of $C$ and $D$ under $\pi$, respectively. Then $h_\tau(f(F')) = \bar{C} + \bar{D}$ consists of two smooth rational curves meeting transversally at one point, and $\tilde{R}$ meets $\bar{C}$ (resp. $\bar{D}$) transversally at six (resp. two) points. By the choice of $\phi\colon P_\tau \to P$, $\phi$ contracts $\bar{D}$. Thus there is only one singular point of order 2 of $R$ on the image of $F'$ and hence by (1.4.1) $s_i(F') = 0$ for $i \geq 3$.)

Admitting (3.8.1) for the moment, we have that $f$ has no essential fibers, and hence by (1.4)

$$K_S^2 = \frac{8}{3}\chi(\mathcal{O}_S) - \frac{32(g(B) - 1)}{3}\,.$$

On the other hand, by (1.6.1), $K_S^2 \geq 4(\chi(\mathcal{O}_S) - 2)$, a contradiction.

It remains to prove (3.8.1). Let $\alpha \in D_6$ (resp. $\sigma \in D_6$) be an element of order 3 (resp. 2). By the proof of Lemma 2.5, $H$ is $\alpha$-fixed. Let $F'$ be a singular fiber of $f$. Let $C < F'$ be an irreducible component such that $CH \neq 0$, and $\tilde{C}$ the normalization of $C$. Then $C$ is $\alpha$-invariant by Lemma 3.1, and the homomorphism $h$ of $G$ into $\operatorname{Aut} C$ is injective. (Otherwise, $\operatorname{Ker} h = \langle\alpha\rangle$ or $G$ since the nontrivial normal subgroup of $G$ is $\langle\alpha\rangle$. Hence $\alpha$ is trivial on $C + H$, which is impossible by (1.11).) We distinguish two cases according to whether $f_H\colon H_{\mathrm{red}} \to B$ is étale at $H \cap F'$ or not.

*Case* 1. $f_H$ is étale at $H \cap F'$. In this case $H \cap F'$ consists of two points, say $p_1$ and $p_2$. Since $HF' = 4$, by Remark 2.6, $F'$ is smooth at these points. Since $H$ is $\alpha$-fixed, $p_1$ and $p_2$ are $\alpha$-fixed. By the choice of $C$, there are at least two $\alpha$-fixed points ($p_1$ and $p_2$) on it, and $C$ is smooth at $p_i$ for $i = 1$ and 2.

If $g(\tilde{C}) = 2$, then by the proof of Lemma 3.6, we have that $F'$ is (i) or (iii).

If $g(\tilde{C}) = 1$, then by Lemma 3.3 (ii), we get a contradiction.

Now we assume $g(\tilde{C}) = 0$. We show that in this case either $F'$ is of type (ii) or there exists a $\sigma$-fixed point $p \in C$ with $2 \nmid \operatorname{mult}_p F'$. We consider three cases according to the singularities of $C$.

(i) *There is a point* $p \in C_{\mathrm{sing}}$ *with* $\operatorname{mult}_p C \geq 3$. Then $p \in C$ is an ordinary triple point and $C \setminus \{p\}$ is smooth and $F' = C$. So $p$ is $\sigma$-fixed and $\operatorname{mult}_p F' = 3$.

(ii) $C_{\mathrm{sing}} \neq \emptyset$ *and for any point* $p \in C_{\mathrm{sing}}$, *with* $\operatorname{mult}_p C = 2$. Since $p_1$ and $p_2$ are $\alpha$-fixed and $\alpha$ has exactly two fixed points on $\tilde{C} \simeq \boldsymbol{P}^1$, we have $\alpha(p) \neq p$ if $p \in C_{\mathrm{sing}}$. Hence either $F'$ is of type (ii), or $F' = C$ is an irreducible curve with three cusps (say $q_1, q_2$ and $q_3$) and the normalization of $F'$ is isomorphic to $\boldsymbol{P}^1$. In the latter case, let $\tilde{q}_i$ ($i = 1,\ 2,\ 3$) be the inverse image of $q_i$ under the normalization map $\tilde{C} \to C$. Since $\{q_1, q_2, q_3\}$ is $\sigma$-invariant and there are exactly two $\sigma$-fixed points on $\tilde{C} \simeq \boldsymbol{P}^1$, there must be a point $\tilde{p} \in \tilde{C} \setminus \{q_1, q_2, q_3\}$ which is $\sigma$-fixed. Let $p$ be the image of $\tilde{p}$ under the normalization map. Then $p$ is $\sigma$-fixed and $\operatorname{mult}_p F' = 1$.

