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# AUTOMORPHISMS OF AN IRREGULAR SURFACE OF GENERAL TYPE ACTING TRIVIALLY IN COHOMOLOGY, II

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**Abstract.** Let $S$ be a complex nonsingular minimal projective surface of general type with $q(S) = 2$, and let $G$ be the group of the automorphisms of $S$ acting trivially on $H^2(S, \boldsymbol{Q})$. In this note we classify explicitly pairs $(S, G)$ with $G$ of order four.

**Introduction.** Let $S$ be a complex minimal nonsingular projective surface of general type, and let $G \subset \mathrm{Aut}S$ be the subgroup of automorphisms of $S$ inducing trivial actions on $H^2(S, \boldsymbol{Q})$. In [Ca1], we proved that $|G| \leq 4$ provided $\chi(\mathcal{O}_S) > 188$. In this note, we continue the classification of the pairs $(S, G)$ with $|G| = 4$, started in [Ca2]. Whereas there we considered the case $q(S) \geq 3$, here we study the case $q(S) = 2$. Our main result is the following.

THEOREM 0.1 (Theorems 2.3 and 3.1). *Let $S$ be a complex nonsingular minimal projective surface of general type with $q(S) = 2$. Assume that there is a subgroup $G \subset \mathrm{Aut}S$, of order* 4, *acting trivially in $H^2(S, \boldsymbol{Q})$. If $p_g(S) > 61$, then $S$ is isogenous to a product of curves*; *in particular*, *it satisfies $K_S^2 = 8\chi(\mathcal{O}_S)$. Explicitly*, *the pair $(S, G)$ is as in one of Examples* 1.1, 1.2 *and* 1.3.

NOTATIONS. We use standard notations as in [Ha].

For a finite Abelian group $G$, we denote by $\widehat{G}$ the character group of $G$. For a representation $V$ of $G$ and a character $\chi \in \widehat{G}$, we let

$$V_G^\chi = \{v \in V;\ g \cdot v = \chi(g)v \text{ for all } g \in G\}\,.$$

If $G$ is a cyclic group generated by $\sigma$, we shall also use the notation $V_\sigma^c$ to denote $V_G^\chi$, where $c = \chi(\sigma)$. If moreover $\sigma$ is of order two, $V_\sigma^{\pm 1}$ is also denoted by $V_\sigma^\pm$.

The symbol $\boldsymbol{Z}_n$ denotes the cyclic group of order $n$.

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**1. Examples.** In this section, we construct explicitly pairs $(S, G)$ with $|G| = 4$, where $S$ is a complex nonsingular minimal projective surface of general type with $q(S) = 2$ and $G$ is the subgroup of automorphisms of $S$ acting trivially on $H^2(S, \boldsymbol{Q})$. These surfaces are isogenous to products of curves; in particular, they satisfy $K_S^2 = 8\chi(\mathcal{O}_S)$.

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EXAMPLE 1.1 ($G \simeq \boldsymbol{Z}_2^{\oplus 2}$). Let $\tilde{B}$ be a hyperelliptic curve of genus $\tilde{g}$ and $\tau$ the hyperelliptic involution of $\tilde{B}$. Suppose there is a curve $F$ of genus $g = 3$ with involutions $\iota$, $\sigma_{1F}$ and $\sigma_{2F}$ such that

(i) the subgroup of Aut$F$ generated by $\iota$, $\sigma_{1F}$ and $\sigma_{2F}$ is isomorphic to $\boldsymbol{Z}_2^{\oplus 3}$;

(ii) $\iota$ has no fixed points;

(iii) for $i = 1$ and 2, $\sigma_{iF}$ induces the identity on $H^0(\Omega_F^1)_\iota^-$.

Let $S = (\tilde{B} \times F)/\langle \tau \times \iota \rangle$, and $\pi : \tilde{B} \times F \to S$ the quotient map. Then $S$ is a smooth surface with $p_g(S) = \tilde{g}$, $q(S) = 2$ and $K_S^2 = 8(\tilde{g} - 1)$.

Let $\sigma_i$ be the automorphism of $S$ induced by $\mathrm{id}_{\tilde{B}} \times \sigma_{iF} \in \mathrm{Aut}(\tilde{B} \times F)$. We have that the group $G$ generated by $\sigma_i$ ($i = 1$ and 2) is isomorphic to $\boldsymbol{Z}_2^{\oplus 2}$ and acts trivially on $H^2(S, \boldsymbol{Q})$. Indeed, (iii) implies that $(\mathrm{id}_{\tilde{B}} \times \sigma_{iF})^* = \mathrm{id}$ on $H^1(\tilde{B}) \otimes H^1(F)_\iota^-$ and hence on $H^2(\tilde{B} \times F)_{\tau\times\iota}^1$. Since $\pi^* : H^2(S) \to H^2(\tilde{B} \times F)_{\tau\times\iota}^1$ is an isomorphism and $\pi^* \circ \sigma_i^* = (\mathrm{id}_{\tilde{B}} \times \sigma_{iF})^* \circ \pi^*$, we have that $\sigma_i^* = \mathrm{id}$ on $H^2(S, \boldsymbol{Q})$.

**1.1.1.** *A curve $F$ of genus* 3 *with involutions $\iota$, $\sigma_{1F}$ and $\sigma_{2F}$ satisfying conditions* (i)–(iii) *in Example* 1.1.

Let $0, \infty, 1, b_1$ and $b_2$ be different points of $B := \boldsymbol{P}^1$. For $i = 1, 2$, let $\hat{\pi}_i : \hat{E}_i \to B$ be the double cover branched along points $0, \infty, 1, b_i$. Using $\hat{\pi}_i$ instead of $\pi_i$, we may modify the construction in [Ca2, 1.1.1] to give a curve $F$ of genus 3 with involutions $\iota$, $\sigma_{1F}$ and $\sigma_{2F}$ satisfying conditions (i)–(iii) in Example 1.1.

EXAMPLE 1.2 ($G \simeq \boldsymbol{Z}_4$). Let $\tilde{B}$ be a hyperelliptic curve of genus $\tilde{g}$ and $\tau$ the hyperelliptic involution of $\tilde{B}$. Suppose there is a curve $F$ of genus 3 with automorphisms $\iota$, $\sigma_F$ such that

(i) the subgroup of Aut$F$ generated by $\iota$ and $\sigma_F$ is isomorphic to $\boldsymbol{Z}_2 \oplus \boldsymbol{Z}_4$;

(ii) $\iota$ has no fixed points;

(iii) $\sigma_F$ induces the identity on $H^0(\Omega_F^1)_\iota^-$.

Let $S = (\tilde{B} \times F)/\langle \tau \times \iota \rangle$. Then $S$ is a smooth surface with $p_g(S) = \tilde{g}$, $q(S) = 2$ and $K_S^2 = 8(\tilde{g} - 1)$.

Let $\sigma$ be the automorphism of $S$ induced by $\mathrm{id}_{\tilde{B}} \times \sigma_F \in \mathrm{Aut}(\tilde{B} \times F)$. One checks easily as in Example 1.1 that the group $G$ generated by $\sigma$ is isomorphic to $\boldsymbol{Z}_4$ and acts trivially on $H^2(S, \boldsymbol{Q})$.

**1.2.1.** *A curve $F$ of genus* 3 *with automorphisms $\iota$, $\sigma_F$ satisfying conditions* (i)–(iii) *in Example* 1.2.

