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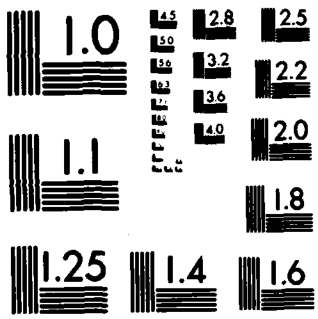

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BY

MOSHE POLLAK

TECHNICAL REPORT NO. 22
SEPTEMBER 1983

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AVERAGE RUN LENGTHS OF AN OPTIMAL METHOD OF DETECTING A CHANGE IN DISTRIBUTION

by

Moshe Pollak
The Hebrew University of Jerusalem

ABSTRACT

Suppose one is able to observe sequentially a series of independent observations $X_1, X_2, \ldots,$ such that $X_1, X_2, \ldots, X_{\nu-1}$ are i.i.d. with known density f_0 and $X_\nu, X_{\nu+1}, \ldots,$ are i.i.d. with density f_θ where ν is unknown. Define

$$R_n^{\{\theta\}} = \sum_{k=1}^{n} \prod_{i=k}^{n} \frac{f_\theta(X_i)}{f_0(X_i)} .$$

It is known that rules which call for stopping and raising an alarm the first time n that $R_n^{\{\theta\}}$ or a mixture thereof exceeds a prespecified level A are optimal methods of detecting that the density of the observations is not f_0 any more.

Practical applications of such stopping rules require knowledge of their operating characteristics, whose exact evaluation is difficult. Here are presented asymptotic ($A \to \infty$) expressions for the expected stopping times of such stopping rules (a) when $\nu = \infty$ and (b) when $\nu = 1$. We assume that the densities f_θ form an exponential family and that the distribution of $\log(f_\theta(X_1)/f_0(X_1))$ is (strongly) non-lattice.

Monte Carlo studies indicate that the asymptotic expressions are very good approximations even when the expected sample sizes are small.

I. INTRODUCTION

Suppose one accumulates independent observations from a certain process. Initially, the process is at State #0. At some unknown point in time something occurs (e.g., a "breakdown") which puts the process in State #1, and consequently the stochastic behavior of the observations changes. It is of interest to declare that a change took place (to "raise an alarm") as soon as possible after its occurrence, subject to a restriction on the rate of false detections. It is assumed that the aforementioned observations are the only information one has about the process, and the problem is to construct a good detection scheme.

Practical examples of this problem arise in areas such as health, quality control, ecological monitoring, etc. For instance, consider surveillance for congenital malformations in newborn infants. Under normal circumstances, the percentage of babies born with a certain type of malformation has a known value, p_θ. Should something occur (such as an environmental change, the introduction of a new drug to the market, etc.) the percentage may increase. ~~(e.g., the thalidomide episode of the 1960's).~~ One would want to raise an alarm as quickly as possible after a change would have taken place, subject to an acceptable rate of false alarms. Generally, the problem arises wherever surveillance is being done.

A solution to the problem depends on what is known in advance about the distributions of the observations. Let f_0 denote the density of observations with respect to a σ-finite measure μ when the process is in State #0, let f_θ denote the density of observations with respect to μ when the process is in State #1, and let ν denote the unknown point in time when the first observation from State #1 is made. Thus one has a sequence of independent observations $X_1, X_2, \ldots,$ such that $X_1, X_2, \ldots, X_{\nu-1}$ are i.i.d. with density f_0 and $X_\nu, X_{\nu+1}, \ldots,$ are i.i.d. with density f_θ where $1 \leq \nu \leq \infty$ is unknown. It will be assumed here that f_0, f_θ belong to an exponential family of distributions and that f_0 is known.

Solutions for the problem which are in current use are known as CUSUM procedures. For a survey see, for instance, Johnson and Leone (1962). (See also Weatherall and Haskey (1976).) Lorden (1971) proved a first-order asymptotic optimality property of a certain class of procedures for reacting to a change in distribution. When f_θ is known, this class includes most of the standard appropriate CUSUM techniques as special cases. When f_θ is unknown, Lorden (1971) suggests a first-order asymptotically optimal procedure. (Asymptotic operating characteristics of this and related procedures are given in Pollak and Siegmund (1975). Further refinements can be obtained using results of Lai and Siegmund (1977).)

Shiryayev (1963, 1978) solved the problem in a Bayesian framework in the case that f_θ is known.

An optimal solution in a classical framework is presented in Pollak (1983). Asymptotic operating characteristics of this and related procedures are the subject under study here.

Without loss of generality, let the assumed exponential family be defined by

$$f_y(x) = e^{yx-\psi(y)}, \quad y \in \Omega$$

where Ω is an interval on the real line, $0 = \psi(0) = \psi'(0)$. Let F be a probability measure on Ω with $F(\{0\}) = 0$. Let $0 < A < \infty$. Define

$$R_n^{\{y\}} = \sum_{k=1}^{n} \prod_{i=k}^{n} \frac{f_y(X_i)}{f_0(X_i)} = \sum_{k=1}^{n} e^{y\sum_{i=k}^{n} X_i-(n-k+1)\psi(y)}$$

$$R_n^F = \int R_n^{\{y\}} \, dF(y)$$

$$N_A^{\{y\}} = \min\{n | R_n^{\{y\}} \geq A\}$$

$$N_A^F = \min\{n | R_n^F \geq A\} \ .$$

Raising an alarm at time $N_A^{\{\theta\}}$ is an optimal procedure when the value θ (of the parameter of the distribution after a change occurred) is known and raising an alarm at time N_A^F has optimality properties when θ is unknown (Pollak (1983)).

In order to evaluate and compare between procedures one needs to formalize a restriction on false detections as well as to formalize an expression for the speed of detection of a change after its occurrence. The restriction on false detections is usually formalized as a requirement that the expected number of observations until a false alarm (assuming that $\nu = \infty$) exceed a prespecified value B. This suggests a need for evaluating $E(N_A^{\{y\}} | \nu = \infty)$, $E(N_A^F | \nu = \infty)$. The quality of a procedure with regard to the speed of detection of a change after its occurrence is often measured by the supremum (or essential supremum) of the expected number of observations that it takes to detect a change after its occurrence, given that no false alarms have previously been raised (see Lorden (1971), Pollak and Siegmund (1975)). This suggests a need for evaluating $E(N_A^{\{\theta\}} - \nu | \nu = 1, \theta)$, $E(N_A^F - \nu | \nu = 1, \theta)$. These operating characteristics are difficult to compute. For simulations see Roberts (1966).

In this article, asymptotic expressions ($A \to \infty$) for these operating characteristics are presented. Monte Carlo studies indicate that these expressions are very good approximations even when the expected samples sizes are small.

II. THE AVERAGE RUN LENGTH WHEN $\nu = \infty$

Denote by $P_\nu^{(y)}$, $E_\nu^{(y)}$ the probability, expectation respectively when $1 \le \nu \le \infty$, $X_1, \ldots, X_{\nu-1}$ are i.i.d. with density f_0 and are independent of $X_\nu, X_{\nu+1}, \ldots$, which are i.i.d. with density f_y. Let P_0, E_0 denote probability, expectation respectively when $\nu = \infty$. Let F be a probability measure on Ω with $F(\{0\}) = 0$. Denote

$$Z_i^{\{y\}} = \log \frac{f_y(X_i)}{f_0(X_i)} = yX_i - \psi(y)$$

$$I(\theta) = E_1^{(\theta)} Z_1^{\{\theta\}} = E_0 Z_1^{\{\theta\}} e^{Z_1^{\{\theta\}}}$$

$$M_B^y = \min\{n \mid \Sigma_{i=1}^n Z_i^{\{y\}} \ge B\},\ M_B^y = \infty \text{ if no such n exists}$$

$$C_0^y = 1/\lim_{B\to\infty} E_1^{(y)} e^{-\left\{\Sigma_{i=1}^{M_B^y} Z_i^y - B\right\}}$$

$$C_0^F = 1/\int (1/C_0^y)\, dF(y) .$$

The computations of C_0^y and C_0^F are applications of renewal theory and have been calculated in other contexts. (See Siegmund (1975), Lai and Siegmund (1977), Theorem 6.2 of Woodroofe (1982).)