(iii) *$C$ is a smooth rational curve.* Since $F'$ is 1-connected, there is an irreducible curve $D < F'$ such that $DC > 0$. Since $D \cap C$ is $\alpha$-invariant by Lemma 3.1, if $\#(D \cap C) \not\equiv 0 \pmod 3$, $\alpha$ has at least there fixed points ($p_1$, $p_2$ and a point in $D \cap C$) on $C$. This implies $\alpha$ is trivial on $C$ and hence on $C + H$, which is impossible by (1.11). So we can assume $\#(D \cap C) \equiv 0 \pmod 3$. Since $p_a(C + D) \leq 3$, we have $DC \leq 4$. So $\#(D \cap C) = 3$. Since $D \cap C$ is $\sigma$-invariant and there are exactly two $\sigma$-fixed points on $\tilde{C} \simeq \boldsymbol{P}^1$, there is a point $p \in C \setminus D \cap C$ which is $\sigma$-fixed. We claim that $\operatorname{mult}_p F' = 1$. Otherwise, there is an irreducible curve $D' < F'$ passing through $p$. By the above argument, we can assume that $\#(D' \cap C) \equiv 0 \pmod 3$. This implies $p_a(C + D + D') > 3$, a contradiction.

Now by the above argument, we have that either $F'$ is of type (ii) or there is a $\sigma$-fixed point $p \in C$ with $2 \nmid \operatorname{mult}_p F'$. In the latter case, let $u\colon \tilde{S} \to S$ be as in (1.2). If $p$ is an isolated $\sigma$-fixed point, then the inverse image $E = u^{-1}(p)$ of $p$ is a $\sigma$-fixed $(-1)$-curve, and the coefficient of $E$ in $(f \circ u)^*(f(F'))$ is not divisible by 2. This is impossible by (1.2.1). So there is a $\sigma$-fixed curve $D$ passing through $p$. Clearly $D \neq C$ by (1.11). By (1.14.1),

$D < \mathrm{Bs}|K_S|$, and hence $D < H$. Since $\alpha$ and $\sigma$ generate $G$, this implies there is a $G$-fixed point $p' \in H \cap F$, and thus $G$ is cyclic by (1.12), a contradiction.

*Case* 2. $f_H$ is not étale at $H \cap F'$. Then $H \cap F'$ consists of one point, say $p$. By the choice of $C$, $C$ passes through $p$. Since $G \hookrightarrow \mathrm{Aut}\ C$, by (1.12), $p \in C$ is a singular point.

Since $HF' = 4$, we have $\mathrm{mult}_C\, F' = 1$ and $\mathrm{mult}_p\, C = 2$. If $p \in C$ is a cusp, it is easy to see $G$ is cyclic by (1.12). So we can assume that $p \in C$ is a node. Blowup $S$ at $p$, and let $E$ be the exceptional curve and $\tilde{H}$ the strict transform of $H$. If $p$ is an ordinary node of $C$, then $\alpha$ preserves the local branches of $C$ at $p$ since the order of $\alpha$ is 3. So $\alpha$ preserves the three local branches of $C + H$ at $p$. This implies $E$ and hence $E + \tilde{H}$ is $\alpha$-fixed. By (1.11) $\alpha$ is trivial on $S$, a contradiction. Now we can assume that $p \in C$ is a node which can be resolved by at least two successive blowups. Then $p_a(C) \geq 2$ and $g(\tilde{C}) \leq 1$, where $\tilde{C}$ is the normalization of $C$. If $g(\tilde{C}) = 1$, by Lemma 3.3(ii), $\alpha$ is a translation of $\tilde{C}$, which is impossible since $\alpha$ preserves the local branches of $C$ at $p$. Now we assume $g(\tilde{C}) = 0$. If $F'$ is reducible, let $D$ be an irreducible curve $D < F'$ such that $DC > 0$. Since $D \cap C$ is $\alpha$-invariant by Lemma 3.1 and there are exactly two $\alpha$-fixed points on $\tilde{C} \simeq \boldsymbol{P}^1$, we have $\#(D \cap C) \equiv 0 \pmod 3$. This implies $p_a(C + D) > 3$, a contradiction. Now we can assume $F' = C$. If there is a point $q \in C_{\mathrm{sing}} \setminus \{p\}$, then $q$ is $\alpha$-fixed and $\mathrm{milt}_q\, C = 2$ since $p_a(C) = 3$, and hence there are at least four $\alpha$-fixed points on $\tilde{C}$. This implies $\alpha$ is trivial on $\tilde{C}$ and hence on $C$, a contradiction. So we can assume that $C \setminus \{p\}$ is smooth. Then $p$ is $\sigma$-fixed. If $\sigma$ preserves the local branches of $C$ at $p$, then $G$ also does. This implies $G$ is cyclic by (1.12), a contradiction. So we can assume $\sigma$ exchanges the local branches of $C$ at $p$. This implies there are two $\sigma$-fixed points on $C \setminus \{p\}$. Now by the same argument as in the last paragraph of Case 1, we get a contradiction. This completes the proof of (3.8.1). □

