

## AUTOMORPHISMS OF $L^*$ -ALGEBRAS\*

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In this paper we are concerned with some properties of (algebraic)\*-automorphisms and \*-isomorphisms of semi-simple  $L^*$ -algebras. As a consequence of the inner product uniqueness theorem for  $L^*$ -algebras established earlier ([4], see Theorem 1 below), it follows that every \*-isomorphism  $\varphi$  of a semi-simple  $L^*$ -algebra  $L$  is necessarily topological and moreover  $\varphi$  is a semi- $L^*$ -isomorphism if  $L$  is simple (Corollary to Theorem 1). From these results we deduce that a \*-isomorphism of a semi-simple  $L^*$ -algebra can be expressed in terms of partial semi- $L^*$ -isomorphisms (Theorem 2).

We give some conditions under which a \*-automorphism is automatically unitary. While a \*-automorphism of any finite-dimensional simple  $L^*$ -algebra is unitary (Corollary to Proposition 2), this result holds for an infinite-dimensional simple  $L^*$ -algebra provided it is of classical type (Theorem 3). Under additional conditions on the automorphism, the same result holds also for the general simple  $L^*$ -algebra (see §2). Actually, it is our conjecture that the result is valid even without the additional conditions.

We introduce a notion of regularity for automorphisms of semi-simple  $L^*$ -algebras and show by means of a category argument that such automorphisms exist whenever the  $L^*$ -algebras are separable (Theorem 4). For automorphisms which are inner, a criterion for regularity is obtained (Proposition 7) which coincides with the one given by Gantmacher for the regularity of automorphisms of semisimple Lie algebras.

**1. Preliminaries and structure of \*-isomorphisms.** Let  $L$  be a real or complex Lie algebra of arbitrary dimension.  $L$  is called an  $L^*$ -algebra if (i)  $L$  is equipped with an inner product relative to which it is a Hilbert space; (ii)  $L$  is closed for a \*-operation  $x \rightarrow x^*$  which satisfies the connecting relation

$$\langle [x, y], z \rangle = \langle y, [x^*, z] \rangle,$$

where  $[\cdot]$  as usual stands for the Lie bracket.

If the centre of  $L$  (as a Lie algebra) is zero,  $L$  is called semi-simple.  $L$  is called simple if it is of dimension greater than one and contains no closed ideals other than  $\{0\}$  and  $L$ .

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If  $L$  is a real semi-simple  $L^*$ -algebra, its complexification  $\tilde{L} = L + \sqrt{-1}L$  can be made into a complex (semi-simple)  $L^*$ -algebra by extending to  $\tilde{L}$  the operations of  $L$  in the following way. If  $z_i = x_i + \sqrt{-1}y_i$ , ( $i=1, 2$ ),  $z = x + \sqrt{-1}y$  belong to  $\tilde{L}$ , we set

- (a)  $[z_1, z_2] = [x_1, x_2] - [y_1, y_2] + \sqrt{-1} \{[x_1, y_2] + [y_1, x_2]\}$
- (b)  $\langle z_1, z_2 \rangle = \langle x_1, x_2 \rangle + \langle y_1, y_2 \rangle + \sqrt{-1} \{ \langle y_1, x_2 \rangle - \langle x_1, y_2 \rangle \}$
- (c)  $z^* = x^* - \sqrt{-1}y^*$ .

$L$  is called a real form of the complex algebra  $\tilde{L}$ .

Every semi-simple  $L^*$ -algebra  $L$  has an orthogonal decomposition  $L = \Sigma \oplus L_i$ , with  $L_i$  simple. (This has been established for complex  $L$  by Schue [8]. Though his proof (involving theory of complex Banach algebras) cannot apparently be adapted for real  $L$ , a proof for this case is easily obtained by using the simple decomposition of the complexification  $\tilde{L}$ .)