THEOREM 1. (i) $E_0 N_A^{\{y\}} \ge A$ for all $y \in \Omega$. If $I(y) < \infty$, then for any $A_0 > 0$ there exists a constant $0 < C_y^{(A_0)} < \infty$ such that $E_0 N_A^{\{y\}} \le C_y^{(A_0)} A$ whenever $A \ge A_0$.

(ii) If $y \in \Omega$, $I(y) < \infty$ and the $P_1^{\{y\}}$-distribution of $\log(f_y(X_1)/f_0(X_1))$ is non-lattice, then

$$E_0 N_A^{\{y\}} = A C_0^y (1 + o(1)) ,$$

where $o(1) \to 0$ as $A \to \infty$.

THEOREM 2. Suppose that the $P_1^{(y)}$-distribution of X_1 is strongly non-lattice (see Stone (1965)) for all $y \in \Omega$. Then

(i) $E_0 N_A^F \geq A$. If $F(\{y|I(y) < \infty\}) = 1$, then for any $A_0 > 0$ there exists a constant $0 < C_F^{A_0} < \infty$ such that $E_0 N_A^F \leq C_F^{A_0}$ whenever $A \geq A_0$.

(ii) If $F(\{y|I(y) < \infty\}) = 1$, then

$$E_0 N_A^F = A C_0^F (1 + o(1)) ,$$

where $o(1) \to 0$ as $A \to \infty$.

III. PROOFS

The proof of Theorems 1 and 2 is based on the observation that (under P_0) $R_n^F - n$ is a martingale with zero expectation with respect to $\mathfrak{F}(X_1,\ldots,X_n)$, so that for stopping times N which are well-behaved $E_0 N = E_0 R_N^F$. The proof becomes an analysis of the asymptotic behavior of $E_0 R^F_{N^F_A}$.

For any m, r

$$(1) \qquad R^F_{m+r} = \int e^{\sum_{i=m+1}^{m+r} Z_i^{\{y\}}} \sum_{k=1}^{m} e^{\sum_{i=k}^{m} Z_i^{\{y\}}} \, dF(y) + \int \sum_{k=m+1}^{m+r} e^{\sum_{i=k}^{m+r} Z_i^{\{y\}}} \, dF(y) \; .$$

Make note of the following three observations: (I) Since $E_0 Z_i^{\{y\}} < 0$, the first expression on the right side of equation (1) becomes negligible as r becomes large. (II) The second expression on the right side of equation (1) when regarded as a process in r has the same stochastic P_0-behavior as the original process R_n^F. (III) If the value of R_m^F is large, the process R_n^F behaves approximately like the first expression on the right side of (1) for n = m+r closely following m.

The idea of the proof can now be described as follows. Let c be a large constant, and let A be much larger than c. Regard the stopping time N_A^* which at first tells one to continue sampling until $N^F_{A/c}$. If "soon" thereafter $R_n^F \geq A$, let $N_A^* = N_A^F$. If not, forget the first $N^F_{A/c}$ obse vations and reapply $N^F_{A/c}$ to the sequence of observations following $N^F_{A/c}$. Repeat this until the first time that $R_n^F \geq A$

"soon" after $N^F_{A/c}$. This first time defines N^*_A. By virtue of observation (I) it will be shown that the asymptotics of $E_0 R^F_{N^F_A}$ are the same as those of $E_0 R^F_{N^*_A}$.

The repeated applications of $N^F_{A/c}$ (conditional on their existence) will be shown to be approximately independent of each other. By virtue of observations (I) and (II), it will be shown that $E_0 R^F_{N^*_A}$ is approximately equal to

(2) $E_0(R^F_{N^*_A} | R^F_n \geq A$ "soon" after the first application of $N^F_{A/c})$.

Letting $m = N^F_{A/c}$ in equation(1), note that the first expression on the right side of equation (1) is equal to $R^F_{N^F_{A/c}} \times W_{N^F_{A/c}+r}$ where

$$W_{N^F_{A/c}+r} = \int e^{\sum_{i=N^F_{A/c}+1}^{N^F_{A/c}+r} Z_i\{y\}} dF_1(y) ,$$

$$dF_1(y) = \sum_{k=1}^{N^F_{A/c}} e^{\sum_{i=k}^{N^F_{A/c}} Z_i\{y\}} dF(y)/R^F_{N^F_{A/c}} .$$

By virtue of observation (III) it will be shown that $\{R^F_n > A$ "soon" after the first application of $N^F_{A/c}\}$ is approximately equal to $\{R^F_{N^F_{A/c}} \; W_{N^F_{A/c}+r} \geq A$ for some $1 \leq r < \infty\}$. Let $H_1 = \min\{r | R^F_{N^F_{A/c}} \; W_{N^F_{A/c}+r} \geq A\}$, $H_1 = \infty$ if no such r exists. It follows from (2) that

$$E_0 R^F_{N^*_A} \;\overset{\text{approximately}}{=\!=\!=}\; E_0(R^F_{N^F_{A/c}} \; W_{N^F_{A/c}+H_1} | H_1 < \infty)$$

$$= \frac{E_0(R^F_{N^F_{A/c}} \; W_{N^F_{A/c}+H_1} ; H_1 < \infty)}{P_0(H_1 < \infty)} .$$

Conditional on $\mathfrak{F}(X_1,\ldots,X_{N^F_{A/c}})$, $W_{N^F_{A/c}+r}$ is a P_0-martingale (with unit expectation with respect to $\mathfrak{F}(X_{N^F_{A/c}+1},\ldots,X_{N^F_{A/c}+r})$. Therefore the numerator in (3) is equal to $E_0 R^F_{N^F_{A/c}}$.

Results of Lai and Siegmund (1977) yield that, with $K(y) = 1/C_0^y$,

$$P_0(H_1 < \infty | \mathfrak{F}(X_1,\ldots,X_{N^F_{A/c}})) = P_0(W_{N^F_{A/c}+r} \geq A/R^F_{N^F_{A/c}}$$

$$\text{for some } 1 \leq r < \infty | \mathfrak{F}(X_1,\ldots,X_{N^F_{A/c}}))$$

$$\underset{\text{approximately}}{=} \frac{\int K(y)\, dF_1(y)}{A/R^F_{N^F_{A/c}}} .$$

Therefore

$$(4) \quad P_0(H_1 < \infty) \underset{\text{approximately}}{=} E_0 A^{-1} \int K(y)\, R^F_{N^F_{A/c}}\, dF_1(y)$$

$$= A^{-1} \int K(y)\, E_0(R^F_{N^F_{A/c}}\, dF_1(y))$$

$$= A^{-1} \int K(y)\, E_0 N^F_{A/c}\, dF(y)$$

$$= \frac{E_0 R^F_{N^F_{A/c}}}{A} \int K(y)\, dF(y)$$

where the equality (4) follows from the definition of $dF_1(y)$ and the fact that $\sum_{k=1}^{n} e^{\sum_{i=k}^{n} Z_i^y} - n$ is a P_0-martingale (with zero expectation) with respect to $\mathfrak{F}(X_1,\ldots,X_n)$. It now follows from (3) that

$$E_0 N^F_A = E_0 R^F_{N^F_A} \underset{\text{approximately}}{=} A / \int K(y)\, dF(y)$$

which is the heart of the content of Theorem 2.

The turning of these heuristic arguments into a rigorous proof requires the ten lemmas presented in the sequel. The method involved is linear and nonlinear renewal theory (cf. Feller (1971), Stone (1965), Woodroofe (1976), Lai and Siegmund (1977)). For a survey see Woodroofe (1982).