Let $F$ be the hyperelliptic curve given by the equation

$$y^2 = (x^4 + 1)(x^4 + a)\,,$$

where $a \in \boldsymbol{C} \setminus \{0, 1\}$. Let $\tau_F$ be the hyperelliptic involution (given by $(x, y) \mapsto (x, -y)$), and $\alpha$ the automorphism given by $(x, y) \mapsto (\sqrt{-1}x, y)$. Note that $\omega_j := x^j \mathrm{d}x/y$ ($j = 0,\ 1,\ 2$) is a basis of $H^0(\Omega_F^1)$. We have that $\alpha^*\omega_j = \sqrt{-1}^{\,j+1}\omega_j$. So $(\tau_F\alpha^2)^*\omega_j = (-1)^j\omega_j$ and

$(\tau_F\alpha)^*\omega_1 = \omega_1$. One checks easily that $\iota := \tau_F\alpha^2$ and $\sigma_F := \tau_F\alpha$ have the desired properties (i)–(iii) in Example 1.2.

EXAMPLE 1.3 ($G \simeq \boldsymbol{Z}_2^{\oplus 2}$). Suppose there is a curve $F$ of genus 5 with automorphisms $\beta_1, \beta_2, \sigma_{1F}, \sigma_{2F}$ such that

(i) the subgroup of Aut$F$ generated by $\beta_1, \beta_2, \sigma_{1F}$ and $\sigma_{2F}$ is isomorphic to $\boldsymbol{Z}_2^{\oplus 4}$;

(ii) $g(F/A) = 2$, where $A := \langle \beta_1, \beta_2 \rangle$;

(iii) for $i = 1$ and 2, $\sigma_{iF}$ induces the identity on $H^0(\Omega_F^1)_A^{\chi_j}$ ($j = 1$ and 2), where $\chi_j$ is the character of $A$ with Ker$\chi_j = \langle \beta_j \rangle$.

Let $\tilde{B}$ be a hyperelliptic curve of genus $\tilde{g}$ with a faithful action of the group $A$ such that $\beta_3 := \beta_1\beta_2$ is the hyperelliptic involution of $\tilde{B}$. (In other words, $A$ is isomorphic to the subgroup of automorphisms generated by a non-hyperelliptic involution and the hyperelliptic involution of $\tilde{B}$.)

Let $S = (\tilde{B} \times F)/A$, where the action of $A$ on $\tilde{B} \times F$ is the diagonal action. Then $S$ is a smooth surface with $p_g(S) = \tilde{g}$, $q(S) = 2$ and $K_S^2 = 8(\tilde{g} - 1)$.

For $i = 1, 2$, let $\sigma_i$ be the automorphism of $S$ induced by $\mathrm{id}_{\tilde{B}} \times \sigma_{iF} \in \mathrm{Aut}(\tilde{B} \times F)$.

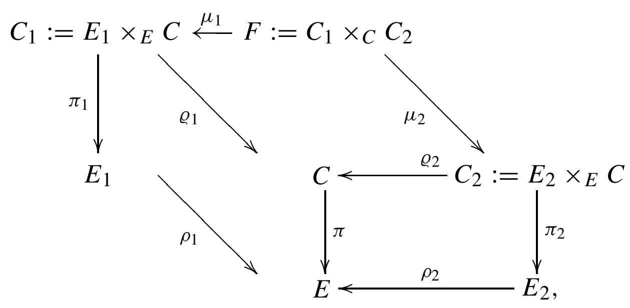
We have that the group $G$ generated by $\sigma_i$ ($i = 1$ and 2) is isomorphic to $\boldsymbol{Z}_2^{\oplus 2}$ and acts trivially on $H^2(S, \boldsymbol{Q})$. Indeed, let $\chi_3 := \chi_1\chi_2$, since Ker$\chi_3 = \langle \beta_3 \rangle$ and $\beta_3$ is the hyperelliptic involution of $\tilde{B}$, we have $H^1(\tilde{B})_A^{\chi_3} = 0$. So

$$H^2(\tilde{B} \times F)_A^1 = W \oplus H^1(\tilde{B})_A^{\chi_1} \otimes H^1(F)_A^{\chi_1} \oplus H^1(\tilde{B})_A^{\chi_2} \otimes H^1(F)_A^{\chi_2},$$

where $W = H^0(\tilde{B}) \otimes H^2(F) \oplus H^2(\tilde{B}) \otimes H^0(F)$. Now (iii) implies that $(\mathrm{id}_{\tilde{B}} \times \sigma_{iF})^* = \mathrm{id}$ on $H^2(\tilde{B} \times F)_A^1$. By the argument as in Example 1.1, we have that $\sigma_i^* = \mathrm{id}$ on $H^2(S, \boldsymbol{Q})$.

**1.3.1.** *A curve $F$ of genus* 5 *with automorphisms $\beta_1$, $\beta_2$, $\sigma_{1F}$, $\sigma_{2F}$ satisfying conditions* (i)–(iii) *in Example* 1.3.

Let $E$ be an elliptic curve, and $\pi : C \to E$ be a double cover branched along two points. Let $\delta_1, \delta_2$ be different non-trivial 2-torsion elements of Pic$^0E$. We have a commutative diagram




where $\rho_i : E_i \to E$ ($i = 1,\ 2$) is the double cover defined by $\delta_i^{\otimes 2} = \mathcal{O}_E$.

We have that $F$ is an irreducible (smooth) curve of genus 5. Indeed, $\varrho_i : C_i \to C$ is the double cover defined by $(\pi^*\delta_i)^{\otimes 2} = \mathcal{O}_C$. Since $\pi^* : \mathrm{Pic}^0 E \to \mathrm{Pic}^0 C$ is injective, we have $\pi^*\delta_1 \not\simeq \pi^*\delta_2$. So $C_1$ is not isomorphic to $C_2$ over $C$, which implies $F$ is irreducible.

Let $\tau_i$ (resp. $\tau$) be the hyperelliptic involution of $C_i$ (resp. $C$). Then $\tau_i$ is the lift of $\tau$, that is, we have $\tau \circ \varrho_i = \varrho_i \circ \tau_i$. One checks easily that $F$ is $\tau_1 \times \tau_2$-invariant.

Let $\alpha_i, \gamma_i$ (resp. $\gamma$) be the involutions of $C_i$ (resp. $C$) corresponding to the double covers $\varrho_i, \pi_i$ (resp. $\pi$). Then $\gamma_i$ is the lift of $\gamma$, that is, we have $\gamma \circ \varrho_i = \varrho_i \circ \gamma_i$. One checks easily that $F$ is $\gamma_1 \times \gamma_2$-invariant.

By the construction of $C_i$, we have $\alpha_i\gamma_i = \gamma_i\alpha_i$. Since $\tau_i$ is in the center of $\mathrm{Aut}(C_i)$, we have $\alpha_i\tau_i = \tau_i\alpha_i$ and $\gamma_i\tau_i = \tau_i\gamma_i$. So $\alpha_1 \times \mathrm{id}_{C_2}$, $\mathrm{id}_{C_1} \times \alpha_2$, $\gamma_1 \times \gamma_2$ and $\tau_1 \times \tau_2$ mutually commute.

