Proceedings of the
44th IEEE Conference on Decision and Control, and
the European Control Conference 2005
Seville, Spain, December 12-15, 2005

Average consensus problems in networks of agents with delayed communications

Pierre-Alexandre Bliman and Giancarlo Ferrari-Trecate

Abstract— **The present paper is devoted to the study of average consensus problems for undirected networks of dynamic agents having communication delays. The accent is put here on the study of the time-delays influence: both constant and time-varying delays are considered, as well as uniform and non uniform repartitions of the delays in the network. The main results provide sufficient conditions (also necessary in most cases) for existence of average consensus under bounded, but otherwise unknown, communication delays. Simulations are provided that show adequation with these results.**

I. INTRODUCTION

In the last few years, the study of multi-agent systems has received a major attention within the control community. Driving applications include unmanned aerial vehicles, satellite clusters, automated highways and mobile robots. In all cases the aim is to control a group of agents connected through a wireless network. More precisely, rather than stabilizing the movement of each agent around a given set point, the goal is to understand how to make the agents coordinate and self-organize in moving formations. This problem becomes even more challenging under partial communication protocols, i.e. when each agent exchanges information only with few others.

Many works in the literature focused on conditions for guaranteeing that the agents asymptotically reach a *consensus*, i.e. they agree upon a common value of a quantity of interest [1], [2], [3], [4], [5], [6]. As an example, in a network of moving vehicles a form of consensus is represented by alignment, that happens when all vehicles asymptotically move with the same velocity. In the aforementioned papers, consensus problems have been studied under a variety of assumptions on the network topology (fixed/switching), the communication protocol (bidirectional or not), additional performance requirements (e.g. collision avoidance, obstacle avoidance, cohesion), and the control scheme adopted (also termed *consensus protocol*). So far, just few works considered consensus problems when communication is affected by time-delays. Some results for discrete-time agent models are given in [7] and [8]. Two different consensus protocols for continuous-time agent dynamics have been investigated in [9] and [4]. More specifically, assuming that agents behave like integrators and that communication delays are constant in time and uniform (i.e. they have the same value in all channels), an analysis of the maximal delay that can be tolerated without compromising consensus has been performed in [9] and [4]. In particular, the protocol adopted in [4] is capable to guarantee *average consensus* (i.e. the state of each agent converges, asymptotically, to the average of the initial agent states rather than to an arbitrary constant) and the authors provide an explicit formula for the largest transmission delay.

In the present work we generalize the results of [4] in various ways. First, we consider uniform and unknown *time-varying* delays and provide upper bounds to the maximal delay that does not prevent from achieving average consensus. Second, we derive similar conditions for networks affected by *non uniform*, constant or time-varying delays. In the case of non uniform and constant delays, we also show that if the communication delay between two agents is equal to zero, then average consensus may achieved irrespectively of the magnitude of all others delays.

The network of agents is modeled in the framework of Partial difference Equations (PdEs) introduced in [10] and used in [6] analyzing the property of various linear and nonlinear consensus protocols. PdEs are models that mimic Partial Differential Equations (PDEs) and provide a mathematical description of the agents network where "spatial" interactions (due to the network structure) and "temporal" ones are kept separated and described by operators acting either on space or time. Section II provides an introduction to PdEs. The main results are presented through Sections III-VI and three simulation experiments are discussed in Section VII.

II. TOOLS FOR FUNCTIONS ON GRAPHS

The communication network is modeled through an undirected weighted graph G defined by a set $\mathscr{N} = \{1, 2, \ldots, N\}$ of *nodes* and a set $\mathscr{E} \subset \mathscr{N} \times \mathscr{N}$ of *edges*. Each node represents an agent and an edge (x, y) means that the agents x and y share the information about their states. Agents linked by an arc are called *neighbors*. The neighboring relation is denoted with $x \sim y$ and we assume that $x \sim x$ always holds. Two nodes x and y are connected by a *path* if there is a finite sequence $x_0 = x, x_1, \ldots, x_n = y$ such that $x_{i-1} \sim x_i$. The graph G is *connected* when each pair of nodes $(x, y) \in G \times G$ is connected by a path and *complete* if $\mathscr{E} = G \times G$.