PROOF OF THEOREM 1(i), THEOREM 2(i). Note that under P_0 both $R_n^{\{y\}} - n$ and $R_n^F - n$ are martingales with zero expectation with respect to $\mathfrak{F}(X_1,\ldots,X_n)$. Denote

$$\pi_A^{\{y\}} = \min\left\{n \,\middle|\, \max_{k=1,\ldots,n} \exp\left\{\sum_{i=k}^{n} Z_i^{\{y\}}\right\} \geq A\right\}$$

$$\pi_A^F = \min\left\{n \,\middle|\, \max_{k=1,\ldots,n} \int \exp\left\{\sum_{i=k}^{n} Z_i^{\{y\}}\right\} dF(y) \geq A\right\} .$$

It is well known that $E_0\, \pi_A^{\{y\}} < \infty$, $E_0\, \pi_A^F < \infty$ (cf. Lorden (1971)). Since $N_A^{\{y\}} \leq \pi_A^{\{y\}}$ and $N_A^F \leq \pi_A^F$ it follows that $E_0\, N_A^{\{y\}} < \infty$ and $E_0\, N_A^F < \infty$. Hence $E_0(R^{\{y\}}_{N_A^{\{y\}}} - N_A^{\{y\}})$ and $E_0(R^F_{N_A^F} - N_A^F)$ exist. Since $|R_n^{\{y\}}| < A$, $|R_n^F| < A$ on $\{N_A^{\{y\}} > n\}$, $\{N_A^F > n\}$ respectively, it is easy to see that

$$\liminf_{n\to\infty} \int_{\{N_A^{\{y\}} > n\}} |R_n - n| \, dP_0 = 0 \quad , \quad \liminf_{n\to\infty} \int_{\{N_A^F > n\}} |R_n^F - n| \, dP_0 = 0 \quad .$$

Hence by the martingale optional stopping theorem (cf. Chow, Robbins, Siegmund (1971), Theorem 2.3 (p. 23)) $E_0(R^{\{y\}}_{N_A^{\{y\}}} - N_A^{\{y\}}) = 0$ and

$E_0(R^F_{N^F_A} - N^F_A) = 0$. Therefore, $E_0 \, N^{\{y\}}_{N^{\{y\}}_A} \geq A$ and $E_0 \, N^F_A = E_0 \, R^F_{N^F_A} \geq A$. This completes the proof of the first parts of Theorem 1(i) and Theorem 2(i).

For the second part of Theorem 1(i), let $S_0 = 0$ and define S_i recursively for $i \geq 1$ by

$$S_i = \min\{n \mid n > S_{i-1}, \sum_{j=S_{i-1}+1}^{n} Z_j^{\{y\}} \notin (0, \log A)\} .$$

Then $\pi_A^{\{y\}} \leq \Sigma_{i=1}^{C} S_i$ where $C = \min\{i \mid \Sigma_{j=S_{i-1}+1}^{S_i} Z_j^{\{y\}} \in [\log A, \log(2A)]\}$. By Wald's lemma,

$$E_0 \, \pi_A^{\{y\}} \leq E_0 S_1 / P_0 \, (\sum_{j=1}^{S_1} Z_j^{\{y\}} \in [\log A, \log (2A)]) .$$

Now

$$P_0(\sum_{j=1}^{S_1} Z_j^{\{y\}} \in \log A, \log (2A)]) = \sum_{n=1}^{\infty} \int_{\{S_1=n, C=1\}} f_0(x_1, \ldots, x_n) dx_1 \cdots dx_n$$

$$\geq (1/(2A)) \sum_{n=1}^{\infty} \int_{\{S_1=n, C=1\}} f_y(x_1, \ldots, x_n) dx_1 \cdots dx_n$$

$$= (1/(2A)) P_1^{(y)} \, (\sum_{j=1}^{S_1} Z_j^{\{y\}} \in [\log A, \log (2A)]) .$$

As $A \to \infty$, $\limsup E_0 S_1 < \infty$, and, by the renewal theorem

$$\liminf P_1^{(y)} \, (\sum_{j=1}^{S_1} Z_j^{\{y\}} \in [\log A, \log (2A)]) > 0 ,$$

from which Theorem 1(i) now follows.

To prove the second part of Theorem 2(i), choose ω_1,ω_2 in the interior of Ω such that $F([\omega_1,\omega_2]) > 0$. Without loss of generality assume that $\omega_1 > 0$. Denote: $\Gamma_0 = 0$, $\Gamma_i = \min\{n|\Sigma_{j=\Gamma_{i-1}+1}^{n} Z_j^{\{\omega_1\}} \leq 0$ or $\int_{\omega_1}^{\omega_2} \exp\{\Sigma_{j=\Gamma_{i-1}+1}^{n} Z_j^{\{y\}}\}dF(y) \geq A\}$, $Y = \min\{i|\int_{\omega_1}^{\omega_2} \exp\{\Sigma_{j=\Gamma_{i-1}+1}^{n} Z_j^{\{y\}} dF(y) \geq A\}$. Clearly $N_A^F \leq \Sigma_{i=1}^{Y} \Gamma_i$. Hence

(5) $$E_0 R_{N_A^F}^F = E_0 N_A^F \leq E_0\Gamma_1 E_0 Y \ .$$

Now

(6) $$E_0\Gamma_1 \leq E_0 \min\{n|\sum_{j=1}^{n} Z_j^{\{\omega_1\}} \leq 0\} < \infty \ .$$

In a manner similar to Theorem 1 of Pollak (1983) one can show that

(7) $$AP_0(Y=1) \underset{A\to\infty}{\longrightarrow} \int_{\omega_1}^{\omega_2} (1/C_0^y)P_1^{(y)} (\sum_{i=1}^{n} Z_i^{\{\omega_1\}} > 0,\ n=1,2,\ldots)dF(y) \ .$$

Therefore for given A_0 there exists a constant $C_F^{A_0}$ such that if $A \geq A_0$ then

(8) $$P_0(Y=1) \geq E_0\Gamma_1/(AC_F^{A_0}) \ .$$

Note that

(9) $$E_0Y = 1/P_0(Y=1) \ .$$

Now (5), (6), (8), and (9) complete the proof of Theorem 2(i).

PROOF OF THEOREM 1(ii), THEOREM 2(ii). Let $A > C_A > 0$ be fixed. Define: $L_0 = 0$. For j=1,2,..., define:

$$L_j = \min\left\{n \mid n > L_{j-1},\ \int \sum_{i=L_{j-1}+1}^{n} e^{\Sigma_{i=k}^{n} Z_i^{\{y\}}}\, dF(y) \geq \frac{A}{C_A}\right\}$$

$$N_j = L_j - L_{j-1}$$

$$V_{j,n} = \begin{cases} \int e^{\Sigma_{i=L_j+1}^{n} Z_i^{\{y\}}} \sum_{k=L_{j-1}+1}^{L_j} e^{\Sigma_{i=k}^{L_j} Z_i^{\{y\}}}\, dF(y) & \text{if} \quad n > L_j \\ \int \sum_{k=L_{j-1}+1}^{L_j} e^{\Sigma_{i=k}^{L_j} Z_i^{\{y\}}}\, dF(y) & \text{if} \quad n = L_j \end{cases}$$

$$H_j = \min\{n \mid n \geq L_j,\ V_{j,n} \geq A\}$$

$$= \infty \quad \text{if no such n exists}$$

$$M_j = H_j \wedge L_{j+1}$$

$$J = \min\{j \mid V_{j,M_j} \geq A\}$$

$$N_A^* = M_J$$

$$R_{j,n} = \int \sum_{k=L_{j-1}+1}^{n} e^{\Sigma_{i=k}^{n} Z_i^{\{y\}}}\, dF(y) \quad \text{for} \quad n > L_{j-1}$$

$$dF_j(y) = \sum_{k=L_{j-1}+1}^{L_j} e^{\Sigma_{i=k}^{L_j} Z_i^{\{y\}}}\, dF(y) \Big/ \int \sum_{k=L_{j-1}+1}^{L_j} e^{\Sigma_{i=k}^{L_j} Z_i^{\{s\}}}\, df(s)$$

$\delta_{\{y\}}$ = probability measure with unit mass at $\{y\}$

$1(\Theta)$ = indicator function of the set Θ

$$\tau_A^F = \min\{n \mid \int \exp\{\sum_{i=1}^{n} Z_i^{\{y\}}\} dF(y) \geq A,\ n \geq 1\}$$

$$= \infty \text{ if no such } n \text{ exists}$$

$$I_A^F - AP_0(\tau_A^F < \infty)$$

$$K(y) = 1/C_0^y$$

$$G_F = \int K(y) dF(y) .$$

By Theorem 3 of Pollak (1983), $I_A^F \to G_F$ as $A \to \infty$.

