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Average orders of multiplicative arithmetical functions of integer matrices

by

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1. Introduction. This paper is an attempt at being a part of the process of systematizing the study of the arithmetic of integer matrices. In this study a major departure from the case of multiplicative arithmetic on integers is the *absence of commutativity* which is both a cause of difficulty and interest.

One way of studying this arithmetic is to look at multiplicative arithmetical functions of integer matrices. In general, the pointwise evaluation of such functions is not simple, though certain functions have been completely (e.g. Euler ϕ function [3], [6]) or partially (e.g. the divisor functions [6], [7], [2]) evaluated. In special cases considered earlier, the analytic questions are often related to those raised in different contexts in arithmetic of integers (see, e.g. [4], [5]). Here we use the available knowledge on pointwise behaviour to study average orders of multiplicative functions, a problem for which we present some tools and ask some questions.

In Section 2 we present some necessary background material.

In Section 3 we determine the *Dirichlet series* associated with the *convolution product* of multiplicative functions on 2×2 matrices. The formula obtained (Theorem 1) is complicated. As a special case we are able to recover a previous result for the divisor function ([1], Theorem 5). We are also able to obtain average orders of other "natural" functions like the Euler ϕ function and the general divisor functions (see Corollaries 1–3).

We are unable to obtain an analogue of Theorem 1 for matrices of higher dimensions. Hence we concentrate on a more restricted problem which we expect to be generic, i.e. the *average order* of the *divisor function*. Here we have at our disposal a recurrence formula (Theorem 2). In Section 4 we use this to evaluate the divisor function pointwise for 3×3 matrices (Proposition 3) and the Dirichlet series associated with the function (Proposition 4).

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In Section 5 we use the same recurrence formula to obtain the *abscissa of convergence* of the Dirichlet series associated with the divisor function of matrices of arbitrary size (Theorem 3).

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2. Preliminaries. Most of the concepts mentioned here are on the lines of Nanda [7], [8]. Some of them have already appeared in print, but we include them nevertheless for the sake of readability. Latin upper case italics denote matrices and small Greek letters are used for functions.

Let $\mathbb{Z}(r,r)$ be the set of $r\times r$ non-singular integer matrices. For the purpose of arithmetic $\mathrm{GL}_r(\mathbb{Z})$, the group of units of $\mathbb{Z}(r,r)$, have to be handled carefully.

For $M,N\in\mathbb{Z}(r,r)$ to be considered *equivalent* (notation: $M\sim N$) there must exist $U,V\in\mathrm{GL}_r(\mathbb{Z})$ such that

$$M=UNV\,. \tag{2.1}$$

They are considered *right equivalent* if

$$M=UN\,. \tag{2.2}$$

Classical results due to H. J. S. Smith (1861) and C. Hermite (1851) assure us that every $M\in\mathbb{Z}(r,r)$ is

(a) equivalent to a *unique* diagonal matrix $S=(s_{ij})$ in $\mathbb{Z}(r,r)$ where $s_{kk}=m_1(M)m_2(M)\dots m_k(M)$, with $m_i(M)\in\mathbb{Z}^+$. We call $m_i(M)$ the ith *Smith invariant* of M and S the *Smith Normal Form* (SNF) of M; and

(b) right equivalent to a unique lower triangular matrix $H=(h_{ij})$ in $\mathbb{Z}(r,r)$ where $h_{kk}\in\mathbb{Z}^+$ and $0\le h_{k+t,k}<h_{kk}$. We call H the *Hermite Normal Form* (HNF) of M.

An *arithmetical function* χ is a mapping from $\mathcal{M}_{\mathbb{Z}}$, the set of integer matrices, to $\mathbb{C}$ such that

$$\begin{gathered}\chi(M)=\chi(N)\quad \text{if } M\sim N\,;\quad \text{and}\\ \chi\begin{pmatrix}1&0\\0&M\end{pmatrix}=\chi(M)\,.\end{gathered} \tag{2.3}$$

After preliminary considerations, it is enough to restrict the domain of χ to $\mathbb{Z}(r,r)$.

We note that arithmetical functions are completely determined pointwise if their values on SNF matrices are known.

A (canonical) *factorization* of M is defined as

$$M = M_1 M_2 \tag{2.4}$$

where M_2 is in HNF. We call M_2 a *divisor* of M and use the notation $M_2 \mid M$.

The *divisor function* $\tau^{(r)}(M) = \tau(M)$ counts the number of (inequivalent) factorizations of $M \in \mathbb{Z}(r,r)$, i.e.

$$\tau^{(r)}(M) = \tau(M) = \sum_{M_2 \mid M} 1\,. \tag{2.5}$$

It has been proved that τ is finite. We note that τ is a special member of a *class of divisor functions* $\sigma_a(M),\ a \in \mathbb{C}$, defined by

$$\sigma_a^{(r)}(M) = \sigma_a(M) = \sum_{M_2 \mid M} (\det M_2)^a\,. \tag{2.6}$$

A *multiplicative function* χ is distributive over matrices with co-prime determinants, i.e.

$$\chi(MN) = \chi(M)\chi(N) \quad \text{whenever } (\det M, \det N) = 1\,. \tag{2.7}$$

We let Γ be the set of all multiplicative arithmetical functions.

For pointwise evaluation of multiplicative functions it is enough to consider only diagonal matrices $F_r^{(p)} = F_r = (f_{ij})$ in $\mathbb{Z}(r,r)$ where p is a given prime number and $f_{kk} = p^{f_1+f_2+\ldots+f_k}$ for a positive integer f_1, and non-negative integers $f_2,\ldots,f_k$. We use the notation $\langle f_1, f_2, \ldots, f_r\rangle$ for F_r. Another matrix often used in pointwise evaluation of divisor functions is $\langle f_1, f_2, \ldots, f_r - 1\rangle$ and is denoted by G_r.