Let $\beta_1, \beta_2, \tilde{\gamma}$ and $\tilde{\tau}$ be the restriction of $\alpha_1 \times \mathrm{id}_{C_2}$, $\mathrm{id}_{C_1} \times \alpha_2$, $\gamma_1 \times \gamma_2$ and $\tau_1 \times \tau_2$ to $F$, respectively. Let $\Delta$ be the subgroup of $\mathrm{Aut}F$ generated by $\beta_1, \beta_2, \tilde{\gamma}$ and $\tilde{\tau}$. Then $\Delta \simeq \mathbf{Z}_2^{\oplus 4}$.

Let $A = \{\mathrm{id_F}, \beta_1, \beta_2, \beta_3 := \beta_1\beta_2\}$. For $j = 1, 2, 3$, let $\chi_j$ be the character of $A$ with $\mathrm{Ker}\chi_j = \langle \beta_j \rangle$. Let $V = H^0(\omega_F)$. By the construction of $F$, we have that $V_A^1 = (\varrho_i \circ \mu_i)^* H^0(\omega_C)$ is of dimension two, and $\dim V_A^{\chi_j} = 1$ for all $j$.

Let $(V_A^1)^+ = (\varrho_i \circ \mu_i)^* H^0(\omega_C)^+_\gamma$ and $(V_A^1)^- = (\varrho_i \circ \mu_i)^* H^0(\omega_C)^-_\gamma$. We have $\dim(V_A^1)^+ = \dim(V_A^1)^- = 1$.

By the construction of $F$, we have that there are exactly eight $\tilde{\gamma}$-fixed points on $F$. Indeed, $\gamma_1 \times \gamma_2$ has $4 \times 4 = 16$ fixed points, eight of which belong to $F$. So $\tilde{\gamma}$ is a bi-elliptic involution. Since $\tilde{\gamma}$ is the lift of $\gamma$, we have that $\tilde{\gamma}$ induces id on $(V_A^1)^+$.

For $i = 1, 2$, since $\tilde{\tau}$ is the lift of $\tau_i$, which is the hyperelliptic involution of $C_i$, we have that $\tilde{\tau}$ induces $-\mathrm{id}$ on $V_A^1 \oplus V_A^{\chi_i}$. So $g(F/\langle\tilde{\tau}\rangle) \le 1$. On the other hand, since $\Delta/\langle\tilde{\tau}\rangle \simeq \mathbf{Z}_2^{\oplus 3}$ is isomorphic to a subgroup of $\mathrm{Aut}(F/\langle\tilde{\tau}\rangle)$, $F/\langle\tilde{\tau}\rangle$ can not be rational. So $\tilde{\tau}$ is a bi-elliptic involution.

In sum, we have that the generators $\beta_1, \beta_2, \tilde{\gamma}, \tilde{\tau}$ of $\Delta$ acting on $V$ are as follows:

| | $(V_A^1)^+$ | $(V_A^1)^-$ | $V_A^{\chi_1}$ | $V_A^{\chi_2}$ | $V_A^{\chi_3}$ |
|---|---|---|---|---|---|
| $\beta_1$ | 1 | 1 | 1 | −1 | −1 |
| $\beta_2$ | 1 | 1 | −1 | 1 | −1 |
| $\tilde{\gamma}$ | 1 | −1 | −1 | −1 | −1 |
| $\tilde{\tau}$ | −1 | −1 | −1 | −1 | 1 |

One checks easily that $\beta_1$, $\beta_2$, $\sigma_{1F} := \tilde{\gamma}\tilde{\tau}$ and $\sigma_{2F} := \tilde{\gamma}\beta_1\beta_2$ have the desired properties (i)–(iii) in Example 1.3.

**2. $\phi_S$ is generically finite.** In this section, we prove Theorem 0.1 in case that the canonical map $\phi_S$ of $S$ is generically finite. We begin with the following lemmas.

LEMMA 2.1. *Let $S$ be a complex nonsingular projective surface, and $f : S \to B$ be a fibration of genus $g \ge 2$. Let $\sigma$ be a non-trivial automorphism of $S$ with $f \circ \sigma = f$. If $\sigma$ induces a trivial action on $H^0(S, \omega_S)$, then $g(B) \le 1$.*

PROOF. Consider the induced action of $\sigma$ on $f_*\omega_S$, which is a locally free sheaf of rank $g$. We have $f_*\omega_S = \mathcal{E} \oplus \mathcal{F}$, where $\mathcal{E}$ is the eigen-subsheaf of $f_*\omega_S$ with eigenvalue 1, and $\mathcal{F}$ is the direct sum of eigen-subsheaves of $f_*\omega_S$ with eigenvalue $\neq 1$. We claim that $\mathcal{F} \neq 0$ and hence $r := \operatorname{rank} \mathcal{F} > 0$. Otherwise, since the natural map $f_*\omega_S \otimes \boldsymbol{C}(p) \to H^0(F, \omega_F)$ is an isomorphism, where $p = f(F)$ (cf. [Ha, Chap. III, Corollary 12.9]), we have that $\sigma$ induces a trivial action on $H^0(F, \omega_F)$, which implies $\sigma_{|F}$ and hence $\sigma$ must be trivial, a contradiction.

Let $\mathcal{E}' \subset f_*\omega_S$ be the subsheaf generated by global sections of $f_*\omega_S$. The assumption that $\sigma$ induces a trivial action on $H^0(S, \omega_S)$ implies that $\mathcal{E}' \subseteq \mathcal{E}$. So $h^0(B, \mathcal{E}) = h^0(B, f_*\omega_S)$ and hence $h^0(B, \mathcal{F}) = 0$. So by the Riemann-Roch, we have

$$\deg \mathcal{F} + r(1 - g(B)) = -h^1(B, \mathcal{F}) \leq 0\,.$$

Since $f_*\omega_S \otimes \omega_B^{-1}$ is semi-positive by a theorem of Fujita [Fu], we have

$$\deg \mathcal{F} - 2r(g(B) - 1) = \deg(\mathcal{F} \otimes \omega_B^{-1}) \geq 0\,.$$

Combining the two inequalities above, we have $g(B) \leq 1$. □

LEMMA 2.2. *Let $S$ be a complex nonsingular minimal projective surface of general type with $q(S) = 2$. Let $G \subset \operatorname{Aut}S$ be a subgroup of order 4 acting trivially in $H^2(S, \boldsymbol{Q})$. Assume that the Albanese map* alb : $S \to \operatorname{Alb}(S)$ *of $S$ is surjective. Then $H^0(\Omega_S^1) = H^0(\Omega_S^1)_G^\chi$ for some $\chi \in \hat{G}$ of order at most 2.*

PROOF. Let $V = H^0(\Omega_S^1)$. It is enough to exclude the following two possibilities:

(i) $V = V_G^{\chi_1} \oplus V_G^{\chi_2}$, where $\chi_1 \neq \chi_2 \in \widehat{G}$, and both $V_G^{\chi_1}$ and $V_G^{\chi_2}$ are of dimension one;

(ii) $V = V_G^\chi$, where $\chi \in \widehat{G}$ is of order 4.