Until further notice, we will assume that the support of F is contained in a compact interval $[a,b]$, $0 < a < b < \infty$, $I(b) < \infty$.

LEMMA 1. For arbitrary $0 < \eta < 1$, and arbitrary probability measure ϕ whose support is contained in $[a,b]$, $0 < a < b < \infty$, there exists $B_0 > 0$ independent of ϕ such that if $B \geq B_0$ then

$$1-\eta < \frac{I_B^\phi}{G_\phi} < 1+\eta . \tag{10}$$

PROOF. This is the content of Theorem 1 of Pollak (1983).

LEMMA 2. For arbitrary $0 < \eta < 1$, $0 < \varepsilon < 1$ there exists $A_1 = A_1(\eta,\varepsilon)$ and $C = C(\eta,\varepsilon)$ such that if $A \geq A_1$ and one chooses $C_A = C$, then

$$P_0\left(1 - \eta < \frac{P_0(H_j < \infty \mid \mathfrak{F}_{L_j})}{G_{F_j} R_{j,L_j}/A} < 1 + \eta\right) > 1 - \varepsilon .$$

PROOF.

$$(11)\qquad P_0(H_j<\infty|\mathfrak{F}_{L_j}) = \begin{cases} 1 & \text{if} \quad R_{j,L_j} \geq A \\ I^{F_j}_{A/R_{j,L_j}}\, R_{j,L_j}/A & \text{if} \quad R_{j,L_j} < A \end{cases} .$$

In a manner analogous to Theorem 1 of Pollak (1983b), replacing ω_1,ω_2 by a,b respectively in (7), one gets the confergence in (7) to be uniform in measures F whose support is contained in [a,b]. Therefore, the constant $C^{A_0}_{F_J}$ in Theorem 2(i) can be replaced by a constant $C(A_0)$ which is independent of F_j (it is only dependent on a,b). Hence for $\Delta > 0$

$$P_0(R_{j,L_j} > \Delta A/C_A) \leq \frac{E_0\, R_{j,L_j}}{\Delta A/C_A} \leq \frac{C(A_0)}{\Delta} .$$

Choosing Δ to be large enough, Lemma 1 in conjunction with (11) complete the proof of Lemma 2.

LEMMA 3. For any $\varepsilon^* > 0$ there exists $\delta > 0$ such that if one chooses $C_A = C$ and if $A \geq C$, then

$$E_0\left(R_{j,L_j};\ R_{j,L_j} \geq \delta \frac{A}{C_A}\right) \leq \varepsilon^* \frac{A}{C_A} .$$

PROOF. Let X be distributed as X_1 under P_0.

$$E_0\left(R_{j,L_j};\ R_{j,L_j} \geq \delta\,\frac{A}{C_A}\right) \leq E_0\left(\int_a^b e^{bX}\left[1 + \sum_{k=L_{j-1}+1}^{L_j-1} e^{\sum_{i=k}^{L_j-1} Z_i\{y\}}\right] dF(y);\right.$$

$$\left.\int_a^b e^{bX}\left[1 + \sum_{k=L_{j-1}+1}^{L_j-1} e^{\sum_{i=k}^{L_j-1} Z_i\{y\}}\right] dF(y) \geq \delta\,\frac{C_A}{A}\right)$$

$$\leq E_0\left(e^{bX}\left[1 + \frac{A}{C_A}\right];\ e^{bX}\left[1 + \frac{A}{C_A}\right] \geq \delta\,\frac{A}{C_A}\right)$$

$$= \left(1 + \frac{A}{C_A}\right) E_0\left(e^{bX};\ e^{bX} \geq \frac{\delta}{1 + C_A/A}\right) .$$

This can be made to be less than $\varepsilon^* A/C_A$ by choosing δ to be large enough.

LEMMA 4. Let $U \sim U(0,1)$ be independent of $X_1, X_2, \ldots$. For $\varepsilon > 0$ let $Q_{\varepsilon,A} = (R_{j,L_j} > \gamma^{(1)}_{\varepsilon,A}) \cup (R_{j,L_j} = \gamma^{(1)}_{\varepsilon,A},\ U > \gamma^{(2)}_{\varepsilon,A})$ where $\gamma^{(1)}_{\varepsilon,A}$, $\gamma^{(2)}_{\varepsilon,A}$ are defined by $P_0(Q_{\varepsilon,A}) = \varepsilon$. Then for $\lambda > 0$ there exists an $\varepsilon > 0$ such that $E_0(R_{j,L_j};\ Q_{\varepsilon,A})/E_0 R_{j,L_j} < \lambda$ uniformly for all A, C_A such that $A > C_A$.

PROOF. Choose $\varepsilon^* < \lambda$. Let δ be as in Lemma 3. Let $\varepsilon > 0$ satisfy $\varepsilon\delta + \varepsilon^* < \lambda$. Then

$$E_0(R_{j,L_j};\ Q_{\varepsilon,A}) = E_0\left(R_{j,L_j};\ \left(\left(R_{j,L_j} < \delta\,\frac{A}{C_A}\right) \cup \left(\delta\,\frac{A}{C_A} \leq R_{j,L_j}\right)\right) \cap Q_{\varepsilon,A}\right)$$

$$\leq \varepsilon \cdot \delta\,\frac{A}{C_A} + \varepsilon^*\,\frac{A}{C_A} < \lambda\,\frac{A}{C_A} < \lambda E_0\, R_{j,L_j} .$$

LEMMA 5. For arbitrary $0 < \eta < 1$ there exist $A_2 = A_2(\eta)$ and $C = C(\eta)$ such that if $A \geq A_2$ and one chooses $C_A = C$, then

$$(1-\eta)\ \frac{G_F}{A/E_0 R_{j,L_j}} < P_0(H_j < \infty) < (1+\eta)\ \frac{G_F}{A/E_0 R_{j,L_j}} \ .$$

PROOF. Choose $0 < \alpha < \eta$. By Lemma 4, one can choose $\varepsilon > 0$ such that whenever $A > C_A$

$$\text{(12)} \qquad \frac{E_0(R_{j,L_j};\ Q_{\varepsilon,A})}{G_F E_0\ R_{j,L_j}} < \eta-\alpha \ ,$$

where $Q_{\varepsilon,A}$ is as defined in Lemma 4. By Lemma 2, there exist A_1 and C such that if $A > \max(A_1, C)$ and one chooses $C_A = C$, then

$$K_\varepsilon = \left\{ 1-\alpha < \frac{P_0(H_j < \infty | \mathfrak{F}_{L_j})}{G_{F_j}\ R_{j,L_j}/A} < 1+\alpha \right\}$$

has a P_0-probability $P_0(K_\varepsilon) \geq 1-\varepsilon$. Note that since $P_0(Q_{\varepsilon,A}) \geq P_0((K_\varepsilon)^c)$, for any set $S \subseteq (K_\varepsilon)^c \cap (Q_{\varepsilon,A})^c$ there exists a set $S^* \subseteq Q_{\varepsilon,A} \cap K_\varepsilon$ such that $P_0(S) \leq P_0(S^*)$. Obviously R_{j,L_j} on S^* is larger than R_{j,L_j} on S, and therefore

$$\text{(13)} \qquad E_0(R_{j,L_j};\ (K_\varepsilon)^c) \leq E_0(R_{j,L_j};\ Q_{\varepsilon,A}) \ .$$

