

AVERAGE ORDER OF THE DIVISOR FUNCTIONS WITH NEGATIVE POWER WEIGHT

Dedicated to Professor Katsumi Shiratani on his 60th birthday

By

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1. Introduction.

In this paper we are primarily concerned with the study of the sums of the sum-of-divisors function $\sigma_a(n)$ with negative power weight n^{-t} ($t > 0$), i. e. the sums of the form

$$\sum_{n \leq x} n^{-t} \sigma_a(n)$$

and we also study the averages of associated error terms. Throughout the paper, we shall refer to [6] as I and whose results we cite e. g. as I-Theorem 1. First we consider the case $0 \leq a - t \in \mathbf{Z}$, where \mathbf{Z} denotes the set of all rational integers, and prove Theorem 1 which generalizes and in some cases corrects MacLeod's Theorem 8[8]. This case is easier to handle although the needed calculations are rather long. And the special case $a = t$ of this is the starting point of the investigation of the case $a < t$. In this case our approach, which depends on MacLeod's back-track method (Lemma 1 below), is not so effective for a large, and we have to restrict ourselves to the narrower range $0 \leq a \leq 3$ which, however, covers and interpolates all the formulas obtained by MacLeod. In the case of general t we appeal to induction, and in order to guess the forms of the formulas, we have to calculate out all the cases $t = a + 1$, $t = a + 2$, $t = a + 3$, the last being the initial value of t for induction. Here we take the instructive standpoint and calculated out all these three cases successively and then give the form for $t \geq a + 3$, since each independent formula seems to have its own interest. Except for integral values of a , our interpolating formulas involve various negative powers of x with extremely complicated and clumsy coefficients, but in some cases they are absorbed in the error terms by just multiplying the log-factor. The main reasons why we restrict ourselves to $0 \leq a \leq 3$ are the complication of these coefficients as well as inapplicability of Lemma 8. However, we state the formulas for $a > 3$ as well, only for $t =$

$a+1$ (Theorem 2), though, for want of a more effective method to compute the sums in question. Of course, in principle, we can continue to calculate further cases $t=a+2, a+3, \dots$ starting from the case $t=a+1$, using the back-track method. But the effort made does not seem to deserve it since the main difficulty lies in the explicit determination of the coefficients. Our Theorem 2 covers and at some points corrects Theorem 10, and the first half of Theorem 12, of MacLeod [8], and Corollary to it covers Corollary to his Theorem 10 and gives a new result as a counterpart of the possible Corollary (which is non-existent) to the first half of Theorem 12. In particular, it follows from our Theorem 2 and [5] that

$$\sum_{n \leq x} \frac{d(n)}{n} = \frac{1}{2} \log^2 x + 2\gamma \log x + \gamma^2 + 2\gamma_1 + O(x^{-15/22}(\log x)^{69/22}).$$

which improves a result of Amitsur [1]. We note that our formulas in the Corollary are proved while MacLeod's are conjectured, since by Lemma 8 an extended form of MacLeod's conjecture is proved. Lemma 8 seems interesting in its own right, indicating a great deal of cancellation of $G_{a,k}(x)$ for x large. Our subsequent Theorem 3-(1) reduces to Theorem 4-(1). However, for the formulas (2) and (3), the case $t=a+2$ is still exceptional, and by only looking at the case $t=a+3$ we can guess the general form of main terms. As in [6], we note that the error estimates claimed in MacLeod's Theorem 12 are not yet proved since they depend on Segal's yet unproven estimates [17], and the error estimates claimed in our Theorems and Corollaries are the best that are known to date.

As in [6], main results of the paper are asymptotic formulas for our sums in terms of $G_{a,k}$ -functions. For completeness' sake we collect some information on $G_{a,k}$. They are defined as

$$G_{a,k}(x) = \sum_{n \leq \sqrt{x}} n^a \bar{B}_k\left(\frac{x}{n}\right).$$

for $a \in \mathbf{R}$, $k \in \mathbf{N}$, where $B_k(y)$ is the k -th Bernoulli polynomial, $\bar{B}_k(y) = B_k(y - [y])$, with $[y]$ the integral part of y , \mathbf{R} and \mathbf{N} denoting the set of all real numbers and the set of all natural numbers respectively. As regards the order estimates of $G_{a,k}(x)$, there is a famous conjecture due to Chowla and Walum saying (in a more precise form)

$$\alpha_k(a) = \begin{cases} a/2 + 1/4 & (a \geq -1/2), \\ 0 & (a \leq -1/2), \end{cases} \quad (\text{S})$$

where $\alpha_k(a)$ denotes the least α for which $G_{a,k}(x) = O(x^{\alpha+\epsilon})$ for each $\epsilon > 0$

(For detailed introduction to these functions we refer the reader to Pétermann [12, 13]). The looser form ($S \leq$) of the conjecture (S), where we replace the equality sign in (S) by \leq , has been known for $k \geq 2$, $a \geq 1/2$, and non-trivial estimates have been known for $k \geq 2$, $0 \leq a < 1/2$, $k=1$, $-1 < a < 0$. Very recently, Pétermann has succeeded in sharpening the last estimates by applying the theory of one dimensional exponent pairs. We will quote these estimates from [13], referring the details to it. E is the set of all exponent pairs (κ, λ) ($0 \leq \kappa \leq 1/2 \leq \lambda \leq 1$) and S is Rankin's set $\subset E$ obtained by applying Theorem 4.1 [13] a finite (but arbitrary) number of times starting with the trivial exponent pair $(0, 1)$. S may be seen as the convex hull $\text{conv } T$ of T obtainable from $(0, 1)$ by applying to it all the finite compositions built with operators A and BA . Also, if M denotes the set $\{h(\varepsilon) \mid 0 < \varepsilon \leq 5/56\}$ of exponent pairs $h(\varepsilon) = (9/56 + \varepsilon, 1/2 + 9/56 + \varepsilon)$ found by Huxley and Watt, we construct the set $S_1 = S_1(M)$ by applying Theorem 4.1 to M repeatedly. If $R = \partial S$, ∂S_1 denotes the borders of S, S_1 respectively, the infimum $I(R, \theta)$ on R of some function θ gives the optimal result.

Now define the function $\Gamma: [0, 1/2] \rightarrow [1/2, 1]$ by

$$\partial S_1 := \{(\kappa, \Gamma(\kappa)), \kappa \in [0, 1/2]\}.$$

Then Γ is continuous, decreasing and convex. Also, with

$$\theta_2(\kappa, \lambda) = \frac{\kappa + \lambda}{2(\kappa + 1)},$$

define $\alpha := I(S_1, \theta_2) = 0.32894 \dots$ and define β to be the larger one of α and $(\lambda - \kappa)/(\kappa + 1)$ for (κ, λ) for which $\theta(\kappa, \lambda) = \alpha$ (see [12] for the numerical values of α and β).

O-THEOREM (Huxley [5], Pétermann [12, 13], Walfisz [18])

$$(1.1) \quad \alpha_1(a) \leq \begin{cases} \phi(a), & -1 < a < -\beta, \\ a/2 + \alpha, & a \geq -\beta, a \neq 0, \end{cases}$$

$$(1.2) \quad G_{0,1}(x) = O(x^{7/22}(\log x)^{69/22}), \quad G_{-1,1}(x) = O((\log x)^{2/3}),$$

and for $k \geq 2$,

$$(1.3) \quad \alpha_k(a) \leq \begin{cases} \phi(a), & -1 \leq a < 1/2, \\ a/2 + 1/4, & a \geq 1/2, \end{cases}$$

where $\phi(a)$ (resp. $\psi(a)$) is the value of $\kappa/(\kappa + 1)$ (resp. κ) at the unique argument κ satisfying $a = (\kappa - \Gamma(\kappa))/(\kappa + 1)$ (resp. $a = 2\kappa - \Gamma(\kappa)$).