Let  $L, L'$  be two semi-simple  $L^*$ -algebras (both real or both complex). A Lie algebra isomorphism  $\varphi$  of  $L$  onto  $L'$  is called a  $*$ -isomorphism if  $\varphi$  is a  $*$ -map, i. e.,  $(\varphi x)^* = \varphi x^*$  for all  $x$  in  $L$ . An isomorphism  $\varphi$  is called a *semi- $L^*$ -isomorphism* if there exists a (positive) constant  $k$  such that

$$\langle \varphi x, \varphi y \rangle = k \langle x, y \rangle \text{ for all } x, y \text{ in } L;$$

$g = k$  is called the *gauge* of  $\varphi$ .

A semi- $L^*$ -isomorphism is automatically a  $*$ -isomorphism (cf. [3, Lemma 4]). If  $k$  (or  $g$ ) = 1, the semi- $L^*$ -isomorphism is called an  *$L^*$ -isomorphism*. Note that the  $L^*$ -isomorphisms of a semi-simple  $L^*$  algebra  $L$  are just its unitary Lie isomorphisms.

**THEOREM 1.** *Let  $L$  be a real or complex centre-free Lie algebra closed for a  $*$ -operation. Let  $\langle \cdot \rangle_1, \langle \cdot \rangle_2$  be two inner products on  $L$  such that relative to  $\langle \cdot \rangle_1$  ( $\langle \cdot \rangle_2$ )  $L$  is a (semi-simple)  $L^*$ -algebra  $L_1$  ( $L_2$ ). Then  $\langle \cdot \rangle_1$  and  $\langle \cdot \rangle_2$  are topologically equivalent. Further, if  $L_1$  is simple so is  $L_2$  and the two inner products are multiples of each other.*

The proof of these assertions when  $L$  is complex will be found in [4]. The extension of the first of these assertions for the case where  $L$  is real is easily obtained by passing to the complexification  $\tilde{L}$ , while that for the second, though not deducible<sup>(1)</sup> from that for  $\tilde{L}$ , can be proved *ab initio* in exactly the same way as for the complex case.

<sup>(1)</sup> because, when  $L$  is simple,  $L$  need not always be simple.

COROLLARY. Let  $L, L'$  be two semi-simple  $L^*$ -algebras. Every  $*$ -isomorphism  $\varphi$  of  $L$  onto  $L'$  is topological. Also, if  $L$  is simple  $\varphi$  is a semi- $L^*$ -isomorphism.

PROOF. Introduce in  $L$  a second inner product  $\langle \cdot, \cdot \rangle_1$  by setting  $\langle x, y \rangle_1 = \langle \varphi x, \varphi y \rangle$ . Then, by Theorem 1,  $\langle \cdot, \cdot \rangle_1$  is equivalent to the original inner product of  $L$ , which means  $\varphi$  is topological. The second assertion of the corollary obviously follows from the corresponding assertion of Theorem 1.

DEFINITION 1. Let  $L, L'$  be two semi-simple  $L^*$ -algebras (both real or both complex). A map  $\varphi$  of  $L$  into  $L'$  is called a *partial semi- $L^*$ -isomorphism* if there exists a closed ideal  $I$  of  $L$  such that the restriction  $\varphi_I$  (of  $\varphi$  to  $I$ ) is a semi- $L^*$ -isomorphism and if further  $\varphi$  maps the orthogonal complement  $I^\perp$  to  $\{0\}$ .

PROPOSITION 1. A *partial semi- $L^*$ -isomorphism*  $\varphi$  of  $L$  is a  $*$ -homomorphism of  $L$  which is bounded;  $\|\varphi\| = g, g$  being the gauge of  $\varphi_I$ .

PROOF. Since  $I$ , as a closed ideal of  $L$ , is a semi-simple  $L^*$ -subalgebra, it follows that  $\varphi_I$ , and hence  $\varphi$ , is a  $*$ -map. Further, since  $[I, I^\perp] = \{0\}$ , if

$$z_i = x_i + y_i (x_i \in I, y_i \in I^\perp), i = 1, 2$$

then

$$\varphi[x_1 + y_1, x_2 + y_2] = \varphi([x_1, x_2] + [y_1, y_2]).$$

The homomorphism property of  $\varphi$  now readily follows from this relation.