Weights on the communication links are defined by a function $\omega : \mathscr{N} \times \mathscr{N} \to \mathbb{R}^+$ with the properties

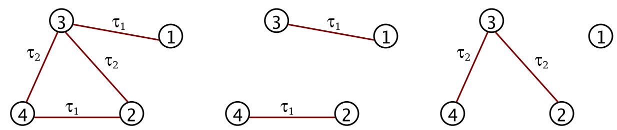
$$\omega(x, y) = \omega(y, x) \tag{1a}$$

$$\omega(x, y) > 0 \Leftrightarrow x \sim y \tag{1b}$$

Corresponding author: P.-A. Bliman

P.-A. Bliman and G. Ferrari-Trecate are with the Institut National de Recherche en Informatique et en Automatique, Domaine de Voluceau, B.P.105, 78153, Le Chesnay Cedex, France. Email: {Pierre-Alexandre.Bliman, Giancarlo.Ferrari-Trecate}@inria.fr. Tel: +33 (0)1 39 63 59 21, Fax: +33 (0)1 39 63 55 68.

This work has been partially done in the framework of the HYCON Network of Excellence, contract number FP6-IST-511368



(a) Graph G. (b) Subgraph G_1. (c) Subgraph G_2.

Fig. 1. A graph and the subgraphs associated to delays τ_1 and τ_2

Time-varying delays in communications, are elements of the set, denoted $\mathscr{D}$, of piecewise continuous functions $\mathbb{R}_+ \to \mathbb{R}_+$. A delay is associated to each edge through an onto function $T : \mathscr{E} \to \mathscr{D}$ verifying $T(x,y) = T(y,x)$. The last equality amounts to consider delays that are symmetric, i.e. the lags in transmissions from x to y and from y to x do coincide. We denote r the number of independent time-delays affecting the communication links, and $\mathscr{I} = \{1,\ldots,r\}$. By construction, the bound $r \leq \frac{N(N-1)}{2}$ is valid.

Agents linked with the same delay $\tau_i(\cdot)$, define a subgraph $G_i = (\mathscr{N}, T^{-1}(\tau_i))$ with associated weights

$$\omega_i(x,y) = \begin{cases} \omega(x,y) & \text{if } T(x,y) = \tau_i \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

An example is reported in Figure 1. We highlight that the subgraphs G_i may be disconnected even if G is connected. Moreover, as shown in Figure 1(c), some nodes can be isolated.

We consider now vector functions $f : \mathscr{N} \to \mathbb{R}^d$ defined over a graph G. For instance, $f(x)$ may represent the position or the velocity of the agent x at a fixed time-instant. Following [10], the *partial derivative* of f is defined as

$$\partial_y f(x) \doteq f(y) - f(x) \tag{3}$$

and enjoys the following basic properties:

$$\partial_y f(x) = -\partial_x f(y) \tag{4a}$$
$$\partial_x f(x) = 0 \tag{4b}$$
$$\partial_y^2 f(x) = \partial_y f(y) - \partial_y f(x) = -\partial_y f(x). \tag{4c}$$

The integral and average of f are defined, respectively, as

$$\int_G f \; dx \doteq \sum_{x \in \mathscr{N}} f(x), \quad \langle f \rangle \doteq \frac{1}{N} \int_G f \; dx. \tag{5}$$

Note that, in (5), "dx" just indicates the integration variable. The Laplacian of f is given by

$$\Delta f(x) \doteq -\sum_{y \sim x} \omega(x,y) \partial_y^2 f(x) = +\sum_{y \sim x} \omega(x,y) \partial_y f(x). \tag{6}$$

where the last identity follows from (4c). In an equivalent way, the Laplacian can be written as

$$\Delta f(x) = \int_G \omega(x,y) \partial_y f(x) \; dy. \tag{7}$$

The Laplacian operator associated to a subgraph G_i is

$$\Delta_i f(x) \doteq \int_G \omega_i(x,y) \partial_y f(x) \; dy. \tag{8}$$

Since the subsets $\{T^{-1}(\tau_i)\}_{i \in \mathscr{I}}$ provide a partition of $\mathscr{E}$, it is immediate to verify that

$$\omega(x,y) = \sum_{i \in \mathscr{I}} \omega_i(x,y) \quad \text{and} \quad \Delta f = \sum_{i \in \mathscr{I}} \Delta_i f. \tag{9}$$

In the sequel we summarize the main properties of the Laplacian operator stated in [10]. The driving idea is to mimic functional analysis tools for studying the classic Laplacian defined on Sobolev spaces, (see [10] and [6] for further details).

