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# AUTOMORPHISMS OF NORMAL TRANSFORMATION SEMIGROUPS

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## 1. Introduction and preliminaries

Let $X$ be an infinite set, $\mathscr{G}_X$ be the group of all bijections of $X$ and $S$ be a semigroup of total transformations of $X$ with the composition of transformations $f$ and $g$ in $S$ defined by the formula

$$fg(x)=f(g(x)), \quad \text{where} \quad x\in X.$$

We say that $S$ is a $\mathscr{G}_X$-*normal* semigroup if

$$hSh^{-1}=S, \quad \text{for all} \quad h\in\mathscr{G}_X.$$

The full transformation semigroup $T_X$, the semigroups of all 1–1 and all onto transformations and the group $\mathscr{G}_X$ itself, are examples of $\mathscr{G}_X$-normal semigroups.

If $S$ is a $\mathscr{G}_X$-normal semigroup, then for each $h\in\mathscr{G}_X$, the map $\phi$ of $S$ given by

$$\phi(f)=hfh^{-1} \ (f\in S)$$

is an automorphism of $S$, specifically an *inner automorphism* of $S$. Our purpose is to prove the following:

**Theorem 1.1.** *Every automorphism of a* $\mathscr{G}_X$*-normal semigroup is inner.*

The subject of this paper was suggested to the author by G. R. Wood.

The question of whether inner automorphisms exhaust all automorphisms of a $\mathscr{G}_X$-normal semigroup has attracted the attention of a number of authors. In 1937 Schreier [**10**] was the first to give a positive answer for $T_X$. Then Malcev [**6**] extended this result to every ideal of $T_X$. Next Sullivan [**12**] generalized this work and confirmed that if a semigroup contains all constant transformations (in particular if a $\mathscr{G}_X$-normal semigroup contains a constant transformation) then it possesses only inner automorphisms, while Fitzpatrick and Symons [**3**] showed this for a semigroup containing $\mathscr{G}_X$. Schein [**8,9**] discovered that a $\mathscr{G}_X$-normal semigroup of 1–1 transformations has only inner automorphisms (see [**4**] for the special case of Baer–Levi semigroups).

Our result subsumes all previously stated results for $\mathscr{G}_X$-normal semigroups and describes completely all automorphisms of every $\mathscr{G}_X$-normal transformation semigroup.

In this paper we use a technique which differs from those used by Sullivan [**12**] and Schein [**8, 9**]. The essence is the production of certain maximal right (Section 2) and left (Section 3) ideals. We note a remarkable duality between properties of these right and left ideals.

For the purpose of our proof we partition all $\mathcal{G}_X$-normal semigroups into three types:

1. Semigroups containing a constant map; and constant-free semigroups into:
2. Semigroups of 1–1 transformations; and
3. Constant-free semigroups containing a transformation which is not 1–1.

All automorphisms of semigroups of the first type are inner [**12**, Theorem 1], so we can restrict our attention to constant-free semigroups.

We begin with some general notes on $\mathcal{G}_X$-normal semigroups.

For a function $f:X\to X$ we denote the *range* of $f$ by $R(f)$ $(=f(X))$ and the *partition* of $f$ by $\pi(f)$ $(=\{f^{-1}(x): x\in R(f)\})$.

If $S$ is an arbitrary semigroup of transformations, let

$$R(S)=\{R(f): f\in S\} \quad \text{and} \quad \pi(S)=\{\pi(f): f\in S\}.$$

We say that $R(S)$ $(\pi(S))$ is *normal* if for each $h\in\mathcal{G}_X$

$$h(R(S))=R(S) \quad (h(\pi(S))=\pi(S)),$$

(by $h(R(S))$ we mean $\{h(A): A\in R(S)\}$ and by $h(\pi(S))$ we mean $\{h(\mathscr{A}): \mathscr{A}\in\pi(S)\}$, where $h(\mathscr{A})=\{h(A): A\in\mathscr{A}\}$).

**Lemma 1.2.** *If $S$ is a $\mathcal{G}_X$-normal semigroup, then $R(S)$ and $\pi(S)$ are normal.*

The proof is straightforward. □

We say that a semigroup $S$ is *trivial* if $S=\{\Delta_X\}$, where $\Delta_X$ is the identity transformation of $X$. In what follows $S$ is non-trivial.

**Result 1.3.** *Every $\mathcal{G}_X$-normal semigroup $S$ is transitive.*

**Proof.** Take arbitrary $x, y$ in $X$. We construct $f$ in $S$ such that $f(x)=y$.

Firstly let $x$ and $y$ be distinct and suppose there exists a $g\in S$ with $g(x)=z\neq x$. If $z=y$ we let $f=g$, otherwise $(y,z)g(y,z)$ is the required $f$ ($(y,z)$ denotes the transposition interchanging $y$ and $z$). To construct $g$, observe that since $S$ is non-trivial there exists a $q\in S$ together with distinct $u$ and $v$ in $X$ such that $q(u)=v$. If $u=x$ we let $g=q$, otherwise $g=(u,x)q(u,x)$.

Now suppose $y=x$, choose any $p$ in $S$ and let $p(x)=w$. If $w=x$ we let $f=p$. Otherwise choose $t\in S$ with $t(w)=x$ (using the first part of the proof), then $f=tp$ takes $x$ to $x$ as required. □

**Remark 1.4.** We exclude from our consideration $\mathcal{G}_X$-normal subsemigroups of $\mathcal{G}_X$, since they are all subgroups of $\mathcal{G}_X$, and hence have only inner automorphisms [**11**].

## 2. $\mathscr{G}_X$-normal semigroups of 1–1 transformations

In this section $S$ denotes a $\mathscr{G}_X$-normal semigroup of 1–1 transformations.

**Definition 2.1.** Let $x \in X$ and

$$\mathscr{R}_x = \{r \in S: x \in X \backslash R(r)\}.$$

Then $\mathscr{R}_x$ is a right ideal of $S$, which we call a *point right ideal.* □

We will use the following observation based on the normality of $R(S)$ (Lemma 1.2) and the fact that $S$ is not a subsemigroup of $\mathscr{G}_X$, that is $R(S)$ contains proper subsets of $X$.

**Remark 2.2.** Given $x, y \in X$ with $x \neq y$ there exists an $A$ in $R(S)$ with $x \in X \backslash A$ and $y \in A$. □

**Lemma 2.3.** *Given $x, y \in X$ the following three statements are equivalent:*

(i) $\mathscr{R}_x \subseteq \mathscr{R}_y$;
(ii) $x = y$;
(iii) $\mathscr{R}_x = \mathscr{R}_y$.

**Proof.** Implications (ii)⇒(iii) and (iii)⇒(i) are trivial. We show (i)⇒(ii). Suppose $x \neq y$ and choose an $A \in R(S)$ with $x \in X \backslash A$, $y \in A$ (Remark 2.2). If $f \in S$ with $R(f) = A$, then $f \in \mathscr{R}_x \backslash \mathscr{R}_y$, so $\mathscr{R}_x \nsubseteq \mathscr{R}_y$, proving (i)⇒(ii). □

Define a map $\theta: X \to \{\mathscr{R}_x: x \in X\}$ via $\theta(x) = \mathscr{R}_x$, each $x \in X$.