Finally,

$$\frac{\|\varphi z\|^2}{\|z\|^2} = \frac{\|\varphi_I x\|^2}{\|x\|^2 + \|y\|^2} \leq \frac{\|\varphi_I x\|^2}{\|x\|^2} \leq g^2,$$

and consequently  $\|\varphi\| = g$ .

THEOREM 2. A  $*$ -isomorphism  $\varphi$  of a semi-simple  $L^*$ -algebra  $L$  has the form

$$\varphi = \sum \varphi_i$$

where  $\varphi_i$  are partial semi- $L^*$ -isomorphisms.

PROOF. Let  $L = \Sigma \oplus L_i$  be the orthogonal decomposition of  $L$  with  $L_i$  simple. By Theorem 1,  $\varphi$  is topological and its restriction to  $L_i$  is a semi- $L^*$ -isomorphism. Now define a linear mapping by setting

$$\varphi_i x = \varphi x \quad \text{if } x \in L_i, \quad \varphi_i x = 0 \quad \text{if } x \perp L_i.$$

Then it is clear that  $\varphi_i$  is a partial semi- $L^*$ -isomorphism of  $L$  and  $\varphi = \Sigma \varphi_i$ .

## 2. Unitariness conditions for \*-automorphisms. We begin with

PROPOSITION 2. *If an automorphism  $\varphi$  of a finite-dimensional simple  $L^*$ -algebra  $L$  leaves the class of Cartan subalgebras (in the  $L^*$ -sense)<sup>(2)</sup> invariant, then  $\varphi$  is \*-preserving and unitary.*

PROOF. First of all, since  $L$  is simple, by a result due to Schue [8, 2.5], the inner product  $\langle \cdot, \cdot \rangle$  of  $L$  and the Cartan scalar product  $(\cdot)$  are connected by the relation

$$(1) \quad \langle x, y^* \rangle = \varepsilon(x, y), \quad (x, y \in L)$$

where  $\varepsilon$  is some positive number independent of  $x, y$ . (Though this result has been established by Schue only when  $L$  is complex, his proof applies equally to the real case.) We next observe that if  $L$  is real then

$$(1') \quad \langle z, w^* \rangle = \varepsilon(z, w), \quad (z, w \in \tilde{L})$$

even though  $\tilde{L}$  may fail to be simple. This observation follows from (1) and the first part of Lemma 6.1 [7, p. 154].

We now make the following notational convention.  $\tilde{L}$  will denote the complexification of  $L$  if  $L$  is real and  $L$  itself if  $L$  is complex.  $\tilde{\varphi}$  will denote accordingly the extension of  $\varphi$  to  $\tilde{L}$  ( $\tilde{\varphi}(x + \sqrt{-1}y) = \varphi x + \sqrt{-1}\varphi y$ ) or  $\varphi$  itself. It is now clearly sufficient to prove the assertions of Proposition 2 for  $\tilde{\varphi}$ .

To prove  $\tilde{\varphi}$  is \*-preserving it is enough, in view of linearity of  $\tilde{\varphi}$ , to show that  $\tilde{\varphi}$  maps self-adjoint elements into self-adjoint elements. Let  $z$  be a self-adjoint element of  $\tilde{L}$ . Then there exists a Cartan subalgebra  $\tilde{H}$  containing  $z$ . Let  $\Delta = \{\alpha\}$  be the root system relative to  $\tilde{H}$ . Then there are elements  $h_\alpha, \hat{h}_\alpha \in H$  with

$$(2) \quad \alpha(h) = \langle h, h_\alpha \rangle = (h, \hat{h}_\alpha),$$

<sup>(2)</sup> these are also Cartan subalgebras in the Lie algebra sense (see [8, p. 71]).