We denote with $L^2(G|\mathbb{R}^d)$ the Hilbert space composed by all functions $f : \mathscr{N} \to \mathbb{R}^d$ equipped with the scalar product and the norm

$$(f,g)_{L^2} = \int_G f \cdot g, \quad \|f\|_{L^2}^2 = \int_G \|f\|^2 \tag{10}$$

where $\cdot$ and $\|\cdot\|$ represent the scalar product and the euclidean norm on $\mathbb{R}^d$, respectively. Let $H^1(G|\mathbb{R}^d)$ be the space collecting all functions in $L^2(G|\mathbb{R}^d)$ with zero average. We will use the shorthand notation L^2 and H^1 when there is no ambiguity on the underlying domain and range of the functions. If G is connected, H^1 is an Hilbert space [10] endowed with scalar product

$$(f,g)_{H^1} = \int_G \int_G \omega(x,y) \partial_y f(x) \cdot \partial_y g(x) \; dxdy. \tag{11}$$

Apparently, $H^1_\perp$ is the space of *constant* functions on G and $\dim(H^1_\perp) = d$. Moreover, the decomposition $L^2 = H^1 \oplus H^1_\perp$ is direct. The L^2 orthogonal projection operators on H^1 and $H^1_\perp$ will be denoted as P_{H^1} and $P_{H^1_\perp}$, respectively.

The eigenstructure of the Laplacian is completely characterized by the next Theorem proved in [10].

Theorem 1: Let G be a connected graph. Then,

1) the operator $\Delta : H^1 \to H^1$ is symmetric, it has $(N-1)d$ strictly negative eigenvalues[1] and the corresponding eigenfunctions form a basis for H^1;
2) for $f \in L^2$, $\Delta f = 0$ if and only if $f \in H^1_\perp$.

Theorem 1 highlights that the Laplacian is invertible on the subspace H^1. Note that when Δ is defined on L^2, it has Nd eigenvalues. In particular, in view of the decomposition $L^2 = H^1 \oplus H^1_\perp$, $(N-1)d$ eigenvalues are those considered in point (1) of Theorem 1 and the remaining d are zeros (this property follows directly from point (2) of Theorem 1).

The next theorem characterizes the eigenvalues of the operators Δ_i.

Theorem 2: The operators $\Delta_i : H^1 \to H^1$, $i \in \mathscr{I}$, are symmetric and negative-semidefinite.

Proof: The proof is reported in [11]. ■

We stress that all the spaces so far introduced are finite dimensional. This can be seen by noting that the lifting operator $\mathscr{L} : L^2(G|\mathbb{R}^d) \to \mathbb{R}^{Nd}$ defined as

$$\mathscr{L}(f) \doteq \left[\begin{array}{ccc} f(1)^T & \cdots & f(N)^T \end{array} \right]^T \tag{12}$$

is an isometry (i.e. bijective and $\|f\|_{L^2} = \|\mathscr{L}(f)\|$) so showing that L^2 is isomorphic to $\mathbb{R}^{Nd}$. Roughly speaking, this means that all concepts introduced in the present section could be re-written in terms of vector and matrices over $\mathbb{R}^{Nd}$.

[1] Such eigenvalues will be termed "the eigenvalues of Δ on H^1".

Definition 1: Consider the linear operator $A : L^2(G|\mathbb{R}^d) \to L^2(G|\mathbb{R}^d)$. Its matrix representation is the unique matrix $\mathscr{M}(A) \in \mathbb{R}^{Nd\times Nd}$ that verifies $\mathscr{L}(Af) = \mathscr{M}(A)\mathscr{L}(f)$, $\forall f \in L^2(G|\mathbb{R}^d)$.

The matrix representation of an operator can be used, for instance, for computing the eigenvalues of A, since they coincide with the eigenvalues of $\mathscr{M}(A)$, up to their multiplicity. The operator Δ is strongly related to the Laplacian matrix of the graph G, defined next (see also [12]). In the sequel, the (x,y) element of a matrix B will be denoted with $(B)_{x,y}$.

Definition 2: For a graph G, the adjacency matrix $A(G)$ is an $N\times N$ matrix with entries

$$(A(G))_{x,y} \doteq \begin{cases} \omega(x,y) & \text{if } x\sim y \text{ and } x\neq y \\ 0 & \text{otherwise} \end{cases} \tag{13}$$

The valency matrix $V(G)$ is an $N\times N$ diagonal matrix with entries $(V(G))_{x,x} \doteq \sum_{y\sim x}\omega(x,y)$ and the Laplacian matrix is $L(G) \doteq A(G) - V(G)$.

It is easy to verify that $\mathscr{L}(\Delta f(x)) = (L(G)\otimes I_d)\mathscr{L}(f)$, where $\otimes$ is the Kronecker product and I_d the identity matrix of order d Then, $\mathscr{M}(\Delta) = L(G)\otimes I_d$.