**Lemma 2.4.** *$\theta$ is a bijection.*

**Proof.** Clearly $\theta$ is onto and Lemma 2.3 ensures $\theta$ is 1–1. □

**Definition 2.5.** Given distinct $f_1, f_2 \in S$ let

$$\mathscr{R}_{f_1,f_2} = \{r \in S: f_1 r = f_2 r\}.$$

Then $\mathscr{R}_{f_1,f_2}$ is a right ideal of $S$ (possibly empty), which we call a *function right ideal.* □

We will show (Result 2.8) that there always exist distinct $f_1, f_2$ in $S$ such that $\mathscr{R}_{f_1,f_2}$ is non-empty. However $\mathscr{R}_{f_1,f_2}$ may be empty. Observe that given $f_1$ and $f_2$,

$$r \in \mathscr{R}_{f_1,f_2} \quad \text{iff} \quad R(r) \subseteq \{x \in X: f_1(x) = f_2(x)\}.$$

Hence if we choose $f_1$ and $f_2$ which are never equal, then $\mathscr{R}_{f_1,f_2} = \Phi$.

Let $S$, for example, be the Baer–Levi semigroup of type $(|X|, |X|)$ [2], that is the semigroup of all 1–1 transformations $f$ such that $|R(f)| = |X \backslash R(f)| = |X|$. Note that $S$ is

$\mathcal{G}_X$-normal and choose $f_1 \in S$, then $X \backslash R(f_1) \in R(S)$ (Lemma 1.2). If $f_2 \in S$ with $R(f_2) = X \backslash R(f_1)$, then $\mathcal{R}_{f_1, f_2} = \Phi$.

The following notation applies to an arbitrary $\mathcal{G}_X$-normal semigroup $S$.

**Notation 2.6.** Let $f_1, f_2$ be distinct transformations in $S$. Then

$$\mathcal{D}_{f_1, f_2} = \{x \in X: f_1(x) \neq f_2(x)\}$$

and

$$D_{f_1, f_2} = \{\{f_1(x), f_2(x)\}: x \in \mathcal{D}_{f_1, f_2}\}. \qquad \square$$

Returning to semigroups of 1–1 transformations, we now derive relationships between point right ideals and function right ideals.

**Result 2.7.** *Let $f_1, f_2 \in S$ with $\mathcal{R}_{f_1, f_2} \neq \Phi$. Then*

$$\mathcal{R}_{f_1, f_2} = \bigcap_{x \in \mathcal{D}_{f_1, f_2}} \mathcal{R}_x.$$

**Proof.** Let $r \in \mathcal{R}_{f_1, f_2}$, that is $f_1 r = f_2 r$. If $x \in \mathcal{D}_{f_1, f_2}$, or $f_1(x) \neq f_2(x)$, then $x \in X \backslash R(r)$, so $r \in \mathcal{R}_x$, and since this is true for each $x \in \mathcal{D}_{f_1, f_2}$ we conclude

$$r \in \bigcap_{x \in \mathcal{D}_{f_1, f_2}} \mathcal{R}_x$$

or

$$\mathcal{R}_{f_1, f_2} \subseteq \bigcap_{x \in \mathcal{D}_{f_1, f_2}} \mathcal{R}_x.$$

Conversely, if

$$r \in \bigcap_{x \in \mathcal{D}_{f_1, f_2}} \mathcal{R}_x,$$

then for each $y$ in $R(r)$ we have $y \in X \backslash \mathcal{D}_{f_1, f_2}$, or $f_1(y) = f_2(y)$ and hence $f_1 r = f_2 r$, that is $r \in \mathcal{R}_{f_1, f_2}$, so

$$\bigcap_{x \in \mathcal{D}_{f_1, f_2}} \mathcal{R}_x \subseteq \mathcal{R}_{f_1, f_2},$$

which proves the desired equality. $\square$

**Result 2.8.** *Given $x \in X$ there exist $f_1, f_2 \in S$ such that $\mathcal{R}_x = \mathcal{R}_{f_1, f_2}$.*

**Proof.** On account of Result 2.7 it is sufficient to construct $f_1, f_2$ such that $\mathcal{D}_{f_1, f_2} = \{x\}$.

Observe that there exists an $f$ in $S$ with

$$|X \backslash R(f)| \geq 2.$$

(For an arbitrary $f$ in $S\backslash\mathscr{G}_X$

$$|X\backslash R(f^2)|=|X\backslash R(f)|+|X\backslash R(f)|$$

and we replace $f$ with $f^2$).

Using the normality of $R(S)$ (Lemma 1.2) choose an $f$ in $S$ with

$$x\in X\backslash R(f) \quad \text{and} \quad |X\backslash R(f)|\geqq 2.$$

Let $f(x)=y$ and $z\in X\backslash R(f)$, $z\neq x$. If

$$g=(x,z)f(x,z)$$

then $g(z)=y$ and $z\in X\backslash R(g)$. We let

$$h=(y,z),\ f_1=gf \text{ and } \quad f_2=hgh^{-1}f.$$

Then for each $u\neq x$:

$$\begin{aligned} f_1(u)=gf(u)=gh^{-1}f(u), &\quad \text{since} \quad f(u)\neq y \quad \text{for} \quad u\neq x \\ &\quad \text{and} \quad z\notin R(f); \\ =hgh^{-1}f(u), &\quad \text{since} \quad gh^{-1}f(u)\neq y \\ &\quad \text{for} \quad f(u)\neq y \\ &\quad \text{and} \quad z\notin R(g); \\ =f_2(u). & \end{aligned}$$

However

$$f_1(x)=gf(x)=g(y)$$

while

$$f_2(x)=hgh^{-1}f(x)=hgh^{-1}(y)=hg(z)=h(y)=z\neq g(y),$$

since $z\in X\backslash R(g)$. Hence $f_1(x)\neq f_2(x)$ and $\mathscr{D}_{f_1,f_2}=\{x\}$. □

**Result 2.9.** *Given $f_1$ and $f_2$ in $S$, $\mathscr{R}_{f_1,f_2}$ is a maximal function right ideal if and only if $|\mathscr{D}_{f_1,f_2}|=1$.*

**Proof.** Suppose $\mathscr{R}_{f_1,f_2}$ is a maximal function right ideal, while $x,y\in\mathscr{D}_{f_1,f_2}$, $x\neq y$.

Then

$$\mathscr{R}_{f_1,f_2}=\bigcap_{z\in\mathscr{D}_{f_1,f_2}}\mathscr{R}_z \qquad \text{(Result 2.7)}$$

$$\subseteqq \mathscr{R}_x\cap\mathscr{R}_y$$

$$\subsetneqq \mathscr{R}_x \qquad \text{(Lemma 2.3).}$$

It follows from Result 2.8 that there exist $g_1$ and $g_2$ with

$$\mathscr{R}_{g_1,g_2}=\mathscr{R}_x,$$

and so

$$\mathscr{R}_{f_1,f_2}\subsetneqq\mathscr{R}_x=\mathscr{R}_{g_1,g_2},$$

a contradiction to the maximality of $\mathscr{R}_{f_1,f_2}$. Hence $|\mathscr{D}_{f_1,f_2}|=1$.