Also note that because of the martingale property of $\Sigma^n_{k=L_{j-1}+1} \exp\{\Sigma^n_{i=k} z_i^{\{y\}}\} - (n - L_{j-1})$ under P_0(given $\mathfrak{F}_{L_{j-1}}$, for $n > L_{j-1}$),

it follows that

$$(14)\quad E_0\, R_{j,L_j}\, G_{F_j} = E_0\, R_{j,L_j} \int K(y)\,dF_j(y)$$

$$= E_0 \int K(y) \sum_{k=L_{j-1}+1}^{L_j} \exp\left\{\sum_{i=k}^{L_j} Z_i^{\{y\}}\right\} dF(y)$$

$$= E_0(L_j - L_{j-1}) \int K(y)\,dF(y)$$

$$= E_0\, R_{j,L_j}\, G_F \quad .$$

Therefore, by (12), (13), and (14),

$$P_0(H_j < \infty) = E_0\left(P_0(H_j < \infty | \mathfrak{F}_{L_j});\ K_\varepsilon \cup (K_\varepsilon)^c\right)$$

$$\leq (1+\alpha)E_0 R_{j,L_j}\, G_{F_j}/A + E_0\left(R_{j,L_j}/A;\ (K_\varepsilon)^c\right)$$

$$\leq (1+\alpha)E_0 R_{j,L_j}\, G_F/A + E_0(R_{j,L_j}/A;\ Q_{\varepsilon,A})$$

$$< (1+\alpha)E_0 R_{j,L_j}\, G_F/A + (\eta-\alpha)E_0 R_{j,L_j}\, G_F/A$$

$$= (1+\eta)\,\frac{G_F}{A/E_0\, R_{j,L_j}} \quad .$$

Likewise,

$$P_0(H_j < \infty) \geq E_0(P_0(H_j < \infty | \mathfrak{F}_{L_j});\ K_\varepsilon)$$

$$> (1-\alpha)E_0(R_{j,L_j}\, G_{F_j}/A;\ K_\varepsilon)$$

$$\geq (1-\alpha)E_0(R_{j,L_j}\, G_{F_j}/A) - E_0\left(R_{j,L_j}/A;\ (K_\varepsilon)^c\right)$$

$$\geq (1-\alpha)E_0 R_{j,L_j}\, G_F/A - E_0(R_{j,L_j}/A;\ Q_{\varepsilon,A})$$

$$> (1-\alpha)E_0 R_{j,L_j} G_F/A - (\eta-\alpha)E_0 R_{j,L_j} G_F/A$$

$$= (1-\eta) \frac{G_F}{A/E_0 R_{j,L_j}} .$$

LEMMA 6. For arbitrary $\eta > 0$ there exist $A_3 = A_3(\eta)$ and $C = C(\eta)$ such that if $A \geq A_3$ and one chooses $C_A = C$, then

$$1 - \eta < \frac{E_0(V_{j,H_j} | H_j < \infty)}{A/G_F} < 1 + \eta .$$

PROOF. Note that $E_0(V_{j,H_j}; H_j < \infty) = E_0 R_{j,L_j}$ and so $E_0(V_{j,H_j} | H_j < \infty) = E_0 R_{j,L_j} / P_0(H_j < \infty)$. An application of Lemma 5 completes the proof of Lemma 6.

LEMMA 7. For arbitrary $0 < \eta < 1$ there exist $A_4 = A_4(\eta)$ and $C = C(\eta)$ such that if $A \geq A_4$ and one chooses $C_A = C$, then

$$1 - \eta < \frac{E_0(V_{j,M_j} | V_{j,M_j} \geq A)}{A/G_F} < 1 + \eta .$$

PROOF. Let A be large enough.

$$(15) \quad E_0(V_{j,M_j} | V_{j,M_j} \geq A) = \frac{E_0(V_{j,M_j}; V_{j,M_j} \geq A)}{P_0(V_{j,M_j} \geq A)} .$$

Clearly, $E_0(V_{j,M_j}; V_{j,M_j} \geq A) \leq E_0(V_{j,M_j}; H_j < \infty)$, $P_0(V_{j,M_j} \geq A) \leq P_0(H_j < \infty)$, and $P_0(V_{j,M_j} \geq A) = P_0(H_j < \infty) - P_0(L_{j+1} < H_j < \infty)$. Denote

$Q_G(\cdot) = \int_a^b P_1^{(y)}(\cdot)dG(y)$. For $x > 0$

$$P_0(x + L_j < H_j < \infty) = E_0 P_0(x + L_j < H_j < \infty | \mathfrak{F}_{L_j})$$

$$\leq E_0 Q_{F_j}(\tau^{F_j}_{A/R_{j,L_j}} > x) \; R_{j,L_j}/A$$

$$= o(1) \; E_0 \; R_{j,L_j}/A = o(1) \; P_0(H_j < \infty) \quad ,$$

where $o(1) \to 0$ as $x \to \infty$ uniformly in $\mathfrak{F}_{L_j}$, A for fixed $C_A = C$. Also, $P_0(L_{j+1} \leq L_j + x) = P_0(L_1 \leq x) = \Sigma_{n=1}^x \; P_0(R_n > A/C_A) \leq x^2 \; C_A/A$. So, for $x > 0$

$$P_0(L_{j+1} \leq H_j < \infty) \leq P_0(x \leq L_{j+1} \leq H_j < \infty) + P_0(L_{j+1} \leq x \leq H_j < \infty)$$

$$+ P_0(L_{j+1} \leq H_j \leq x)$$

$$\leq 2P_0(x \leq H_j < \infty) + P_0(L_{j+1} \leq x)$$

$$\leq o(1) \; P_0(H_j < \infty) + x^2 \; C_A/A \quad .$$

Since (by Lemma 5 and Theorem 2(i)) $P_0(H_j < \infty)$ is of the order of magnitude of $1/C_A$, choosing A, C_A large enough will cause $P_0(L_{j+1} < H_j < \infty)/P_0(H_j < \infty)$ to be arbitrarily small, i.e., $P_0(V_{j,M_j} \geq A)/P_0(H_j < \infty)$ to be arbitrarily close to 1. Similarly,

$$E_0(V_{j,M_j}; \; V_{j,M_j} < A, \; H_j < \infty) = E_0 \; Q_{F_j}\left(\tau^{F_j}_{A/R_{j,L_j}} > N_1\right) \; R_{j,L_j}$$

$$= o(1) \; P_0(H_j < \infty) \cdot A \quad ,$$

where $o(1) \to 0$ as $A \to \infty$. Therefore, choosing large C_A and very large A one can get $E_0(V_{j,M_j}; V_{j,M_j} \geq A)/E_0(V_{j,H_j}; H_j < \infty)$ to be arbitrarily close to 1. Hence, one can make

$$(16) \qquad \frac{E_0(V_{j,M_j}; V_{j,M_j} \geq A)}{P_0(V_{j,M_j} \geq A)} \Bigg/ \frac{E_0(V_{j,H_j}; H_j < \infty)}{P_0(H_j < \infty)}$$

be arbitrarily close to 1. Lemma 7 now follows from (15), (16), and Lemma 6.

LEMMA 8. For arbitrary $0 < \eta < 1$ there exist $A_5 = A_5(\eta)$ and $C = C(\eta)$ such that if $A \geq A_5$ and one chooses $C_A = C$, then

$$1 - \eta < \frac{E_0 V_{J,M_J}}{A/G_F} < 1 + \eta \quad .$$

PROOF. Denote $V_{0,M_0} = V_{-1,M_{-1}} = 0$.

$$(17) \qquad E_0 V_{J,M_J} = \sum_{j=1}^{\infty} \int_{J=j} V_{j,M_j}\, dP_0$$

$$= \sum_{j=1}^{\infty} \int_{\{V_{i,M_i} < A,\ i=0,\dots,j-1;\ V_{j,M_j} \geq A\}} V_{j,M_j}\, dP_0$$

$$= \sum_{j=1}^{\infty} \int_{\{V_{i,M_i} < A,\ i=-1,0,\dots,j-2;\ V_{j,M_j} \geq A\}} V_{j,M_j}\, dP_0$$

$$- \sum_{j=2}^{\infty} \int_{\{V_{i,M_i}<A,\ i=0,\ldots,j-2;\ V_{j-1,M_{j-1}} \geq A, V_{j,M_j} \geq A\}} V_{j,M_j}\, dP_0 \quad .$$