where  $h_\alpha$  is known to be self-adjoint [8, p. 72]. Since (1'), (2) imply  $\widehat{h}_\alpha = \varepsilon h_\alpha$ , it follows that  $\widehat{h}_\alpha$  is also self-adjoint. Now  $\widetilde{\varphi}$  being an automorphism,  $\widetilde{\varphi}\widehat{h}_\alpha = \widehat{h}_{\alpha'}$  where  $\alpha'$  is a root relative to  $\widetilde{\varphi}\widetilde{H}$ . Thus  $\widetilde{\varphi}\widehat{h}_\alpha$  is selfadjoint for each  $\widehat{h}_\alpha$ , and since the  $\widehat{h}_\alpha$  span  $\widetilde{H}$ , it is clear that  $\widetilde{\varphi}z$  is self-adjoint, as we wished to show.

It remains to prove that  $\widetilde{\varphi}$  is unitary. But this now readily follows from (1') since  $\widetilde{\varphi}$  is  $*$ -preserving.

COROLLARY. *Every  $*$ -automorphism of  $L$  is unitary.*

The rest of the present section is concerned with some generalisations of the above corollary to infinite-dimensional simple  $L^*$ -algebras.

THEOREM 3. *Let  $L$  be either a complex simple  $L^*$ -algebra of classical type or a real form of such an algebra. Then a  $*$ -automorphism  $\varphi$  of  $L$  is unitary.*

PROOF. We adopt the notational convention introduced in Proposition 2.  $\widetilde{L}$  is therefore a complex simple  $L^*$ -algebra of classical type and so, by definition, is semi- $L^*$ -isomorphic (say under a map  $\psi$ ) to one of the standard algebras  $L_A, L_B, L_C$ . (For the definitions of the standard algebras see [5], or [8, Theorem 3] (separable case).) By Theorem 1,  $\widetilde{\varphi}$  is a semi- $L^*$ -automorphism of  $\widetilde{L}$ .

Let  $\Delta = \{\alpha\}$  be the root system of  $\widetilde{L}$  relative to a Cartan subalgebra  $\widetilde{H}$  of  $\widetilde{L}$ , and  $g_0$  the gauge of  $\psi$ . Denote by  $\rho(\widetilde{H})$  the range of values of  $\|\alpha\|$  ( $= \|h_\alpha\|$ ) as  $\alpha$  varies in  $\Delta$ . Then, using explicitly the root systems for  $L_A, L_B, L_C$  determined in [5], we obtain

$$\rho(\widetilde{H}) = \begin{cases} \left( \frac{\sqrt{2}}{g_0} \right) & \text{if } \widetilde{L} \text{ is of type } A, \\ \left( \frac{1}{g_0}, \frac{1}{g_0\sqrt{2}} \right) & \text{if } \widetilde{L} \text{ is of type } B \text{ and } \widetilde{H} \text{ of type } 1, \\ \left( \frac{1}{g_0} \right) & \text{if } \widetilde{L} \text{ is of type } B \text{ and } \widetilde{H} \text{ of type } 2, \\ \left( \frac{1}{g_0}, \frac{\sqrt{2}}{g_0} \right) & \text{if } \widetilde{L} \text{ is of type } C. \end{cases}$$

Since with  $\widetilde{H}$ ,  $\widetilde{\varphi}\widetilde{H}$  is also a Cartan subalgebra (of the same type too), it follows that

$$(1) \quad \rho(\widetilde{\varphi}\widetilde{H}) = \rho(\widetilde{H}).$$

On the other hand, if  $\tilde{\varphi}h_\alpha = h_{\alpha'}$ , then

$$(2) \quad \|\alpha\| = g\|\alpha'\|,$$

where  $g$  is the gauge of  $\varphi$ . The relations (1), (2) can clearly subsist only if  $g=1$ . This means  $\tilde{\varphi}$ , and hence  $\varphi$ , is unitary.