For the converse, suppose $\mathscr{D}_{f_1,f_2}=\{x\}$, some $x\in X$, while there exist $g_1,g_2\in S$ such that

$$\mathscr{R}_{g_1,g_2}\supseteq\mathscr{R}_{f_1,f_2}.$$

Since

$$\mathscr{R}_{g_1,g_2}=\bigcap_{y\in\mathscr{D}_{g_1,g_2}}\mathscr{R}_y \qquad \text{(Result 2.7)}$$

we have

$$\bigcap_{y\in\mathscr{D}_{g_1,g_2}}\mathscr{R}_y=\mathscr{R}_{g_1,g_2}\supseteq\mathscr{R}_{f_1,f_2}=\mathscr{R}_x \qquad \text{(Result 2.7 again),}$$

and so Lemma 2.3 ensures $\mathscr{D}_{g_1,g_2}=\{x\}$, that is

$$\mathscr{R}_{g_1,g_2}=\mathscr{R}_x=\mathscr{R}_{f_1,f_2}. \qquad \square$$

**Corollary 2.10.** *Given $f_1$ and $f_2$ in S, $\mathscr{R}_{f_1,f_2}$ is a maximal function right ideal if and only if $\mathscr{R}_{f_1,f_2}=\mathscr{R}_x$, some $x\in X$.*

**Proof.** Follows from Results 2.7 and 2.9. □

We show now that each automorphism $\phi$ of $S$ permutes point right ideals.

**Result 2.11.** *Given $x\in X$,*

$$\phi(\mathscr{R}_x)=\mathscr{R}_y,$$

*for some $y\in X$.*

**Proof.** Choose $f_1$ and $f_2$ in $S$ such that $\mathscr{R}_{f_1,f_2}=\mathscr{R}_x$ (Result 2.8), then

$$\begin{aligned}\phi(\mathscr{R}_x)=\phi(\mathscr{R}_{f_1,f_2})&=\phi(\{r: f_1r=f_2r\})\\&=\{\phi(r): \phi(f_1r)=\phi(f_2r)\}\\&=\{\phi(r): \phi(f_1)\phi(r)=\phi(f_2)\phi(r)\}\\&=\{r': \phi(f_1)r'=\phi(f_2)r'\}\\&=\mathscr{R}_{\phi(f_1),\phi(f_2)}.\end{aligned}$$

Now Corollary 2.10 ensures $\mathscr{R}_{f_1,f_2}$ is a maximal function right ideal, hence $\mathscr{R}_{\phi(f_1),\phi(f_2)}(=\phi(\mathscr{R}_{f_1,f_2}))$ is a maximal function right ideal, so there exists $y\in X$ such that

$$\mathscr{R}_{\phi(f_1),\phi(f_2)}=\mathscr{R}_y \qquad \text{(Corollary 2.10)}$$

and thus

$$\phi(\mathscr{R}_x)=\mathscr{R}_{\phi(f_1),\phi(f_2)}=\mathscr{R}_y. \qquad \square$$

Define a map

$$\eta: \{\mathscr{R}_x: x\in X\}\rightarrow\{\mathscr{R}_x: x\in X\}$$

via $\eta(\mathscr{R}_x)=\phi(\mathscr{R}_x)$, each $\mathscr{R}_x\subseteq S$.

**Lemma 2.12.** *$\eta$ is a bijection.*

**Proof.** That $\eta$ is a mapping is the content of Result 2.11. Similarly by considering the automorphism $\phi^{-1}$ we define a map

$$\zeta: \{\mathscr{R}_x: x\in X\}\rightarrow\{\mathscr{R}_x: x\in X\}$$

via $\zeta(\mathscr{R}_x)=\phi^{-1}(\mathscr{R}_x)$, each $\mathscr{R}_x\subseteq S$.
Certainly, $\zeta$ is the inverse of $\eta$ and so $\eta$ is a bijection. $\square$

We now define a map

$$h: X\rightarrow X \quad \text{via} \quad h(x)=y, \quad \text{where} \quad \eta(\mathscr{R}_x)=\mathscr{R}_y, \quad \text{each} \quad x\in X.$$

It is clear, that

$$h=\theta^{-1}\eta\theta,$$

and so Lemmas 2.4 and 2.12 ensure $h$ is a bijection of $X$. We call $h$ the *bijection associated with $\phi$*.

**Lemma 2.13.** *Given* $f \in S$,

$$R(\phi(f)) = h(R(f)).$$

**Proof.** Observe that to show $R(\phi(f)) = h(R(f))$ it is sufficient to show that

$$X \backslash R(\phi(f)) = h(X \backslash R(f)),$$

because for the bijection $h$, $h(X \backslash R(f)) = X \backslash h(R(f))$.

Now if $x \in X \backslash R(f)$, that is $f \in \mathscr{R}_x$, then $\phi(f) \in \eta(\mathscr{R}_x) = \mathscr{R}_{h(x)}$, so $h(x) \in X \backslash R(\phi(f))$, or

$$h(X \backslash R(f)) \subseteq X \backslash R(\phi(f)).$$

To show the reverse inclusion is true, observe that $h^{-1} = \theta^{-1}\eta^{-1}\theta$ is the bijection associated with $\phi^{-1}$ and so the first part of the proof implies that given $g \in S$,

$$h^{-1}(X \backslash R(g)) \subseteq X \backslash R(\phi^{-1}(g)).$$

In particular taking $g = \phi(f)$ we have $h^{-1}(X \backslash R(\phi(f))) \subseteq X \backslash R(\phi^{-1}(\phi(f)))$, or

$$h(X \backslash R(f)) \supseteq X \backslash R(\phi(f)),$$

and the equality follows. □

We complete our study of automorphisms of $\mathscr{G}_X$-normal semigroups of 1–1 transformations, that is, semigroups of Type 2, by presenting the following result.

**Result 2.14.** *Let $S$ be a $\mathscr{G}_X$-normal semigroup of 1–1 transformations ($S \nsubseteq \mathscr{G}_X$). Then each automorphism $\phi$ of $S$ is inner, that is, for some $h \in \mathscr{G}_X$*

$$\phi(f) = hfh^{-1}, \quad \textit{for each} \quad f \in S.$$

**Proof.** Consider the bijection $h$ associated with $\phi$ as defined prior to Lemma 2.13. Take an arbitrary $f \in S$, $x \in X$ and let $f(x) = y$. Choose $A$ in $R(S)$ with $A \neq X$ and $x \in A$. Let $z \in X \backslash A$ and $B = (A \backslash \{x\}) \cup \{z\} \in R(S)$ (Lemma 1.2). Choose $p$ and $q$ in $S$ such that $R(p) = A$ and $R(q) = B$.

Now $R(p) \backslash R(q) = A \backslash B = \{x\}$, thus $R(fp) \backslash R(fq) = \{f(x)\} = \{y\}$. Using Lemma 2.13 we have:

$$R(\phi(p)) \backslash R(\phi(q)) = \{h(x)\}$$

and

$$R(\phi(fp)) \backslash R(\phi(fq)) = \{h(y)\}.$$

However

$$R(\phi(fp)) \backslash R(\phi(fq)) = R(\phi(f)\phi(p)) \backslash R(\phi(f)\phi(q))$$

$$= \{\phi(f)h(x)\},$$

so

$$\phi(f)h(x)=h(y)=hf(x), \quad \text{that is}$$

$$\phi(f)=hfh^{-1}. \qquad \square$$

**Remark 2.15.** The fact that every $\mathcal{G}_X$-normal semigroup of 1–1 transformations possesses only inner automorphisms was first established by B. M. Schein [**8, 9**]. We understand that his proof, based on the study of ordered sets of ranges, is quite different from ours. $\square$

## 3. $\mathcal{G}_X$-normal constant-free semigroups containing a transformation which is not 1–1

Let $S$ be a $\mathcal{G}_X$-normal constant-free semigroup containing a transformation which is not 1–1. We prove that all automorphisms of $S$ are inner. We start by showing that $R(S)$ contains only sets of cardinality $|X|$.