## 4. Proof of Proposition 2.7, the case $g = 4, 5$.

LEMMA 4.1. *Let $f : S \to B$, $H$, $\Gamma_i$, $g$ and $G$ be as in* (2.1). *Assume that $g = 4$ and $|G| = 6$. Then $H$ is reduced and $\Gamma_i$ is nonsingular for every $i$.*

PROOF. Let $F$ be a general fiber of $f$. Using the Hurwitz formula for $\pi : F \to F/G$, we get that $g(F/G) = 1$ (note that $g(F/G) \geq 1$ by (1.15)) and $\pi$ has six ramification points. By (2.4.1), we have $\#(H \cap F) \geq 6$. This implies $H$ is reduced. Since $\#(H \cap F) \leq 2g - 2 = 6$, we have $R_F = H \cap F$. By (2.4.2), $\Gamma_i$ is nonsingular for every $i$. □

LEMMA 4.2. *Let $S$ be a minimal surface whose canonical map is composed with a pencil, and $f : S \to B$ the associated canonical fibration of genus $g$. Assume that $g = 4$, and that the horizontal part $H$ of the fixed part of $|K_S|$ is reduced and each irreducible component of $H$ is nonsingular. Then $\chi(\mathcal{O}_S) \leq 188$.*

PROOF. Let the notation be as in (2.1). Under the assumption, we have

$$K_S \equiv M + \sum_{i=1}^{t} \Gamma_i + V\,, \quad (t \leq 6)\,.$$

Let $g_i = g(\Gamma_i)$. From $K_S\Gamma_i \geq M\Gamma_i + \Gamma_i^2$ and the adjunction formula for $\Gamma_i$, we get

$$K_S\Gamma_i \geq \frac{M\Gamma_i}{2} + g_i - 1\,. \tag{1}$$

So

$$\begin{aligned} K_S^2 &\geq K_S M + \sum K_S\Gamma_i \geq 6d + \frac{\sum M\Gamma_i}{2} + \sum(g_i - 1) \\ &= 9d + \sum(g_i - 1)\,. \qquad \Big(\sum M\Gamma_i = MH = dFH = 6d\Big) \end{aligned} \tag{2}$$

On the other hand, using the logarithmic Miyaoka-Yau inequality for $(S, \Gamma_i)$ (1.8), we have $K_S^2 \leq 9\chi(\mathcal{O}_S) + (g_i - 1) - K_S\Gamma_i/4$ for every $i$. Hence

$$\begin{aligned} K_S^2 &\leq 9\chi(\mathcal{O}_S) + \frac{\sum(g_i - 1)}{t} - \frac{\sum K_S\Gamma_i}{4t} \\ &\leq 9\chi(\mathcal{O}_S) + \frac{3\sum(g_i - 1)}{4t} - \frac{3d}{4t} \quad \text{(by (1))}\,. \end{aligned} \tag{3}$$

Combining (2) and (3), we get

$$3d \leq (4t - 3)\sum(1 - g_i) + 36t(\chi(\mathcal{O}_S) - d)\,.$$

Note that $t \leq 6$, and $d = \chi(\mathcal{O}_S)$ if $g(B) = 1$ and $d = \chi(\mathcal{O}_S) - 2 + q(S)$ if $g(B) = 0$ (2.1.2). Hence we get $\chi(\mathcal{O}_S) \leq 188$. □

PROOF OF PROPOSITION 2.7, THE CASE $g = 4$. Let $f : S \to B$ be the canonical fibration associated with $\phi_S$, $F$ the general fiber of $f$, and $H$ the horizontal part of the fixed part of $|K_S|$. We have that $G$ induces the trivial action on $B$, and $G \hookrightarrow \operatorname{Aut} F$ by (2.2). By Lemma 2.5, if $|G| > 4$ then either $|G| = 6$ or $G$ is a nonabelian group of order 8 or 12 ($|G| \neq 5$ by the Hurwitz formula).