We define the *primes of* $\mathbb{Z}(r,r)$ with respect to prime numbers p and positive integers j and $r,\ j \le r$, as

$$P_{j,r}^{(p)} = P_r = \begin{pmatrix} E_{r-j} & 0 \\ 0 & pE_j \end{pmatrix} \tag{2.8}$$

where E and 0 denote identity and zero matrices respectively.

The *Dirichlet convolution* $*$ of two functions χ_1 and χ_2 is defined as

$$(\chi_1 * \chi_2)(M) = \sum_{M_2 \mid M} \chi_1(M M_2^{-1})\chi_2(M_2)\,. \tag{2.9}$$

It has been proved that $(\Gamma, *)$ is an abelian group with its *identity* η defined as

$$\eta(M) = \begin{cases} 1, & M \sim E\,, \\ 0, & \text{otherwise}\,. \end{cases} \tag{2.10}$$

The *unit function* $\mathbf{1}$ is defined as

$$\mathbf{1}(M) = 1 \quad \forall M\,,$$

so that

$$\tau = \mathbf{1} * \mathbf{1}\,. \tag{2.11}$$

The *Möbius function* μ is defined as

$$\mu * \mathbf{1} = \eta\,. \tag{2.12}$$

For explicit evaluation, see [6].

We let $\nu(M) = \det M$, and define the *Euler ϕ function* as

$$\phi = \mu * \nu\,. \tag{2.13}$$

3. Dirichlet series. We can understand the convolution product by studying the associated Dirichlet series. We formally define $D_H(\chi; s_1, s_2, \ldots, s_r)$ as

$$\begin{aligned} D_H(\chi; s_1, s_2, \ldots, s_r) &= \sum_{H \text{ in HNF}} \frac{\chi(H)}{H^{s_1,\ldots,s_r}} \\ &= \sum_{H \text{ in HNF}} \frac{\chi(H)}{m_1(H)^{s_1} \ldots m_r(H)^{s_r}}\,. \end{aligned} \tag{3.1}$$

The drawback of this definition is that, for $r \geq 2$, we do not have, in general, the property

$$D_H(\chi_1 * \chi_2; s_1, \ldots, s_r) = D_H(\chi_1; s_1, \ldots, s_r) D_H(\chi_2; s_1, \ldots, s_r)\,, \tag{3.2}$$

which is a serious handicap. However, (3.2) holds if $s_1 = r s_r$, $s_2 = (r-1)s_r, \ldots, s_{r-1} = 2s_r$, and this enabled us [2] to evaluate

$$\sum_{\substack{H \text{ in HNF} \\ \det H \leq x}} \chi(H)$$

for some functions χ.

However, since the value of χ depends only on SNF matrices, we can instead study

$$D_S(\chi; s_1, \ldots, s_r) = \sum_{S \text{ in SNF}} \frac{\chi(S)}{S^{s_1,\ldots,s_r}}\,. \tag{3.3}$$

Clearly,

$$D_H(\chi; s_1, \ldots, s_r) = D_S(\chi h; s_1, \ldots, s_r) \tag{3.4}$$

where $h(S)$ is the number of HNF matrices equivalent to S. We now determine $D_S(\chi_1 * \chi_2; s_1, s_2)$.

3.1. *Convolution product for $r = 2$.* Here we obtain the Dirichlet series $D_S(\chi_1 * \chi_2; s_1, s_2)$ for two multiplicative arithmetical functions χ_1 and χ_2. Since $\chi_1 * \chi_2$ is multiplicative, it is enough to consider its value on F_2 (see

Section 1). We first obtain all divisors of F_2 and introduce this in the "p component" of $D_S(\chi_1 * \chi_2; s_1, s_2)$.

We require the following notations. The pair $(\langle k,s\rangle_p, \langle l,r\rangle_p)$ is said to be a *divisor couple* of $F_2^{(p)}$ with *multiplicity* m (the parameter p is usually suppressed) if there exist m distinct factorizations AB of F_2 such that $\mathrm{SNF}(A) = \langle k,s\rangle$ and $\mathrm{SNF}(B) = \langle l,r\rangle$.

We now recall (see e.g. [9]) the SNF of a given matrix.

Lemma 3.1.

$$\mathrm{SNF}\begin{pmatrix} u & 0 \\ v & w \end{pmatrix} = \begin{pmatrix} \gcd(u,v,w) & 0 \\ 0 & uw/\gcd(u,v,w) \end{pmatrix}. \tag{3.5}$$

We now study the divisor couples of F_2. We introduce the notation

$$\phi_+(n) = n \prod_{p\mid n} \frac{p+1}{p}.$$

Proposition 1. *Let k,l,s,r be non-negative integers and p a prime number. Then $(\langle k,s\rangle_p, \langle l,r\rangle_p)$ is a divisor couple of $F_2^{(p)}$ if and only if $k+l \le f_1$, and one of the following sets of conditions holds*:

Case 1: $f_2 = 0; k+l+r = f_1; s = r$. *Here the multiplicity is $\phi_+(p^r)$.*

Case 2: $f_2 \ne 0$ *and*

(a) $k+l+r = f_1; s = f_2 + r$. *Here the multiplicity is p^r.*

(b) $k+l+r = f_1 + t$ *for* $0 < t < f_2; s = f_2 + r - 2t$. *Here the multiplicity is $\phi(p^{r-t})$.*

(c) $k+l+r = f_1 + f_2; s = r - f_2$. *Here the multiplicity is p^s.*

Proof. Let us consider all factorizations of F_2, i.e.

$$F_2 = \begin{pmatrix} p^{f_1} & 0 \\ 0 & p^{f_1+f_2} \end{pmatrix} = \begin{pmatrix} p^{f_1-a} & 0 \\ -xp^{f_1+f_2-b-a} & p^{f_1+f_2-b} \end{pmatrix} \begin{pmatrix} p^a & 0 \\ x & p^b \end{pmatrix} \tag{3.6}$$

with $0 \le x < p^a$ and $xp^{f_1+f_2-b-a}$ integral.