Ω -THEOREM (Hafner [3, 4], Pétermann [10, 11, 14, 15])

$$(1.4) \quad G_{a,1}(x) = \begin{cases} \Omega_+((x \log x)^{1/4}(\log \log x)^{(3+2 \log 2)/4} \exp(-A \sqrt{\log \log \log x})) \\ \Omega_-(x^{1/4} \exp(c(\log \log x)^{1/4}(\log \log \log x)^{-3/4}), & a=0, \\ \Omega_{\pm}(\exp(c_a \frac{(\log x)^{a+1}}{\log \log x})), & -1 < a < 0, \\ \Omega_{\pm}(\log \log x), & a = -1, \\ \Omega_{\pm}(1), & a < -1, \end{cases}$$

$$(1.5) \quad G_{a,2}(x) = \begin{cases} \Omega_{\pm}(x^{a/2+1/4} g_a(x)), & 0 < a < 1/2, \\ \Omega_+(x^{a/2+1/4} f_a(x)), & 0 \leq a \leq 1/2, \\ \Omega_{\pm}(x^{a/2+1/4} f_a(x)), & -1/2 \leq a < 0, \\ \Omega_+(\exp(c_a \frac{(\log x)^{a+1}}{\log \log x})), & -1 < a < -1/2, \\ \Omega_{\pm}(\log \log x), & a = -1, \\ \Omega_{\pm}(1), & a < -1, \end{cases}$$

for some positive constant A, c, c_a , where

$$f_a(x) = \begin{cases} (\log x)^{1/4-a/2}, & -1/2 \leq a < 1/2, \\ \log \log x, & a = 1/2, \end{cases}$$

and

$$g_a(x) = \exp\left(c \frac{(\log \log x)^{1/4-a/2}}{(\log \log \log x)^{3/4+a/2}}\right).$$

The interested reader should ask the author for a detailed version of the paper at the address indicated at the end.

2. Statement of results.

THEOREM 1. *Let $t > 0$ and $0 \leq a - t \in \mathbf{Z}$. Then*

(1)

$$\sum_{n \leq x} n^{-t} \sigma_a(n) = \frac{\zeta(a+1)}{a-t+1} x^{a-t+1} + \sum_{r=1}^{a-t+1} \frac{(-1)^r}{r} \binom{a-t}{r-1} x^{a-t+1-r} G_{r-a-1, r}(x) \\ + \begin{cases} \zeta(1-a) \log x + \gamma \zeta(1-a) + \zeta(1-a), & t=1, \\ \zeta(t) \zeta(t-a) + \frac{\zeta(t-a)}{1-t} x^{1-t} & t \neq 1, \end{cases} \\ - x^{-t} G_{a,1}(x) + O(x^{a/2-t}),$$

where γ denotes Euler's constant.

(2) Define $E_{a,t}^{\alpha}(x)$ by

$$E_{-t}^a(x) = \sum_{n \leq x} n^{-1} \sigma_a(n) - g_{-t}^a(x),$$

where

$$g_{-t}^a(x) = \frac{\zeta(a+1)}{a-t+1} x^{a-t+1} + \begin{cases} 0, & t > 2 \text{ or } 0 < t \leq 2, t \neq 1 \text{ or } t=1, a > 2, \\ \zeta(1-a) \log x, & t=1, a \leq 2. \end{cases}$$

and let $G_{-t}^a(x) = \sum_{n \leq x} E_{-t}^a(n)$.

Then

$$(i) \quad G_{-a}^a(x) = -\frac{1}{2x^a} G_{a+1,2}(x) + \frac{1}{2} G_{1-a,2}(x) + O(x^{(1-a)/2})$$

$$+ \begin{cases} \frac{\zeta(1-a)}{(1-a)(2-a)} x^{2-a} + \frac{\zeta(a+1) - \zeta(a)}{2} x + \left(\frac{1}{2} - \bar{B}_1(x)\right) \frac{\zeta(1-a)}{1-a} x^{1-a}, & 0 < a=t < 1, \\ \frac{1}{2}(\zeta(2) - \gamma - \log 2\pi)x - \frac{1}{4} \log x - \left(\frac{1}{2} - \bar{B}_1(x)\right) G_{-1,1}(x), & a=t=1, \\ \frac{\zeta(a+1) - \zeta(a)}{2} x + \frac{\zeta(1-a)}{(1-a)(2-a)} x^{2-a} + \frac{\zeta(a-1) + \zeta(a+1)}{12} - \frac{1}{2} \left(\frac{1}{2} - \bar{B}_1(x)\right) \zeta(a) \\ - \frac{\zeta(a+1)}{2} \bar{B}_2(x) - \left(\frac{1}{2} - \bar{B}_1(x)\right) G_{-a,1}(x), & 1 < a=t < 2, \\ \frac{\zeta(3) - \zeta(2)}{2} x + \frac{1}{12} \log x + \frac{\gamma}{12} + A_1 + \frac{1}{24} + \frac{\zeta(3)}{12} - \frac{\zeta(3)}{2} \bar{B}_2(x) \\ - \frac{1}{2} \left(\frac{1}{2} - \bar{B}_1(x)\right) \zeta(2) - \left(\frac{1}{2} - \bar{B}_1(x)\right) G_{-2,1}(x), & a=t=2, \end{cases}$$

where $A_1 = -\zeta'(-1)$ (see I-Theorem 1).

$$(ii) \quad G_{-t}^a(x) = \frac{\zeta(a+1)}{2(a-t+1)} x^{a-t+1} + \sum_{m=1}^{a-1} \frac{(-1)^m}{m+1} \binom{a-t}{m-1} x^{a-t+1-m} G_{m-a, m+1}(x)$$

$$+ \frac{\zeta(a+1)}{a-t+1} \sum_{m=1}^{a-t+1} \frac{(-1)^m}{m+1} \binom{a-t+1}{m} \bar{B}_{m+1}(x) x^{a-t+1-m}$$

$$+ \left(\frac{1}{2} - \bar{B}_1(x)\right) \frac{1}{a-t+1} \sum_{m=1}^{a-t+1} (-1)^m \binom{a-t+1}{m} x^{a-t+1-m} G_{m-a-1, m}(x)$$

$$- \frac{1}{2x^t} G_{a+1,2}(x) + \zeta(t-a) \zeta(t) x + \frac{(-1)^{a-t+1}}{a-t+2} G_{1-t, a-t+2}(x)$$

$$+ \frac{B_{a-t+2}}{a-t+1} \zeta(t-1) + \frac{B_{a-t+2}}{(a-t+1)(a-t+2)} \zeta(a+1) + \left(\frac{1}{2} - \bar{B}_1(x)\right) \zeta(t-a) \zeta(t)$$

$$+ \begin{cases} F_2 x^{2-t}, & 2 < t < 3, \\ 0, & t \geq 0, \end{cases} + O(x^{(a+1)/2-t}),$$

where

$$F_1 = -\frac{1}{a(a-t+1)(a-t+2)} - \frac{(-1)^{a-t}}{a-t+2} \left(\sum_{u=0}^{a-t+1} \binom{a-t+2}{u} \frac{B_u}{u-a} - \frac{\zeta(1-a)}{\binom{2-t}{a-t+2}} \right) + \frac{\zeta(t-a)}{1-t}.$$

$$(iii) \quad G_{a+1}^a(x) = \frac{\zeta(a+1)}{4} x^2 + \frac{\zeta(1-a)}{(a-2)(a-3)} x^{3-a} - \left(\frac{\zeta(a-1)}{12} + \frac{\zeta(a+1)}{2} \bar{B}_2(x) \right) x \\ + \left(\frac{1}{2} - \bar{B}_1(x) \right) \frac{\zeta(1-a)}{2-a} x^{2-a} - \frac{x}{2} G_{1-a,2}(x) - \frac{x^{1-a}}{2} G_{1+a,2}(x) \\ - \left(\frac{1}{2} - \bar{B}_1(x) \right) x G_{-a,1}(x) + O(x^{(3-a)/2}), \quad 0 < t < 1, \quad a-t=1,$$

$$(iv) \quad G_t^a(x) \\ = \frac{\zeta(a+1)}{2(a-t+1)} x^{a-t+1} + F_1 x^{2-t} - \frac{\zeta(a+1)}{a-t+1} \sum_{m=1}^{a-t-1} \frac{(-1)^{m+1}}{m+1} \binom{a-t+1}{m} \bar{B}_{m+1}(x) x^{a-t+1-m} \\ + \sum_{m=1}^{a-t} \frac{(-1)^m}{m+1} \binom{a-t}{m-1} x^{a-t+1-m} G_{m-a, m+1}(x) - \frac{1}{2x^t} G_{1+a,2}(x) \\ + \left(\frac{1}{2} - \bar{B}_1(x) \right) \frac{x^{1-t}}{a-t+1} \sum_{m=1}^{a-t-1} (-1)^m \binom{a-t+1}{m} x^{a-m} G_{m-a-1, m}(x) + O(x^{(a+1)/2-t}) \\ 0 < t < 1, \quad a-t \geq 2,$$

where

$$F_2 = (-1)^{a-t+1} \sum_{u=0}^{a-t} \binom{a-t+1}{u} \frac{B_u}{u-a} \left(\frac{1}{a-t+2-u} - \frac{a-t}{a-t+1} + \frac{(a-t-1)(a-t+1-u)}{2(a-t+1)} \right) \\ + (-1)^{a-t} \zeta(1-a) \left(\frac{1-t}{a-t+1} \right)^{-1} \left(\frac{1}{2-t} - \frac{a-t}{a-t+1} - \frac{(a-t-1)(1-t)}{a-t+1} \right),$$