**Lemma 3.1.** *If $S$ is a $\mathcal{G}_X$-normal constant-free semigroup, then $|R(f)|=|X|$, each $f\in S$.*

**Proof.** Suppose there is an $f$ in $S$ with $|R(f)|=\alpha<|X|$, that is $|\pi(f)|=|R(f)|=\alpha$. We show that there exists an $A\in\pi(f)$ with $|A|\geqq\alpha$. The result is clear when $\alpha$ is finite. Hence assume $\alpha$ is infinite and denote by $\alpha^+$ the cardinal successor of $\alpha$. Then either $\alpha^+=|X|$ (and so $|X|$ is regular [7, 21.14]) or there exists $\beta<|X|$, $\beta=\alpha^+$ (and so $\beta$ is regular [7, 21.14]). The assumption that each $A\in\pi(f)$ has a cardinality less than $\alpha$ implies that $|\cup\pi(f)|<|X|$ or $|\cup\pi(f)|<\beta<|X|$ respectively [7, 21.18], a contradiction. Hence we can choose an $A\in\pi(f)$ with $|A|\geqq\alpha$ and a $B\in R(S)$ with $B\subseteq A$ and $|B|=\alpha$ (Lemma 1.2) together with a $g\in S$ such that $R(g)=B$. Then $|R(fg)|=1$, so that $fg$ is a constant map in $S$, a contradiction which proves $|R(f)|=|X|$. $\square$

Let $\mathcal{P}_2$ be the set of all doubletons in $X$.

**Definition 3.2.** Given $A\in\mathcal{P}_2$, $A=\{a_1,a_2\}$, let

$$\mathcal{L}_A=\{l\in S: l(a_1)=l(a_2)\}.$$

Then $\mathcal{L}_A$ is a left ideal of $S$ which we call a *point left ideal.* $\square$

**Lemma 3.3.** *For each $A\in\mathcal{P}_2$, $\mathcal{L}_A\neq\Phi$.*

**Proof.** Choose a map $f$ in $S$ which is not 1–1, say $f(x)=f(y)$ for distinct $x,y\in X$. If $h\in\mathcal{G}_X$ is such that $\{h(x),h(y)\}=A$ then $hfh^{-1}\in\mathcal{L}_A$. $\square$

**Lemma 3.4.** *Given $A, B \in \mathscr{P}_2$, the following three statements are equivalent:*

(i) $\mathscr{L}_A \subseteq \mathscr{L}_B$;
(ii) $A = B$;
(iii) $\mathscr{L}_A = \mathscr{L}_B$.

**Proof.** Implications (ii)⇒(iii) and (iii)⇒(i) are trivial. We show (i)⇒(ii).

Let $B = \{b_1, b_2\}$ and suppose $A \neq B$, say $b_1 \in B \backslash A$. Choose an $l \in \mathscr{L}_A$ (Lemma 3.3) and let $x \in R(l) \backslash l(A \cup B)$ (note: $|X| = |R(l)| > |l(A \cup B)|$, Lemma 3.1). If $y \in X$ is such that $l(y) = x$, let $h = (b_1, y)$ and $f = hlh^{-1}$. We show $f \in \mathscr{L}_A \backslash \mathscr{L}_B$. That $f \in \mathscr{L}_A$ follows from the fact that $h$ moves only points $b_1$ and $y$, which are not in $A$. To show that $f \notin \mathscr{L}_B$, observe that $f(b_1) = hlh^{-1}(b_1) = hl(y) = h(x)$, while $f(b_2) = hlh^{-1}(b_2) = hl(b_2)$, because $b_2 \neq y$ (else $x = l(y) = l(b_2) = l(B)$, contrary to the choice of $x$). Hence $f(b_2) \neq h(x)$, because $l(b_2) = l(B) \neq x$. Thus $f(b_1) \neq f(b_2)$ and $f \in \mathscr{L}_B$.

Define a map $\delta: \mathscr{P}_2 \to \{\mathscr{L}_A : A \in \mathscr{P}_2\}$ via $\delta(A) = \mathscr{L}_A$, each $A \in \mathscr{P}_2$.

**Lemma 3.5.** *$\delta$ is a bijection.*

**Proof.** Clearly $\delta$ is onto and Lemma 3.4 ensures $\delta$ is 1–1. □

**Definition 3.6.** Given distinct $f_1$, $f_2 \in S$ let

$$\mathscr{L}_{f_1, f_2} = \{l \in S : lf_1 = lf_2\}.$$

Then $\mathscr{L}_{f_1, f_2}$ is a left ideal of $S$ (possibly empty, see Example 3.7 below), which we call a *function left ideal.* □

We will show (Result 3.10) that for each $\mathscr{G}_X$-normal constant-free semigroup $S$ containing a transformation which is not 1–1 there exist $f_1, f_2 \in S$ with $\mathscr{L}_{f_1, f_2} \neq \Phi$. In general, the question of whether $f_1$, $f_2 \in S$ generate a non-empty $\mathscr{L}_{f_1, f_2}$ is the question of whether the equation $lf_1 = lf_2$ has a solution $l$ in $S$. The example below illustrates that $\mathscr{L}_{f_1, f_2}$ may be empty.

**Example 3.7.** Let $S$ be the dual Baer-Levi semigroup of the type $(|X|, |X|)$ **[1]**, that is the semigroup of all onto mappings $f$ such that $|f^{-1}(x)| = |X|$ for each $x \in X$. Certainly $S$ is $\mathscr{G}_X$-normal. Assume $X = \mathbb{N}$, so that $|X| = \aleph_0$. Fix an arbitrary $f_1 \in S$ and let

$$\mathscr{A} = \pi(f_1) = \{A_1, A_2, A_3, \ldots\}.$$

Partition each $A_i \in \mathscr{A}$ such that $A_i = A_i' \dot{\cup} A_i''$, $|A_i'| = A_i''| = \aleph_0$. Let $\mathscr{B}$ be the partition of $X$

given by

$$\mathscr{B}=\{A_1', A_1'' \dot{\cup} A_2', A_2'' \dot{\cup} A_3', \ldots\}.$$

Since $\mathscr{B}$ is a partition of $X$ into $\aleph_0$ sets, each of cardinality $\aleph_0$, $\mathscr{B}\in\pi(S)$, and so there exists $f_2\in S$ with $\pi(f_2)=\mathscr{B}$. Suppose $l\in\mathscr{L}_{f_1,f_2}$ that is $lf_1=lf_2$ and let $l_1(A_1)=x$. Then because of the choice of $\mathscr{B}$ we have the following chain of equalities:

$$x=lf_1(A_1)=lf_1(A_1'')=lf_2(A_1'')=lf_2(A_2')=lf_1(A_2')=lf_1(A_2)=\ldots$$

thus

$$x=lf_1(A_1)=lf_1(A_2)=\ldots,$$

that is $R(lf_1)=\{x\}$ and $lf_1$ is a constant in $S$, contradicting the construction of $S$, so that $\mathscr{L}_{f_1,f_2}=\Phi$. □

Recall that $\mathscr{D}_{f_1,f_2}$ and $D_{f_1,f_2}$ (Notation 2.6) were defined for an arbitrary $\mathscr{G}_X$-normal semigroup $S$ ($f_1, f_2\in S$). The following remark is an immediate consequence of the definition of $D_{f_1,f_2}$.

**Remark 3.8.** Let $f_1, f_2\in S$, then $D_{f_1,f_2}\subseteq\mathscr{P}_2$. □

We proceed with two results deriving relationships between point left ideals and function left ideals.