In case (i), for $i = 1, 2$, let $\omega_i \in V_G^{\chi_i}$ be a non-zero holomorphic 1-form. Since the Albanese map alb : $S \to \operatorname{AlB}(S)$ is surjective, by [BPV, p.11, Corollary 1.2], $H^2(\operatorname{AlB}(S), \boldsymbol{C}) \to H^2(S, \boldsymbol{C})$ is injective. This implies the natural map induced by cup product $\wedge^2 H^1(S, \boldsymbol{C}) \to H^2(S, \boldsymbol{C})$ is injective. So $\omega_1 \wedge \omega_2 \neq 0$, $\omega_1 \wedge \overline{\omega_2} \neq 0$ in $H^2(S, \boldsymbol{C})$, where complex conjugation acts naturally on

$$H^1(S, \boldsymbol{R}) \otimes \boldsymbol{C} = H^1(S, \boldsymbol{C}) = H^0(\Omega_S^1) \oplus H^1(S, \mathcal{O}_S)\,.$$

Since $G$ acts trivially on $H^2(S, \boldsymbol{C})$, from $\alpha^*(\omega_1 \wedge \omega_2) = \chi_1(\alpha)\chi_2(\alpha)\omega_1 \wedge \omega_2$ for each $\alpha \in G$, we have $\chi_1\chi_2 = 1$ in $\widehat{G}$. Since $\chi_1 \neq \chi_2$, we have that $\chi_i$ is of order 4. Then $G \simeq \boldsymbol{Z}_4$. Let $\sigma$ be the generator of $G$, such that $\chi_1(\sigma) = \sqrt{-1}$ and $\chi_2(\sigma) = -\sqrt{-1}$. We have

$$\sigma^*(\omega_1 \wedge \overline{\omega_2}) = \chi_1(\sigma)\overline{\chi_2(\sigma)}\omega_1 \wedge \overline{\omega_2} = -\omega_1 \wedge \overline{\omega_2}\,,$$

which is a contradiction since $\sigma$ acts trivially on $H^2(S, \boldsymbol{C})$.

In case (ii), we have $G \simeq \boldsymbol{Z}_4$. Let $\sigma$ be the generator of $G$ such that $\chi(\sigma) = \sqrt{-1}$. Let $\omega_1,\ \omega_2 \in V_G^\chi$ be linearly independent holomorphic 1-forms. We have $\sigma^*(\omega_1 \wedge \omega_2) = -\omega_1 \wedge \omega_2$. By the argument as above, we get a contradiction. □

THEOREM 2.3. *Let $S$ be a complex nonsingular minimal projective surface of general type with $q(S) = 2$ and $p_g(S) > 61$. Let $G \subset \operatorname{Aut}S$ be a subgroup of order 4 acting trivially* *on* $H^2(S, \boldsymbol{Q})$. *If the canonical map* $\phi_S$ *of* $S$ *is generically finite*, *then the pair* $(S, G)$ *is as in Example* 1.3.

PROOF. Thanks to [X2], by the argument as in [Ca2, 2.3], we have that, if $p_g(S) > 61$, then $S$ has a fibration

$$f : S \to B$$

of genus $g = 5$ or 6, and $\phi_S$ separates fibers of $f$ and maps them onto a pencil of straight lines on $\mathrm{Im}\phi_S$, which is ruled over $B$, and the numerical invariants of $S$ and $B$ satisfy

$$K_S^2 \geq \frac{2g-2}{2g-5}(gp_g(S) - 6g + 20)\,, \tag{2.3.1}$$

$$g(B) \leq 1\,. \tag{2.3.2}$$

Since $G$ induces trivial actions on $\mathrm{Im}\phi_S$, and hence on $B$, $G$ is included in $\mathrm{Aut}F$ for a general fiber $F$ of $f$. □

**2.4.** The case $g = 6$ is excluded provided $p_g(S) \geq 36$ as in [Ca2, 2.8]. Indeed, by the argument in loc. cit., we may assume that $G \simeq \boldsymbol{Z}_4$. Let $\sigma$ be the element of $G$ of order 2. We may estimate the upper bound of $H^2$ for each $\sigma$-fixed curve $H$ and apply [Ca2, Lemma 2.1] to obtain an upper bound for $K_S^2$. In our case $q(S) = 2$ the inequality in loc. cit. reads

$$K_S^2 \leq \frac{480}{59}(p_g(S) - 1) + \frac{40}{59}\,.$$

While (2.3.1) gives

$$K_S^2 \geq \frac{10}{7}(6p_g(S) - 16)\,.$$

Combining the two inequalities above, we get $p_g(S) < 36$, a contradiction provided $p_g(S) \geq 36$.

**2.5.** From now on, we assume that $g = 5$. By [Ca2, Lemma 2.4], $g(F/G) = 2$. So $G$ acts freely on $F$.

**2.6.** Let $\pi : S \to S/G$ be the quotient map, and $T'$ the minimal desingularization of $S/G$. Let $h\colon T \to B$ be the relatively minimal fibration of the (induced) fiber space $T' \to B$.

LEMMA 2.7. *We have* $g(B) = 0$.

PROOF. Otherwise, by (2.3.2), $g(B) = 1$. Consider the canonical map

$$\phi_S : S \dashrightarrow \Sigma := \mathrm{Im}\phi_S \subset \boldsymbol{P}^{p_g(S)-1}\,.$$

Since $\Sigma$ is ruled over $B$, we have $q(\Sigma) = g(B) = 1$. By the classification of nondegenerate surfaces of minimal degree in $\boldsymbol{P}^{p_g(S)-1}$, we have that $\deg \Sigma > \mathrm{codim}\,\Sigma + 1 = p_g(S) - 2$. So

$$K_S^2 \geq \deg\phi_S \deg\Sigma \geq 8\chi(\mathcal{O}_S)\,.$$

On the other hand, by the argument as in [Ca2, 3.1], we have

$$K_S^2 \leq 8\chi(\mathcal{O}_S)\,.$$

Combining the two inequalities above, we have $K_S^2 = 8\chi(\mathcal{O}_S)$ and $K_S^2 = \deg\phi_S \deg\Sigma$, which implies $|K_S|$ is base-locus free. Consequently, we have

(2.7.1) for each id $\neq \sigma \in G$, since every $\sigma$-fixed curve is contained in the fixed part of $|K_S|$ (cf. [Ca1, 1.14.1]), $\sigma$ has no fixed curves.

(2.7.2) $S/G$ has at most rational double singularities since $G$ acts trivially on $H^0(\omega_S)$.

Let $T$, $T'$ be as in 2.6. By (2.7.1) and (2.7.2), we have that $K_S = \pi^* K_{S/G}$, $T'$ is minimal and $T = T'$. So $K_T^2 = 2\chi(\mathcal{O}_S) = 2\chi(\mathcal{O}_T)$. On the other hand, the assumption $g(B) = 1$ implies that the Albanese map of $S$ is generically finite. Since $G$ induces trivial actions on $B$, we have $0 \neq f^* H^0(\omega_B) \subset H^0(\Omega_S^1)_G^1$. By Lemma 2.2, we have that $G$ induces trivial action on $H^0(\Omega_S^1)$. So $q(T) = 2$. By a theorem of Debarre (cf. [De, Theorem 6.1]), we have $K_T^2 \geq 2p_g(T) = 2\chi(\mathcal{O}_T) + 2$, a contradiction. □

Let $C$ be the image of the Albanese map alb : $S \to \mathrm{Alb}(S)$.

LEMMA 2.8. *$C$ is a curve of genus* 2.

