AVERAGE RENEWAL LOSS RATES[1]

By M. V. Johns, Jr. and R. G. Miller, Jr.

Stanford University

1. Introduction and summary. Many problems in the theory of decision processes give rise to an unending sequence of cycles whose lengths are given by a sequence of independent and identically distributed positive random variables $X_1, X_2, \cdots$, constituting a renewal process. Typically X_i will represent the number of items passing an inspection point in a production process or the length of time elapsing in some continuous time process until a decision point is reached. In many such problems another sequence of random variables, say $Y_1, Y_2, \cdots$, arises where Y_i is the profit or loss associated with the ith cycle. Examples of problems having this structure may be found, for instance, in [1], [3], [6], [9] and [10]. In most cases X_i and Y_i will not be independent for the same index i and in this note we will permit Y_i to depend on $X_{i-q}, \cdots, X_{i-1}, X_i$ for any fixed finite q.

In all such problems the appropriate index of merit for the decision procedure under consideration is the "average" profit (or loss) per unit time (or per item) for a large number of cycles. The notion of "average" profit rate can be mathematically defined in four distinct and apparently equally plausible ways, and it is the purpose of this note to show that the various definitions are not necessarily equivalent and to determine the conditions under which they are equivalent.

First, the profit rate up to time t may be defined in terms of $N(t)$, the number of cycles completed by time t, as $(1/t)\sum_{i=1}^{N(t)} Y_i$. Then the average profit rate may be defined as either the limit of the expected value of the profit rate, or as the almost sure limit of the profit rate, as $t \to \infty$. The use of the expected value definition is consistent with the principles of utility theory and the notions of risk that underlie the formulation of problems in decision theory. The almost sure limit definition has, of course, considerable intuitive appeal. It is shown in this note that, as an immediate consequence of well-known results from renewal theory, both of the above definitions lead to the average profit rate η/μ where $EX = \mu$ and $EY = \eta$, assuming that η is finite.

Alternately, the profit rate may be defined for n cycles as $\sum_{i=1}^{n} Y_i / \sum_{i=1}^{n} X_i$ and again the average profit rate may be defined as either the limit of the expected values, or the almost sure limit of this ratio, as $n \to \infty$. By the strong law of large numbers the almost sure limit is always η/μ but in Section 2 an example is given for which the limit of the expected values is not η/μ. In Section

Received May 10, 1962.

[1] This work was supported in part by Office of Naval Research Contract Nonr-225(52) at Stanford University. Reproduction in whole or in part is permitted for any purpose of the United States Government.

3 necessary and sufficient conditions are obtained for the limit of the expected values of the ratios to be η/μ if η is finite.

2. An example. For the processes $\{X_1, X_2, \cdots\}$ and $\{Y_1, Y_2, \cdots\}$ where the X_i are independent, identically distributed with cdf $F(F(0+) = 0)$ and $Y_i \equiv 1$, $i = 1, 2, \cdots$, with probability one, the profit rate, or in this case renewal intensity, is n/t where n is the number of cycles or renewals in time t. For fixed time t the quantity n/t is a random variable in the number of renewals, i.e., $N(t)/t$, and for a fixed number of renewals it is a random variable in time, i.e., $n/T(n)$. These random variables are defined by

$$X_1 + \cdots + X_{N(t)} \leqq t < X_1 + \cdots + X_{N(t)+1}, \qquad T(n) = X_1 + \cdots + X_n.$$

Asymptotically these two definitions are equivalent in the sense of a.s. convergence since by the strong law of large numbers $n/T(n) \to 1/\mu$, a.s., as $n \to +\infty$, where $\mu = E(X) \leqq +\infty$, and by Doob [2] $N(t)/t \to 1/\mu$, a.s., as $t \to +\infty$. For non-random t the expectation of the renewal intensity agrees with the a.s. limit; i.e., $E[N(t)/t] \to 1/\mu$ as $t \to +\infty$ (see Feller [4], [5], Doob [2]). However, the example of this section will show that this is not necessarily so for non-random n; i.e., $E[n/T(n)] \nrightarrow 1/\mu$ as $n \to +\infty$. In fact, an example will be constructed in which

$$E[n/T(n)] \equiv +\infty \qquad \text{for all } n, \tag{2.1}$$

but

$$E[N(t)/t] \to 0 \quad \text{as} \quad t \to +\infty; \tag{2.2}$$

i.e., in terms of expectation the renewal intensities are at opposite ends of the scale.