**Result 3.9.** *Let $f_1$ and $f_2$ be distinct elements of S, and $\mathscr{L}_{f_1,f_2}\neq\Phi$. Then*

$$\mathscr{L}_{f_1,f_2}=\bigcap_{A\in D_{f_1,f_2}}\mathscr{L}_A.$$

**Proof.** Let $l\in\mathscr{L}_{f_1,f_2}$, that is, $lf_1=lf_2$ and so for each $x\in\mathscr{D}_{f_1,f_2}$ we have $lf_1(x)=lf_2(x)$ (recall $f_1(x)\neq f_2(x)$) so $l\in\mathscr{L}_{\{f_1(x),f_2(x)\}}$ and since this is true for each $x\in\mathscr{D}_{f_1,f_2}$ we conclude

$$l\in\bigcap_{x\in\mathscr{D}_{f_1,f_2}}\mathscr{L}_{\{f_1(x),f_2(x)\}}=\bigcap_{A\in D_{f_1,f_2}}\mathscr{L}_A,\quad\text{or}\quad\mathscr{L}_{f_1,f_2}\subseteq\bigcap_{A\in D_{f_1,f_2}}\mathscr{L}_A.$$

Conversely, let $l\in\bigcap_{A\in D_{f_1,f_2}}\mathscr{L}_A$, then for each $x\in\mathscr{D}_{f_1,f_2}$, $lf_1(x)=lf_2(x)$. Now for each $y\notin\mathscr{D}_{f_1,f_2}$ we have $f_1(y)=f_2(y)$, so we deduce $lf_1=lf_2$. That is,

$$l\in\mathscr{L}_{f_1,f_2}\quad\text{and}\quad\bigcap_{A\in D_{f_1,f_2}}\mathscr{L}_A\subseteq\mathscr{L}_{f_1,f_2},$$

which proves the desired equality. □

**Result 3.10** *Given an $A \in \mathscr{P}_2$, there exist $f_1$ and $f_2$ is $S$ such that*

$$\mathscr{L}_A = \mathscr{L}_{f_1,f_2}.$$

**Proof.** On account of Result 3.9 it is sufficient to construct $f_1$ and $f_2$ such that

$$D_{f_1,f_2} = \{A\}.$$

Choose an $f$ in $\mathscr{L}_A$ (Lemma 3.3) and let $f(A) = z$. Let $A = \{a_1, a_2\}$. Since $S$ is transitive (Result 1.3) there exists $g$ in $S$ such that $g(z) = a_1$. Let $h = (a_1, a_2)$ and

$$f_1 = gf; \quad f_2 = hf_1h^{-1}.$$

Since $h$ moves only points in $A$ and $f_1 \in \mathscr{L}_A$ ($\mathscr{L}_A$ is a left ideal), we conclude $f_2 = hf_1$. For each $x \in X \backslash f_1^{-1}(A)$ we have:

$$f_1(x) = hf_1(x) = f_2(x),$$

so $\mathscr{D}_{f_1,f_2} \subseteq f_1^{-1}(A)$. Now if $x \in f_1^{-1}(A)$, that is $f_1(x) = a_i$, $i = 1, 2$, then

$$f_1(x) = a_i \neq h(a_i) = hf_1(x) = f_2(x),$$

hence $\mathscr{D}_{f_1,f_2} \supseteq f_1^{-1}(A)$. We conclude

$$\mathscr{D}_{f_1,f_2} = f_1^{-1}(A).$$

Thus

$$\begin{aligned} D_{f_1,f_2} &= \{\{f_1(x), f_2(x)\} : x \in \mathscr{D}_{f_1,f_2}\} \quad \text{(Notation 2.6)} \\ &= \{\{f_1(x), f_2(x)\} : x \in f_1^{-1}(A)\} \\ &= \{\{a_i, h(a_i)\} : \ i = 1, 2\} \\ &= \{\{a_1, a_2\}\} \\ &= \{A\}, \end{aligned}$$

as required. □

**Result 3.11.** *Given distinct $f_1$ and $f_2$ in $S$, $\mathscr{L}_{f_1,f_2}$ is a maximal function left ideal if and only if $|D_{f_1,f_2}| = 1$.*

**Proof.** Let $\mathscr{L}_{f_1,f_2}$ be a maximal function left ideal and suppose $A, B \in D_{f_1,f_2}$, $A \neq B$.

Then $A, B \in \mathscr{P}_2$ (Remark 3.8). Hence

$$\mathscr{L}_{f_1,f_2} = \bigcap_{C \in D_{f_1,f_2}} \mathscr{L}_C \quad \text{(Result 3.9)}$$

$$\subseteqq \mathscr{L}_A \cap \mathscr{L}_B$$

$$\subsetneqq \mathscr{L}_A \qquad \text{(Lemma 3.4)}$$

$$= \mathscr{L}_{g_1,g_2} \qquad \text{(Result 3.10)},$$

for some distinct $g_1, g_2 \in S$, contradicting the maximality of $\mathscr{L}_{f_1,f_2}$. Hence $|D_{f_1,f_2}| = 1$.

Conversely, suppose $D_{f_1,f_2} = \{A\}$, some $A \in \mathscr{P}_2$, while there exists a function left ideal $\mathscr{L}_{g_1,g_2}$ $(g_1, g_2 \in S)$ such that

$$\mathscr{L}_{g_1,g_2} \supseteqq \mathscr{L}_{f_1,f_2}.$$

Since $\mathscr{L}_{g_1,g_2} = \bigcap_{B \in D_{g_1,g_2}} \mathscr{L}_B$ (Result 3.9) we have

$$\bigcap_{B \in D_{g_1,g_2}} \mathscr{L}_B = \mathscr{L}_{g_1,g_2} \supseteqq \mathscr{L}_{f_1,f_2} = \mathscr{L}_A \quad \text{(Result 3.9 again)},$$

and so Lemma 3.4 ensures $D_{g_1,g_2} = \{A\}$, that is

$$\mathscr{L}_{g_1,g_2} = \mathscr{L}_A = \mathscr{L}_{f_1,f_2}. \qquad \square$$

**Corollary 3.12.** *Given $f_1$ and $f_2$ is $S$, $\mathscr{L}_{f_1,f_2}$ is a maximal left function ideal if and only if $\mathscr{L}_{f_1,f_2} = \mathscr{L}_A$, some $A \in \mathscr{P}_2$.*

**Proof.** Follows from Results 3.9 and 3.11. $\square$

We show now that each automorphism $\phi$ of $S$ permutes point left ideals.

**Result 3.13.** *Given $A \in \mathscr{P}_2$,*

$$\phi(\mathscr{L}_A) = \mathscr{L}_B,$$

*for some $B \in \mathscr{P}_2$.*

**Proof.** Choose $f_1$ and $f_2$ in $S$ such that $\mathscr{L}_{f_1,f_2}=\mathscr{L}_A$ (Result 3.10), then

$$\begin{aligned}\phi(\mathscr{L}_A)=\phi(\mathscr{L}_{f_1,f_2})&=\phi(\{l: lf_1=lf_2\})\\&=\{\phi(l): \phi(lf_1)=\phi(lf_2)\}\\&=\{\phi(l): \phi(l)\phi(f_1)=\phi(l)\phi(f_2)\}\\&=\{l': l'\phi(f_1)=l'\phi(f_2)\}\\&=\mathscr{L}_{\phi(f_1),\phi(f_2)}.\end{aligned}$$

Now Corollary 3.12 ensures $\mathscr{L}_{f_1,f_2}$ is a maximal function left ideal, hence $\mathscr{L}_{\phi(f_1),\phi(f_2)}$ $(=\phi(\mathscr{L}_{f_1,f_2}))$ is a maximal function left ideal, so there exists $B\in\mathscr{P}_2$ such that

$$\mathscr{L}_{\phi(f_1),\phi(f_2)}=\mathscr{L}_B \quad \text{(Corollary 3.12)}.$$

We conclude

$$\phi(\mathscr{L}_A)=\mathscr{L}_{\phi(f_1),\phi(f_2)}=\mathscr{L}_B. \qquad \square$$

Define a map

$$\mu: \{\mathscr{L}_A: A\in\mathscr{P}_2\}\to\{\mathscr{L}_A: A\in\mathscr{P}_2\}$$

via $\mu(\mathscr{L}_A)=\phi(\mathscr{L}_A)$, each $\mathscr{L}_A\subseteq S$.