Note that

(18)
$$\sum_{j=2}^{\infty} \int_{\{V_{i,M_i}<A,\ i=0,\ldots,j-2;\ V_{j-1,M_{j-1}} \geq A, V_{j,M_j} \geq A\}} V_{j,M_j}\, dP_0$$

$$= \sum_{j=2}^{\infty} P_0\{V_{i,M_i} < A,\ i=0,\ldots,j-2;\ V_{j-1,M_{j-1}} \geq A\} = 1 \quad .$$

Now,

(19)
$$\sum_{j=1}^{\infty} \int_{\{V_{i,M_i}<A,\ i=-1,0,\ldots,j-2;\ V_{j,M_j} \geq A\}} V_{j,M_j}\, dP_0$$

$$= \sum_{j=1}^{\infty} E_0(V_{j,M_j} | V_{j,M_j} \geq A) P_0(V_{i,M_i} < A,\ i=-1,0,\ldots,j-2;\ V_{j,M_j} \geq A)$$

$$= E_0(V_{1,M_1} | V_{j,M_j} \geq A) \left[1 + \sum_{j=2}^{\infty} P_0\left(V_{i,M_i} < A,\ i=-1,0,\ldots,j-2;\right.\right.$$

$$\left.\left. V_{j-1,M_{j-1}} \geq A,\ V_{j,M_j} \geq A\right)\right] \quad .$$

$$(20)\quad \sum_{j=2}^{\infty} P_0(V_{i,M_i} < A,\ i=-1,0,\ldots,j-2;\ V_{j-1,M_{j-1}} \geq A,\ V_{j,M_j} \geq A)$$

$$\leq \left[\sum_{j=2}^{\infty} P_0(V_{i,M_i} < A,\ i=-1,0,\ldots,j-3)\right] P_0(V_{1,M_1} \geq A,\ V_{2,M_2} \geq A)$$

$$= [1+E_0J]P_0(V_{1,M_1} \geq A,\ V_{2,M_2} \geq A) \quad .$$

Denote: $J_{odd} = \min\{n|n \text{ odd},\ V_{n,M_n} \geq A\}$, $J_{even} = \min\{n|n \text{ even},\ V_{n,M_n} \geq A\}$. Since $J = \min\{J_{odd}, J_{even}\} \leq J_{odd} + J_{even}$,

$$(21)\quad E_0J \leq E_0J_{odd} + E_0J_{even} \leq \frac{4}{P_0(V_{1,M_1} \geq A)} \quad .$$

Therefore, because of (17)-(21), it only remains to show that $P_0(V_{1,M_1} \geq A,\ V_{2,M_2} \geq A)$ can be made to be sufficiently small.

$$(22)\quad P_0(V_{1,M_1} \geq A,\ V_{2,M_2} \geq A) = P_0(V_{1,M_1} \geq A,\ R_{2,M_1} < \frac{A}{C_A},\ V_{2,M_2} \geq A)$$

$$+ P_0(V_{1,M_1} \geq A,\ R_{2,M_1} \geq \frac{A}{C_A},\ V_{2,M_2} \geq A) \quad .$$

Suppose that $A/C_A \geq A_0$ where A_0 is a constant, as in Theorem 2(i). Note that on $\{R_{2,M_1} < A/C_A\}$

$$E_0(R_{2,L_2}|R_{2,M_1},\ V_{1,M_1} > A)$$

$$= E_0\left(\int\left(\sum_{k=L_1+1}^{M_1} e^{\sum_{i=k}^{L_2} Z_i^{\{y\}}} + \sum_{k=M_1+1}^{L_2} e^{\sum_{i=k}^{L_2} Z_i^{\{y\}}}\right) \cdot dF(y) \,\middle|\, R_{2,M_1},\ V_{1,M_1} \geq A\right)$$

$$\leq R_{2,M_1} + E_0 R_{2,L_2}$$

$$\leq \frac{A}{C_A}\left[1 + C(A_0)\right] \quad ,$$

where $C(A_0)$ is a constant as in the proof of Lemma 2. Therefore,

(23) $$P_0(V_{1,M_1} \geq A,\ R_{2,M_1} < \frac{A}{C_A},\ V_{2,M_2} \geq A)$$

$$\leq P_0(V_{2,M_2} \geq A \mid V_{1,M_1} \geq A,\ R_{2,M_1} < \frac{A}{C_A}) P_0(V_{1,M_1} \geq A)$$

$$\leq E_0\left[\frac{R_{2,L_2}}{A}\middle| V_{1,M_1} \geq A,\ R_{2,M_1} < \frac{A}{C_A}\right] P_0(V_{1,M_1} \geq A)$$

$$\leq \frac{1 + C(A_0)}{C_A} P_0(V_{1,M_1} \geq A) \quad .$$

Now for any $x > 0$

(24) $$P_0(V_{1,M_1} \geq A,\ R_{2,M_1} \geq \frac{A}{C_A},\ V_{2,M_2} \geq A)$$

$$\leq P_0(V_{1,M_1} \geq A,\ R_{2,M_1} \geq \frac{A}{C_A},\ M_1 \leq L_1 + x)$$

$$+ P_0(V_{1,M_1} \geq A,\ R_{2,M_1} \geq \frac{A}{C_A},\ M_1 > L_1 + x)$$

$$\leq P_0(R_{2,M_1} \geq \frac{A}{C_A},\ M_1 \leq L_1 + x) + P_0(V_{1,M_1} \geq A,\ M_1 > L_1 + x) \quad ,$$

(25) $$P_0(R_{2,M_1} \geq \frac{A}{C_A},\ M_1 \leq L_1 + x) \leq \sum_{n=1}^{x} P_0(R_n^F \geq \frac{A}{C_A}) \leq \frac{x^2}{A/C_A} \quad ,$$

and as in the proof of Lemma 7,

$$(26) \qquad P_0(V_{1,M_1} \geq A,\ M_1 > L_1+x) \leq P_0(L_1+x < H_j < \infty) = o(1)\ P_0(H_j < \infty) ,$$

where $o(1) \to 0$ as $x \to \infty$ uniformly in $\mathfrak{F}_{L_j}$, large A, for fixed $C_A = C$. Now (17)-(26) in conjunction with Lemma 5 and Lemma 7 and its proof complete the proof of Lemma 8.

LEMMA 9. Let $\lambda > 0$. There exist $C = C(\lambda)$ and $A_6 = A_6(\lambda)$ such that if $A \geq A_6$ and one chooses $C_A = C$ then

$$(27) \qquad E_0 \int \sum_{k=L_J+1}^{M_J} \exp\{\sum_{i=k}^{M_J} Z_i^{\{y\}}\} dF(y) \leq \lambda A .$$

The sum in (27) is understood to be zero if $M_J = L_J$.

PROOF. It is enough to prove that under the conditions described

$$(28) \qquad E_0(R_{2,M_1} | V_{1,M_1} \geq A) \leq \lambda A ,$$

for then

$$E_0 \int \sum_{k=L_J+1}^{M_J} \exp\{\sum_{i=k}^{M_j} Z_i^{\{y\}}\} dF(y)$$

$$\leq E_0 \int \sum_{k=L_{J_{odd}}+1}^{M_{J_{odd}}} \exp\{\sum_{i=k}^{M_{J_{odd}}} Z_i^{\{y\}}\} dF(y)$$

$$+ E_0 \sum_{k=L_{J_{even}}+1}^{M_{J_{even}}} \exp\{\sum_{i=k}^{M_{J_{even}}} Z_i^{\{y\}}\} dF(y)$$

$$\leq 2\lambda A$$

where J_{odd} and J_{even} are as in the proof of Lemma 8.

On $\{A/C_A \leq R_{1,L_1} < A\}$

(29) $$E_0(R_{2,M_1}; V_{1,M_1} \geq A) = E_0(R_{2,M_1}; H_1 \leq L_2) .$$

Let $x > 1$. Note that $\{H_1 \leq L_2\} = \{H_1 \leq L_1+x < L_2\} \cup \{H_1 \leq L_2 \leq L_1+x\}$ $\cup \{L_1+x < H_1 < L_2\} \cup \{L_1+x < H_1 = L_2\}$. We will analyze the expectation in (29) on each of these four events separately. Note that $R_{2,H_1} < A/C_A \leq R_{2,L_1}$ on $\{H_1 < L_2\}$.