PROOF. Suppose alb is surjective. By Lemma 2.2, $H^0(\Omega_S^1) = H^0(\Omega_S^1)_G^\chi$ for some $\chi \in \hat{G}$ of order at most 2. If $\chi = 1$, let $h : T \to B$ be as in 2.6, then $q(T) = 2$. By [Be2, Lemma, p. 345], $h$ is trivial, and so $p_g(T) = 0$. This is absurd since $p_g(T) = p_g(S) > 0$.

If $\chi$ is of order 2, then the kernel Ker$(\chi)$ of $\chi : G \to \boldsymbol{C}^*$ is not trivial. Let $\sigma$ be the generator of Ker$(\chi)$. Let $V = H^0(\Omega_F^1)$. Then $V_G^1 \oplus V_G^\chi = V_\sigma^1$. Since $\dim V_G^1 = g(F/G) = 2$, this implies $\dim V_G^\chi = 1$. On the other hand, let $r : H^0(\Omega_S^1) \to H^0(\Omega_F^1)$ be the restriction map, and $W$ be its image. We have $\dim W = 2$ (since $F$ is a general fiber of $f$, if $r(\varpi) = 0$ for some holomorphic 1-form $\varpi$ of $S$, $\varpi = f^*\varpi'$ for some holomorphic 1-form $\varpi'$ of $B$) and $W \subseteq V_G^\chi$. This is a contradiction. □

**2.9.** For each $\sigma \in G$, denote by $\bar{\sigma}$ the automorphism of $C$ induced by $\sigma$. The homomorphism from $G$ to Aut $C$, sending $\sigma$ to $\bar{\sigma}$, is injective by Lemma 2.1. Let $\bar{G}$ be its image in Aut $C$. Then $\bar{G} \simeq G$.

LEMMA 2.10. *$f$ has constant moduli.*

PROOF. By Lemma 2.8, we have that $\mu := \mathrm{alb}_{|F} : F \to C$ is a finite morphism. Let $d = \deg \mu$. By the Hurwitz formula, we have $2 \leq d \leq 4$.

We show that $d = 4$, which implies $\mu$ is étale, and so $f$ has constant moduli.

*Case* 1. $G \simeq \boldsymbol{Z}_4$. Let $\sigma \in G$ be a generator of $G$. By the Hurwitz formula, there exists a $\bar{\sigma}$-fixed point $x$ on $C$. Since $\bar{\sigma} \circ \mu = \mu \circ \sigma$, $\mu^{-1}(x)$ is $\sigma$-invariant. Since $\sigma$ has no fixed points on $F$ (cf. 2.5), we have that $\#\mu^{-1}(x)$ divides by 4 and hence $d = 4$.

*Case* 2. $G \simeq \boldsymbol{Z}_2^2$. Assume $d \leq 3$. We will get a contradiction. Since $\bar{G} \simeq \boldsymbol{Z}_2^2$ in this case, there exist $\sigma \in G$ such that $\bar{\sigma}$ is the hyperelliptic involution of $C$. By the Hurwitz formula, there is a point $x \in C$ such that $x$ is $\bar{\sigma}$-fixed and $\mu$ is étale over $x$. So $\mu^{-1}(x)$ is $\sigma$-invariant and $d = \#\mu^{-1}(x)$. This implies $d$ divides by 2 since $\sigma$ has no fixed points on $F$ (cf. 2.5). Hence $d = 3$ does not occur.

Now we assume $d = 2$. Then $f \times \mathrm{alb} : S \to P := B \times C$ is generically finite of degree 2. Let $S \to S' \stackrel{\pi}{\to} P$ be the Stein factorization of $f \times \mathrm{alb}$. Let $(\Delta,\ \delta)$ be the (singular) double cover data corresponding to $\pi$. Let $l = B \times \mathrm{pt}$ and $l' = \mathrm{pt} \times C$. We have $\Delta l' = 4$

and $\delta \equiv 2l + ml'$ for some $m$. We show that each singular point of $\Delta$ is either a double point or a triple point with at least two different tangents, and hence $S'$ has at most canonical singularities. Indeed, if there exists a point $x := (b, c) \in B \times C$ with $\mathrm{mult}_x \Delta_1 \geq 3$, where $\Delta_1$ is the horizontal part of $\Delta$ w.r.t. the projection $P \to B$, then $c$ must be $\bar{G}$-fixed since $\Delta_1$ is $\mathrm{id}_B \times \bar{G}$-invariant and $\Delta_1 l' = 4$. This is absurd since $\bar{G} \simeq G$ is not cyclic. Now by the double cover formula, we have that

$$K_S^2 = 16(m-2)\,, \quad \chi(\mathcal{O}_S) = 3m - 4\,.$$

So $S$ satisfies $K_S^2 = 16(\chi(\mathcal{O}_S) - 2)/3$, contrary to (2.3.1). □

**2.11.** By Lemma 2.10, there exists a finite group $A$ acting faithfully on a general fiber $F$ of $f$ and on some smooth curve $\tilde{B}$ such that $f$ is equivalent to the fiber surface

$$p : (\tilde{B} \times F)/A \to \tilde{B}/A\,,$$

where the action of $A$ on $\tilde{B} \times F$ is the diagonal action and $p$ is the projection to the first factor (cf. e.g., [Se]).

We have $g(F/A) = q(S) = 2$. This implies the projection

$$q : (\tilde{B} \times F)/A \to F/A$$

is equivalent to the Albanese map $\mathrm{alb} : S \to C$. We have $|A| = 4$ since the degree of $\mathrm{alb}_{|F} : F \to C$ is 4 by the proof of Lemma 2.10. So $A$ acts freely on $F$ and $S \simeq (\tilde{B} \times F)/A$. In particular, we have $g(\tilde{B}) = p_g(S)$.

**2.12.** Let $V = H^0(\omega_F)$ and $W = H^0(\omega_{\tilde{B}})$. We have

$$H^0(\omega_S) \simeq \oplus_{\chi \in \widehat{A}} V_A^{\chi} \otimes W_A^{\chi^{-1}}\,. \tag{2.12.1}$$

Since $\phi_S$ separates fibers of $f$ and maps them onto a pencil of straight lines on $\mathrm{Im}\phi_S$, we have that the image of $H^0(\omega_S)$ in $H^0(\omega_F)$ is of dimension two. This implies that, among the direct sum factors of the right side of (2.12.1), there are exactly two factors having positive dimension. So

$$H^0(\omega_S) \simeq V_A^{\chi_1} \otimes W_A^{\chi_1^{-1}} \oplus V_A^{\chi_2} \otimes W_A^{\chi_2^{-1}} \tag{2.12.2}$$

for some $\chi_1, \chi_2 \in \widehat{A}$. Since $\dim W_A^1 = g(\tilde{B}/A) = g(B) = 0$ (Lemma 2.7), we have that $\chi_j \neq 1$ (the idenity character) for $j = 1, 2$.

**2.13.** For each $\sigma \in G$, $\sigma$ induces an automorphism of $\tilde{B} \times_B S$, which is of the form $\mathrm{id}_{\tilde{B}} \times \sigma_F$ for some $\sigma_F \in \mathrm{Aut}(F)$ under the identification of $\tilde{B} \times_B S$ with $\tilde{B} \times F$. We have that $\mathrm{id}_{\tilde{B}} \times \sigma_F$ is a lift of $\sigma$ to $\tilde{B} \times F$, and

$$\mathrm{alb}_{|F} \circ \sigma_F = \bar{\sigma} \circ \mathrm{alb}_{|F}\,, \tag{2.13.1}$$

where $\bar{\sigma}$ is as in 2.9.