III. Delayed multi-agent models and PdEs

Let $v(x,t)\in\mathbb{R}^d$ and $u(x,t)\in\mathbb{R}^d$, $x\in\mathscr{N}$, $t\in\mathbb{R}^+$ denote the state and control input of agent x at time t, respectively. When each agent behaves as an integrator, the collective dynamics is described by the (open-loop) equation $\dot{v}(x,t) = u(x,t)$, where the dot indicates time-derivative. In this paper we consider *delayed Laplacian protocols* of the type $u = \sum_{i\in\mathscr{I}}\Delta_i v(x,t-\tau_i(t))$, thus yielding the closed-loop collective dynamics

$$\dot{v} = \sum_{i\in\mathscr{I}}\Delta_i v(x,t-\tau_i(t)). \tag{14}$$

Formula (14) defines a *time-delay Partial difference Equation (PdE)* (see [6] for a general definition of PdEs) whose solution depends on the initial condition. As for linear time-delay systems, if all delays are bounded by a constant $\bar{\tau}$, the latter may be given as a function $\tilde{v}\in\mathscr{C}([-\bar{\tau},0],L^2)$.

As shown in [6], PdEs can be always recast into Ordinary Differential Equations by using the lifting operator (12). Then, it is not surprising that linear time-delay PdEs inherit all the properties of linear time-delay systems. As an example, if all delays are constant in time, the characteristic equation associated to (14), is

$$E(s) \doteq sI - \sum_{i\in\mathscr{I}} e^{-s\tau_i}\Delta_i = 0, \quad s\in\mathbb{C} \tag{15}$$

where I is the identity operator on L^2. Then, many properties of the network of agents can be characterized in terms of the *poles* of (14), i.e. the roots of (15). We outline that if the delays are constant, model (14) coincides with the network dynamics considered in Section 10 of [4].

The main goal of the present work is to investigate when (14) guarantees average consensus.

Definition 3: The network dynamics achieves average consensus if $v\to\langle v(\cdot,0)\rangle$ as $t\to+\infty$.

In absence of delays, u results in the *Laplacian protocol*, and the PdE (14) reduces to the *heat equation*

$$\dot{v} = \Delta v \quad v(\cdot,0) = \tilde{v}\in L^2. \tag{16}$$

The consensus properties of Laplacian protocols have been analyzed in various works. In particular, A. Jadbabaie *et al.* [1] proved that the Laplacian protocol is able to guarantee average consensus under various assumption on the network topology. A formal analysis of the PdE (16) has been carried out in [6], where it has been also shown that the Laplacian protocol can guarantee consensus even when the agent dynamics are perturbed by exponentially decreasing errors and/or an agent acts as the leader of the group.

In order to highlight the rationale we will use for analyzing the PdE (14), let us summarize the main results of [6] for the collective dynamics (16). Decomposing the state as $v(\cdot,t) = v_1(\cdot,t)+\bar{v}(\cdot,t)$, $v_1(\cdot,t)\in H_1$, $\bar{v} = \langle v(\cdot,t)\rangle\in H_\perp^1$, one can show, through a simple variational technique, that the velocity components fulfill the dynamics

$$\dot{\bar{v}} = 0 \tag{17a}$$

$$\dot{v}_1 = \Delta v_1 \tag{17b}$$

thus proving that the spaces H_1 and $H_\perp^1$ are positively invariant for (16). In particular, equation (17a) highlights that the average velocity of the agents is constant in time. Then an *exponentially stable* average consensus is achieved if the origin of (17b) is exponentially stable, a fact that can be easily shown by exploiting the characterization of the eigenvalues of Δ on H^1 given in Theorem 1. In [6] it is also shown that average consensus can be intuitively expected on the basis of the physical analogy between (16) and the classic heat equation.

For the delayed model (14), we will adopt a similar argument. The next Lemma provides the dynamics of the v_1 and $\bar{v}$ components.

Lemma 1: Given $\tilde{v}\in\mathscr{C}([-\bar{\tau},0],L^2)$, the function v is solution to the PdE (14) if and only if v_1 and $\bar{v}$, are solutions to the PdEs

$$\Sigma_1:\ \dot{v}_1 = \sum_{i\in\mathscr{I}}\Delta_i v_1(x,t-\tau_i(t)), \qquad \bar{\Sigma}:\ \dot{\bar{v}} = 0 \tag{18}$$

equipped with the initial conditions $v_1(\cdot,t)|_{[-\bar{\tau},0]} = P_{H^1}\tilde{v}(\cdot,t)$, $\bar{v}(t)|_{[-\bar{\tau},0]} = P_{H_\perp^1}\tilde{v}(\cdot,t)$ for $t\in[-\bar{\tau},0]$.