**Lemma 3.14.** *$\mu$ is a bijection.*

**Proof.** That $\mu$ is a mapping is the content of Result 3.13. Similarly by considering the automorphism $\phi^{-1}$ we define a map

$$\xi: \{\mathscr{L}_A: A\in\mathscr{P}_2\}\to\{\mathscr{L}_A: A\in\mathscr{P}_2\}$$

via $\xi(\mathscr{L}_A)=\phi^{-1}(\mathscr{L}_A)$, each $\mathscr{L}_A\subseteq S$. Certainly, $\xi$ is the inverse of $\mu$ and so $\mu$ is a bijection. $\square$

We now define a map

$$\lambda: \mathscr{P}_2\to\mathscr{P}_2 \quad \text{via} \quad \lambda(A)=B, \quad \text{where} \quad \mu(\mathscr{L}_A)=\mathscr{L}_B, \quad \text{each} \quad A\in\mathscr{P}_2.$$

It is clear that

$$\lambda=\delta^{-1}\mu\delta,$$

and so Lemmas 3.5 and 3.14 ensure $\lambda$ is a bijection of $\mathscr{P}_2$. We call $\lambda$ the *bijection of* $\mathscr{P}_2$ *associated with* $\phi$.

We show that $\lambda$ is *induced* by a bijection $h$ of $X$, that is

$$\lambda(A) = h(A),$$

for each $A \in \mathscr{P}_2$. Note here that not every bijection of $\mathscr{P}_2$ is induced, as shown in Example 3.15 below.

**Example 3.15.** Fix $A$ and $C$ in $\mathscr{P}_2$, $A \neq C$ and let $\lambda$ be a bijection of $\mathscr{P}_2$, which interchanges $A$ and $C$ and the identity otherwise. Choose $B \in \mathscr{P}_2$, $B = \{x, y\}$ such that $x \in A \backslash C$ and $y \in X \backslash (A \cup C)$. Note $A \cap B = \{x\}$ and $B \cap C = \Phi$. Suppose $\lambda$ is induced by $h \in \mathscr{G}_X$, then

$$h(x) = h(A \cap B) = h(A) \cap h(B) = \lambda(A) \cap \lambda(B) = C \cap B = \Phi.$$

Thus $\lambda$ is not induced. □

Observe that in the example above we had $\lambda$, a bijection of $\mathscr{P}_2$, such that

$$|A \cap B| \neq |\lambda(A) \cap \lambda(B)|,$$

for some $A, B$ in $\mathscr{P}_2$. This leads us to a criterion for a bijection $\lambda$ of $\mathscr{P}_2$ to be induced.

**Result 3.16.** *Let* $\lambda$ *be a bijection of* $\mathscr{P}_2$. *Then* $\lambda$ *is induced if and only if* $|A \cap B| = |\lambda(A) \cap \lambda(B)|$, *for every* $A, B \in \mathscr{P}_2$.

**Proof.** If $\lambda$ is induced by an $h \in \mathscr{G}_X$, then for every $A, B \in \mathscr{P}_2$, $|A \cap B| = |h(A \cap B)| = |h(A) \cap h(B)| = |\lambda(A) \cap \lambda(B)|$.

For the converse suppose that $\lambda$ is a bijection of $\mathscr{P}_2$ such that for every $A, B \in \mathscr{P}_2$

$$|A \cap B| = |\lambda(A) \cap \lambda(B)|. \tag{*}$$

We show that $\lambda$ is induced. This is done in the following three steps.

1. *Given* $x \in X$ *there exists a unique* $y \in X$ *such that for every* $A, B \in \mathscr{P}_2$ *with* $A \cap B = \{x\}$ *we have* $\lambda(A) \cap \lambda(B) = \{y\}$.

Take a pair $A, B$ in $\mathscr{P}_2$ with $A \cap B = \{x\}$, then by the assumption (*) $\lambda(A) \cap \lambda(B) = \{y\}$, for some $y \in X$.

Take any other pair $C, D$ in $\mathscr{P}_2$ with $|C \cap D| = 1$ and let $\mathscr{F} \subseteq \mathscr{P}_2$ be such that:

($\alpha$) for every distinct $F_1, F_2 \in \mathscr{F}, |F_1 \cap F_2| = 1$;

($\beta$) for any $F \in \mathscr{F}, |A \cap F| = |B \cap F| = |C \cap F| = |D \cap F| = 1$.

We show:

$$C \cap D = \{x\} \text{ iff there exists an } \mathscr{F} \text{ (as described above) with } |\mathscr{F}| = |X|.$$

Let $A \cup B \cup C \cup D = E$, then $|E| \leq 8$ and $|X \backslash E| = |X|$.

Assume firstly that $C \cap D = \{x\}$ and let $\mathscr{F} = \{\{x, y\} : y \in X \backslash E\}$. Then $\mathscr{F}$ satisfies ($\alpha$) and ($\beta$) and $|\mathscr{F}| = |X \backslash E| = |X|$.

For the converse assume $C \cap D = \{z\}$, $z \neq x$ and $\mathscr{F} \subseteq \mathscr{P}_2$ satisfies ($\alpha$) and ($\beta$). For each $F \in \mathscr{F}$ we have $|E \cap F| > 1$. (If not, then

$$|E \cap F| = |(A \cup B \cup C \cup D) \cap F|$$
$$= |(A \cap F) \cup (B \cap F) \cup (C \cap F) \cup (D \cap F)| \leqq 1.$$

Using condition ($\beta$) we conclude:

$$A \cap F = B \cap F = C \cap F = D \cap F = A \cap B = \{x\},$$

or $C \cap D = \{x\}$, a contradiction).

Define a map $\nu: \mathscr{F} \to \mathscr{P}(E)$, where $\mathscr{P}(E)$ is the power set of $E$, via $\nu(F) = E \cap F$, each $F \in \mathscr{F}$. We show $\nu$ is 1–1. Suppose $F_1, F_2 \in \mathscr{F}$ with $\nu(F_1) = \nu(F_2)$. Then

$$1 < |E \cap F_1| = |E \cap F_1 \cap F_2| \leqq |F_1 \cap F_2|,$$

so that $|F_1 \cap F_2| > 1$, thus $F_1 = F_2$ (condition ($\alpha$)). However $\mathscr{P}(E)$ is finite, so $|\mathscr{F}| \leqq |\mathscr{P}(E)| < \aleph_0$, or $|\mathscr{F}| < |X|$. We conclude $C \cap D = \{x\}$.