We use the notation $\langle\langle u,v\rangle\rangle$ for $\langle u, v-2u\rangle$ and write x as $x = p^{a-d}x'$, with $(x',p) = 1$, $0 \le x' < p^d$, $0 \le d \le a$, $f_1+f_2-b-d \ge 0$ to get all divisor couples of F_2. (When $x = 0$ we fix $d = 0 = x'$.) We notice that all divisor couples are given by

$$\begin{aligned}(\langle\langle \min(f_1+f_2-b-d, f_1-a), 2f_1+f_2-a-b\rangle\rangle, \langle\langle \min(b, a-d), a+b\rangle\rangle)\\ = (\langle\langle \min(f_2-b-d+a, 0) + f_1 - a, 2f_1+f_2-a-b\rangle\rangle,\\ \langle\langle \min(f_2-b-d+a, f_2) + b - f_2, a+b\rangle\rangle)\end{aligned}$$

with multiplicity $\phi(p^d)$.

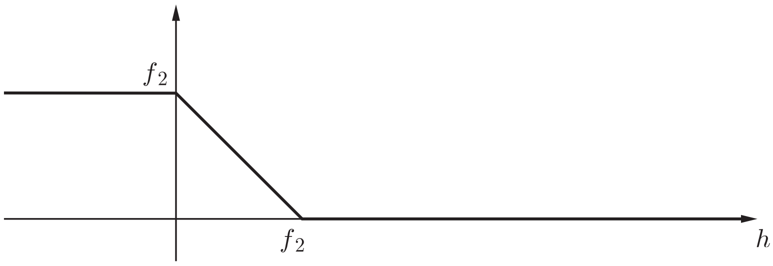
Now we look at the problem of determining the multiplicity of a divisor couple $(\langle k,s\rangle,\langle l,r\rangle)$ of F_2. We notice that for non-zero multiplicity

$$k+l\le f_1,\qquad 2(k+l)+r+s=2f_1+f_2\,.$$

Now let $h=f_2-b-d+a$. Then we get

$$k+l+r=\min(h,0)-\min(h,f_2)+f_1+f_2\,. \tag{3.7}$$

Since the graph of $h\to\min(h,0)-\min(h,f_2)+f_2$ is



we consider the possibilities for $k+l+r-f_1$ corresponding to the various cases of our proposition.

Case 2: $f_2\ne 0$ and

(c) $k+l+r=f_1+f_2$. Then

$$\begin{aligned}k&=f_1+f_2-b-d\,,\\ s&=-f_2+b+2d-a\,,\\ l&=a-d\,,\\ r&=b-a+2d\,,\end{aligned}$$

with $d\ge f_2-b+a$; and the multiplicity is $\phi(p^d)$. This implies that $s=r-f_2$ and for all d in $[0,f_1-(k+l)]$ there exist solutions.

(b) $k+l+r=f_1+t, 0<t<f_2$. Here $d=f_1-(k+l)$.

(a) $k+l+r=f_1$. This situation is similar to case 2(c).

Case 1: $f_2=0$. We take all solutions of cases 2(a) and 2(c) and subtract the solutions of case 2(b) from it. ■

Remark. Proposition 1 gives us another method of evaluating $\tau(F_2)$.

We now reach the main result of this section.

Theorem 1. *Let χ_1 and χ_2 be two multiplicative arithmetical functions on $M_2(\mathbb{Z})$ and p a prime number. Let*

$$\Xi_1(r) = \sum_{k\ge 0} \chi_1\langle k, r\rangle_p X^k\,, \tag{3.8}$$

$$\Xi_2(r) = \sum_{l\ge 0} \chi_2\langle l, r\rangle_p X^l \tag{3.9}$$

for any non-negative integer r *and an indeterminate* X*. Then*

$$\begin{aligned}\sum_{f_1,f_2\ge 0} (\chi_1 * \chi_2)F_2^{(p)} X^{f_1}Y^{f_2} &= \Big(\sum_{r\ge 0} \Xi_1(r)Y^r\Big)\Big(\sum_{s\ge 0}\Xi_2(s)Y^s\Big)\\ &\quad + \left(1-\frac{Y^2}{p^2X}\right)\sum_{t\ge 1}\left(\frac{pX}{Y^2}\right)^t \Big(\sum_{r\ge t}\Xi_1(r)Y^r\Big)\Big(\sum_{s\ge t}\Xi_2(s)Y^s\Big)\\ &\quad + \frac{1}{p}\Big(\sum_{r\ge 1}\Xi_1(r)Y^r\Big)\Big(\sum_{s\ge 1}\Xi_2(s)Y^s\Big)\\ &\quad + \frac{1}{p}\sum_{t\ge 1}\left(\frac{pX}{Y^2}\right)^t \Xi_1(t)Y^t\Xi_2(t)Y^t\,.\end{aligned} \tag{3.10}$$