Proof: To prove the result, we use a variational argument by testing each side of (14) against all $c\in H_\perp^1$. This means that we take the integrals

$$\int_G c\cdot\dot{v}\ dx = \int_G c\cdot\sum_{i\in\mathscr{I}}\Delta_i v(x,t-\tau_i(t))\ dx. \tag{19}$$

By using (8), the right side of (19) can be written as $\sum_{i\in\mathscr{I}} S_i$, where

$$S_i = \int_G c\cdot\int_G \omega_i(x,y)\partial_y v(x,t-\tau_i(t))\ dydx. \tag{20}$$

From (1a) and (4a), the functions $g_i(x,y) = \omega_i(x,y)\partial_y v(x,t-\tau_i(t))$ are antisymmetric, i.e. $g_i(x,y) = -g_i(y,x)$. Then, each integral S_i can be expanded into sums containing only terms of the type $c\cdot(g_i(x,y)+g_i(y,x))$ that are all identically equal to zero. The fact that $\int_G c \cdot \dot{v}\ dx = 0$, $\forall c \in H^1_\perp$ corresponds to the condition $P_{H^1_\perp} \dot{v} = 0$, or, equivalently, to $\dot{\bar{v}} = 0$, thus obtaining the dynamics $\bar{\Sigma}$. From (14) we have

$$\dot{v}_1 + \dot{\bar{v}} = \sum_{i \in \mathscr{I}} \Delta_i v_1(x, t - \tau_i(t)) + \sum_{i \in \mathscr{I}} \Delta_i \bar{v}(x, t - \tau_i(t)) \quad (21)$$

and the dynamics Σ_1 follows from $\dot{\bar{v}} = 0$ and $\Delta_i \bar{v} = 0$. ■

Lemma 1 shows that the spaces H^1 and $H^1_\perp$ are positively invariant for the PdE (14). Moreover, as for (16), the average state $\bar{v}$ is constant in time and equal to $\langle \tilde{v}(\cdot, 0) \rangle$. Then, the problem of checking average consensus is reduced to proving that $v_1 \to 0$ as $t \to \infty$.

We say that average consensus is *globally exponentially*, resp. *asymptotically stable*, if the zero solution to Σ_1 enjoys the same property, i.e. if it is exponentially, resp. asymptotically stable.

For subsequent use, we introduce the operator norm $\|\Delta\| \doteq \max_{u \in H^1} \frac{(f, \Delta f)}{(f, f)} = |\lambda_{min}| = -\lambda_{min} > 0$ where λ_{min} is the minimal eigenvalue of the Laplacian on H^1. Similarly, by recalling that Δ is invertible on H^1, one has $\|\Delta^{-1}\|^{-1} = |\lambda_{max}| = -\lambda_{max} > 0$.

IV. The Case of Uniform Delays

In this section, we analyze the stability properties of the dynamics Σ_1 when the delay is *uniform* in the network, i.e. when $\mathscr{I}$ is a singleton. We start with the simpler case of time-invariant delays, considered also in [4]. The results of the next Theorem coincide with those of Theorem 10 in [4], but are proved through a different argument, i.e. the diagonalization of the Laplacian operator on H^1.

Theorem 3 (Constant delay): The zero solution is a globally exponentially stable solution to the PdE

$$\dot{v}_1(x,t) = \Delta v_1(x, t - \tau), \quad P_{H^1_\perp} v_1(\cdot, t]|_{[-\bar{\tau}, 0]} \equiv 0 \quad (22)$$

for all possible $0 \leq \tau \leq \bar{\tau}$, *if and only if*

$$\bar{\tau} < \frac{\pi}{2\|\Delta\|} \ . \quad (23)$$

Proof: In view of Theorem 1, the Laplacian can be diagonalized on H^1. Let $\{\psi_i\}_{i=1}^{(N-1)d}$ be an orthonormal set of eigenfunctions of Δ forming a basis for H^1 and associated to the eigenvalues $\{\lambda_i\}_{i=1}^{(N-1)d}$. Then $v_1(x,t) = \sum_{i=1}^{(N-1)d} \alpha_i(t) \psi_i(x)$ for suitable functions $\alpha_i : \mathbb{R}^+ \to \mathbb{R}$. By testing each side of (22) against ψ_j we form the integrals

$$\int_G \left(\sum_{i=1}^{(N-1)d} \dot{\alpha}_i(t) \psi_i(x) \right) \cdot \psi_j(x)\ dx =$$

$$= \int_G \left(\sum_{i=1}^{(N-1)d} \alpha_i(t - \tau) \Delta \psi_i(x) \right) \cdot \psi_j(x)\ dx \quad (24)$$