Let $G_F = \langle \sigma_F;\ \sigma \in G \rangle$. Clearly, $G_F \simeq G$. Since $\mathrm{id}_{\tilde{B}} \times \sigma_F$ acts trivially on the right side of (2.12.2) for each $\sigma_F \in G_F$, we have that $G_F$ induces trivial action on $V_A^{\chi_1} \oplus V_A^{\chi_2}$, where $\chi_1, \chi_2$ are as in (2.12).

**2.14.** Let $\Xi$ be the subgroup of Aut$F$ generated by $A$ and $G_F$. Then $V_A^{\chi_1} \oplus V_A^{\chi_2}$ is a $\Xi$-submodule of $V$. Let $\rho : \Xi \to \mathrm{GL}(V_A^{\chi_1} \oplus V_A^{\chi_2})$ be the corresponding linear representation. By (2.13), we have $G_F \subseteq \mathrm{Ker}\rho$. We show that $\rho_{|A} : A \to \mathrm{GL}(V_A^{\chi_1} \oplus V_A^{\chi_2})$ is injective: indeed, since both $V_A^1$ and $V_A^{\chi_1} \oplus V_A^{\chi_2}$ are contained in $V^1_{\mathrm{Ker}(\rho_{|A})}$, $\dim V^1_{\mathrm{Ker}(\rho_{|A})} \geq \dim V_A^1 + \dim(V_A^{\chi_1} \oplus V_A^{\chi_2}) = g(F/A) + 2 = 4$ (cf. (2.11)). This implies $\mathrm{Ker}(\rho_{|A})$ must be trivial. So $G_F = \mathrm{Ker}\rho$, and hence $G_F$ is a normal subgroup of $\Xi$. Note that $A$ is a normal subgroup of $\Xi$. We have that $\Xi$ is the internal direct product of $G_F$ and $A$; in particular, $\Xi$ is an Abelian group.

Now we distinguish four cases according to $A$ and $G$.

**2.15.** $A \simeq \boldsymbol{Z}_4$ and $G \simeq \boldsymbol{Z}_2^2$. We show that this case does not occur. Otherwise, let $\beta$ be a generator of $A$. Let $V$ be as in 2.12. We have $\dim V_\beta^1 = g(F/A) = 2$. By the holomorphic Lefschetz formula, $\dim V_\beta^{-1} = \dim V_\beta^i = \dim V_\beta^{-i} = 1$.

We have $\bar{G} \simeq \boldsymbol{Z}_2^2$ (cf. (2.9)). So there is an involution $\sigma \in G$ such that $\bar{\sigma}$ is the hyperelliptic involution of $C$. The operation of $\sigma^*$ and $(\sigma\beta)^*$ acting on eigenspaces of $\beta^*$ is as follows:

| | $V_\beta^1$ | $V_\beta^{-1}$ | $V_\beta^i$ | $V_\beta^{-i}$ |
|---|---|---|---|---|
| $\sigma^*$ | $-1$ | $1$ | $1$ | $1$ |
| $(\sigma\beta)^*$ | $-1$ | $-1$ | $i$ | $-i$ |

Indeed, since $\Xi$ is Abelian (cf. 2.14), the eigenspace of each eigenvalue of $\beta^*$ is $\Xi$-invariant. The equality $\sigma^* = -\mathrm{id}$ on $V_\beta^1$ follows by (2.13.1), and $\sigma^* = \mathrm{id}$ on the others since $g(F/\sigma) = 3$ (cf. (2.5)).

By the above table, we have

$$\mathrm{tr}(\sigma\beta|\bar{V}) = -(\dim V_\beta^1 + \dim V_\beta^{-1}) - i \dim V_\beta^i + i \dim V_\beta^{-i} = -3\,.$$

Applying the holomorphic Lefschetz formula to $\sigma\beta$, we have

$$1-(-3) = 1 - \mathrm{tr}(\sigma\beta|\bar{V}) = \frac{a}{1-i} + \frac{b}{1+i}\,, \tag{2.15.1}$$

where $a$ (resp. $b$) is the number of fixed points of $\sigma\beta$ such that the induced action of $\sigma\beta$ on the tangent space at each of these points is given by $v \mapsto iv$ (resp. $v \mapsto -iv$). So $a+b=8$. Applying the Riemann-Hurwitz formula to $F \to F/\langle\sigma\beta\rangle$, we have $8 = 2g(F) - 2 \geq 4(-2 + (1-1/4)(a+b)) = 16$, a contradiction.

**2.16.** $A \simeq \boldsymbol{Z}_4 \simeq G$. Let $\gamma$ be a generator of $G$. By (2.9), $\bar{\gamma}$ is of order 4, and so $g(C/\bar{\gamma}) = 0$. Applying the topological Lefschetz formula to $\bar{\gamma}$, we have that $\bar{\gamma}$ has $2 + 2\dim H^0(\omega_C)^-_{\bar{\gamma}}$ fixed points. Applying the Riemann-Hurwitz formula to $C \to C/\bar{\gamma}$, we have

$$2 = 2g(C) - 2 \geq 4\left(-2 + \left(1 - \frac{1}{4}\right)(2 + 2\dim H^0(\omega_C)^-_{\bar{\gamma}})\right).$$

This implies $\dim H^0(\omega_C)^-_{\bar{\gamma}} = 0$. So $\bar{\gamma}^2$ induces $-\mathrm{id}$ on $H^0(\omega_C)$, and hence $\gamma^2$ induces $-\mathrm{id}$ on $H^0(\omega_F)^1_\beta$. Now by the argument as in 2.15 (consider $\gamma^2\beta$ instead of $\sigma\beta$), we get a contradiction.

**2.17.** $A \simeq \boldsymbol{Z}_2^2 \simeq G$. Let $\chi_1, \chi_2$ be as in 2.12, and let $\chi_3 = \chi_1\chi_2$. For $j = 1, 2, 3$, let $\beta_j$ be the generator of $\mathrm{Ker}\chi_j$. Then $\beta_j$ $(j = 1, 2, 3)$ are non-unit elements of $A$. Note that $V_{\beta_j}^1 = V_A^1 \oplus V_A^{\chi_j}$, $\dim V_A^1 = g(F/A) = 2$, and $\dim V_{\beta_j}^1 = g(F/\langle\beta_j\rangle) = 3$. So $\dim V_A^{\chi_j} = 1$ for $j = 1, 2, 3$, and the action of generators of $A$ on $V = H^0(F, \omega_F)$ is as follows:

| | $V_A^1$ | $V_A^{\chi_1}$ | $V_A^{\chi_2}$ | $V_A^{\chi_3}$ |
|---|---|---|---|---|
| $\beta_1$ | 1 | 1 | $-1$ | $-1$ |
| $\beta_2$ | 1 | $-1$ | 1 | $-1$ |