P r o o f. From Proposition 1 we get

$$\begin{aligned}\sum_{f_1,f_2\ge 0} ((\chi_1 * \chi_2)F_2)X^{f_1}Y^{f_2} = \sum_{k,r,l,s\ge 0} \chi_1\langle k,r\rangle\chi_2\langle l,s\rangle X^{k+l}Y^{r+s}\Big\{\sum\nolimits_1 + \sum\nolimits_2 + \sum\nolimits_3 + \sum\nolimits_4\Big\}\end{aligned} \tag{3.11}$$

where

$$\sum\nolimits_1 = \sum_{\substack{f_1\ge 0\\ f_2=0}} X^rY^{-2r}\phi_+(p^r) \quad \text{if } r=s \quad \text{and } 0 \text{ if } r\ne s\,,$$

$$\sum\nolimits_2 = \sum_{\substack{f_1\ge 0, f_2>0\\ k+l+r=f_1+f_2,\, s=r-f_2}} p^sX^sY^{-2s}\,,$$

$$\sum\nolimits_3 = \sum_{\substack{f_1\ge 0, f_2>0\\ k+l+r=f_1,\, s=r+f_2}} p^rX^rY^{-2r}$$

and

$$\sum\nolimits_4 = \sum_{\substack{f_1\ge 0, f_2>0\\ 0<m<f_2,\, k+l+r=f_1+m,\, s=f_2-2m+r}} \phi(p^{r-m})X^{r-m}Y^{2(m-r)}\,.$$

We can thus write:

$$\sum\nolimits_1 = \begin{cases} \sum X^r Y^{-2r}\{2p^r - \phi(p^r)\} & \text{if } r = s\,,\\ 0 & \text{if } r \neq s\,,\end{cases} \tag{3.12}$$

$$\sum\nolimits_2 = \begin{cases} \sum \left(\frac{pX}{Y^2}\right)^s & \text{if } r > s\,,\\ 0 & \text{if } r \leq s\,,\end{cases} \tag{3.13}$$

$$\sum\nolimits_3 = \begin{cases} \sum \left(\frac{pX}{Y^2}\right)^r & \text{if } r < s\,,\\ 0 & \text{if } r \geq s\,,\end{cases} \tag{3.14}$$

and

$$\sum\nolimits_4 = \begin{cases} \displaystyle\sum_{t=0}^{\min(r,s)-1} \frac{\phi(p^t)}{p^t}\left(\frac{pX}{Y^2}\right)^t & \text{if } \min(r,s) > 0\,,\\ 0 & \text{otherwise.}\end{cases} \tag{3.15}$$

We can verify that

$$\sum_{t=0}^{y} \frac{\phi(p^t)}{p^t} T^t = \frac{1-T^{y+1}}{1-T} - \frac{T}{p(1-T)}(1-T^y)\,. \tag{3.16}$$

Thus

$$\begin{aligned} &\sum_{f_1,f_2\geq 0} (\chi_1 * \chi_2) F_2 X^{f_1} Y^{f_2} \\ &= -\frac{1}{p} \sum_{\substack{k,l,r,s\geq 0\\ \min(r,s)=0}} \chi_1\langle k,r\rangle X^k Y^r \chi_2\langle l,s\rangle X^l Y^s \\ &\quad + \frac{1}{p} \sum_{\substack{k,l\geq 0\\ r>0}} \chi_1\langle k,r\rangle X^k Y^r \chi_2\langle l,r\rangle X^l Y^r \left(\frac{pX}{Y^2}\right)^r \\ &\quad + \frac{1-X/Y^2}{1-pX/Y^2}\Big(\sum_{k,r\geq 0}\chi_1\langle k,r\rangle X^k Y^r\Big)\Big(\sum_{l,s\geq 0}\chi_2\langle l,s\rangle X^l Y^s\Big) \\ &\quad - \frac{pX/Y^2 - 1/p}{1-pX/Y^2} \sum_{k,l,r,s\geq 0} \chi_1\langle k,r\rangle X^k Y^r \chi_2\langle l,s\rangle X^l Y^s \left(\frac{pX}{Y^2}\right)^{\min(r,s)}. \end{aligned} \tag{3.17}$$

Now, after introducing $\Xi_1(r), \Xi_2(s)$ and $t = \min(r,s)$ and going through some cumbersome manipulations we get the required formula. ■

We can use Theorem 1 to obtain Dirichlet series and average orders of various multiplicative functions. We give a few examples here:

If $\chi_1 = \mathbf{1}$, then

$$\Xi_1(r) = \frac{1}{1-X}\,.$$

If $\chi_1 = \mu$, then

$$\Xi_1(r) = \begin{cases} 1+pX & \text{if } r=0\,, \\ -1 & \text{if } r=1\,, \\ 0 & \text{if } r>1\,. \end{cases}$$

If $\chi_1 = \nu^a$, then

$$\Xi_1(r) = \frac{p^{ar}}{1-p^{2a}X}\,.$$

From these we can verify that

$$D_S(\eta; s_1, s_2) = 1\,, \tag{3.18}$$

and obtain some corollaries.

Corollary 1.

$$D_S(\tau; s_1, s_2) = \zeta(s_1)^2\zeta(s_2)^2\zeta(s_1-1)\prod_p\left(1+\frac{1}{p^{s_1}}-\frac{2}{p^{s_1+s_2}}\right). \tag{3.19}$$

Corollary 2.

$$D_S(\phi; s_1, s_2) = \zeta(s_1-2)\zeta(s_2-1)\prod_p\left(1-\frac{1}{p^{s_1}}-\frac{1}{p^{s_2}}+\frac{1}{p^{s_1+s_2+1}}\right), \tag{3.20}$$

which further gives

$$\sum_{\det S\le x}\phi(S) \sim \frac{1}{2}\prod_p\left(1-\frac{1}{p^2}-\frac{1}{p^4}+\frac{1}{p^5}\right)x^2\log x\,. \tag{3.21}$$

Corollary 3.

$$\begin{aligned} &D_S(\sigma_a; s_1, s_2) \\ &= \zeta(s_1)\zeta(s_1-2a)\zeta(s_2)\zeta(s_2-a)\zeta(s_1-a-1)\prod_p\left(1+\frac{1}{p^{s_1-a}}-\frac{p^a+1}{p^{s_1+s_2-a}}\right). \end{aligned} \tag{3.22}$$

4. The divisor function for $r = 3$. For matrices of size greater than 2, we concentrate only on the divisor function. We use a recurrence formula we have earlier obtained to explicitly evaluate $\tau(F_3)$ and then find a Dirichlet series corresponding to it. In more complicated cases, this recurrence does not seem to yield a closed formula. Its repeated use, however, helps us obtain the abscissa of convergence of the Dirichlet series for arbitrary r, which we do in the next section.