$$(v) \quad G_{-1}^2(x) = \frac{\zeta(3)}{4} x^2 - \left(\frac{1}{24} + \frac{7}{12} + A_1 + \frac{\zeta(3)}{2} \bar{B}_2(x) \right) x - \frac{x}{2} G_{-1,2}(x) \\ - \frac{1}{2x} G_{3,2}(x) - \left(\frac{1}{2} - \bar{B}_1(x) \right) x G_{-2,1}(x) + O(x^{1/2}), \quad t=1, \quad a=2,$$

$$(vi) \quad G_t^a(x) = \frac{\zeta(a+1)}{2(a-t+1)} x^{a-t+1} + \sum_{m=1}^{a-1} \frac{(-1)^m}{m+1} \binom{a-t}{m+1} x^{a-t+1-m} G_{m-a, m+1}(x) \\ + \frac{\zeta(a+1)}{a-t+1} \sum_{m=1}^{a-t} \frac{(-1)^m}{m+1} \binom{a-t+1}{m} \bar{B}_{m+1}(x) x^{a-t+1-m} \\ + \left(\frac{1}{2} - \bar{B}_1(x) \right) \frac{1}{a-t+1} \sum_{m=1}^{a-t} (-1)^m \binom{a-t+1}{m} x^{a-t+1-m} G_{m-a-1, m}(x) - \frac{1}{2x^t} G_{a+1,2}(x)$$

$$+ \begin{cases} 0, & t=1, a-t \geq 2, \\ F_1 x^{2-t} + \zeta(t-a)\zeta(t)x, & 1 < t < 2, a-t \geq 1, +O(x^{(a+1)/2-t}), \\ -\frac{B_{a-1}}{a-1}\zeta(2)x + \frac{(-1)^{a+1}}{a}G_{-1,a}(x), & t=2, a-t \geq 1, \end{cases}$$

(3) We have

$$(i) \int_1^x E_{-a}^a(u)du = -\frac{1}{2x^a}G_{a+1,2}(x) - \frac{1}{2}G_{1-a,2}(x) + O(x^{(a+1)/2-t})$$

$$+ \begin{cases} \left. \begin{aligned} &\frac{\zeta(1-a)}{(1-a)(2-a)}x^{2-a} - \frac{\zeta(a)}{2}x, & 0 < a=t < 1, \\ &-\frac{1}{2}(\log 2\pi + \gamma)x, & a=t=1, \end{aligned} \right\} \\ \left. \begin{aligned} &-\frac{\zeta(a)}{2}x + \frac{\zeta(1-a)}{(1-a)(2-a)}x^{2-a} + \frac{\zeta(a-1)}{12} + \frac{\zeta(a+1)}{2}, & 1 < a=t < 2, \\ &-\frac{\zeta(2)}{2}x + \frac{1}{12}\log x + \frac{\gamma}{12} + A_1 + \frac{1}{24} + \frac{\zeta(3)}{2}, & a=t=2, \end{aligned} \right\}$$

$$(ii) \int_1^x E_{-t}^a(u)du$$

$$= \sum_{m=1}^{a-1} \frac{(-1)^m}{m+1} \binom{a-t}{m-1} x^{a-t+1-m} G_{m-a, m+1}(x) - \frac{1}{2x^t} G_{a+1,2}(x) + \zeta(t-a)\zeta(t)x$$

$$+ \frac{B_{a-t+2}}{a-t+2} \zeta(t-1) + \frac{\zeta(a+1)}{(a-t+1)(a-t+2)} + \frac{(-1)^{a-t+1}}{a-t+2} G_{1-t, a-t+2}(x)$$

$$+ \begin{cases} 0, & t \geq 3, \\ F_1 x^{2-t}, & 2 < t < 3, \end{cases} + O(x^{(a+1)/2-t}),$$

$$(iii) \int_1^x E_{-a+1}^a(u)du = -\frac{x}{2}G_{1-a,2}(x) - \frac{x^{1-a}}{2}G_{1+a,2}(x) + \frac{1}{3}G_{2-a,3}(x)$$

$$+ \frac{\zeta(1-a)}{(a-2)(a-3)}x^{3-a} - \frac{\zeta(a-1)}{12}x + O(x^{(a+1)/2-t}),$$

$0 < t < 1, a-t=1,$

$$(iv) \int_1^x E_{-t}^a(u)du = \sum_{m=1}^{a-1} \frac{(-1)^m}{m+1} \binom{a-t}{m-1} x^{a-t+1-m} G_{m-a, m+1}(x) - \frac{1}{2x^t} G_{a+1,2}(x)$$

$$+ F_2 x^{2-t} + O(x^{(a+1)/2-t}), \quad 0 < t < 1, a-t \geq 2,$$

$$(v) \int_1^x E_{-1}^2(u)du$$

$$= -\frac{x}{2}G_{-1,2}(x) - \left(\frac{1}{24} + \frac{\gamma}{12} + A_1\right)x - \frac{1}{2x}G_{3,2}(x) + O(\sqrt{x}), \quad t=1, a=2,$$

$$(vi) \int_1^x E_{-t}^a(u) du = \sum_{m=1}^{a-t} \frac{(-1)^m}{m+1} \binom{a-t}{m-1} x^{a-t+1-m} G_{m-a, m+1}(x) - \frac{1}{2x^t} G_{a+1, 2}(x)$$

$$+ \begin{cases} 0, & t=1, a-t \geq 2, \\ F_3 x^{2-t} + \zeta(t-a)\zeta(t)x, & 1 < t < 2, a-t \geq 1, \\ \zeta(2-a)\zeta(2)x + (-1)^{a+1} a^{-1} G_{-1, a}(x), & t=2, a-t \geq 1, \end{cases} + O(x^{(a+1)/2-t})$$

where

$$F_3 = -\frac{1}{a(a-t+1)(a-t+2)} - \frac{\zeta(t-a)}{1-t} - \frac{\zeta(t-1-a)}{2-t}$$

$$- \frac{(-1)^{a-t}}{a-t+2} E_{1-t, a-t+2} + (-1)^{a-t} \frac{a-t}{a-t+1} E_{-t, a-t+1}.$$

Theorem 1, (1.1)-(1.5) imply the following Corollary.

COROLLARY 1. For every $\epsilon > 0$, we have

$$(i) \sum_{n \leq x} \frac{\sigma_a(n)}{n^a} = \begin{cases} x \log x + (2\gamma - 1)x, & ([6]) & a=0, \\ \zeta(a+1)x - \frac{\zeta(a)}{2} + \frac{\zeta(1-a)}{1-a} x^{1-a}, & 0 < a < 1, \\ \zeta(2)x - \frac{1}{2} \log x - \frac{1}{2} (\log 2\pi + \gamma), & a=1, \end{cases}$$

$$+ \begin{cases} O(x^{-a/2+\alpha+\epsilon}), & 0 \leq a \leq \beta, \\ O(x^{\phi(-a)+\epsilon}), & \beta < a < 1, \\ O((\log x)^{2/3}), & a=1. \end{cases}$$

In particular

$$\sum_{n \leq x} \frac{\sigma(n)}{n} - \frac{\pi^2}{6} x + \frac{1}{2} \log x = \Omega_+(\log \log x), \quad a=1.$$

$$(ii) \sum_{n \leq x} E_{-1}^1(n) = \frac{1}{2} (\zeta(2) - \gamma - \log 2\pi)x + \begin{cases} O(x^{\phi(0)+\epsilon}), \\ \Omega_+(x^{1/4}(\log x)^{1/4}), \end{cases}$$

$$\int_1^x E_{-1}^1(u) du = -\frac{1}{2} (\log 2\pi + \gamma)x + \begin{cases} O(x^{\phi(0)+\epsilon}), \\ \Omega_+(x^{1/4}(\log x)^{1/4}), \end{cases}$$

$$(iii) \sum_{n \leq x} E_{-2}^2(n) = \frac{\zeta(3) - \zeta(2)}{2} x + \begin{cases} O(x^\epsilon) \\ \frac{1}{12} \log x + \Omega_+(\log \log x) \end{cases}$$

$$\int_1^x E_{-2}^2(u) du = -\frac{\zeta(2)}{2} x + \begin{cases} O(x^\epsilon), \\ \frac{1}{12} \log x + \Omega_+(\log \log x), \end{cases}$$

$$(iv) \sum_{n \leq x} E_{-1}^2(n) = \frac{\zeta(3)}{4} x^2 + \begin{cases} O(x^{1+\epsilon}), \\ \Omega_+(x \log \log x). \end{cases}$$

$$\int_1^x E_{-1}^2(u) du = \begin{cases} O(x^{1+\epsilon}), \\ \Omega_+(x \log \log x). \end{cases}$$

Similar results hold for other values of a .