(30) $$E_0(R_{2,H_1}; H_1 \leq L_1+x < L_2) \leq E_0(R_{2,H_1}; H_1 \leq L_1+x) \leq E_0(R_{2,L_1+x}) = x .$$

(31) $$E_0(R_{2,H_1}; H_1 \leq L_2 \leq L_1+x) \leq E_0(R_{2,L_2}; H_1 \leq L_2 \leq L_1+x) \leq E_0(R_{2,L_1+x}) = x .$$

(32) $$E_0(R_{2,H_1}; L_1+x < H_1 < L_2) \leq (A/C_A)\ P_0(L_1+x < H_1 < \infty) .$$

(Later we will let x be large and will evaluate (32) as in the proof of Lemma 7.) Denote: $\Xi_{k,x} = \{(L_1+x) \vee (k-1) < H_1-L_2\}$. Given $L_1,X_1,\ldots,X_{L_1}$:

(33) $$E_0(R_{2,H_1}; L_1+x < H_1 = L_2) = E_0 \int \sum_{k=L_1+1}^{\infty} \exp\{\sum_{i=k}^{H_1} Z_i^{\{y\}}\} 1(\Xi_{k,x}) dF(y) .$$

(34) $$E_0 \int \exp\{\sum_{i=k}^{H_1} Z_i^{\{y\}}\} 1(\Xi_{k,x}) dF(y)$$

$$= \iint \exp\{-\sum_{i=L_1+1}^{k-1} Z_i^{\{y\}}\} \exp\{\sum_{i=L_1+1}^{H_1} Z_i^{\{y\}}\} 1(\Xi_{k,x}) dP_0\ dF(y)$$

$$= \int E_{L_1+1}^{(y)} \exp\{-\sum_{i=L_1+1}^{k-1} Z_i^{\{y\}}\} 1(\Xi_{k,x}) dF(y)$$

$$= \int E^{(y)}_{L_1+1} E^{(y)}_{L_1+1} (\exp\{- \sum_{i=L_1+1}^{k-1} Z_i^{\{y\}}\} 1(\Xi_{k,x}) | X_{L_1+1}, \ldots X_{k-1}) dF(y)$$

$$= \int E^{(y)}_{L_1+1} \exp\{- \sum_{i=L_1+1}^{k-1} Z_i^{\{y\}}\} P^{(y)}_{L_1+1}(\Xi_{k,x} | X_{L_1+1}, \ldots, X_{k-1}) dF(y)$$

$$= \int E_0 P^{(y)}_{L_1+1}(\Xi_{k,x} | X_{L_1+1}, \ldots, X_{k-1}) dF(y) .$$

Let $\psi(a) > h > 0$. Let $\varepsilon_0 > 0$, $j_0 = (\log A)^{1+\varepsilon_0}$. For large enough A there exists $\varepsilon_1 > 0$ such that for all $j > j_0$ $P_0(\Sigma_{i=L_1+1}^{L_1+j-1} X_i > (j-1)h/b) < \exp\{-\varepsilon_1 j\}$. On $\{\Sigma_{i=L_1+1}^{L_1+j-1} X_i \leq (j-1)h/b\}$: for $n \geq j-1 + L_1$, $V_{1,n} \leq \exp\{j(h-\psi(a)\} \int \exp\{y\Sigma^n_{i=L_1+j} - (n-j-L_1+1)\psi(y)\} dF_1(y) R_{1,L_1}$. Let $k = L_1+j$. Let $H_{1,k} = \min\{n | n \geq k, V_{1,n} \geq A\}$. $E^{(y)}_{L_1+1}(H_{1,k} - k) \geq (j-1)(\psi(a) - h)/\psi(y)b)$. So, for large enough A, for $a \leq y \leq b$, there exists $\varepsilon_2 > 0$ such that for all $j \geq j_0$ $P^{(y)}_{L_1+1}(H_{1,k} - k < \frac{1}{2}(j-1)(\psi(a) - h)/(\psi'(b)b) \leq \exp\{-\varepsilon_2 j\}$ for all $a \leq y \leq b$ (see Pollak and Siegmund (1975) for an example of the considerstions involved). Since $L_2 \leq \min\{n | \int_a^b \exp\{y \sum_{i=k}^{n} X_i - (n-k\_1)\psi(y)\} dF(y) \geq A/C_A\}$, there exists $\varepsilon_2 > 0$ such that for all $j \geq j_0$ $P^{(y)}_{L_1+1}(L_2-k > \frac{1}{2}(j-1)(\psi(a)-h)/(\psi'(b)b) \leq \exp\{-\varepsilon_3 j\}$ for all $a \leq y \leq b$ if A is large enough. Therefore, for $j \geq j_0$, $k = L_1+j$, if A is large enough,

$$E_0 P^{(y)}_{1_1+1} (\Xi_{k,x} | X_{L_1+1}, \ldots, X_{k-1})$$

$$\leq E_0 P^{(y)}_{L_1+1} ((k-1) \vee (L+x) < L_2 = H_{1,k} | X_{L_1+1}, \ldots, X_{k-1})$$

$$\leq \exp\{-\varepsilon_1 j\} + \exp\{-\varepsilon_2 j\} + \exp\{-\varepsilon_3 j\}$$

and so

(36) $E_0(R_{2,H_1}: L_1+x < H_1=L_2) \leq j_0 + \sum_{j=j_0}^{\infty} (\exp\{-\epsilon_1 j\} + \exp\{-\epsilon_2 j\} + \exp\{-\epsilon_3 j\})$.

By letting x be large enough - such as $(\log A)^{1+\epsilon_0}$ - one gets by virtue of (30) - (36) that $E_0(R_{2,M_1}; V_{1,M_1} \geq A)/A$ is arbitrarily small for large enough A, from which (28) follows.

LEMMA 10. Let $\lambda > 0$. There exists $C = C(\lambda)$ and $A_7 = A_7(\lambda)$ such that if $A \geq A_7$ and one chooses $C_A = C$, then

$$0 \leq ER^F_{N^*_A} - E_0 V_{J,M_J} \leq \lambda A .$$

PROOF. Clearly,

$$(37) \quad R_{N^*_A} = V_{J,M_J} + \int \sum_{k=L_J+1}^{M_J} e^{\sum_{i=k}^{M_J} Z_i^{\{y\}}} dF(y) + \sum_{j=1}^{J-1} V_{j,M_j} .$$

Therefore, by Lemma 9, it suffices to show that $E_0 \sum_{j=1}^{J-1} V_{j,M_j} < \lambda A$ for appropriately chosen C. Let J_{odd} and J_{even} be as in the proof of Lemma 8.

$$(38) \quad E_0 \sum_{j=1}^{J-1} V_{j,M_J} = \sum_{j=1}^{\infty} E_0(V_{j,M_J}; j \leq J - 1)$$

$$= \sum_{j=1}^{\infty} E_0(V_{j,M_j}; j \leq J - 1)$$

$$= E_0 \sum_{j=1}^{J-1} V_{j,M_j}$$

$$= E_0 \sum_{\substack{j=1 \\ j_{odd}}}^{J-1} V_{j,M_j} + E_0 \sum_{\substack{j=2 \\ j_{even}}}^{J-1} V_{j,M_j}$$

$$\leq E_0 \sum_{\substack{j=1 \\ j_{odd}}}^{J_{odd}-1} V_{j,M_j} + E_0 \sum_{\substack{j=2 \\ j_{even}}}^{J_{even}-1} V_{j,M_j}$$

$$= E_0 \sum_{\substack{j=1 \\ j_{odd}}}^{J_{odd}-2} V_{j,M_j} + E_0 \sum_{\substack{j=2 \\ j_{even}}}^{J_{even}-2} V_{j,M_j}$$