We recall that the matrix G_r which occurs in the recurrence formula is a notation we use for a matrix of the form $\langle f_1, f_2, \ldots, f_r - 1\rangle_p$. Our *recurrence formula* is:

THEOREM 2 [2].

$$\sigma_a(F_r) - p^a\sigma_a(G_r) = \sigma_{a+1}(F_{r-1})\,. \tag{4.1}$$

We now use this theorem for $a = 0, r = 3$. But first we require the following proposition.

PROPOSITION 2.

$$\sigma_1(F_2) = (f_1+1)\frac{p^{2f_1+f_2+1}}{p-1} - \frac{p^{2f_1}}{p-1} - \frac{p^{2f_1}-1}{(p-1)(p^2-1)}\,. \tag{4.2}$$

Proof. We consider the factorization of F_2:

$$F_2 = \begin{pmatrix} p^{f_1} & 0 \\ 0 & p^{f_1+f_2} \end{pmatrix} = \begin{pmatrix} p^{f_1-t} & 0 \\ y & p^{f_1+f_2-u} \end{pmatrix}\begin{pmatrix} p^t & 0 \\ x & p^u \end{pmatrix} \tag{4.3}$$

with $0 \le x < p^t$, $yp^t + xp^{f_1+f_2-u} = 0$.

The number of solutions of this system is $(p^t, p^{f_1+f_2-b})$. We identify the following two cases:

Case 1: $t + u \le f_1 + f_2$. Here the contribution to $\sigma_1(F_2)$ comes from

$$\sum_{t=0}^{f_1}\sum_{u=0}^{f_1+f_2-t} p^{2t+u}\,. \tag{4.4}$$

Case 2: $t + u > f_1 + f_2$. The contribution to $\sigma_1(F_2)$ is now made by

$$\sum_{t=1}^{f_1}\sum_{u=f_1+f_2+1-t}^{f_1+f_2} p^{t+f_1+f_2}\,. \tag{4.5}$$

The sum of expressions (4.4) and (4.5) gives

$$\sigma_1(F_2) = (f_1+1)p^{2f_1}\sum_{n=0}^{f_2} p^n + (1+p^{-1})\sum_{n=0}^{f_1}(f_1-n)p^{2f_1-2n-1}\,. \tag{4.6}$$

But

$$\sum (f_1-n)p^{2f_1-2n-1} = f_1p^{2f_1-1}\sum p^{-2n} - p^{2f_1-1}\sum np^{-2n}\,.$$

We now use the fact that

$$\sum_{m=1}^{t} mx^m = x\sum_{m=1}^{t} mx^{m-1} = x\frac{d}{dx}\left\{\frac{1-x^{t+1}}{1-x}\right\} \tag{4.7}$$
$$= x\frac{1-x^{t+1}}{(1-x)^2} - (t+1)\frac{x^{t+1}}{1-x}$$

to get the result. ∎

Proposition 3.

$$\tau(F_3) = (f_1+1)(f_3+1)\frac{p^{2f_1+f_2+1}-p^{2f_1}}{p-1} + 2(f_1+1)\frac{p^{2f_1+f_2+1}-p^{2f_1}}{(p-1)^2} - 2(f_1+1)(f_2+1)\frac{p^{2f_1}}{p-1} + \left\{\left(\frac{1}{p}+1\right)(3f_1+2f_2+f_3+2)+\frac{2}{p}\right\} \times \left\{p\frac{1-p^{2f_1+2}}{(p^2-1)^2} + \frac{(f_1+1)p^{2f_1+1}}{p^2-1}\right\} - 3(p+1)\left\{\frac{2p^{2f_1+4}}{(p^2-1)^2} - (2f_1+3)\frac{p^{2f_1+2}}{(p^2-1)^2} + (f_1+1)^2\frac{p^{2f_1}}{p^2-1} - \frac{p^2+1}{(p^2-1)^3}\right\}. \tag{4.8}$$

Proof. We use Theorem 2 to express $\tau^{(3)} = \sigma_0^{(3)}$ in terms of $\sigma_1^{(2)}$. We have

$$\tau(F_3) = \tau\langle f_1, f_2-1, 1\rangle + (f_3+1)\sigma_1(F_2)\,. \tag{4.9}$$

But

$$\tau\langle f_1, f_2-t, 1\rangle = \tau\langle f_1, f_2-t-1, 1\rangle + 2\sigma_1\langle f_1, f_2-t\rangle \tag{4.10}$$

for $0 < t \le f_2$.

Further, for $0 < u < f_1$,

$$\tau\langle f_1-u, 0, 1\rangle = \tau\langle f_1-u-1, 1, 0\rangle + \sigma_1\langle f_1-u, 1\rangle + 2\sigma_1\langle f_1-u, 0\rangle\,. \tag{4.11}$$

We use the above three substitutions to get

$$\begin{aligned}\tau(F_3) = {} & (f_3+1)\sigma_1(F_2) \\ & + 2\{\sigma_1\langle f_1, f_2-1\rangle + \sigma_1\langle f_1, f_2-2\rangle + \ldots + \sigma_1\langle f_1, 0\rangle\} \\ & + \sigma_1\langle f_1-1, 1\rangle + 2\sigma_1\langle f_1-1, 0\rangle + \ldots + \sigma_1\langle 1, 1\rangle \\ & + 2\sigma_1\langle 1, 0\rangle + \tau\langle 1, 0\rangle.\end{aligned} \tag{4.12}$$