Let $\bar{\sigma}_1, \bar{\sigma}_2 \in \bar{G}$ be bi-elliptic involutions of $C$, and $\sigma_{1F}, \sigma_{2F} \in G_F$ be their corresponding elements, where $\bar{G}$ is as in 2.9 and $G_F$ is as in 2.13. For $l = 1, 2$, let $\bar{v}_l$ be a basis of $H^0(C, \omega_C)_{\bar{\sigma}_l}^+$, and $v_l \in V_A^1$ the corresponding element of $\bar{v}_l$ under the identification of $V_A^1$ with $H^0(C, \omega_C)$ (cf. 2.11). Then $v_1$ and $v_2$ is a basis of $V_A^1$. Note that the action of $G_F$ on $V_A^1$ is the same as that of $\bar{G}$ on $H^0(C, \omega_C)$ by (2.13.1), and $G_F$ acts trivially on $V_A^{\chi_1}$ and $V_A^{\chi_2}$ (cf. 2.13). So the action of generators of $G_F$ on $V = H^0(F, \omega_F)$ is as follows:

| | $v_1$ | $v_2$ | $V_A^{\chi_1}$ | $V_A^{\chi_2}$ | $V_A^{\chi_3}$ |
|---|---|---|---|---|---|
| $\sigma_{1F}$ | 1 | $-1$ | 1 | 1 | $-1$ |
| $\sigma_{2F}$ | $-1$ | 1 | 1 | 1 | $-1$ |

Combining $V_A^{\chi_3} \neq 0$ with (2.12.2), we have $W_A^{\chi_3} = 0$, and hence $g(\tilde{B}/\beta_3) = 0$, i.e., $\tilde{B}$ is hyperelliptic with the hyperelliptic involution $\beta_3$. So $(S, G)$ is as in Example 1.3.

**2.18.** $A \simeq \boldsymbol{Z}_2^2$ and $G \simeq \boldsymbol{Z}_4$. Note that $G$ acts freely on $F$ (cf. 2.5), and that $A$ induces a faithful action on $F/G$ (cf. 2.14). Observing that the proof of the case $A \simeq \boldsymbol{Z}_4$ and $G \simeq \boldsymbol{Z}_2^2$ uses only the properties of representations of $G$ and $A$ on $V$, by the argument as in 2.15 with the role of $G$ and $A$ being transposed, we have that this case does not occur.

This completes the proof of Theorem 2.3. □

**3. $\phi_S$ is composed with a pencil.** In this section, we prove Theorem 0.1 in the case that the canonical map $\phi_S$ of $S$ is composed with a pencil.

THEOREM 3.1. *Let $S$ be a complex nonsingular minimal projective surface of general type with $q(S) = 2$ and $p_g(S) \geq 23$. Let $G \subset \mathrm{Aut}S$ be a subgroup of order 4 acting trivially in $H^2(S, \boldsymbol{Q})$. If the canonical map $\phi_S$ of $S$ is composed with a pencil, then the pair $(S, G)$ is as in Example 1.1 or Example 1.2 depending on $G \simeq \boldsymbol{Z}_2^{\oplus 2}$ or $\boldsymbol{Z}_4$.*

PROOF. By [Be1, Prop. 2.1], the moving part of $|K_S|$ has no base points. Let

$$\phi_S = \varphi \circ f : S \to B \to \mathrm{Im}\phi_S \subset \boldsymbol{P}^{p_g(S)-1}$$

be the Stein factorization of $\phi_S$, and let $F$ be a general fiber of $f$. Let $g$ be the genus of a general fiber of $f$. One has $2 \leq g \leq 5$ (cf. [Be1]) and $g(B) = 0$ (cf. [X1]).

Since $G$ acts trivially on $H^0(S, \omega_S)$, we have that $G$ induces the trivial action on $B$, and the inclusion $G \hookrightarrow \mathrm{Aut}F$ (cf. [Ca1, 2.2]). In particular, we have that any section of $f$ is $G$-fixed.

Let $C$ be the image of the Albanese map of $S$.

LEMMA 3.2. *If $g \leq 4$, then $C$ is a curve (of genus 2).*

PROOF. If the Albanese map of $S$ is surjective, by Lemma 2.2, $H^0(\Omega_S^1) = H^0(\Omega_S^1)_G^\chi$ for some $\chi \in \hat{G}$ of order at most 2. Then the kernel Ker($\chi$) of $\chi : G \to \boldsymbol{C}^*$ is not trivial. Let $\sigma \in$ Ker($\chi$) be an element of order 2. Then $H^0(\Omega_S^1)_G^\chi \subseteq H^0(\Omega_S^1)_\sigma^1$, and so $q(S/\sigma) = 2$. The assumption $g \leq 4$ implies that $S/\sigma \to B$ is a fiber space of genus $g' \leq 2$. Hence we have that $g' = q(S/\sigma) - g(B)$. This implies $S/\sigma \to B$ is trivial by [Be2, Lemma, p. 345], and so $p_g(S/\sigma) = 0$, a contradiction since $p_g(S/\sigma) = p_g(S) > 0$. □

LEMMA 3.3. *The cases $g = 2, 4$ and $5$ do not occur.*

PROOF. Let $M$ and $Z$ be the moving part and the fixed part of $|K_S|$, respectively. We write $Z = H + V$, and $H = n_1\Gamma_1 + n_2\Gamma_2 + \cdots$ with $n_1 \geq n_2 \geq \cdots$, where $H$ (resp. $V$) is the horizontal part (resp. the vertical part) of $Z$ with respect to $f$, and $\Gamma_i$ $(i = 1, 2, \ldots)$ are the irreducible components of $H$, with $n_i$ the multiplicity of $\Gamma_i$ in $H$.

Since $M \equiv \chi(\mathcal{O}_S)F$ (cf. e.g. [Ca1, 2.1.2]), we have

$$K_S^2 = K_S(M + H + V) \geq (2g - 2)\chi(\mathcal{O}_S) + K_S H\,. \tag{3.3.1}$$

We distinguish three cases according to $g$.

**3.3.1.** $g = 5$. In this case we have that

$$K_S H \geq \frac{8}{5}(\chi(\mathcal{O}_S) - 8)\,. \tag{3.3.2}$$

Indeed, since $n_1 K_{S/B} + H + V$ is nef, from

$$((n_1 + 1)K_S - M + 2n_1 F)H = (n_1 K_{S/B} + H + V)H \geq 0\,,$$

we get $K_S H \geq 8(\chi(\mathcal{O}_S) - 2n_1)/(n_1 + 1)$. So if $n_1 < 5$, we obtain (3.3.2).

Now we can assume that $n_1 \geq 5$. Then $\Gamma_1$ is a section of $f$. This implies $\Gamma_1$ and hence the point $F \cap \Gamma_1 \in F$ is $G$-fixed. So $G$ is cyclic (of order four).

Let $R_F$ be the set of ramified points of the quotient map $F \to F/G$. Using the Hurwitz formula for $F \to F/G$ (note that $g(F/G) \geq 1$ and $F \cap \Gamma_1$ is a ramification point of index 4 of the quotient map), we have that $R_F$ consists of four points and among them there are exactly two $G$-fixed points. Since $R_F \subseteq H_{\mathrm{red}} \cap F$ (cf. [Ca1, 2.4.1]) and $(H - n_1\Gamma_1)F = 8 - n_1 \leq 3$, we have $\#(H_{\mathrm{red}} \cap F) = 4$ and $H = 5\Gamma_1 + \Gamma_2 + \Gamma_3$ with $\Gamma_2 F = 1$ and $\Gamma_3 F = 2$.