In the example the cdf F will be chosen absolutely continuous with density p. In this case $E[n/T(n)] = n \int_0^\infty (1/x) p^{(n)}(x)\, dx$, where $p^{(n)}$ is the n-fold convolution of p. For $x < \exp\{-2\}$ let $F(x) = (-\log x)^{-1}$ with density $p(x) = [x(\log x)^2]^{-1}$, and for the present let F be unspecified for $x > \exp\{-2\}$. Differentiation readily verifies that in a neighborhood of the origin p is monotonically decreasing so for x in this neighborhood

$$p^{(2)}(x) = \int_0^x p(x-y)p(y)\, dy > p(x) \int_0^x p(y)\, dy.$$

Similarly, differentiation of $p(x)(F(x))^k$ verifies that for each $k > 0$ this function is monotonically decreasing in some neighborhood of the origin so by induction $p^{(n)}(x) > p(x)(F(x))^{n-1}$ in some neighborhood of the origin. Therefore, for ϵ in this neighborhood

$$\int_0^\epsilon \frac{1}{x} p^{(n)}(x)\, dx > \int_0^\epsilon \left(\frac{1}{x}\right) \left(\frac{1}{x(-\log x)^{n+1}}\right) dx.$$

Since $x(-\log x)^{n+1} \to 0$ as $x \to 0$ the latter integral is divergent which establishes (2.1).

So far F has been specified only below its median value. To simultaneously satisfy (2.2) simply choose the density for the upper tail of F so that $\mu = +\infty$.

3. Equivalence theorems. The example of the last section depended upon the cdf F placing enough mass near the origin so that $E[1/T(n)]$ was infinite for all n. The question naturally arises as to whether the two expectations can have different finite limits. The answer is that they cannot.

THEOREM 1. *If for some* n_0, $E[1/T(n_0)] < +\infty$, *then* $E[1/T(n)] < +\infty$ *for all* $n > n_0$ *and* $\lim_{n\to+\infty} E[n/T(n)] = 1/\mu$.

The proof of this theorem depends on the following convergence lemma which may be found in Johns [7]:

LEMMA A. *If* $(\Omega, \mathfrak{A}, \nu)$ *is a measure space,* $\{f_n\}$ *and* $\{g_n\}$ *are sequences of non-negative integrable functions,* f *is an integrable function, and* g *is a function such that*

(i) $\lim_{n\to\infty} f_n = f$, a.e.; $\lim_{n\to\infty} g_n = g$, a.e.,

(ii) $g_n \leqq f_n$, *all* n,

(iii) $\limsup_{n\to\infty} \int f_n \, d\nu \leqq \int f \, d\nu$,

then g *is integrable and* $\lim_{n\to\infty} \int g_n \, d\nu = \int g \, d\nu$.

The extended Lebesgue dominated convergence theorem of Pratt [8] which could also have been used in this connection is a consequence of this lemma.

PROOF OF THEOREM 1. The first conclusion of the theorem follows immediately from $T(n) > T(n_0)$, a.s., for $n > n_0$.

Let $Z_i = \sum_{j=(i-1)n_0+1}^{in_0} X_j$. Then $\{Z_1, Z_2, \cdots\}$ is a renewal process with cdf $F^{(n_0)}$. The assumption of the theorem is that $E(1/Z_i) < +\infty$. Since $Z_i > 0$,

$$(3.1) \quad \left(\sum_1^m 1/Z_i\right)\left(\sum_1^m Z_i\right) \geqq m^2 \quad \text{or} \quad \left((1/m) \sum_1^m 1/Z_i\right) \geqq m/\sum_1^m Z_i,$$

by the Cauchy-Schwarz inequality (i.e., the arithmetic mean is greater than the harmonic mean). Since $(\sum 1/Z_i)/m \to E(1/Z)$, a.s., as $m \to +\infty$,

$$E\left(m/\sum_1^m Z_i\right) \to 1/E(Z) = 1/n_0\,\mu,$$

as $m \to +\infty$ by (3.1) and Lemma A. Hence, $E[n/T(n)] \to 1/\mu$, as $n \to +\infty$ through the sequence $n = kn_0$, $k = 1, 2, \cdots$.

Convergence for all n is obtained from the simple inequality

$$(n + 1)/T(n + 1) = [(n + 1)/n]\cdot n/T(n + 1) \leqq [1 + (1/n)][n/T(n)].$$

This gives for $n = kn_0 + l$, $0 < l < n_0$,

$$\prod_{h=0}^{l-1}\left(1 + \frac{1}{kn_0 + h}\right) E\left(\frac{kn_0}{T(kn_0)}\right) \geqq E\left(\frac{n}{T(n)}\right)$$
$$\geqq \prod_{h=l}^{n_0-1}\left(1 + \frac{1}{kn_0 + h}\right)^{-1} E\left(\frac{(k + 1)n_0}{T((k + 1)n_0)}\right),$$

which implies the desired result since the bounding expectations both tend to $1/\mu$ and the product terms tend to 1 as $k \to +\infty$.