Substituting the value of σ_1 from Proposition 2 now gives

$$\begin{aligned}\tau(F_3) = {} & (f_1+1)\sum_{m=0}^{f_2}(f_3+2m+1)p^{2f_1+f_2-m} \\ & + \sum_{n=0}^{f_1-1}\{(f_1-n)((2f_2+f_3+3n+2)p^{2f_1-2n-1} \\ & \qquad + (2f_2+f_3+3n+4)p^{2f_1-2n-2})\}\,.\end{aligned} \tag{4.13}$$

We write

$$\tau(F_3)=(f_1+1)(f_3+1)p^{2f_1}\sum_{t=0}^{f_2}p^t+2(f_1+1)p^{2f_1+f_2}\sum_{t=0}^{f_2}tp^{-t}$$
$$+\{(3f_1+2f_2+f_3+2)+p^{-1}(3f_1+2f_2+f_3+4)\}\sum_{t=0}^{f_1}tp^{2t-1}$$
$$-3(1+p^{-1})\sum_{t=0}^{f_1}t^2p^{2t-1}\,. \tag{4.14}$$

We now use (4.7) with $x=p^{-1}$ and p^2 to evaluate the second and third sums respectively. For the last sum we use the fact that

$$\sum_{m=1}^{t}m^2x^m=x\frac{d^2}{dx^2}\sum_{m=1}^{t}x^{m+1}-\sum_{m=1}^{t}mx^m$$
$$=2x\frac{1-x^{t+2}}{(1-x)^3}-2(t+2)\frac{x^{t+2}}{(1-x)^2}$$
$$-\frac{(t+1)(t+2)x^{t+1}}{1-x}-\frac{x(1-x^{t+1})}{(1-x)^2}+(t+1)\frac{x^{t+1}}{1-x}\,. \tag{4.15}$$ ■

We now obtain the Dirichlet series corresponding to the divisor function for $r=3$. We use the notation:

$$D(\tau^{(3)})=\sum\tau(F_3^{(p)})X^{f_1}Y^{f_2}Z^{f_3} \tag{4.16}$$

for indeterminates X,Y and Z, a prime number p and non-negative integers f_1,f_2,f_3. We prove that

Proposition 4.

$$D(\tau^{(3)})=\frac{(1+Y+Z-YZ)(1+pX-p(p+1)XY)}{(1-X)(1-p^2X)^2(1-Y)(1-Z)}$$
$$+\frac{(p+3)X+2pX^2-(1+2(p+1)X+pX^2)Y}{1-X}\,. \tag{4.17}$$

Proof. We break $D(\tau^{(3)})$ into several parts and use the recursion formula to make the computations possible, i.e.

$$D(\tau^{(3)})=\sum_{\substack{f_1,f_2\ge0\\ f_3>0}}\tau\langle f_1,f_2,f_3\rangle X^{f_1}Y^{f_2}Z^{f_3}$$
$$+\sum_{\substack{f_1\ge0\\ f_2>0}}\tau\langle f_1,f_2,0\rangle X^{f_1}Y^{f_2}+\sum_{f_1\ge0}\tau\langle f_1,0,0\rangle X^{f_1}\,. \tag{4.18}$$

We denote the three sums on the right hand side of (4.18) by $H^{(3)},H_0^{(2)}$ and

$H_{00}^{(1)}$ respectively. Now,

$$H^{(3)} = \sum \tau\langle f_1, f_2, f_3 - 1\rangle X^{f_1} Y^{f_2} Z^{f_3} + \frac{Z}{1-Z} \sum \sigma_1\langle f_1, f_2\rangle X^{f_1} Y^{f_2} = ZH^{(3)} + ZH_0^{(2)} + ZH_{00}^{(1)} + \frac{Z}{1-Z} K^{(2)} \tag{4.19}$$

where $K^{(2)} = \sum_{f_1, f_2 \geq 0} \sigma_1\langle f_1, f_2\rangle X^{f_1} Y^{f_2}$. Thus

$$H^{(3)} = \frac{Z}{1-Z}(H_0^{(2)} + H_{00}^{(1)}) + \frac{Z}{(1-Z)^2} K^{(2)} \,. \tag{4.20}$$

But $H_0^{(2)} = \sum \tau\langle f_1, f_2 - 1, 1\rangle X^{f_1} Y^{f_2} + K^{(2)} - K_0^{(1)}$.

Writing

$$H_1^{(2)} = \sum_{f_1, f_2 \geq 0} \tau\langle f_1, f_2, 1\rangle X^{f_1} Y^{f_2}$$

and

$$K_0^{(1)} = \sum_{f_1 \geq 0} \sigma_1\langle f_1, 0\rangle X^{f_1} \,,$$

we get

$$H_0^{(2)} = H_1^{(2)} Y + K^{(2)} - K_0^{(1)} \tag{4.21}$$

and

$$H_1^{(2)} = H_0^{(2)} + K^{(2)} - K_0^{(1)} \,. \tag{4.22}$$

Simplifying, we get

$$H_0^{(2)} = \frac{1+Y}{1-Y} K^{(2)} - \frac{1}{1-Y} K_0^{(1)} \,. \tag{4.23}$$

Using (4.20) and (4.23) we can write

$$D(\tau^{(3)}) = \left(\frac{1+Y}{1-Y} \cdot \frac{1}{1-Z} + \frac{Z}{1-Z}\right) K^{(2)} - \frac{1}{1-Z} \cdot \frac{1}{1-Y} K_0^{(1)} + \frac{1}{1-Z} H_0^{(1)} \,. \tag{4.24}$$

With similar manipulations, we can express $H_{00}^{(1)}$ in terms of the K's. In fact, with $K_1^{(1)} = \sum_{f_1 \geq 0} \sigma_1\langle f_1, 1\rangle X^{f_1}$, we get

$$H_{00}^{(1)} = \frac{1+X}{1-X} K_0^{(1)} + \frac{X}{1-X} K_1^{(1)} \,, \tag{4.25}$$

so that $D(\tau^{(3)})$ can now be written in terms of the K's.