Observe now that the definition of the set $\mathscr{F}$ depends on the sets $A, B, C$ and $D$. We denote this dependence by $\mathscr{F} = \mathscr{F}(A, B, C, D)$. Hence $C \cap D = \{x\}$ iff $\exists\, \mathscr{F}(A, B, C, D)$ with $|\mathscr{F}(A, B, C, D)| = |X|$ iff $\exists\, \mathscr{F}(\lambda(A), \lambda(B), \lambda(C), \lambda(D))$ with $|\mathscr{F}(\lambda(A), \lambda(B), \lambda(C), \lambda(D))| = |X|$ (assumption (*))

$$\text{iff } \lambda(C) \cap \lambda(D) = \{y\}.$$

Now define a map

$$h: X \to X \text{ via } \{h(x)\} = \lambda(A) \cap \lambda(B), \text{ where } \{x\} = A \cap B, \text{ for } A, B \in \mathscr{P}_2 \text{ and each } x \in X.$$

2. *$h$ is a bijection of $X$.*

That $h$ is well-defined is the content of step 1. Observe that the bijection $\lambda^{-1}$ of $\mathscr{P}_2$ is associated with the automorphism $\phi^{-1}$. By considering $\phi^{-1}$ and $\lambda^{-1}$ instead of $\phi$ and $\lambda$ we define a map $k: X \to X$ via $\{k(x)\} = \lambda^{-1}(A) \cap \lambda^{-1}(B)$, where $\{x\} = A \cap B$, for $A, B \in \mathscr{P}_2$ and each $x \in X$. Then for each $x \in X$

$$\{kh(x)\} = k(\lambda(A) \cap \lambda(B)), \quad \text{where} \quad A \cap B = \{x\}$$
$$= \lambda^{-1}\lambda(A) \cap \lambda^{-1}\lambda(B)$$
$$= A \cap B$$
$$= \{x\}.$$

Similarly we can show $hk(x) = x$, for each $x \in X$. Thus $k$ is the inverse of $h$, and so $h$ is a bijection of $X$.

3. *$\lambda$ is induced by $h$.*

To show $\lambda$ is induced by $h$ we must show $\lambda(A)=h(A)$ for each $A\in\mathscr{P}_2$. From the definition of $h$ we at once have $h(A)\subseteq\lambda(A)$. Take $y\in\lambda(A)$ and let $B\in\mathscr{P}_2$ be such that $\lambda(A)\cap\lambda(B)=\{y\}$. Then $A\cap B=\{x\}$, some $x\in A$, so $h(x)=y$ and $h(A)\supseteq\lambda(A)$. The equality follows. □

**Remark 3.17.** In view of Result 3.16 our aim now is to show that for every $A,B\in\mathscr{P}_2$

$$|A\cap B|=|\lambda(A)\cap\lambda(B)| \tag{*}$$

where $\lambda$ is the bijection of $\mathscr{P}_2$ associated with $\phi$ as defined prior to Example 3.15. Observe that (*) is equivalent to the statement

$$|A\cap B|=1 \quad \text{if and only if} \quad |\lambda(A)\cap\lambda(B)|=1, \tag{**}$$

for each $A,B\in\mathscr{P}_2$.

Indeed (*) certainly implies (**). We show the reverse implication.

Assume (**) holds. If $|A\cap B|=2$, that is $A=B$, then $\lambda(A)=\lambda(B)$, and so $|\lambda(A)\cap\lambda(B)|=2$. If $|A\cap B|=1$, then by our assumption $|\lambda(A)\cap\lambda(B)|=1$. The case $|A\cap B|=0$ follows by elimination. □

The next lemma illustrates the fact that the existence of a transformation $f$ in $S$ which is not 1–1 provides an extensive variety of elements in $\pi(S)$.

**Lemma 3.18.** *Given $B_1,B_2\subseteq X$ with $B_1\cap B_2=\Phi$ and $|B_1|=|B_2|=3$ there exists an $\mathscr{A}\in\pi(S)$ with $B_1\subseteq A_1\in\mathscr{A}$, $B_2\subseteq A_2\in\mathscr{A}$.*

**Proof.** Suppose that there exists a transformation $f$ in $S$ such that:

$$C_1,C_2\in\pi(f) \quad \text{and} \quad |C_1|,|C_2|\geqq 3.$$

Choose a bijection $p$ of $X$ with

$$B_1\subseteq p(C_1) \quad \text{and} \quad B_2\subseteq p(C_2).$$

Certainly $pfp^{-1}\in S$. Let

$$\mathscr{A}=\pi(pfp^{-1})(=p(\pi(f))), A_1=p(C_1) \quad \text{and} \quad A_2=p(C_2),$$

then $A_1,A_2\in\mathscr{A}\in\pi(S)$ and $B_1\subseteq A_1$, $B_2\subseteq A_2$.

To construct such an $f$ as used above we first show that there exists a $g$ in $S$ such that

$$g(x_1)=g(x_2)=g(x_3)=x_1, \quad \text{for some distinct } x_1,x_2,x_3\in X.$$

Choose a $t$ in $S$ not 1–1 and let $x, x_1, x_2 \in X$ be such that

$$t(x_1) = t(x_2) = x.$$

We assume $x = x_1$ (for if $x \neq x_1$ choose $s \in S$ such that $s(x) = x_1$ (Result 1.3) and replace $t$ by $st$). Let $x_4 \in R(t) \backslash \{t^{-1}(x_1)\}$ (note: $R(t) \backslash \{t^{-1}(x_1)\} \neq \Phi$, else $t^2$ is a constant in $S$) and let $x_3 \in X$ such that $t(x_3) = x_4$. Then

$$g = (x_2, x_4)t(x_2, x_4)t$$

is such that $g(x_1) = g(x_2) = g(x_3) = x_1$.

To accomplish the construction of the above $f$ choose distinct $z_1, z_2, z_3$ in $R(g) \backslash \{g^{-1}(x_1)\}$ together with $y_1, y_2, y_3 \in X$ such that $g(y_i) = z_i$, $i = 1, 2, 3$. Let

$$k = (x_1, z_1)(x_2, z_2)(x_3, z_3) \in \mathscr{G}_X \quad \text{and} \quad f = kgk^{-1}g.$$

Let $kg(z_1) = z_4$. Then

$$f(x_1) = f(x_2) = f(x_3) = z_4$$

and

$$f(y_1) = f(y_2) = f(y_3) = z_1.$$

Now $z_1 \neq z_4$ (else $kg(z_1) = z_1$ implies $g(z_1) = x_1$ or $z_1 \in g^{-1}(x_1)$, contrary to the choice of $z_1$). Let $C_1 = f^{-1}(z_1)$, $C_2 = f^{-1}(z_4)$. Then $|C_1|, |C_2| \geqq 3$ and $C_1, C_2 \in \pi(f)$ as required. □

**Remark 3.19.** It easily follows from Lemma 3.18 that

$$\mathscr{L}_A \cap \mathscr{L}_B \neq \Phi,$$

for every $A, B \in \mathscr{P}_2$. □

**Lemma 3.20.** *Let $A, B \in \mathscr{P}_2$, $A \neq B$. Then $|A \cap B| = 1$ iff there is a $C$ in $\mathscr{P}_2$, $C \neq A$ or $B$, such that $\mathscr{L}_A \cap \mathscr{L}_B \subseteq \mathscr{L}_C$.*

**Proof.** Assume $|A \cap B| = 1$ and let $C = (A \cup B) \backslash (A \cap B)$. For each $l$ in $\mathscr{L}_A \cap \mathscr{L}_B$ (Remark 3.19):

$$l(A) = l(A \cap B) = l(B) = l(A \cup B) = l(C),$$

so that $l \in \mathscr{L}_C$ and $\mathscr{L}_A \cap \mathscr{L}_B \subseteq \mathscr{L}_C$.