First we suppose that $|G| = 6$ or 12. Then by Lemmas 2.5 and 4.1, we have that $H$ is reduced and each irreducible component of $H$ is smooth. So by Lemma 4.2, $\chi(\mathcal{O}_S) \leq 188$, contradicting the assumption.

Second, we suppose that $G$ is a nonabelian group of order 8. Then either $G \simeq D_8$ or $G \simeq Q_8$.

(i) The case $G \simeq D_8$ does not occur.

Otherwise, $D_8 \hookrightarrow \operatorname{Aut} F$ for a general fiber $F$ of $f$. By (1.9), we have $4 + 2g(F/D_8) = g(F/\langle\alpha\rangle) + g(F/\langle\beta_1\rangle) + g(F/\langle\beta_2\rangle)$, where $\alpha$, $\beta_i$ are as in (1.9). But this is impossible since $g(F/D_8) = 1$, $g(F/\langle\alpha\rangle) = 1$, and $g(F/\langle\beta_i\rangle) \leq 2$ for every $i$ by the Hurwitz formula.

(ii) The case $G \simeq Q_8$ does not occur.

Otherwise, let $\sigma$ be a generator of $\operatorname{stab}(p)$ for some point $p \in H \cap F$. By the proof of Lemma 2.5, $\sigma$ is of order 4. Consider the commutative diagram

$$\begin{array}{ccc} F & \xrightarrow{\pi} & F/\langle\sigma\rangle \\ \big\downarrow & & \big\uparrow{\scriptstyle\lambda} \\ C & \xrightarrow{:=} & F/\langle\sigma^2\rangle. \end{array}$$

Since the ramification index of $\pi$ at $p \in F$ is 4, λ cannot be étale. This implies $g(C) = 2$.

Since $Q_8$ has only one element of order 2, $\langle\sigma^2\rangle$ is a normal subgroup of $Q_8$ and $\overline{G} := Q_8/\langle\sigma^2\rangle \simeq C_2 \times C_2$. Using the Hurwitz formula for $C \to C/G \simeq F/Q_8$, (note that $g(F/Q_8) = 1$,) by (1.13), we get $|\overline{G}| \leq 2$. This is a contradiction. □

PROOF OF PROPOSITION 2.7, THE CASE $g = 5$. Let $f: S \to B$ be the canonical fibration associated to $\phi_S$, and $F$ a general fiber of $f$. Let $M$, $H$, $V$, $\Gamma_i$, $n_i$ and $d$ be as in (2.1). Set $b = g(B)$. First we suppose that $n_1 < g$. Since $n_1 K_{S/B} + H + V$ is nef,

$$((n_1+1)K_S - M - n_1(2b-2)F)H = (n_1 K_{S/B} + H + V)H \geq 0\,.$$

So

$$K_S H \geq \frac{(2g-2)(d+n_1(2b-2))}{n_1+1} \geq \frac{(2g-2)(d+n_1(2b-2))}{g}\,.$$

On the other hand, using the Miyaoka-Yau inequality (cf. [Mi, Y]), we have

$$9\chi(\mathcal{O}_S) \geq K_S^2 = K_S(M+H+V) \geq (2g-2)d + K_S H\,.$$

Combining these two inequalities, we get $\chi(\mathcal{O}_S) \leq 34$, which contradicts the assumption.

Now we can assume that $n_1 \geq g$. Then $\Gamma_1$ is a section of $f$. This implies $\Gamma_1$ and hence the point $F \cap \Gamma_1 \in F$ is $G$-fixed. By (1.12), $G$ is cyclic. Using the Hurwitz formula for $F \to F/G$, (note that $g(F/G) \geq 1$ (1.15) and by (1.13) when $g(F/G) = 1$) we get $G \simeq C_5$ and $\#(R \cap F) = 2$ if $|G| > 4$.