To evaluate the K's we use the recurrence further and express them in terms of $L^{(1)} = \sum \sigma_2\langle f_1\rangle X^{f_1}$.

Thus

$$K_0^{(1)} = pXK_1^{(1)} + L^{(1)} \tag{4.26}$$

and

$$K_1^{(1)} = pK_0^{(1)} + L^{(1)}\,, \tag{4.27}$$

which gives

$$K_0^{(1)} = \frac{1+pX}{1-p^2X}L^{(1)}, \qquad K_1^{(1)} = \frac{1+p}{1-p^2X}L^{(1)}\,. \tag{4.28}$$

Now

$$\begin{aligned} K^{(2)} &= p\sum \sigma\langle f_1, f_2-1\rangle X^{f_1}Y^{f_2} + \frac{Y}{1-Y} + L^{(1)} + K_0^{(1)} \\ &= \frac{1+pX-p(p+1)XY}{(1-pY)(1-Y)(1-p^2X)}L^{(1)}\,. \end{aligned} \tag{4.29}$$

But

$$L^{(1)} = \sum \left(\frac{p^{2f_1+2}-1}{p^2-1}\right)X^{f_1} = \frac{1}{1-X}\cdot\frac{1}{1-p^2X}\,. \tag{4.30}$$

We have now evaluated all the sums. ∎

Remark. From Proposition 4 we can derive that $\sum_{|s|\le x}\tau^{(3)}(S) \sim Ax\log^4 x$ for a positive constant A.

5. A problem in higher rank. We are not able to find the Dirichlet series associated with the convolution product of matrices for $r \ge 3$. Hence we study the divisor function and with the help of the recursion formula (Theorem 2) obtain the abscissa of convergence of the associated Dirichlet series.

We define β_r to be the *abscissa of convergence* of the Dirichlet series with positive coefficients:

$$D_S^0(\tau, s) = \sum_{S\ \mathrm{in\ SNF}} \frac{\tau(S)}{(\det S)^s} = D_S(\tau; rs, (r-1)s, \ldots, s)\,. \tag{5.1}$$

It is simpler to see the relative strength of our results for $D_S^0(\tau, s)$ though the methods employed help us understand $D_S(\tau; s_1, \ldots, s_r)$ as well. Knowing the estimates ([1], [2])

$$\sum_{\substack{\det H\le x\\ H\ \mathrm{in\ HNF}}} \tau(H) \sim C_r x^r\log x\,; \qquad \sum_{\substack{\det H\le x\\ H\ \mathrm{in\ HNF}}} 1 \sim C_r' x^r\,;$$

$$\sum_{\substack{\det S\le x\\ S\ \mathrm{in\ SNF}}} 1 \sim C_r'' x\,; \qquad C_r, C_r', C_r''\ \text{positive constants}\,,$$

one would expect $\sum_{\det S\le x,\, S \text{ in SNF}} \tau(S)$ to be of order $x\log x$. But this argument is misleading even for $r=2$ and further so as $r\to\infty$ (see Theorem 3). We get an *upper bound* for τ in Proposition 5, the proof of which depends on a repeated use of our recurrence formula, which we do in the following lemmas. Henceforth we assume a to be non-negative.

LEMMA 5.1.

$$\tag{5.2} \begin{aligned} &\sigma_a^{(r)}\langle f_1,\dots,f_r\rangle \\ &\quad\le p^{af_r}\sigma_a^{(r)}\langle f_1,\dots,f_{r-1},0\rangle + f_r p^{a(f_r-1)}\sigma_{a+1}^{(r-1)}\langle f_1,\dots,f_{r-1}\rangle\,. \end{aligned}$$

Proof. We use the recurrence formula (Theorem 2) and the fact that σ_a is non-negative. ∎

When necessary, we use the notation 0_m to denote a string of m zeros.

LEMMA 5.2. *For* $t\ge 1$,

$$\tag{5.3} \begin{aligned} &p^{at}\sigma_a^{(r)}\langle f_1,f_2,\dots,f_{s-1},f_s-1,0_{t-1},1,0,\dots,0\rangle \\ &\qquad\qquad\le \sigma_a^{(r)}\langle f_1,\dots,f_s,0,\dots,0\rangle\,. \end{aligned}$$

Proof. We have $p^a\sigma_a^{(r)}\langle f_1,\dots,f_{s-1},f_s-1,1,0,\dots,0\rangle \le \sigma_a^{(r)}\langle f_1,\dots,f_s, 0,\dots,0\rangle$. We now use induction on t. ∎

LEMMA 5.3.

$$\tag{5.4} \begin{aligned} \sigma_a^{(r)}\langle f_1,\dots,f_s,0,\dots,0\rangle &\le p^{a(r-s+1)}\sigma_a^{(r)}\langle f_1,\dots,f_{s-1},f_s-1,0,\dots,0\rangle \\ &\quad + (r-s+1)\sigma_{a+1}^{(r-1)}\langle f_1,\dots,f_s,0,\dots,0\rangle\,. \end{aligned}$$

Proof. A repeated use of the recurrence formula gives

$$\tag{5.5} \begin{aligned} &\sigma_a^{(r)}\langle f_1,\dots,f_s,0,\dots,0\rangle \\ &= p^{a(r-s+1)}\sigma_a^{(r)}\langle f_1,\dots,f_{s-1},f_s-1,0,\dots,0\rangle \\ &\quad + \sigma_{a+1}^{(r-1)}\langle f_1,\dots,f_s,0,\dots,0\rangle + p^a\sigma_{a+1}^{(r-1)}\langle f_1,\dots,f_s-1,1,0,\dots,0\rangle \\ &\quad + \ldots + p^{a(r-s)}\sigma_{a+1}^{(r-1)}\langle f_1,\dots,f_s-1,0,\dots,0\rangle\,. \end{aligned}$$