For the converse suppose $A \cap B = \Phi$ and $C \in \mathscr{P}_2$ is distinct from $A$ and $B$. Let $C = \{c_1, c_2\}$. Since $|A \cap C| \leqq 1$ and $|B \cap C| \leqq 1$ assume without loss of generality that $c_1 \in X \backslash B$ and $c_2 \in X \backslash A$. Choose

$$\mathscr{A} \in \pi(S) \quad \text{with} \quad A \cup \{c_1\} \subseteq A_1 \in \mathscr{A}, B \cup \{c_2\} \subseteq A_2 \in \mathscr{A} \quad \text{and} \quad A_1 \neq A_2 \quad \text{(Lemma 3.18).}$$

If $l \in S$ has $\pi(l) = \mathscr{A}$, then $l \in (\mathscr{L}_A \cap \mathscr{L}_B) \backslash \mathscr{L}_C$.
This confirms that $|A \cap B| = 1$. □

**Lemma 3.21.** *Let* $A, B$ *and* $C$ *be distinct elements of* $\mathscr{P}_2$. *Then*

$$\mathscr{L}_A \cap \mathscr{L}_B \subseteq \mathscr{L}_C \quad \textit{iff} \quad \mathscr{L}_{\lambda(A)} \cap \mathscr{L}_{\lambda(B)} \subseteq \mathscr{L}_{\lambda(C)}.$$

**Proof.** Observe that $\mathscr{L}_A \cap \mathscr{L}_B \neq \Phi$ (Remark 3.19) and

$$\mathscr{L}_A \cap \mathscr{L}_B \subseteq \mathscr{L}_C \quad \text{iff} \quad \phi(\mathscr{L}_A \cap \mathscr{L}_B) \subseteq \phi(\mathscr{L}_C).$$

Now

$$\phi(\mathscr{L}_A \cap \mathscr{L}_B) = \phi(\mathscr{L}_A) \cap \phi(\mathscr{L}_B) = \mathscr{L}_{\lambda(A)} \cap \mathscr{L}_{\lambda(B)},$$

by the definition of $\lambda$. Also $\phi(\mathscr{L}_C) = \mathscr{L}_{\lambda(C)}$, so that

$$\phi(\mathscr{L}_A \cap \mathscr{L}_B) \subseteq \phi(\mathscr{L}_C) \quad \text{iff} \quad \mathscr{L}_{\lambda(A)} \cap \mathscr{L}_{\lambda(B)} \subseteq \mathscr{L}_{\lambda(C)},$$

and the desired equivalence is established. □

**Result 3.22.** *Given* $A$ *and* $B$ *in* $\mathscr{P}_2$,

$$|A \cap B| = 1 \quad \textit{if and only if} \quad |\lambda(A) \cap \lambda(B)| = 1.$$

**Proof.** We have:
$|A \cap B| = 1$ iff $\exists\, C \neq A$ or $B$ such that $\mathscr{L}_A \cap \mathscr{L}_B \subseteq \mathscr{L}_C$ (Lemma 3.20)
iff $\exists\, \lambda(C) \neq \lambda(A)$ or $\lambda(B)$ such that $\mathscr{L}_{\lambda(A)} \cap \mathscr{L}_{\lambda(B)} \subseteq \mathscr{L}_{\lambda(C)}$
($\lambda$ is a bijection and Lemma 3.21)

iff $|\lambda(A) \cap \lambda(B)| = 1$ (Lemma 3.20 again). □

From Results 3.16, 3.22 and Remark 3.17 we readily deduce

**Result 3.23.** $\lambda$ *is induced by a bijection of* $X$. □

Now we are ready to show that a constant-free $\mathscr{G}_X$-normal semigroup containing a transformation which is not 1–1 (that is a semigroup of Type 3), possesses only inner automorphisms.

**Result 3.24.** *Let* $S$ *be a constant-free* $\mathscr{G}_X$*-normal semigroup containing a transformation which is not* 1–1. *Then each automorphism* $\phi$ *of* $S$ *is inner, that is for some* $h \in \mathscr{G}_X$

$$\phi(f) = hfh^{-1}, \quad \textit{for each} \quad f \in S.$$

**Proof.** Let $h$ be the bijection which induces $\lambda$ (Result 3.23). In what follows we use the fact that for any distinct $x_1, x_2 \in X$

$$\phi(\mathscr{L}_{\{x_1,x_2\}}) = \mathscr{L}_{\lambda(\{x_1,x_2\})} = \mathscr{L}_{\{h(x_1),h(x_2)\}}.$$

Take an arbitrary $f \in S$, $x \in X$ and let $y \in X$ with $f(x) \neq f(y)$ (that is $f \notin \mathscr{L}_{\{x,y\}}$). Then

$$\phi(\mathscr{L}_{\{f(x),f(y)\}}) = \mathscr{L}_{\{hf(x),hf(y)\}}.$$

Let $\phi(g) \in \phi(\mathscr{L}_{\{f(x),f(y)\}})$. Then $g \in \mathscr{L}_{\{f(x),f(y)\}}$ or $gf(x) = gf(y)$. It follows that $\phi(gf) \in \mathscr{L}_{\{h(x),h(y)\}}$, hence

$$\phi(g)\phi(f)h(x) = \phi(g)\phi(f)h(y).$$

Note that $f \notin \mathscr{L}_{\{x,y\}}$ implies $\phi(f) \notin \phi(\mathscr{L}_{\{x,y\}})$ or $\phi(f) \notin \mathscr{L}_{\{h(x),h(y)\}}$, that is

$$\phi(f)h(x) \neq \phi(f)h(y).$$

Thus $\phi(g) \in \mathscr{L}_{\{\phi(f)h(x),\phi(f)h(y)\}}$ and we conclude

$$\phi(\mathscr{L}_{\{f(x),f(y)\}}) \subseteq \mathscr{L}_{\{\phi(f)h(x),\phi(f)h(y)\}}.$$

This in turn implies

$$\mathscr{L}_{\{hf(x),hf(y)\}} \subseteq \mathscr{L}_{\{\phi(f)h(x),\phi(f)h(y)\}}.$$

Hence $\{hf(x), hf(y)\} = \{\phi(f)h(x), \phi(f)h(y)\}$ (Lemma 3.4).

Since the choice of $y$ is independent of $x$ (providing $y \neq x$) we conclude

$$\phi(f)h(x) = hf(x), \quad \text{for each} \quad x \in X,$$

so that

$$\phi(f) = hfh^{-1}, \quad \text{as required.}$$

□

**Conclusion**

We return to

**Theorem 1.1.** *Every automorphism of a $\mathscr{G}_X$-normal semigroup $S$ is inner.*

**Proof.** If $S$ is a semigroup of Type 1, that is, contains a constant transformation, then we appeal to Sullivan [**12,** Theorem 1].

If $S$ is a semigroup of Type 2, that is, a semigroup of 1–1 transformations, the result is given in 2.14 and 1.4.

If $S$ is a semigroup of Type 3, that is, a semigroup containing a transformation which is not 1–1, then the result is given in 3.24.

This completes the proof of Theorem 1.1. □

**Remark.** If $X$ is a *finite* set and $S$ is a semigroup of transformations of $X$ which is not contained in $\mathcal{G}_X$, then $S$ is $\mathcal{G}_X$-normal if and only if all automorphisms of $S$ are inner **[13]**.

However, this is not the case for an infinite set $X$. While, as we showed, every $\mathcal{G}_X$-normal semigroup $S$ has only inner automorphisms, there are examples **[5]** of semigroups which are neither subsemigroups of $\mathcal{G}_X$, nor $\mathcal{G}_X$-normal, yet have only inner automorphisms.

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