REMARK 1. These formulas improve and correct MacLeod's corresponding results in [8].

In what follows we shall use the notation $\bar{\delta}_{i,j} = 1 - \delta_{i,j}$, where $\delta_{i,j} = 1$ for $i = j$ and 0 otherwise.

THEOREM 2.

$$(1) \sum_{n \leq x} \frac{\sigma_a(n)}{n^{a+1}} = \begin{cases} \frac{1}{2} \log^2 x + 2\gamma \log x + \gamma^2 + 2\gamma_1, & a=0, \\ \zeta(a+1) \log x + \zeta'(a+1) + \gamma \zeta(a+1), & a>0, \end{cases}$$

$$-x^{-1}G_{-a,1}(x) - x^{-a-1}G_{a,1}(x)$$

$$+ \begin{cases} 0, & a=0, \\ -\frac{\zeta(1-a)}{a} x^{-a}, & 0 < a < 2, \\ -\frac{1}{2x^2} G_{-1,2}(x) & a=2, \\ \left(\frac{\zeta(a+1)}{6} + \frac{\zeta(a)}{4}\right) x^{-2} - \frac{1}{2x^2} G_{1-a,2}(x) & 2 < a < 3, \\ \sum_{1 < r < a/2+2} c_r(x) x^{-1-r} + \sum_{n > x} \frac{G_{1-a,2}(n)}{n(n+1)^2} + 2 \sum_{n > x} \frac{G_{-a,1}(n)}{n(n+1)^2} \\ -\frac{1}{2x^2} G_{1-a,2}(x) - \sum_{n > x} \frac{G_{-a,1}(n)}{n^2(n+1)} + \frac{1}{2} \sum_{n > x} \frac{G_{1-a,2}(n)}{n^2(n+1)^2} \\ + \sum_{n > x} \frac{G_{-a,1}(n)}{n^2(n+1)^2}, & a > 3, \end{cases}$$

$$+ O(x^{-a/2-1}),$$

where $c_1(x) = \frac{\zeta(a+1)}{6} + \frac{\zeta(a)}{4}$,

$$c_2(x) = (-2\bar{B}_3(x) + 3\bar{B}_2(x) - 1) \frac{\zeta(a+1)}{6} + (3\bar{B}_1(x) - 1) \frac{\zeta(a)}{6}$$

$$+ (\bar{B}_2(x) - \bar{B}_1(x) + 3^{-1}) G_{-a,1}(x),$$

$$c_r(x) = \frac{1}{r+1} \left[\sum_{u=0}^r (-1)^{r-u} \binom{r+1}{u} \left\{ \frac{\zeta(a+1)(u-1)(2r-u)}{2(r+2-u)} - \frac{\zeta(a)}{2} \right. \right. \\ \left. \left. + (r-1)G_{-a,1}(x) \right\} \bar{B}_u(x) - 2^{-1} \zeta(a+1)r(r-1)\bar{B}_{r+1}(x) \right], \quad r \geq 3,$$

and γ_1 is the constant term in the Euler-Maclaurin expansion of $\sum_{n \leq x} n^{-1} \log n$ (see I-Lemma 8).

(2) On defining $E_{-a-1}^a(x)$ by

$$E_{-a-1}^a(x) = \sum_{n \leq x} \frac{\sigma_a(n)}{n^{a+1}} - \begin{cases} 2^{-1} \log^2 x + 2\gamma \log x + \gamma^2 + 2\gamma_1, & a=0, \\ \zeta(a+1) \log x + \zeta'(a+1) + \gamma \zeta(a+1), & a>0, \end{cases}$$

we have

$$\sum_{n \leq x} E_{-a-1}^a(n) = \begin{cases} \frac{1}{4} \log^2 x + \gamma \log x + W_{-1}^0 + 2x \sum_{n > x} \frac{G_{1,2}(n)}{n^3} - x^{-1} G_{1,2}(x), & a=0, \\ \frac{\zeta(2)+1}{2} \log x + W_{-2}^1 + \left(\bar{B}_1(x) - \frac{1}{2} \right) x^{-1} G_{-1,1}(x) - \frac{1}{2x} (x^{-1} G_{2,2}(x) + G_{0,2}(x)), & a=1, \\ \left. \begin{aligned} & - \frac{\zeta(1-a)}{a(1-a)} x^{1-a} + \frac{\zeta(1-a)}{a} \left(\bar{B}_1(x) - \frac{1}{2} \right) x^{-a}, & 0 < a < 1, \\ & - \frac{\zeta(1-a)}{a(1-a)} \delta_{a,2} \delta_{a,3} x^{1-a} + f_1(x) x^{-1}, & 1 < a \leq 3, \\ & \sum_{1 \leq r \leq (a+1)/2} f_r(x) x^{-r} + 2x \sum_{n > x} \frac{G_{-a,1}(n)}{n(n+1)^2} \\ & - x \sum_{n > x} \frac{G_{-a,1}(n)}{n^2(n+1)} + x \sum_{n > x} \frac{G_{-a,1}(n)}{n^2(n+1)^2} - \left(\bar{B}_1(x) - \frac{1}{2} \right) \\ & \cdot \left\{ \sum_{n > x} \frac{G_{1-a,2}(n)}{n(n+1)^2} + 2 \sum_{n > x} \frac{G_{-a,1}(n)}{n(n+1)^2} - \sum_{n > x} \frac{G_{-a,1}(n)}{n^2(n+1)} \right. \\ & + \frac{1}{2} \sum_{n > x} \frac{G_{1-a,2}(n)}{n^2(n+1)^2} + \sum_{n > x} \frac{G_{-a,1}(n)}{n^2(n+1)^2} - \frac{1}{2x^2} G_{1-a,2}(x) \\ & \left. + x \sum_{n > x} \frac{G_{1-a,2}(n)}{n(n+1)^2} + \frac{1}{2} x \sum_{n > x} \frac{G_{1-a,2}(n)}{n^2(n+1)^2} \right\}, & a > 3, \end{aligned} \right. \\ \left. \begin{aligned} & - \frac{1}{2x} (x^{-a} G_{1+a,2}(x) + G_{1-a,2}(x)) \\ & + O(x^{-(a+1)/2}), \end{aligned} \right.$$

where,

$$f_1(x) = \frac{\zeta(a+1)}{6} + \frac{\zeta(a)}{4} - \frac{\zeta(a+1)}{2} \bar{B}_2(x) + \left(\bar{B}_1(x) - \frac{1}{2} \right) G_{-a,1}(x),$$

$$\begin{aligned}
 f_2(x) &= (-8\bar{B}_3(x) + 6\bar{B}_2(x) - 2\bar{B}_1(x) - 1) \frac{\zeta(a+1)}{12} + (6\bar{B}_1(x) - 1) \frac{\zeta(a)}{24} \\
 &\quad + (\bar{B}_2(x) - \bar{B}_1(x) + 3^{-1}) G_{-a,1}(x), \\
 f_r(x) &= c_r(x) - (\bar{B}_1(x) - 1/2) c_{r-1}(x), & r \geq 3, \\
 W_{-1}^0 &= \gamma^2 + 2\gamma_1 - \frac{\zeta''(0)}{2} + \frac{3}{4} - \gamma \log 2\pi, & a=0, \\
 W_{-2}^1 &= (\log 2\pi - \zeta(2) \log 2\pi + 2\zeta'(2) + 2\gamma\zeta(2) + \gamma + 1)/2, & a=1, \\
 W_{-a-1}^a &= \frac{\zeta(a)}{2} + \zeta'(a+1) + \gamma\zeta(a+1) - \frac{\zeta(a+1)}{2} \log 2\pi, & a \neq 0, 1,
 \end{aligned}$$