We now use Lemma 5.2. ∎

LEMMA 5.4. *For* $1\le s\le r-1$,

$$\tag{5.6} \begin{aligned} &\sigma_a^{(r)}\langle f_1,\dots,f_s,0,\dots,0\rangle \\ &\quad\le p^{a(r-s+1)f_s}\sigma_a^{(r)}\langle f_1,\dots,f_{s-1},0,\dots,0\rangle \\ &\qquad + (r-s+1)f_s\max(1,p^{(a-r+s)(f_s-1)})\sigma_{a+1}^{(r-1)}\langle f_1,\dots,f_s,0,\dots,0\rangle\,. \end{aligned}$$

Proof. We get this by a repeated use of the previous lemma. ∎

Lemma 5.5.

$$\sigma_a^{(r)}\langle f_1,\ldots,f_r\rangle \le p^{a(f_r+2f_{r-1}+\ldots+rf_1)} + (f_r+2f_{r-1}+\ldots+rf_1)\max_{1\le s\le r}(p^{\Sigma_{t=s}^r(a-r+t)f_t})\sigma_{a+1}^{(r-1)}\langle f_1,\ldots,f_{r-1}\rangle\,. \tag{5.7}$$

Proof. From Lemma 5.1 we get

$$\sigma_a^{(r)}\langle f_1,\ldots,f_r\rangle \le p^{af_r}\sigma_a^{(r)}\langle f_1,\ldots,f_{r-1},0\rangle + f_rp^{af_r}\sigma_{a+1}^{(r-1)}\langle f_1,\ldots,f_{r-1}\rangle\,. \tag{5.8}$$

We then use Lemma 5.4 repeatedly to get the upper bound

$$p^{a(f_r+2f_{r-1}+\ldots+rf_1)} + \sum_{s=1}^{r}(r-s+1)f_s\max(1,p^{(a-r+s)(f_s-1)}) \times p^{a(f_r+2f_{r-1}+\ldots+(r-s)f_{s+1})}\sigma_{a+1}^{(r-1)}\langle f_1,\ldots,f_s,0,\ldots,0),$$

where for $s=r$, $\sigma_{a+1}^{(r-1)}\langle f_1,\ldots,f_s,0,\ldots,0\rangle$ is to be interpreted as $\sigma_{a+1}^{(r-1)}\langle f_1,\ldots,f_{r-1}\rangle$. Now a repeated use of Lemma 5.2 gives

$$\sigma_{a+1}^{(r-1)}\langle f_1,\ldots,f_s,0,\ldots,0\rangle \le p^{(a+1)(f_{r-1}+2f_{r-2}+\ldots+(r-s-1)f_{s+1})}\sigma_{a+1}\langle f_1,\ldots,f_{r-1}\rangle\,. \tag{5.9}$$

Hence the lemma. ∎

We are now ready for our proposition. We use the notations $\lceil x\rceil$ for the integer n satisfying

$$n-1<x\le n\,,$$

and $\alpha_r(t)$ to denote

$$\left\lceil\frac{r+1-t}{2}\right\rceil\left(r+1-t-\left\lceil\frac{r+1-t}{2}\right\rceil\right).$$

Proposition 5.

$$\tau\langle f_1,\ldots,f_r\rangle \le 1+(f_r+2f_{r-1}+\ldots+rf_1)^{r-1}p^{\Sigma_{t=1}^r\alpha_r(t)f_t}\,. \tag{5.10}$$

Proof. We use Lemma 5.5 repeatedly, beginning with $a=0$. The maximum involved is easily computable, and we obtain a sum of various p^m's. These exponents m, for some $0\le k<r$, are

$$\sum_{t=1}^{r}\sum_{\substack{a=(r+1-t)/2\\ a\le k}}^{r-t}(t-r+2a)f_t + \begin{cases}(k+1)(r-k-t)f_t & \text{if } t<r-k\,,\\ 0 & \text{otherwise}\,.\end{cases}$$

If $t \ge r-k$, the coefficient of f_t is given by

$$\left\lceil \frac{r+1-t}{2} \right\rceil \left(r+1-t-\left\lceil \frac{r+1-t}{2} \right\rceil \right) = \alpha_r(t)\,.$$

If $t < r-k$, the coefficient is also

$$\left(k+1-\left\lceil \frac{r+1-t}{2} \right\rceil\right)\left(k+t-r+\left\lceil \frac{r+1-t}{2} \right\rceil\right) + (k+1)(r-k-t) = \alpha_r(t)\,. \ \blacksquare$$

The above proposition, together with a previous result [6] for $\tau(P_r)$, i.e.

$$\tau^{(r)}\langle 1,0,\ldots,0\rangle = \sum_{j=0}^{r} \begin{bmatrix} r \\ j \end{bmatrix}, \tag{5.11}$$

where the $\left[{r \atop j}\right]$ are Gaussian polynomials in p, help us get the abscissa of convergence of $D_S^0(\tau, s)$.

THEOREM 3. *For a positive integer* r,

$$\beta_r = \frac{\alpha_r(1)+1}{r}\,. \tag{5.12}$$

P r o o f. The upper bound for β_r comes from Proposition 5, and the lower bound from (5.11). $\blacksquare$

A closer look at the foregoing proofs in fact shows that

$$x^{\beta_r} \ll \sum_{\substack{\det S \le x \\ S \text{ in SNF}}} \tau(S) \ll x^{\beta_r} \log^{\gamma_r} x\,, \tag{5.13}$$

for some $\gamma_r \ge 0$.

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