Consider now the general case of profit-loss rate per unit time, i.e.,

$$\sum_1^n Y_i/\sum_1^n X_i, \qquad \text{or} \qquad \sum_1^{N(t)} Y_i/t,$$

depending on whether n is fixed or a random variable $N(t)$ defined by the X-process. It will be assumed that the sequence $\{Y_1, Y_2, \cdots\}$ is q-dependent on the process $\{X_1, X_2, \cdots\}$; i.e., Y_i is independent of $X_1, \cdots, X_{i-q}$ and X_{i+1}, $X_{i+2}, \cdots$, but may be dependent on $X_{i-q+1}, \cdots, X_i$. The sequence variables Y_i are assumed to be identically distributed for $i \geqq q$, and Y_i and Y_{i-q} are assumed independent. The exact probability structure of $Y_1, \cdots, Y_{q-1}$ need not be specified because of the asymptotic character of the results.

Without loss of generality it may be assumed that the Y_i are non-negative random variables since the profit-loss rate can be written as

$$\sum_1^n Y_i/\sum_1^n X_i = \sum_1^n Y_i^+/\sum_1^n X_i + \sum_1^n Y_i^-/\sum_1^n X_i, \tag{3.2}$$

or a corresponding expression with denominator t, where $Y_i^+ = \max\{0, Y_i\}$ and $Y_i^- = \min\{0, Y_i\}$. The arguments to be presented below for non-negative random variables can be applied separately to the two ratios on the right in (3.2) to yield the combined result. Also, without loss of generality it can be assumed that $q = 1$; i.e., $\{Y_1, Y_2, \cdots\}$ is 1-dependent on $\{X_1, X_2, \cdots\}$. The profit rate can be written as

$$\sum Y_i/\sum X_i = (Y_1 + Y_{q+1} + \cdots)/\sum X_i$$
$$+ (Y_2 + Y_{q+2} + \cdots)/\sum X_i + \cdots + (Y_q + Y_{2q} + \cdots)/\sum X_i,$$

or with denominator t. Since the sums $Z_i^k = \sum_{j=k+(i-1)q+1}^{k+iq} X_j$ form a renewal process for each $k = 1, 2, \cdots, q$, the arguments given below can be applied to each of the ratios with proper handling of the initial Y and X variables.

By the strong law of large numbers as $n \to +\infty$, $\sum_1^n Y_i/\sum_1^n X_i \to \eta/\mu$, a.s., where $\eta = E(Y)$ is henceforth assumed to be finite. By a trivial adaptation of Doob's argument [2], $\sum_1^{N(t)} Y_i/t \to \eta/\mu$, a.s., as $t \to +\infty$ so the a.s. limits agree with what would be expected.

It is known (see, for example, [1]) that the standard renewal equation results can be adapted to the profit rate case. Let

$$V(t) = E\left\{\sum_1^{N(t)} Y_i\right\}, \qquad W(t) = \int_0^t E\{Y_1 \mid X_1 = u\}\, dF(u).$$

Then the expected profit function V satisfies the renewal equation $V(t) = W(t) + \int_0^t V(t-u)\, dF(u)$. The unique solution of this equation is

$$W(t) * [1 + M(t)]$$

where $M(t) = \sum_1^\infty F^{(n)}(t)$ and $(*)$ denotes convolution, and it is well-known that $\lim_{t\to+\infty} V(t)/t = W(+\infty)/\mu = \eta/\mu$, (see Feller [4], [5], Doob [2], Smith [11]). Thus, the limiting expected profit rate for fixed t agrees with the a.s. limits.

That the limiting expectation for non-random n also agrees with this limit is established in the following theorem. The only condition imposed on the X-process is that $E[1/T(n_0)] < +\infty$ for some n_0 and in view of the preceding section this is a necessary condition.

THEOREM 2. *If* $\eta = E(Y) < +\infty$ *and for some* n_0, $E[1/T(n_0)] < +\infty$, *then*

$$\lim_{n\to\infty} E\left\{\sum_1^n Y_i/\sum_1^n X_i\right\} = \eta/\mu.$$

PROOF. If $E[1/T(n_0)] < +\infty$, then

$$E\left\{\sum_1^{2n_0} Y_i/\sum_1^{2n_0} X_i\right\} = E\left\{\sum_1^{n_0} Y_i/\sum_1^{2n_0} X_i\right\} + E\left\{\sum_{n_0+1}^{2n_0} Y_i/\sum_1^{2n_0} X_i\right\}$$

$$\leqq E\left\{\sum_1^{n_0} Y_i/\sum_{n_0+1}^{2n_0} X_i\right\} + E\left\{\sum_{n_0+1}^{2n_0} Y_i/\sum_1^{n_0} X_i\right\} = 2n_0\,\eta E[1/T(n_0)] < +\infty.$$