$$\begin{aligned}
 (3) \int_1^x E_{-a-1}^a(u) du &= Y_{-a-1}^a - \frac{1}{2x} (x^{-a} G_{1+a,2}(x) + G_{1-a,2}(x)) \\
 &\quad + \left\{ \begin{aligned} &2x \sum_{n>x} \frac{G_{1,2}(n)}{n^3}, & a=0, \\ &\frac{1}{2} \log x, & a=1, \\ &-\frac{\zeta(1-a)}{a(1-a)} \delta_{a,2} \delta_{a,3} x^{1-a} \\ &0, & 0 < a < 1, \\ &\left(\frac{\zeta(a+1)}{6} + \frac{\zeta(a)}{4}\right) x^{-1}, & 1 < a \leq 3, \\ &\sum_{1 \leq r \leq (a+1)/2} c_r(x) x^{-r} + 2x \sum_{n>x} \frac{G_{-a,1}(n)}{n(n+1)^2} \\ &-x \sum_{n>x} \frac{G_{-a,1}(n)}{n^2(n+1)} + x \sum_{n>x} \frac{G_{1+a,2}(n)}{n^{a+1}(n+1)^2} \\ &+x \sum_{n>x} \frac{G_{1-a,2}(n)}{n(n+1)^2} + \frac{1}{2} x \sum_{n>x} \frac{G_{1-a,2}(n)}{n^2(n+1)^2}, & a > 3, \end{aligned} \right. \\
 &\quad + O(x^{-(a+1)/2} (\log x)^{\delta_{a,1}}),
 \end{aligned}$$

where,

$$Y_{-a-1}^a = \begin{cases} \gamma^2 + 2\gamma_1 - 2\gamma + 3/4, & a=0, \\ (\log 2\pi + \gamma + 1)/2 + \zeta'(2) + \gamma\zeta(2) - \zeta(2), & a=1, \\ \zeta'(a+1) + \gamma\zeta(a+1) - \zeta(a+1) + \zeta(a)/2, & a \neq 0, 1, \end{cases}$$

By Lemma 8 and (1.1), (1.2), (1.4), we have

COROLLARY 2. For any $\epsilon > 0$,

$$\begin{aligned}
\text{(i)} \quad \sum_{n \leq x} \frac{d(n)}{n} &= \frac{1}{2} \log^2 x + 2\gamma \log x + \gamma^2 + 2\gamma_1 + O(x^{-15/22+\epsilon}), \\
\text{(ii)} \quad \sum_{n \leq x} E_{-1}^0(n) &= \frac{1}{4} \log^2 x + \gamma \log x + W_{-1}^0 + \begin{cases} O(x^{-1/4}), \\ \Omega_{\pm}(x^{-1/4}), \end{cases} \\
\int_1^x E_{-1}^0(u) du &= Y_{-1}^0 + \begin{cases} O(x^{-1/4}), \\ \Omega_{\pm}(x^{-1/4}), \end{cases} \\
\text{(iii)} \quad \sum_{n \leq x} E_{-2}^1(n) &= \frac{\zeta(2)+1}{2} \log x + W_{-2}^1 + \begin{cases} O(x^{\phi(0)-1+\epsilon}), \\ \Omega_{\pm}(x^{-3/4}(\log x)^{1/4}), \end{cases} \\
\int_1^x E_{-2}^1(u) du &= \frac{1}{2} \log x + Y_{-2}^1 + \begin{cases} O(x^{\phi(0)-1+\epsilon}), \\ \Omega_{\pm}(x^{-3/4}(\log x)^{1/4}), \end{cases} \\
\text{(iv)} \quad \sum_{n \leq x} E_{-3}^2(n) &= \frac{\zeta(3)}{2} \log x + W_{-3}^2 + f_1(x)x^{-1} + \begin{cases} O(x^{-1+\epsilon}), \\ \Omega_{\pm}(x^{-1} \log \log x), \end{cases} \\
\int_1^x E_{-3}^2(u) du &= Y_{-3}^2 + \begin{cases} O(x^{-1+\epsilon}), \\ \Omega_{\pm}(x^{-1} \log \log x). \end{cases}
\end{aligned}$$

Similarly, for other values of a the order of the error term essentially depends on that of $G_{1-a,2}(x)$.

THEOREM 3. (1)

$$\begin{aligned}
\sum_{n \leq x} \frac{\sigma_a(n)}{n^{a+2}} &= \zeta(2)\zeta(a+2) - x^{-2}(x^{-a}G_{a,1}(x) + G_{-a,1}(x)) + O(x^{-a/2-2}) \\
&+ \begin{cases} -x^{-1} \log x - (2\gamma+1)x^{-1}, & a=0, \\ -\zeta(a+1)x^{-1} + \begin{cases} -\frac{\zeta(1-a)}{a+1}x^{-1-a}, & 0 < a < 2, \\ -\frac{1}{x^3}G_{1-a,2}(x) & a=2, \\ \left(\frac{5}{18}\zeta(a+1) + \frac{1}{3}\zeta(a)\right)x^{-3} - \frac{1}{x^3}G_{1-a,2}(x), & 2 < a \leq 3, \end{cases} \end{cases}
\end{aligned}$$

(2) On defining $E_{-a,2}^a(x)$ by

$$E_{-a,2}^a(x) = \sum_{n \leq x} \frac{\sigma_a(n)}{n^{a+2}} - \begin{cases} \zeta^2(2) - x^{-1} \log x - (2\gamma+1)x^{-1}, & a=0, \\ \zeta(2)\zeta(a+2) - \zeta(a+1)x^{-1}, & 0 < a \leq 3, \end{cases}$$

we have

$$\sum_{n \leq x} E_{a-2}^a(n) = W_{a-2}^a - \frac{1}{2x^2} (x^{-a} G_{1+a,2}(x) + G_{1-a,2}(x))$$

$$+ \begin{cases} -\frac{1}{2x} \log x - \left(\gamma + \frac{1}{2}\right) x^{-1}, & a=0, \\ \frac{\zeta(1-a)}{a(1+a)} \bar{\delta}_{a,2} \bar{\delta}_{a,3} x^{-a} - \frac{\zeta(a+1)}{2} x^{-1} \\ \left\{ \frac{\zeta(1-a)}{a+1} \left(\bar{B}_1(x) - \frac{1}{2}\right) x^{-1-a}, \right. & 0 < a < 1, \\ \left. + \left(\bar{B}_1(x) - \frac{1}{2}\right) G_{-a,1}(x) x^{-2}, \right. & a=1, \\ \left. \left\{ \frac{\zeta(a+1)}{9} + \frac{\zeta(a)}{12} - \frac{\zeta(a+1)}{2} \bar{B}_2(x) + \left(\bar{B}_1(x) - \frac{1}{2}\right) G_{-a,1}(x) \right\} x^{-2}, \right. & 1 < a \leq 3, \end{cases}$$

$$+ O(x^{-(a+3)/2}),$$

where

$$W_{a-2}^a = \begin{cases} \zeta^2(2) + \gamma^2 - \gamma - \gamma_1 - 1, & a=0, \\ \zeta(2)\zeta(a+2) - \zeta'(a+1) - \zeta(a+1), & \text{otherwise,} \end{cases}$$

$$(3) \int_1^x E_{a-2}^a(u) du = Y_{a-2}^a - \frac{1}{2x^2} (x^{-a} G_{1+a,2}(x) + G_{1-a,2}(x))$$

$$+ \begin{cases} 0, & a=0, \\ \frac{\zeta(1-a)}{a(1+a)} \bar{\delta}_{a,2} \bar{\delta}_{a,3} x^{-a} + \begin{cases} 0, & 0 < a \leq 1, \\ \left(\frac{\zeta(a+1)}{9} + \frac{\zeta(a)}{12}\right) x^{-2} & 1 < a \leq 3, \end{cases} \\ + O(x^{-(a+3)/2}), \end{cases}$$

where,

$$Y_{a-2}^a = \begin{cases} \zeta^2(2) - \gamma^2 - 2\gamma - 2\gamma_1 - 1, & a=0, \\ \zeta(2)\zeta(a+2) - \zeta(a+1) - \gamma\zeta(a+1) - \zeta'(a+1), & 0 < a \leq 3. \end{cases}$$