Now we prove that the case $G \simeq C_5$ does not occur. Otherwise, by (2.4.1), $\#(R \cap F) \geq 2$. Since $(H - n_1\Gamma_1)F = 8 - n_1 \leq 3$ and $|G| = 5$, we must have $\#(R \cap F) = 2$. So $H = n\Gamma_1 + (8-n)\Gamma_2$ with $5 \leq n \leq 7$ and $\Gamma_2 F = 1$. Since $\Gamma_1 + \Gamma_2$ is $G$-fixed, by (1.11), $\Gamma_1\Gamma_2 = 0$. From $K_S\Gamma_1 = (M+H+V)\Gamma_1 \geq d + n\Gamma_1^2$ and the adjunction formula for $\Gamma_1$, we get

$$K_S\Gamma_1 \geq \frac{d+n(2b-2)}{n+1}\,.$$

Similarly, we have

$$K_S\Gamma_2 \geq \frac{d+(8-n)(2b-2)}{9-n}\,.$$

Using the logarithmic Miyaoka-Yau inequality (1.8), we have

$$\begin{aligned} 9\chi(\mathcal{O}_S) + (b-1) - \frac{1}{4}K_S(\Gamma_1+\Gamma_2) \geq K_S^2 &= K_S(M + n\Gamma_1 + (8-n)\Gamma_2 + V) \\ &\geq (2g-2)d + nK_S\Gamma_1 + (8-n)K_S\Gamma_2\,. \end{aligned}$$

Combining these inequalities, we get $\chi(\mathcal{O}_S) \leq 60$, which contradicts the assumption. □

## REFERENCES

[Ac] R. D. M. ACCOLA, Riemann surfaces with automorphism groups admitting partitions, Proc. Amer. Math. Soc. 21 (1969), 477–482.

[Be] A. BEAUVILLE, L'application canonique pour les surfaces de type qénéral, Invent. Math. 55 (1979), 121–140.

[BR] D. BURNS AND M. RAPOPORT, On the Torelli problem for Kählerian $K3$ surfaces, Ann. Sci. École Norm. Sup. (4)8 (1975) 235–274.

[BPV] W. BARTH, C. PETERS AND A. VAN DE VEN, Compact complex surfaces, Ergeb. Math. Grenzgeb. (3), Springer-Verlag, Berlin, 1984.

[Ca] J.-X. CAI, On Abelian automorphism groups of fiber surfaces of small genus, Math. Proc. Cambridge Philos. Soc. 130 (2001), 161–174.

[Fr] E. FREITAG, Uber die Struktur der Funktionenkörper zu Hyperabelschen Gruppen I, J. Reine Angew. Math. 247 (1971), 97–117.

[Ha] R. HARTSHORNE, Algebraic Geometry, Graduate Texts in Mathematics 52, Springer-Verlag, New York-Heidelberg, 1977.

[Mi] Y. MIYAOKA, On the Chern numbers of surfaces of general type, Invent. Math. 42 (1977), 225–237.

[Mu] D. MUMFORD, Algebraic Geometry I: Complex Projective Varieties, Springer-Verlag, 1976.

[Pe] C. A. M. PETERS, Holomorphic automorphisms of compact Kahler surfaces and their induced actions in cohomology, Invent. Math. 52 (1979), 143–148.

[Re] M. REID, Problems on pencils of small genus, Preprint (1990).

[Sa] F. SAKAI, Semi-stable curves on algebraic surfaces and logarithmic pluri-canonical maps, Math. Ann. 254 (1980), 89–120.

[Ue] K. UENO, A remark on automorphisms of Enriques surfaces, J. Fac. Sci. Univ. Tokyo, Sect. I A Math. 23 (1976), 149–165.

[X1] G. XIAO, $\pi_1$ of elliptic and hyperelliptic surfaces, Internat. J. Math. 2 (1991), 599–615.

[X2] G. XIAO, L'irrégularité des surfaces de type général dont le système canonique est composé d'un pinceau, Compositio Math. (1985), 251–257.

[Y] S.-T. YAU, Calabi's conjecture and some new results in algebraic geometry, Proc. Nat. Acad. Sic. USA 74 (1977), 1789–1799.

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