All subsequent ratio expectations are finite from the inequality

$$E\left\{\sum_1^n Y_i/\sum_1^n X_i\right\} \leqq E\left\{\sum_1^n Y_i/\sum_1^{n-1} X_i\right\} = E\left\{\sum_1^{n-1} Y_i/\sum_1^{n-1} X_i\right\} + \eta E[1/T(n-1)].$$

Let $Z_i = \sum_{j=2(i-1)n_0+1}^{2in_0} X_j$ and $S_i = \sum_{j=2(i-1)n_0+1}^{2in_0} Y_j$. The processes

$$\{Z_1, Z_2, \cdots\} \quad \text{and} \quad \{S_1, S_2, \cdots\}$$

are renewal processes, and the S-process is 1-dependent on the Z-process. Since $Z_i > 0$,

$$\sum_1^m S_i/\sum_1^m Z_i \leqq \left((1/m)\sum_1^m S_i\right)\left((1/m)\sum_1^m 1/Z_i\right), \tag{3.3}$$

(see (3.1)). As $m \to +\infty$, $\sum S_i/m \to 2n_0\eta$, a.s., $(\sum 1/Z_i)/m \to E[1/T(2n_0)]$, a.s., and

$$E\left\{\left(\frac{1}{m}\sum_1^m S_i\right)\left(\frac{1}{m}\sum_1^m \frac{1}{Z_i}\right)\right\} = \frac{1}{m^2}\left\{E\left(\sum_1^m S_i/Z_i\right) + E\left(\sum_{i\neq j}\sum S_i/Z_j\right)\right\}$$

$$= \frac{1}{m}E(S/Z) + \left(\frac{m-1}{m}\right)2n_0\,\eta E[1/T(2n_0)] \to 2n_0\,\eta E[1/T(2n_0)].$$

Therefore, by (3.3) and Lemma A, $E\{\sum_1^n Y_i/\sum_1^n X_i\} \to \eta/\mu$, as $n \to +\infty$ through the sequence $n = 2kn_0$, $k = 1, 2, \cdots$. Convergence for all n is obtained from

$$E\left\{\sum_1^{2kn_0} Y_i/\sum_1^{2kn_0} X_i\right\} + \eta\sum_{h=0}^{l-1} E[1/T(2kn_0+h)] \geqq E\left\{\sum_1^n Y_i/\sum_1^n X_i\right\}$$

$$\geqq E\left\{\sum_1^{2(k+1)n_0} Y_i/\sum_1^{2(k+1)n_0} X_i\right\} - \eta\sum_{h=l}^{2n_0-1} E[1/T(2kn_0+h)],$$

for $n = 2kn_0 + l$, $0 < l < 2n_0$, since by Theorem 1 $E[1/T(n)] = O(1/n)$ as $n \to +\infty$.

REFERENCES

[1] BELL, L. F. (1961). A mathematical theory of guarantee policies. Technical Report No. 49, Department of Statistics, Stanford Univ.

[2] DOOB, J. L. (1948). Renewal theory from the point of view of the theory of probability. *Trans. Amer. Math. Soc.* **63** 422–438.

[3] ELFVING, G. (1962). Quality control for expensive items. Technical Report No. 57, Department of Statistics, Stanford Univ.

[4] FELLER, W. (1941). On the integral equation of renewal theory. *Ann. Math. Statist.* **12** 243–267.

[5] FELLER, W. (1949). Fluctuation theory of recurrent events. *Trans. Amer. Math. Soc.* **67** 98–119.

[6] GIRSHICK, M. A. and RUBIN, H. (1952). A Bayes approach to a quality control model. *Ann. Math. Statist.* **23** 114–125.

[7] JOHNS, M. V., JR. (1957). Non-parametric empirical Bayes procedures. *Ann. Math. Statist.* **28** 649–669.

[8] PRATT, J. W. (1960). On interchanging limits and integrals. *Ann. Math. Statist.* **31** 74–77.

[9] SAVAGE, I. R. (1959). A production model and continuous sampling plan. *J. Amer. Statist. Assoc.* **54** 231–247.

[10] SAVAGE, I. R. (1961). Surveillance Problems. Technical Report No. 14, Department of Statistics, Univ. of Minnesota.

[11] SMITH, W. L. (1954). Asymptotic renewal theorems. *Proc. Roy. Soc. Edinburgh. Sect. A* **64** 9–48.