THEOREM 4. (1) For $t \geq 2$, we have

$$\sum_{n \leq x} \frac{\sigma_a(n)}{n^{a+t}} = \zeta(t)\zeta(a+t)$$

$$+ \begin{cases} -\frac{1}{t-1} x^{-t+1} \log x - \frac{1}{t-1} \left(2\gamma + \frac{1}{t-1}\right) x^{-t+1}, & a=0, \\ \left\{ \begin{array}{l} -\frac{\zeta(1-a)}{a+t-1} x^{-a-t+1}, \\ -\frac{\zeta(a+1)}{t-1} x^{-t+1} + \begin{cases} -\frac{t}{2} x^{-t-1} G_{-1,2}(x), & a=2, \\ (\alpha(t)\zeta(a+1) + \beta(t)\zeta(a)) x^{-t-1} - \frac{t}{2} x^{-t-1} G_{1-a,2}(x), & 2 < a \leq 3, \end{cases} \end{array} \right. & 0 < a < 2, \end{cases}$$

$$- x^{-t} (x^{-a} G_{a,1}(x) + G_{-a,1}(x)) + O(x^{-a/2-t}),$$

where,

$$\alpha(t) = \frac{t^2 + 1}{6(t+1)}, \quad \beta(t) = \frac{t}{2(t+1)}.$$

(2) On defining $E_{a-t}^a(x)$ by

$$E_{a-t}^a(x) = \sum_{n \leq x} \frac{\sigma_a(n)}{n^{a+t}} - g_{a-t}^a(x),$$

where

$$g_{a-t}^a(x) = \zeta(t)\zeta(a+t) + \begin{cases} -\frac{1}{t-1}x^{-t+1} \log x - \frac{1}{t-1}\left(2\gamma + \frac{1}{t-1}\right)x^{-t+1}, & a=0, \\ -\frac{\zeta(a+1)}{t-1}x^{-t+1}, & 0 < a \leq 3, \end{cases}$$

we have, for $t \geq 3$,

$$\begin{aligned} \sum_{n \leq x} E_{a-t}^a(n) &= W_{a-t}^a \\ &+ \begin{cases} -\frac{1}{2(t-1)}x^{-t+1} \log x - \frac{1}{2(t-1)}\left(2\gamma + \frac{1}{t-1}\right)x^{-t+1}, & a=0, \\ -\frac{\zeta(a+1)}{2(t-1)}x^{-t+1} + \frac{\zeta(1-a)}{(a+t-1)(a+t-2)}\bar{\delta}_{a,2}\bar{\delta}_{a,3}x^{-a-t+2} \\ + \begin{cases} \frac{\zeta(1-a)}{a+t-1}\left(\bar{B}_1(x) - \frac{1}{2}\right)x^{-a-t+1}, & 0 < a < 1, \\ \left(\bar{B}_1(x) - \frac{1}{2}\right)G_{-1,1}(x)x^{-t}, & a=1, \\ \left(\alpha'(t)\zeta(a+1) + \beta'(t)\zeta(a) - \frac{\zeta(a+1)}{2}\bar{B}_2(x)\right)x^{-t} + \left(\bar{B}_1(x) - \frac{1}{2}\right)G_{-a,1}(x)x^{-t}, & 1 < a \leq 3, \end{cases} \end{cases} \\ &- \frac{1}{2x^t}(x^{-a}G_{1+a,2}(x) + G_{1-a,2}(x)) + O(x^{-a/2-t+1/2}), \end{aligned}$$

where,

$$W_{a-t}^a = \begin{cases} \zeta^2(t) - \zeta^2(t-1) - \frac{\zeta(t-1)}{t-1}\left(2\gamma + \frac{1}{t-1}\right) - \frac{\zeta(t-1)}{t-1}, & a=0, \\ \zeta(t)\zeta(a+t) - \zeta(t-1)\zeta(a+t-1) + \frac{\zeta(a+1)}{t-1}\zeta(t-1), & \text{otherwise.} \end{cases}$$

(3) For $t \geq 3$,

$$\int_1^x E_{-a-t}^a(u) du = Y_{-a-t}^a + \begin{cases} 0, & a=0, \\ \frac{\zeta(1-a)}{(a+t-2)(a+t-1)} \bar{\delta}_{a,2} \bar{\delta}_{a,3} x^{-a-t+2} \\ + \begin{cases} 0, & 0 \leq a \leq 1, \\ (\alpha'(t)\zeta(a+1) + \beta'(t)\zeta(a))x^{-t}, & 1 < a \leq 3, \end{cases} \end{cases}$$

$$- \frac{1}{2x^t} (x^{-a} G_{1+a,2}(x) + G_{1-a,2}(x)) + O(x^{-a/2-t+1/2}),$$

where,

$$Y_{-a-t}^a = \begin{cases} \zeta^2(t) - \zeta^2(t-1) + \frac{1}{(t-1)(t-2)} \left(2\gamma + \frac{1}{t-1} + \frac{1}{t-2} \right) & a=0, \\ \zeta(t)\zeta(a+t) - \zeta(t-1)\zeta(a+t-1) + \frac{\zeta(a+1)}{(t-1)(t-2)}, & \text{otherwise,} \end{cases}$$

$$\alpha'(t) = \alpha(t) - \alpha(t-1), \quad \beta'(t) = \beta(t) - \beta(t-1).$$

REMARK 2. From (1.1)-(1.5) and [7] the last but one term involving $G_{a,k}$ -functions can be estimated as follows:

(i) $-x^{-t} G_{-a,1}(x) + O(x^{-a/2-t+\alpha+\epsilon})$

$$= \begin{cases} O(x^{-a/2-t+\alpha+\epsilon}), & 0 \leq a \leq \beta, \\ O(x^{\phi(-a)-t+\epsilon}), & \beta < a < 1, \\ O((\log x)^{2/3}), & a = 1, \end{cases}$$

$$= \begin{cases} \Omega_+(x^{-t+1/4}(\log x)^{1/4}(\log \log x)^{(3+2 \log 2)/4} \exp(-A \sqrt{\log \log x \log x})), & a=0, \\ \Omega_-(x^{-t+1/4} \exp(c(\log \log x)^{1/4}(\log \log \log x)^{-3/4})), & a=0, \\ \Omega_{\pm} \left(x^{-t} \exp \left(c_{-a} \frac{(\log x)^{-\alpha+1}}{\log \log x} \right) \right), & 0 < a < 1, \\ \Omega_{\pm}(x^{-t} \log \log x), & a=1, \\ \Omega_{\pm}(x^{-t}), & a > 1. \end{cases}$$

(ii) $-\frac{1}{2x^{-t}} G_{1-a,2}(x) + O(x^{-a/2-t+3/4})$

$$= \begin{cases} O(x^{-a/2-t+3/4}), & 0 \leq a \leq 1/2, \\ O(x^{\phi(1-a)-t+\epsilon}), & 1/2 < a \leq 2, \\ O(x^{-t}), & a > 2, \end{cases}$$

$$= \begin{cases} \mathcal{O}_{\pm}(x^{-a/2-t+3/4}), & 0 \leq a < 1/2, \\ \mathcal{O}_{+}(x^{-a/2-t+3/4} f_a(x)), & 1/2 \leq a \leq 1, \\ \mathcal{O}_{\pm}(x^{-a/2-t+3/4} f_a(x)), & 1 < a \leq 3/2, \\ \mathcal{O}_{+}\left(x^{-t} \exp\left(c_{1-a} \frac{(\log x)^{2-a}}{\log \log x}\right)\right), & 3/2 < a < 2, \\ \mathcal{O}_{\pm}(x^{-t} \log \log x), & a = 2, \\ \mathcal{O}_{\pm}(x^{-t}), & a > 2. \end{cases}$$

3. Preliminaries.

LEMMA 1. (MacLead [8]) For $t > 0$, let

$$\sum_{n \leq x} n^{-t} \sigma_a(n) = g_{-t}^a(x) + E_{-t}^a(x),$$

and suppose the series

$$\sum_{n=1}^{\infty} \frac{E_{-t}^a(n)}{n^2}, \quad \sum_{n=1}^{\infty} \frac{E_{-t}^a(n)}{n^2(n+1)}, \quad \sum_{n=1}^{\infty} \frac{g_{-t}^a(n)}{n^2(n+1)},$$

all converge and that $x^{-2} \sum_{n \leq x} E_{-t}^a(n) \rightarrow 0$ as $x \rightarrow \infty$. Then we have

$$(1) \quad \sum_{n \leq x} n^{-t-1} \sigma_a(n) = \sum_{i=1}^6 S_i(t) + K_{-t}^a,$$

where $S_1(t) = \sum_{n \leq x} \frac{g_{-t}^a(n)}{n^2} + \sum_{n > x} \frac{g_{-t}^a(n)}{n^2(n+1)},$

$$S_2(t) = -2 \sum_{n > x} \frac{G_{-t}^a(n)}{n(n+1)^2} - \sum_{n > x} \frac{G_{-t}^a(n)}{n^2(n+1)^2},$$

$$S_3(t) = G_{-t}^a(x)([x] + 1)^{-2},$$

$$S_4(t) = \sum_{n \leq x} \frac{E_{-t}^a(n)}{n^2(n+1)},$$

$$S_5(t) = (\bar{B}_1(x) - 1/2)x^{-1}([x] + 1)^{-1} \sum_{n \leq x} n^{-t} \sigma_a(n),$$

$$S_6(t) = x^{-1} \sum_{n \leq x} n^{-t} \sigma_a(n),$$

and that

$$(2) \quad \sum_{n \leq x} (x-n)n^{-t-1} \sigma_a(n) = x \sum_{i=1}^5 S_i(t) + xK_{-t}^a,$$

where K_{-t}^a is the sum of the three series, $G_{-t}^a(x)$ is defined by

$$G_{-t}^a(x) = \sum_{n \leq x} E_{-t}^a(n),$$

and the R. H. S. of (2) should be interpreted as the sum of first five terms in (1)

multiplied by x , with an error term better than the error term for the L. H. S. of (1) by $x^{-3/2}$.