**COROLLARY.** *Every \*-automorphism of a separable simple  $L^*$ -algebra is unitary.*

This follows from Theorem 3 and Schue's result that every separable (infinite-dimensional) simple  $L^*$ -algebra is of classical type [8, Theorem 3].

**PROPOSITION 3.** *Let  $\varphi$  be a \*-automorphism of a complex simple  $L^*$ -algebra  $L$  such that  $\varphi$  leaves some Cartan subalgebra  $H$  of  $L$  set-wise invariant,  $H = \varphi H$ . Then  $\varphi$  is unitary.*

**PROOF.** As in Theorem 3, we obtain the relation

$$(2) \quad \|\alpha\| = g\|\alpha'\|$$

where  $\Delta = \{\alpha\}$  is the root system relative to  $\tilde{H} = H$ ,  $g$  the gauge of  $\tilde{\varphi} = \varphi$  and  $\alpha \rightarrow \alpha'$  is now a bijective mapping of  $\Delta$  onto itself. By Corollary 1 to Proposition 2 of [1], we have for any two roots  $\alpha, \beta \in \Delta$  ( $\|\alpha\| \geq \|\beta\|$ )

$$(2') \quad \|\alpha\| = \|\beta\| \text{ or } \sqrt{2} \|\beta\|$$

(assuming here, as we may, that  $L$  is infinite-dimensional). The relations (2), (2') clearly imply that  $g=1$ , i. e., that  $\varphi$  is unitary.

**PROPOSITION 4.** *Let  $L$  be a complex simple  $L^*$ -algebra. A \*-automorphism  $\varphi$  (of  $L$ ) whose spectrum contains a number  $\lambda_0$  of unit modulus is unitary. In particular, any \*-automorphism  $\varphi$  admitting a non-zero fixed point is unitary.*

**PROOF.** By the Corollary to Theorem 1,  $\varphi$  is a semi- $L^*$ -automorphism:

$$\langle \varphi x, \varphi y \rangle = g^2 \langle x, y \rangle, \quad (x, y \in L).$$

It follows that  $\frac{\varphi}{g}$  is unitary, so that  $\varphi\varphi^* = g^2 I = \varphi^*\varphi$ , where  $I$  is the identity operator. The last equations imply that  $\varphi$  is normal. Since  $\lambda_0 \in \sigma(\varphi)$ , the spectrum

of  $\varphi$ , it results from the spectral mapping theorem that

$$|\lambda_0|^2 \in \sigma(\varphi\varphi^*) = \{g^2\}.$$

Therefore  $g = |\lambda_0| = 1$ , whence  $\varphi$  is unitary.

### 3. Semi-regular and regular automorphisms.

DEFINITION 2. Let  $L$  be a semi-simple  $L^*$ -algebra. Let  $D$  be a bounded derivation (in the Lie algebra sense) of  $L$ . We set

$$e^D = I + D + \frac{D^2}{2!} + \dots, \quad (I = \text{identity})$$

Then  $e^D$  is a bounded operator which is moreover, by a standard reasoning, an automorphism of  $L$ . In particular, for  $D = \text{ad } a$  ( $a \in L$ ) we write  $\varphi_a$  for  $e^{\text{ad } a}$ . If  $a$  is a normal element (i. e.,  $[a, a^*] = 0$ ), we call  $\varphi_a$  an *inner* automorphism.

DEFINITION 3. An automorphism  $\varphi$  of  $L$  is called *semi-regular* if 1 is an eigenvalue of  $\varphi$  and further the 1-eigensubspace  $L_1$  contains a maximal abelian subalgebra of  $L$ . (Observe that the 1-eigensubspace of an automorphism is always a subalgebra.)

Let  $\varphi_a$  be an inner automorphism. Since  $a$  is normal, it is contained in a Cartan subalgebra  $H$ . Since  $H$  is abelian it is clear that  $L_1 \supset H$ , and a Cartan subalgebra being maximal abelian [8, p. 70],  $\varphi_a$  is semi-regular. More generally, if  $D$  is a bounded derivation annihilating some Cartan subalgebra, then  $e^D$  is semi-regular.