From $K_S\Gamma_i = (M + H + V)\Gamma_i \geq \chi(\mathcal{O}_S) + n_i\Gamma_i^2$ and the adjunction formula for $\Gamma_i$, we get

$$K_S\Gamma_1 \geq \frac{\chi(\mathcal{O}_S) - 10}{6}\,, \quad K_S\Gamma_i \geq \frac{\chi(\mathcal{O}_S) - 2}{2} \quad \text{for } i = 2,\ 3\,.$$

$K_S H = 5K_S\Gamma_1 + K_S\Gamma_2 + K_S\Gamma_3 \geq (11/6)\chi(\mathcal{O}_S) - 31/3$. This finishes the proof of (3.3.2).

Combining (3.3.1) with (3.3.2), if $\chi(\mathcal{O}_S) \geq 22$, we get $K_S^2 \geq (48/5)\chi(\mathcal{O}_S) - 64/5 > 9\chi(\mathcal{O}_S)$, contrary to the Bogomolov-Miyaoka-Yau inequality.

**3.3.2.** $g = 4$. By Lemma 3.2, we have that $\mathrm{alb}_{|F} : F \to C$ is either an étale cover of degree 3 or a ramified double cover, where $F$ is a general fiber of $f$.

In the former case, we have that $f$ has constant moduli. So it is equivalent to $p : (\tilde{B} \times F)/A \to \tilde{B}/A$ for some $A$, $\tilde{B}$ as in 2.11.

We have $g(F/A) = q(S) = 2$. So $F/A \simeq C$. This implies $|A| = 3$ and $S \simeq (\tilde{B} \times F)/\langle \iota \times \tau \rangle$, where $\iota \in \mathrm{Aut}\tilde{B}$ of order 3 with $g(\tilde{B}/\iota) = 0$ and $\tau \in \mathrm{Aut}F$ of order 3 without fixed points.

By the explicit description of $S$ above, $f$ has multiple fibers with multiplicity 3. So $\Gamma_i F$ divides by 3 for each $i$. Thus there are only three possibilities for $H$:

(a) $H = 2\Gamma_1$ with $\Gamma_1 F = 3$;

(b) $H = \Gamma_1$ with $\Gamma_1 F = 6$;

(c) $H = \Gamma_1 + \Gamma_2$ with $\Gamma_1 F = \Gamma_2 F = 3$.

Let $D$ be the horizontal part (w.r.t. $f$) of the ramification divisor of $S \to S/G$. We have $D < H$ (cf. [Ca1, 2.4]). Using the Hurwitz formula for the quotient map $F \to F/G$, which is ramified exactly at points $D \cap F$, we have either (i) $DF = 2$ and the ramification index of each points of $D \cap F$ is four, or (ii) $DF = 6$ and that of $D \cap F$ is two. Since $D < H$, by the possibilities for $H$ listed above, we see easily that the case (i) does not occur.

Consider therefore the case (ii). Note that $HF = 6$, we have $H = D$. This implies that $H$ is contained in sums of fibers of alb. Indeed, if $\mathrm{alb}_{|\Gamma} : \Gamma \to C$ is surjective for some $\Gamma < H$, let $\alpha \in G$ be a non-trivial automorphism such that $\Gamma$ is $\alpha$-fixed (such an automorphism exists since $\Gamma < D$), then the induced action of $\alpha$ on $C$ is trivial, a contradiction by Lemma 2.1. Since $\mathrm{alb}^*(c)F = 3$ for any point $c \in C$, (b) is ruled out; since $H = D$ is reduced, (a) is ruled out. So $H$ is as in (c) with $\Gamma_1$, $\Gamma_2$ being fibers of alb. Hence $K_S\Gamma_1 = K_S\Gamma_2 = 2g(\tilde{B}) - 2 = 2\chi(\mathcal{O}_S)$. By (3.3.1), $K_S^2 \geq 6\chi(\mathcal{O}_S) + K_S\Gamma_1 + K_S\Gamma_2 = 10\chi(\mathcal{O}_S)$, contrary to the Bogomolov-Miyaoka-Yau inequality.

In the latter case, we have that

$$f \times \mathrm{alb} : S \to T := B \times C$$

is generically finite of degree 2. Let $S \to S' \xrightarrow{\pi} T$ be the Stein factorization of $f \times \mathrm{alb}$. Let $l = B \times \mathrm{pt}$, and $l' = \mathrm{pt} \times C$. Let $(\Delta,\ \delta)$ be the (singular) double cover data corresponding to $\pi$. We have $\Delta l' = 2$, and $\delta \equiv l + ml'$ for some $m$. This implies that each singular point of $\Delta$ is either a double point or a triple point with at least two different tangents, and hence $S'$ has at most canonical singularities. By the double cover formula, we have

$$K_S^2 = K_{S'}^2 = 2(K_T + \delta)^2 = 12(m-2)\,,$$

$$\chi(\mathcal{O}_S) = \chi(\mathcal{O}_{S'}) = 2\chi(\mathcal{O}_T) + \frac{1}{2}\delta(K_T + \delta) = 2m - 3\,.$$

Hence $K_S^2 = 6\chi(\mathcal{O}_S) - 6$, and we get a contradiction by (3.3.1).

**3.3.3.** $g = 2$. Since $p_g(S/G) = p_g(S) > 0$, we have $g(F/G) = 1$. The commutativity of $G$ implies that the quotient map $F \to F/G$ has at least two branch points. Applying the Hurwitz formula to $F \to F/G$, we get a contradiction. □

**3.4.** By Lemma 3.3, we may assume that $g = 3$. Then $\mathrm{alb}_{|F} : F \to C$ is an étale double cover by Lemma 3.2. So $f$ has constant moduli, and it is equivalent to

$$p : (\tilde{B} \times F)/A \to \tilde{B}/A$$

for some $A$, $\tilde{B}$ as in 2.11.

We have $g(F/A) = q(S) = 2$. This implies $|A| = 2$ and $S \simeq (\tilde{B} \times F)/\langle \tau \times \iota \rangle$, where $\tau$ is the hyperelliptic involution of $\tilde{B}$ and $\iota$ is an involution of $F$ without fixed points.

For each $\sigma$ in $G$, since $\sigma$ induces trivial action on $B$, $\tilde{B} \times_B S \subset \tilde{B} \times S$ is $(\mathrm{id}_{\tilde{B}} \times \sigma)$-invariant. Then there is an automorphism $\sigma_F$ of $F$ such that, under the identification of $\tilde{B} \times F$ with $\tilde{B} \times_B S$, $\mathrm{id}_{\tilde{B}} \times \sigma_F$ equals to the restriction of $\mathrm{id}_{\tilde{B}} \times \sigma$ to $\tilde{B} \times_B S$. Clearly, we have $(\mathrm{id}_{\tilde{B}} \times \sigma_F) \circ \pi = \pi \circ \sigma$, where $\pi : \tilde{B} \times F \to S$ is the induced map. Since $\sigma$ induces trivial action on $H^2(S, \boldsymbol{C})$, we have that $\sigma_F$ induces the identity on $H^0(\Omega^1_F)^-_\iota$. So $(S, G)$ is as in Example 1.1 (resp. Example 1.2) provided that $G \simeq \boldsymbol{Z}_2^2$ (resp. $\boldsymbol{Z}_4$).

This completes the proof of Theorem 3.1. □

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