$$(3) \quad \sum_{n \leq x} E^{\alpha}_t(n) = \sum_{n \leq x} (x-n)n^{-t} \sigma_{\alpha}(n) - \sum_{n \leq x} g^{\alpha}_t(n) - (\bar{B}_1(x) - 1/2) \sum_{n \leq x} n^{-t} \sigma_{\alpha}(n),$$

and (Segal [17])

$$(4) \quad \int_1^x E^{\alpha}_t(u) du = \sum_{n \leq x} (x-n)n^{-t} \sigma_{\alpha}(n) - \int_1^x g^{\alpha}_t(u) du.$$

LEMMA 2. For $v > 0$, let

$$L_{-v, -1}(x) = \sum_{n > x} n^{-v}(n+1)^{-1}, \quad M_{-v, -1}(x) = \sum_{n > x} n^{-v}(n+1)^{-1} \log n,$$

and for $v > -1$, let

$$L_{-v, -2}(x) = \sum_{n > x} n^{-v}(n+1)^{-2}, \quad M_{-v, -2}(x) = \sum_{n > x} n^{-v}(n+1)^{-2} \log n.$$

Then for any $N \in \mathbb{N}$,

$$(1) \quad L_{-v, -1}(x) = \sum_{r=0}^N \frac{(-1)^r}{v+r} \sum_{s=0}^r (-1)^s \binom{v+r}{s} \bar{B}_s(x) x^{-v-r} + O(x^{-N-v-1}),$$

$$(2) \quad M_{-v, -1}(x) = \sum_{r=0}^N \frac{(-1)^r}{v+r} \sum_{s=0}^r (-1)^s \binom{v+r}{s} \bar{B}_s(x) (\log x + j_{s-2}(s-v-r-1)) x^{-v-r} + O(x^{-N-v-1} \log x),$$

$$(3) \quad L_{-v, -2}(x) = \sum_{r=0}^N \frac{(-1)^r}{v+r+1} \sum_{s=0}^r (-1)^s (r-s+1) \binom{v+r+1}{s} \bar{B}_s(x) x^{-v-r-1} + O(x^{-N-v-2}),$$

$$(4) \quad M_{-v, -2}(x) = \sum_{r=0}^N \frac{(-1)^r}{v+r+1} \sum_{s=0}^r (-1)^s (r-s+1) \binom{v+r+1}{s} \bar{B}_s(x) (\log x + j_{s-2}(s-v-r-2)) x^{-v-r-1} + O(x^{-N-v-2} \log x),$$

where $j_{n-1}(u) = \sum_{k=0}^{n-1} 1/(u-k)$ for $n \geq 1$, $j_{-1}(u) = 0$, $j_{-2}(u) = -1/(u+1)$.

PROOF. Substituting

$$(n+1)^{-1} = \sum_{r=0}^N (-1)^r n^{-r-1} + O(n^{-N-2}),$$

$$(n+1)^{-2} = \sum_{r=0}^N (-1)^r (r+1) n^{-2-r} + O(n^{-N-3}),$$

in the definitions of L 's and M 's and applying the asymptotic formulas for $L_u(x)$ and $M_u(x)$ contained in I-Lemma 3, 8 we conclude the assertion.

LEMMA 3. Let $S_r(x) = (x-1)^r$. Then we have

$$(1) \quad S_r(x) = \sum_{u=0}^r a_u(r) B_u(x),$$

where $a_u(r) = \frac{(-1)^{r+u}}{r+1} \binom{r+1}{u}$ ($u \leq r$), and 0 otherwise.

$$(2) \quad B_u(x) = \sum_{r=0}^u b_r(u) S_r(x),$$

$$\text{where } b_r(u) = \begin{cases} \binom{u}{r}, & 0 \leq r \leq u-2, \\ u/2, & r = u-1, \\ 1, & r = u. \end{cases}$$

PROOF. These follow from the recurrence relations of Bernoulli numbers and Bernoulli polynomials.

LEMMA 4. For any nonnegative integer N , we have

$$(1) \quad ([x]+1)^{-1} = \sum_{r=0}^N \left(\sum_{u=0}^r \frac{(-1)^{r-u}}{r+1} \binom{r+1}{u} \bar{B}_u(x) \right) x^{-r-1} + O(x^{-N-2}),$$

$$(2) \quad ([x]+1)^{-2} = \sum_{r=1}^N \left(\sum_{u=0}^{r-1} (-1)^{r-u+1} \binom{r}{u} \bar{B}_u(x) \right) x^{-r-1} + O(x^{-N-2}).$$

PROOF. We have

$$(3.1) \quad \begin{aligned} ([x]+1)^{-1} &= x^{-1} (1 + (1 - \{x\})/x)^{-1} \\ &= \sum_{r=0}^N S_r(\{x\}) x^{-r-1} + O(x^{-N-2}), \end{aligned}$$

similarly

$$(3.2) \quad ([x]+1)^{-2} = \sum_{r=1}^N r S_{r-1}(\{x\}) x^{-r-1} + O(x^{-N-2}).$$

Now the result follows from Lemma 3.

LEMMA 5. For any nonnegative integer N , we have

$$(1) \quad (\bar{B}_1(x) - 1/2) x^{-1} ([x]+1)^{-1} = \sum_{r=1}^N \sum_{u=0}^r a_u(r) \bar{B}_u(x) x^{-1-r} + O(x^{-N-2}),$$

$$(2) \quad (\bar{B}_1(x) - 1/2) ([x]+1)^{-2} = \sum_{r=1}^N r \sum_{u=0}^r a_u(r) \bar{B}_u(x) x^{-1-r} + O(x^{-N-2}),$$

$$(3) \quad \begin{aligned} \bar{B}_2(x) ([x]+1)^{-2} &= \sum_{r=1}^N r \left(\sum_{u=0}^{r-1} (a_u(r+1) + a_u(r) + a_u(r-1)/6) \bar{B}_u(x) \right. \\ &\quad \left. + \bar{B}_{r+1}(x) + (1-r) \bar{B}_r(x)/2 \right) + O(x^{-N-2}), \end{aligned}$$

where $a_n(r)$ are defined in Lemma 3.

PROOF. From (3.1) and (3.2) we can expand the L.H.S. of (1), (2) in powers of x with coefficients $S_{r+1}(\{x\})$, which are expressed in terms of $\bar{B}_u(x)$ by Lemma 3, (1). Similarly, for (3) we first express $\bar{B}_2(x)$ in terms of $S_r(\{x\})$ by Lemma 3, (2), and then we can expand the L.H.S. of (3) in powers of x with coefficients involving $S_r(\{x\})$ and again we apply Lemma 3, (1) to express them via $\bar{B}_u(x)$.

LEMMA 6. For $u \in \mathbf{N}$, we have

$$(B_1(x) - 1/2)B_u(x) = \frac{1}{u+1} \sum_{m=0}^u \binom{u+1}{m} B_{u-m+1} B_m(x) + B_{u+1}(x).$$

PROOF. We have

$$\text{L. H. S.} = (x-1)B_n(x) = \sum_{r=0}^u b_r(u) S_{r+1}(x).$$

Using Lemma 3-(1), and interchanging the order of summation gives the result.

LEMMA 7. (cf. Walfisz [18]) Suppose $f(x)$ is a bounded, Riemann integrable function on $[a, b]$ and that it has the bounded, Riemann integrable derivative on $[a, b]$ except at integer points, where it has the right derivative. Then

$$\sum_{a < n \leq b} f(n) = \int_a^b f(y) dy + \bar{B}_1(a)f(a) - \bar{B}_1(b)f(b) + \int_a^b \bar{B}_1(y) f'(y) dy.$$

PROOF. The proof goes on the same lines as those of Lemma 1.3.1 in Walfisz. Indeed, by dividing the interval $(a, b]$ into subintervals, we see that it suffices to prove our formula in the case where there is at most one integer in $(a, b]$. We distinguish three cases. First, if there is no integer in $(a, b]$, our formula follows by integration by parts.