By Theorem 1, formula (24) reduces to

$$\dot{\alpha}_j(t) = \lambda_j \alpha_j(t - \tau) \quad (25)$$

System (25) is a first-order linear time-delay system. Since $\lambda_j < 0$, according to [13, Theorem A.5], system (25) is exponentially stable if and only if $\tau < \frac{\pi}{2|\lambda_j|}$. Then, the PdE (22) is exponentially stable if and only if all systems (25), for $j = 1, \ldots, (N-1)d$ are exponentially stable, i.e. if (23) holds. ■

Remark 1: For $h \geq 0$, it may be of interest to quantify the largest delay $\bar{\tau}_h$ for which an exponential decay rate h is guaranteed for the solutions to (22). Simple calculations detailed in [11] reveal that $\bar{\tau}_h$ is given by

$$\bar{\tau}_h \doteq \min \left\{ \tau \geq 0 \ : \ \|\Delta\| e^{h\tau} \cos\left(\tau \sqrt{\|\Delta\|^2 e^{2h\tau} - h^2} \right) = h \right\} \ .$$

The map $h \mapsto \bar{\tau}_h$ is decreasing, with $\bar{\tau}_0 = \pi / 2\|\Delta\|$, $\bar{\tau}_{\|\Delta\|} = 0$.

We consider now the case of a single *time-varying* delay.

Theorem 4 (Time-varying delay): The zero solution is a globally exponentially stable solution to the PdE

$$\dot{v}_1(x,t) = \Delta v_1(x, t - \tau(t)), \quad P_{H^1_\perp} v_1(\cdot, t]|_{[-\bar{\tau}, 0]} \equiv 0 \quad (26)$$

for all piecewise continuous delays $\tau(t)$ verifying $0 \leq \tau(t) \leq \bar{\tau}$, *if and only if*

$$\bar{\tau} < \frac{3}{2\|\Delta\|} \ . \quad (27)$$

Proof: As in the proof of Theorem 3, diagonalization of the Laplacian on H^1 leads to the study of the first-order systems $\dot{\alpha}_i = \lambda_i \alpha_i(t - \tau(t))$, for any eigenvalue λ_i of Δ on H^1.

The conclusion is then deduced from a classical result initially published in [14] and [15], (see also [13, p. 164] and the references therein). ■

If the nominal collective model is the PdE (16), Theorems 3 and 4 characterize the robustness of average consensus with respect to different delay models. In particular, the bounds given in Theorems (23) and (27) do not depend upon the precise structure of the communication network but only upon the magnitude of $\|\Delta\|$. In other words, by interpreting G as the "spatial" domain of the PdEs (22) and (26), bounds (23) and (27) relate the maximal tolerated delays to a spatial feature. Explicit formulas for $\|\Delta\|$ in the case of complete and loop-shaped networks are provided in [11]. Other results linking the graph structure with the eigenvalues of the Laplacian operator can be found in [16], [17] and [18].

We also outline that the constant in (27) is smaller than the corresponding one in (23), the greater conservativity arising from the time-varying nature of the delay. However, the bound (27) is the best possible one since the corresponding stability condition is *necessary and sufficient*.

V. The Case of Non-Uniform Delays

In this Section, we generalize the results of Section IV to the case where the delays do not take a common value in the whole network. Let us consider first the case of constant delays. The next Theorem provides a robust stability result for *all* possible delays τ_i within the interval $[0, \bar{\tau}]$. Quite remarkably, the bound (23) still gives a necessary and sufficient condition for stability.

Theorem 5 (Constant delays): The zero solution is a globally exponentially stable solution to the PdE

$$\dot{v}_1(x,t) = \sum_{i \in \mathscr{I}} \Delta_i v_1(x, t - \tau_i), \quad P_{H^1_\perp} v_1(\cdot, t]|_{[-\bar{\tau}, 0]} \equiv 0 \quad (28)$$

for all possible $0 \leq \tau_i \leq \bar{\tau}$, $i \in \mathscr{I}$, *if and only if* (23) holds.

Proof: The proof is available in [11]. ■

We stress once more the robustness flavor of Theorem 5, that requires just the knowledge of a common upper bound $\bar{\tau}$ on the (unknown) delays τ_i. On the other hand, there may exist combinations of delays τ_i such that $\tau_i \geq \bar{\tau}$, for some $i \in \mathscr{I}$, but the PdE (28) remains asymptotically stable. An example is provided in Section VII.

The argument used in the proof of Theorem 4 does not seem to extend to the case of non-stationary delays. In this case, the next Theorem provides a sufficient stability condition.