PROPOSITION 5. *Every semi-regular \*-automorphism  $\varphi$  of a semi-simple  $L^*$ -algebra  $L$  is unitary.*

PROOF. The hypothesis on  $\varphi$  clearly implies that the 1-eigensubspace  $L_1$  of  $L$  contains a Cartan subalgebra  $H$ . Then, with the notational convention in Theorem 2,  $\tilde{H}$  is a Cartan subalgebra of  $\tilde{L}$ . Let

$$\tilde{L} = \tilde{H} \oplus \Sigma \oplus \tilde{V}_\alpha \quad (\oplus \text{ denoting orthogonal sum})$$

be the root space (or Cartan) decomposition of  $\tilde{L}$  relative to  $\tilde{H}$  (see [9]). Since  $\tilde{\varphi}$  leaves  $\tilde{H}$  element-wise invariant, it follows that

$$\tilde{\varphi}h_\alpha = h_\alpha, \quad \tilde{\varphi}\tilde{V}_\alpha = \tilde{V}_\alpha,$$

where  $h_\alpha$  is the vector of  $\tilde{H}$  such that  $\chi(h) = \langle h, h_\alpha \rangle$  for all  $h$  in  $\tilde{H}$ . Now choose for each positive root  $\alpha$  a vector  $v_\alpha \in \tilde{V}_\alpha$  with  $\|v_\alpha\| = 1$ , then  $v_\alpha^* \in \tilde{V}_{-\alpha}$  ([8, p. 73]). Let

$$\tilde{\varphi}v_\alpha = \lambda_\alpha v_\alpha, \quad \tilde{\varphi}v_\alpha^* = \lambda_{-\alpha} v_\alpha^*.$$

Then

$$\begin{aligned} \lambda_\alpha \lambda_{-\alpha} [v_\alpha, v_\alpha^*] &= \tilde{\varphi}[v_\alpha, v_\alpha^*] \\ &= \tilde{\varphi}h_\alpha = h_\alpha = [v_\alpha, v_\alpha^*]. \end{aligned}$$

Therefore  $\lambda_\alpha \lambda_{-\alpha} = 1$ . Again  $\tilde{\varphi}v_\alpha^* = (\tilde{\varphi}v_\alpha)^* = \overline{\lambda_\alpha} v_\alpha^*$ , whence  $\lambda_{-\alpha} = \overline{\lambda_\alpha}$ . Hence  $|\lambda_\alpha| = 1$ , which means  $\|\tilde{\varphi}v_\alpha\| = 1$ . On the other hand, since  $\tilde{\varphi}h = h$ , we have trivially  $\|\tilde{\varphi}h\| = \|h\|$  for all  $h \in \tilde{H}$ . These conclusions plus the mutual orthogonality of the  $\tilde{V}_\alpha$  and  $\tilde{H}$  imply that  $\tilde{\varphi}$  (and so  $\varphi$ ) is unitary.

**PROPOSITION 6.** *For an inner automorphism  $\varphi_{h_0}$  of a semi-simple  $L^*$ -algebra  $L$ , the following assertions are equivalent:*

- (i)  $\varphi_{h_0}$  is a  $*$ -map;
- (ii)  $\varphi_{h_0}$  is unitary;
- (iii)  $h_0$  is skew-adjoint.

**PROOF.** That (i)  $\Rightarrow$  (ii) follows from the previous proposition, while (ii)  $\Rightarrow$  (i) is just a particular case of the general fact that an  $L^*$ -isomorphism (or even a semi- $L^*$ -isomorphism) is automatically a  $*$ -map.