Secondly, if there is an integer n such that $a < n < b$, we proceed as in Walfisz to arrive at

$$\int_a^b \bar{B}_1(y) f(y) dy = \int_a^n + \int_n^b = \int_a^{n-0} + \int_n^b.$$

Since f has the right derivative at $x=n$, we may integrate the second as well as the first integral by parts, and so we infer that

$$\begin{aligned} \int_a^b \bar{B}_1(y) f'(y) dy &= \bar{B}_1(n-0) f(n) - \bar{B}_1(a) f(a) - \int_a^n f(y) dy \\ &\quad + \bar{B}_1(b) f(b) - \bar{B}_1(n) f(n) - \int_n^b f(y) dy \\ &= f(n) + \bar{B}_1(b) f(b) - \bar{B}_1(a) f(a) - \int_a^b f(y) dy. \end{aligned}$$

Finally, if b is an integer, we can repeat the argument of Walfisz.

LEMMA 8. Let $a(n)$ be a polynomial in n of degree t with leading coefficient positive and let $a < 2t - 2$. Then

$$T := \sum_{n > x} \frac{G_{a,k}(n)}{a(n)} = \begin{cases} O(x^{a/2-t+1}), & a > -2, \\ O(x^{-t}(\log x)^{\delta_{a,-1}}), & a \leq -2, \end{cases}$$

PROOF. Changing the order of summation, we have

$$T = \sum_{m \geq 1} m^a \sum_{n > x, n \geq m^2} a(n)^{-1} \bar{B}_k\left(\frac{n}{m}\right).$$

Dividing the range of m into two: $m \leq \sqrt{x}$, $m > \sqrt{x}$ and estimating the sum over $m > x$, $n = m^2$ trivially, we obtain

$$T = \sum_{m \leq \sqrt{x}} m^a S(x) + \sum_{m > \sqrt{x}} m^a S(m^2) + O(x^{a/2-t+1/2}),$$

where

$$S(Y) = \sum_{n > Y} a(n)^{-1} \bar{B}_k\left(\frac{n}{m}\right).$$

Now, by Lemma 7 and the second mean value theorem,

$$\begin{aligned} S(Y) &= \int_Y^\infty a(u)^{-1} \bar{B}_k\left(\frac{u}{m}\right) du + a(Y)^{-1} \bar{B}_k\left(\frac{Y}{m}\right) \\ &\quad + \int_Y^\infty \bar{B}_1(u) \left\{ \frac{d}{du} a(u)^{-1} \bar{B}_k\left(\frac{u}{m}\right) + a(u)^{-1} \frac{k}{m} \bar{B}_{k-1}\left(\frac{u}{m}\right) \right\} du \\ &= O(m^2 Y^{-t-1} + m Y^{-t}), \end{aligned}$$

since, by integration by parts

$$\int_Y^\infty \bar{B}_1(u) \bar{B}_k\left(\frac{u}{m}\right) du = O(m^2).$$

Hence

$$T = O \left(\sum_{m \leq \sqrt{x}} m^a (mx^{-t} + m^2 x^{-t-1}) \right) + O \left(\sum_{m > \sqrt{x}} m^{a-2t+1} \right) + O \left(x^{a/2-t+1/2} \right)$$

$$= \begin{cases} O \left(x^{a/2-t+1} \right), & \text{if } a > -2, \\ O \left(x^{-t} (\log x)^{\delta a, -2} \right), & \text{if } a \leq -2, \end{cases}$$

as asserted.

LEMMA 9. (Richert [16, Satz 2]). Let $Z(s) = \sum_{n=1}^{\infty} f(n)n^{-s}$ be a convergent Dirichlet series whose analytic continuation is meromorphic in every subdomain of C and satisfies $|Z(s)| = O(|t|^c)$ uniformly in σ with a constant c . Suppose there exist real numbers $\alpha, \bar{\alpha}$ such that $f(n) = O(n^{\alpha+\epsilon})$ and $Z(s)$ is absolutely convergent for $\sigma > \bar{\alpha}$. Suppose also that we can express $Z(s)$ as

$$Z(s) = C^{\theta s + \theta_1} G(s) Z_1(r-s)$$

with r, θ real, $C > 0$, θ_1 complex, and

$$G(s) = \sum_{j=1}^M \frac{\Gamma(\beta_j - b_j s)}{\Gamma(\delta_j + d_j s)},$$

where $M \in N, \beta_j, \delta_j \in R, b_j, d_j > 0, 1 \leq j \leq M$, and where $Z_1(s)$ is an analytic function defined by a convergent Dirichlet series $Z_1(s) = \sum_{n=1}^{\infty} f_1(n)n^{-s}$. Suppose further that there are two real numbers $\alpha_1, \bar{\alpha}_1$ such that $f_1(n) = O(n^{\alpha_1+\epsilon})$, and $\sum_{n=1}^{\infty} f_1(n)n^{-s}$ is absolutely convergent for $\sigma > \bar{\alpha}_1$. Then, if $\alpha \geq -1$ and $\kappa \geq 0$, we have, as $x \rightarrow \infty$,

$$\Gamma(\kappa+1)^{-1} \sum_{n \leq x} f(n)(x-n)^\kappa$$

$$= F_\kappa(x) + O \left(x^{\alpha+(\kappa+1) \left(1 - \frac{\bar{\alpha}_1 + \alpha + 1 - r}{q(\bar{\alpha}_1 - r) + \lambda + 1/2} \right) + \epsilon} \right) + O \left(x^{\alpha+\epsilon} \right) + O \left(x^{\kappa + \frac{\lambda - 1/2 - \kappa}{q} + \delta} \right),$$

where

$$F_\kappa(x) = \sum_{\min(r - \bar{\alpha}_1, \frac{\lambda - \kappa - 1}{q}) \leq \sigma \leq \alpha + 1} \text{Res } x^{s+\kappa} \frac{\Gamma(s)Z(s)}{\Gamma(s+\kappa+1)},$$

$$q = 2 \sum_{j=1}^M d_j, \quad \lambda = \sum_{j=1}^M (\beta_j - \delta_j).$$

4. Sketch of proofs

The proof of Theorem 1 goes along similar lines to those of proofs of Theorem 3 in [6]. In order to apply I-Lemma 3 without error term we have to restrict ourselves to the case $a-t \in Z$. In what follows, by (Case: $t=b$) we refer to formulas for

$$(1) \sum_{n \leq x} n^{-a-b} \sigma_a(n), \quad (2) \sum_{n \leq x} E_{a-b}^a(n), \quad (3) \int_1^x E_{a-b}^a(u) du.$$

Then, using Lemma 1-6, 8 we deduce Theorem 2 (Case: $t=1$) from Theorem 1 with $t=a$, which however, differs from the corresponding formulas in Theorem 4 (Case: $t \geq 2$). We therefore go on to deduce Theorem 3 (Case: $t=2$). Formula (1) in Theorem 3 coincides with Formula (1) in Theorem 4 with $t=2$, while neither Formulas (2) nor (3) coincide with those in Theorem 4. We then proceed to deduce Formulas (2) and (3) in (Case: $t=3$), which are seen to coincide with Formulas (2) and (3) in Theorem 4 with $t=3$.

To prove Theorem 4-(1) we apply induction on t . The starting point for Formula (1) is $t=2$ and, for Formulas (2) and (3), it is $t=3$. Applying the back-track method to Formulas for some t (which we assume to hold), we can check that the Formulas for $t+1$ are precisely those given in Theorem 4 with $t+1$ in place of t .

In order to derive Formulas (2) and (3), we need Riesz sums, which we calculate with the aid of Lemma 1-(2), Lemma 9. They read

PROPOSITION. For $3 \leq t \leq Z$, we have

$$\begin{aligned} \sum_{n \leq x} (x-n) \frac{\sigma_a(n)}{n^{a+t}} &= \zeta(t)\zeta(a+t)x - \zeta(t-1)\zeta(a+t-1) \\ &+ \begin{cases} \frac{1}{(t-1)(t-2)} x^{-t+2} \log x + \frac{1}{(t-1)(t-2)} \left(2\gamma + \frac{1}{t-2} + \frac{1}{t-1}\right) x^{-t+2}, & a=0, \\ \frac{\zeta(a+1)}{(t-1)(t-2)} x^{-t+2} + \frac{\zeta(1-a)}{(a+t-1)(a+t-2)} \delta_{a,2} \delta_{a,3} x^{-a-t+2} \\ + \begin{cases} 0, & 0 < \alpha \leq 1, \\ (\alpha'(t)\zeta(a+1) + \beta'(t)\zeta(a))x^{-t} & 1 < \alpha \leq 3, \end{cases} \\ - \frac{1}{2x^t} (x^{-a} G_{1+a,2}(x) + G_{1-a,2}(x)) + O(x^{-a/2-t+1/2}), \end{cases} \end{aligned}$$

where

$$\alpha'(t) = \alpha(t) - \alpha(t-1), \quad \beta'(t) = \beta(t) - \beta(t-1).$$

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