Theorem 6 (Time-varying delays): The zero solution is a globally stable solution to the PdE

$$\dot{v_1}(x,t) = \sum_{i\in\mathscr{I}} \Delta_i v_1(x,t-\tau_i(t)), \quad P_{H_\perp^1} v_1(\cdot,t]|_{[-\bar{\tau},0]} \equiv 0$$

for all piecewise continuous delay $\tau_i(t)$ verifying $0 \leq \tau_i(t) \leq \bar{\tau}$, *if*

$$\bar{\tau} < \frac{1}{\sum_{i,i'\in\mathscr{I}} \|\Delta_i\Delta_{i'}\|\|\Delta^{-1}\|} \ . \quad (29)$$

Proof: The proof is reported in [11]. ■

The results of Theorems 3–6 are summarized in Table I.

$\bar{\tau}$	Uniform delays	Non-uniform delays
Time-invariant delays	$\frac{\pi}{2\|\Delta\|}$ (E)	$\frac{\pi}{2\|\Delta\|}$ (E)
Time-varying delays	$\frac{3}{2\|\Delta\|}$ (E)	$\frac{1}{\sum_{i,i'\in\mathscr{I}} \|\Delta_i\Delta_{i'}\|\ \|\Delta^{-1}\|}$ (S)

TABLE I

BOUNDS ON THE WORST-CASE STABILIZING DELAY.

E: EXACT, S: SUFFICIENT.

Remark 2: By comparison with (27), the bound (29) depends in a more involved manner upon the structure of the communication network. However, one may check that (29) is verified for example when

$$\bar{\tau} < \frac{1}{(\mathrm{Tr}\Delta)^2\|\Delta^{-1}\|}$$

where $\mathrm{Tr}\Delta$ is the trace of the Laplacian on H^1. For checking that the results of Theorems 4 and 6 are coherent, one can use the following inequalities

$$\sum_{i,i'\in\mathscr{I}} \|\Delta_i\Delta_{i'}\|\|\Delta^{-1}\| \geq \left\|\sum_{i,i'\in\mathscr{I}} \Delta_i\Delta_{i'}\right\|\|\Delta^{-1}\| =$$

$$= \left\|\left(\sum_{i\in\mathscr{I}} \Delta_i\right)^2\right\|\|\Delta^{-1}\| = \|\Delta^2\|\|\Delta^{-1}\| \geq \|\Delta\| \ ,$$

that imply (29)$\leq \frac{1}{\|\Delta\|} \leq \frac{3}{2\|\Delta\|}$. Also, we highlight the trade-off between stability with large delays on the one hand, and large decay-rate of the solutions on the other hand: the first one requires a small $\|\Delta\|$, whereas the second one requires a large $\|\Delta^{-1}\|^{-1} \leq \|\Delta\|$.

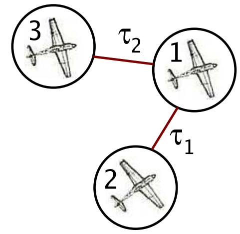


Fig. 2. The multi-agent system, with the communication delays, used in Section VII.

VI. A DELAY-INDEPENDENT CONDITION FOR AVERAGE CONSENSUS

According to the standard terminology in time-delay systems, all the results presented in Sections IV and V are "delay-dependent" in the sense that they guarantee average consensus when all the communication delays are upper-bounded by a suitable value $\bar{\tau}$. Next, we show that if a single delay is zero, average consensus may be achieved irrespectively of the magnitude of all other delays. In this sense, we provide a "delay-independent" condition for average consensus. For two operators A and B from L^2 to L^2, the inequality "$A < B$ on H^1" means that

$$\forall f \in H^1 \setminus \{0\},\ (f,(A-B)f)_{L^2} > 0 \ . \quad (30)$$

Theorem 7: Consider the PdE (28) and assume that $\tau_{i'} \equiv 0$ for an index $i' \in \mathscr{I}$. If

$$\Delta_{i'} < \sum_{i\in\mathscr{I}\setminus\{i'\}} \Delta_i \quad \text{on } H^1,$$

then, the zero solution is a globally exponentially stable solution to (28) for any $\tau_i \geq 0$, $i \in \mathscr{I}\setminus\{i'\}$. Conversely, if the zero solution to system (28) with $\tau_{i'} = 0$ is globally asymptotically stable for any $\tau_i \geq 0$, $i \in \mathscr{I}\setminus\{i'\}$, then

$$\Delta_{i'} \leq \sum_{i\in\mathscr{I}\setminus\{i'\}} \Delta_i \quad \text{on } H^1 \ .$$

Proof: The proof is given in [11]. ■

VII. EXAMPLES

We stress once more that the results in Sections IV and V characterize robustness of average consensus, i.e. average consensus *for any value of the delays less or equal to* $\bar{\tau}$. In order to illustrate this concept, we consider the network of three agents whose communication graph G is represented in Figure 2.