We shall now prove that (i)  $\Rightarrow$  (iii). With the previous notational convention, if (i) holds then  $\varphi_{h_0}$  is a  $*$ -map of  $\tilde{L}$ . Further, it is clear that if  $v_\alpha \in \tilde{V}_\alpha$  then

$$\varphi_{h_0}(v_\alpha) = e^{\alpha(h_0)} v_\alpha, \quad \varphi_{h_0}(v_\alpha^*) = e^{-\alpha(h_0)} v_\alpha^*.$$

But  $\varphi_{h_0}(v_\alpha^*) = (\varphi_{h_0}(v_\alpha))^*$ , so that  $e^{\overline{\alpha(h_0)}} = e^{-\alpha(h_0)}$ . Therefore

$$\alpha(h_0) + \overline{\alpha(h_0)} = 0, \quad \text{or} \quad \alpha(h + h_0^*) = 0.$$

The arbitrariness of the root  $\alpha$  and the 'total' property of the set  $\Delta = \{\alpha\}$  of roots [1, Lemma 6] now imply  $h_0^* = -h_0$ .

To complete the proof of the theorem we have only to show that (iii)  $\Rightarrow$  (ii). But this readily follows since (assuming (iii))

$$\varphi_{h_0}^{-1} = \varphi_{-h_0} = \varphi_{h_0^*} = (\varphi_{h_0})^*. \quad \text{q. e. d.}$$

In view of the above proposition we call an inner automorphism  $\varphi_{h_0}$ , with



$h_0$  skew-adjoint, an inner  $L^*$ -automorphism.

DEFINITION 4. A semi-regular automorphism  $\varphi$  of  $L$  is called *regular* if the 1-eigensubspace  $L_1$  is a maximal abelian subalgebra.

PROPOSITION 7. An inner automorphism  $\varphi_{h_0}$  is regular if and only if the 1-eigensubspace  $L_1$  of  $L$  is abelian (cf. [6, Theorems 5, 8]).

PROOF. Suffices to prove that if  $L_1$  is abelian then  $\varphi_{h_0}$  is regular. Since  $h_0$  is a normal element there exists a Cartan subalgebra  $H$  of  $L$  containing  $h_0$ . Since  $H$  is abelian,  $\varphi_{h_0}$  leaves  $H$  pointwise invariant, and therefore  $H \subset L_1$ . But  $H$  as a Cartan subalgebra is maximal abelian. Consequently  $H = L_1$  and  $\varphi_{h_0}$  is regular.

COROLLARY. For a regular inner automorphism  $\varphi_{h_0}$ , the 1-eigensubspace  $L_1$  is a Cartan subalgebra.

PROPOSITION 8. An inner automorphism  $\varphi_{h_0}$  of a semi-simple  $L$  is regular if and only if for some Cartan subalgebra  $\tilde{H}$  (of  $\tilde{L}$ ) containing  $h_0$  we have

$$\frac{\alpha(h_0)}{2\pi\sqrt{-1}} \equiv 0 \pmod{1}, \text{ for all } \alpha \in \Delta,$$

where  $\Delta$  is the root system of  $\tilde{L}$  relative to  $\tilde{H}$ . In particular, if  $\varphi_{h_0}$  is regular then  $h_0$  is a regular element in the sense of [2] (i.e., the null space  $\tilde{N}_0$  of  $\text{ad } h_0$  in  $\tilde{L}$  is a Cartan subalgebra).

PROOF. Suppose first that  $\varphi_{h_0}$  is regular. Then by Corollary to Proposition 7,  $\tilde{L}_1 = \tilde{H}$  is a Cartan subalgebra of  $\tilde{L}$ . If the condition in Proposition 8 is not satisfied for  $H$ , we must have

$$e^{\alpha(h_0)} = 1 \text{ for some } \alpha \in \Delta.$$

It follows that if  $v_\alpha \in \tilde{V}_\alpha$ , then

$$e^{\text{ad } h_0} v_\alpha = e^{\alpha(h_0)} v_\alpha = v_\alpha.$$

This means  $\tilde{V}_\alpha \subset \tilde{L}_1 = \tilde{H}$ , which is absurd. Hence the condition must hold.

Next, suppose the condition holds relative to some Cartan subalgebra  $\tilde{H}$ . Then in particular,  $\alpha(h_0) \neq 0$  for all  $\alpha \in \Delta$  so that by Theorem 1 of [2],  $h_0$  is

a regular element. Further  $\tilde{N}_0 \supset \tilde{H}$ , whence by maximality property of Cartan subalgebras  $\tilde{N}_0 = \tilde{H}$ . This clearly implies that  $\tilde{L}_1 \supset \tilde{H}$ . Let now

$$\tilde{L} = \tilde{H} \oplus \Sigma \oplus \tilde{V}_\alpha$$

be the root space decomposition of  $\tilde{L}$ . For  $x \in \tilde{L}_1$ , we can write

$$x = h + \Sigma v_\alpha (v_\alpha \in \tilde{V}_\alpha).$$

Then

$$h + \Sigma v_\alpha = x = e^{\text{ad } h_0} x = h + \Sigma e^{\alpha(h_0)} v_\alpha.$$

It follows that  $e^{\alpha(h_0)} v_\alpha = v_\alpha$  for all  $\alpha$ . But by our supposition  $e^{\alpha(h_0)} \neq 1$ . Hence  $v_\alpha = 0$ ,  $x = h \in \tilde{H}$ , and  $\tilde{L}_1 = \tilde{H}$ . Thus  $\varphi_{h_0}$  is regular as we wished to show.

**THEOREM 4.** *Let  $L$  be a separable semi-simple  $L^*$ -algebra. Then there exist regular inner automorphisms of  $L$ . If  $L$  is compact<sup>(3)</sup> or complex, then there exist even regular inner  $L^*$ -automorphisms.*

**PROOF.** First let  $L$  be real and  $\tilde{L}$  be its complexification. Let  $\tilde{H}$  be a Cartan subalgebra which is the complexification of a Cartan subalgebra  $H$  of  $L$ , and  $\Delta$  the root system of  $\tilde{L}$  relative to  $\tilde{H}$ . Since  $L$  is separable, so is  $\tilde{L}$ , and consequently  $\Delta = \{\alpha_i\}$  is countable. Define

$$P_{n,i} = \{h \in H : \alpha_i(h) = 2n\pi\sqrt{-1}\},$$

where  $n$  runs through all integers. Each  $P_{n,i}$  is either empty or a hyperplane of  $H$ . In any case  $P_{n,i}$  is non-dense (see footnote in [2, p.162]). It follows by Baire's category theorem that we can choose an  $h_1 \in H$  with  $\alpha_i(h_1) \neq 2n\pi\sqrt{-1}$  for any  $i$  or  $n$ . Then by Proposition 8,  $\varphi_{h_1}$  is a regular inner automorphism of  $L$ .

Next, let  $L$  be compact. Then  $h_1^* = -h_1$  and  $\varphi_{h_1}$  is a regular inner  $L^*$ -automorphism (Proposition 6). Finally, if  $L$  is complex we take its compact from  $L_k$ , i. e., the real  $L^*$ -algebra  $L_k$  of all skew-adjoint elements of  $L$  ([3, p.523]). Choose a  $h_1 \in L_k$  as above. Then  $\varphi_{h_1} = e^{\text{ad } h_1}$  taken over  $L$ , gives a regular inner  $L^*$ -automorphism of  $L$ .

**REMARK.** It was shown in [2] that a non-separable type  $A$  complex simple  $L^*$ -algebra  $L_A$  contains no regular element. More generally, it can be shown,

<sup>(3)</sup> i. e., every element of  $L$  is skew-adjoint.

using Bessel's inequality and the criterion for regular element [2, Theorem 1], that every complex semi-simple  $L^*$ -algebra admitting an uncountable subset of mutually orthogonal roots contains no regular element. The  $L^*$ -algebras  $L$  admitting such an orthogonal subset of roots include besides the non-separable simple algebras  $L_A, L_B, L_C$  also all semi-simple  $L$  with uncountably many simple components. In view of Proposition 8 none of these algebras — which are all, of course, non-separable — has a regular inner automorphism.

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