We assume that $v(x,t) \in \mathbb{R}^2$, that the weights $\omega(x,y) = 1 \Leftrightarrow x \sim y$ are used, and that the delays $\tau_i > 0$, $i = 1,2$ are constant in time. Moreover, the agents evolve according to the PdE (14) starting from the initial conditions

$$\tilde{v}(1,t) \equiv [2,\ 2]', \quad \tilde{v}(2,t) \equiv [2,\ -2]', \quad \tilde{v}(3,t) \equiv [1,\ 3]'$$

where $t \in [-\max\{\tau_1,\tau_2\},0]$. The average velocity at time $t = 0$ is $\bar{v} = [\frac{5}{3},\ 1]'$.

From Theorem 1, the eigenvalues of Δ on H^1 are the non null eigenvalues of $\mathscr{M}(\Delta)$ (modulus their multiplicity). In our case, one gets $\|\Delta\| = 3$, and the bound (23) is equal to $\pi/6 \simeq 0.524$.

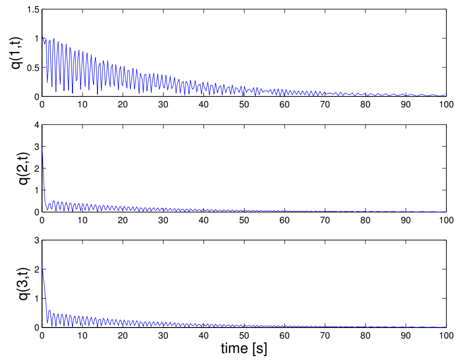


Fig. 3. Time evolution of $q(x,t) = \|v(x,t) - \bar{v}\|$ for the multi-agent system in Figure 2, with $\tau_1 = \tau_2 = 0.51$.

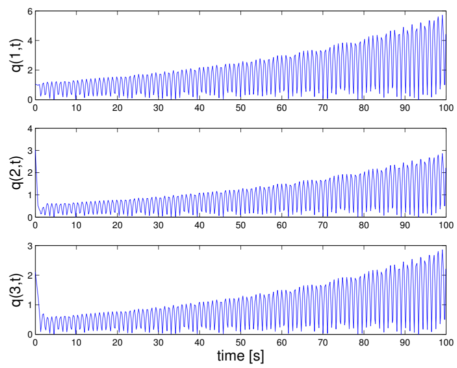


Fig. 4. Time evolution of $q(x,t) = \|v(x,t) - \bar{v}\|$ for the multi-agent system in Figure 2, with $\tau_1 = \tau_2 = 0.53$.

In the first experiment, we choose the delays $\tau_1 = \tau_2 = 0.51$ that are slightly below $\bar{\tau}$. Then, Theorem 3 guarantees average consensus and such a result can be verified from Figure 3, where the evolution of $\|v(x,t) - \bar{v}\|$, $x \in \{1,2,3\}$ is represented. In the second experiment, we use $\tau_1 = \tau_2 = 0.53$, so having $\tau_1 = \tau_2 > \bar{\tau}$. The dynamics of v_1 becomes unstable and average consensus cannot be achieved. This can be clearly seen in Figure 4. Finally, we choose $\tau_1 = 0.1$ and $\tau_2 = 0.7$. In this case, τ_2 violates the bound of Theorem 3. However, $\tau_1 < \bar{\tau}$ and Theorem 3 cannot be used for checking the average consensus property. In the present case, the achievement of average consensus can be verified by simulation, as shown in Figure 5.

VIII. Conclusions

We provided convergence analysis of an average consensus protocol for undirected networks of dynamic agents having communication delays. We considered constant or time-varying delays, uniformly or non uniformly distributed in the network. Sufficient conditions (also necessary in most cases) for existence of average consensus under bounded, but otherwise unknown, communication delays, have been given. Simulations have been provided that demonstrate the correctness of some bounds computed analytically.

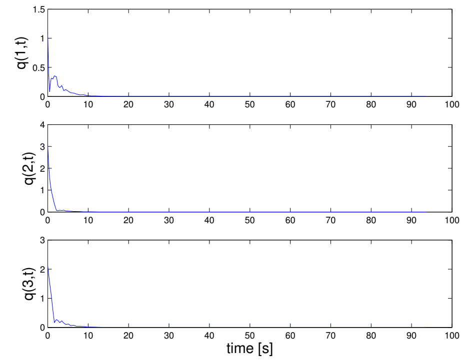


Fig. 5. Time evolution of $q(x,t) = \|v(x,t) - \bar{v}\|$ for the multi-agent system in Figure 2, with $\tau_1 = 0.1$ and $\tau_2 = 0.7$.

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