$$= E_0(V_{1,M_1} | V_{1,M_1} < A) E_0(J_{odd} - 2 + J_{even} - 2)$$

$$\leq E_0(V_{1,M_1} | V_{1,M_1} < A)(E_0 \, J_{odd} + E_0 \, J_{even})$$

$$\leq E_0(V_{1,M_1} | V_{1,M_1} < A) \frac{4}{P_0(V_{1,M_1} \geq A)} \quad .$$

Now

(39) $$E_0(V_{1,M_1}; V_{1,M_1} < A) = E_0 \, V_{1,M_1} - E_0(V_{1,H_1}; H_1 < \infty)$$

$$+ E_0(V_{1,H_1}; H_1 < \infty) - E_0(V_{1,M_1}; V_{1,M_1} \geq A)$$

$$= E_0 R_{1,L_1} - E_0 R_{1,L_1} + E_0(V_{1,H_1}; H_1 < \infty)$$

$$\cdot \left[1 - \frac{E_0(V_{1,M_1}; V_{1,M_1} \geq A)}{E_0(V_{1,H_1}; H_1 < \infty)}\right]$$

$$= (E_0 R_{1,L_1}) \, o(1) \quad ,$$

where $o(1) \to 0$ as $A \to \infty$ as in the proof of Lemma 7. Since (as in the proof of Lemma 7) $P_0(V_{1,M_1} \geq A)/P_0(H_1 < \infty) \to 1$ as $A \to \infty$, (38) and (39) with Theorem 2(i) and Lemma 5 complete the proof of Lemma 10.

PROOF OF THEOREM 2(ii). Since (see (37)) $R^F_{N^*_A} \geq V_{J,M_J} \geq A$, it follows that $N^*_A \geq N^F_A$ and so

$$(40) \qquad E_0\, N^F_A \leq E_0\, N^*_A = E_0\, R^F_{N^*_A} \ .$$

Denote $J^* = \max\{j \mid L_j = N^F_A$ and $V_{j-1,L_j} < A$, or $L_{j-1} < N^F_A\}$.

$$R^F_{N^F_A} = V_{J^*,N^F_A} + \int \sum_{k=L_{J^*}+1}^{N^F_A} e^{\sum_{i=k}^{N^F_A} Z_i^{\{y\}}}\, dF(y) + \sum_{j=1}^{J^*-1} V_{j,N^F_A} \ .$$

Since $V_{j,M_j} < A$ for $j \leq J^*-1$ and since

$E_0(\Sigma_{j=1}^{J^*-1} V_{j,N^F_A};\ N^*_A > N^F_A) = E_0(\Sigma_{j=1}^{J^*-1} V_{j,N^*_A};\ N^*_A > N^F_A)$, it follows (since $M_J = N^*_A$) that

$$E_0 \sum_{j=1}^{J^*-1} V_{j,N^F_A} \leq E_0 \sum_{j=1}^{J-1} V_{j,M_J} \ ,$$

which in turn is bounded as in Lemma 10 (see (38), (39) above) by $A\ o(1)$.

$$E_0 \int \sum_{k=L_{J^*}+1}^{N^F_A} e^{\sum_{i=k}^{N^F_A} Z_i^{\{y\}}}\, dF(y) = E_0\left(\int \sum_{k=L_{J^*}+1}^{N^F_A} e^{\sum_{i=k}^{N^F_A} Z_i^{\{y\}}}\, dF(y);\ N^F_A < N^*_A\right)$$

$$+ E_0\left(\int \sum_{k=L_{J^*}+1}^{N_A^F} e^{\sum_{i=k}^{N_A^F} Z_i^{\{y\}}} dF(y);\right.$$

$$\left. N_A^F = N_A^*\right) .$$

$$E_0\left(\int \sum_{k=L_{J^*}+1}^{N_A^F} e^{\sum_{i=1}^{N_A^F} Z_i^{\{y\}}} dF(y);\ N_A^F < N_A^*\right) < \frac{A}{C_A} = \frac{A}{C} .$$

for large enough A, by virtue of Lemma 9,

$$E_0\left(\int \sum_{k=L_{J^*}+1}^{N_A^F} e^{\sum_{k=1}^{N_A^F} Z_i^{\{y\}}} dF(y);\ N_A^F = N_A^*\right)$$

$$= E_0\left(\int \sum_{k=L_J+1}^{M_J} e^{\sum_{i=k}^{M_J} Z_i^{\{y\}}} dF(y);\ N_A^F = N_A^*\right)$$

$$\leq E_0\left(\int \sum_{k=L_J+1}^{M_J} e^{\sum_{i=k}^{M_J} Z_i^{\{y\}}} dF(y)\right)$$

$$\leq d(C) \quad ,$$

Hence, for large enough A, $E_0(R^F_{N_A^F} - V_{J^*,N_A^F}) \leq 2A/C$.

Let $\varepsilon > 0$, $\lambda = 2/(C\varepsilon)$. Since $R^F_{N_A^F} - V_{J^*,N_A^F} \geq 0$, it follows that $P_0(R^F_{N_A^F} - V_{J^*,N_A^F} \geq \lambda A) \leq 2/(\lambda C) = \varepsilon$. Hence, given $\varepsilon > 0$, by choosing C to be large enough λ would be arbitrarily small, and $P_0(R^F_{N^F_A} - V_{J^*,N^F_A} \geq \lambda A) \leq \varepsilon$ for all large enough A. *I.e.*, for large enough A,

$$\varepsilon \geq P_0(V_{J^*,N^F_A} \leq R^F_{N^F_A} - \lambda A) \geq P_0(V_{J^*,N^F_A} \leq (1-\lambda)A) \quad .$$

Let $N^{**}_{(1-\lambda)A}$ denote $N^*_{(1-\lambda)A}$ when one chooses $C_{(1-\lambda)A} = (1-\lambda)C_A$. It follows that for large enough A, with $C_A = C$ as above,

$$P_0(N^{**}_{(1-\lambda)A} \leq N^F_A) \geq 1 - \varepsilon \quad .$$

Therefore, if C was chosen to be large enough,

$$\begin{aligned}
(41) \quad E_0N^F_A &= E_0(N^F_A;\ N^F_A \geq N^{**}_{(1-\lambda)A}) + E_0(N^F_A;\ N^F_A < N^{**}_{(1-\lambda)A}) \\
&\geq E_0(N^{**}_{(1-\lambda)A};\ N^F_A \geq N^{**}_{(1-\lambda)A}) + E_0(N^F_A;\ N^F_A < N^{**}_{(1-\lambda)A}) \\
&= E_0N^{**}_{(1-\lambda)A} - E_0(N^{**}_{(1-\lambda)A} - N^F_A;\ N^F_A < N^{**}_{(1-\lambda)A}) \\
&\geq E_0N^{**}_{(1-\lambda)A} - \varepsilon[E_0R_{1,L_1} + E_0N^{**}_{(1-\lambda)A}] \\
&\geq (1-2\varepsilon)E_0N^{**}_{(1-\lambda)A} \quad ,
\end{aligned}$$

for all large enough A. Since ε and λ can be arbitrarily small, the fact that $E_0N^{**}_{(1-\lambda)A} = E_0R^F_{N^{**}_{(1-\lambda)A}}$ coupled with (40), (41), Lemma 10, and Lemma 8 complete the proof of Theorem 2(ii) for the case where the support of F is contained in $[a,b]$, $0 < a < b < \infty$.

If $(0,\infty) \subseteq \Omega$, $a = 0$, and/or $b = \infty$: If one replaces dF by $dF^*_n = 1(\frac{1}{n},n)dF$, then $N^F_A \leq N^{F^*_n}_A$ (letting $N^{F^*_n}_A$ have the obvious meaning, despite F_n^* not being a probability distribution) and so $E_0 N_A^F \leq A\, C_0^F(1+o(1))$, where $o(1) \to 0$ as $A \to \infty$, and C_0^F is the constant in Theorem 2(ii) (as described after the statement of the theorem).

For arbitrary $\alpha > 0$ define $F_{n,\alpha} = (1+\alpha)F_n^*$.

$$E_0\left\{\int_0^{1/n} \sum_{k=1}^{N_A^F} e^{\sum_{i=k}^{N_A^F} Z_i^{\{y\}}}\, dF(y) + \int_n^{\infty} \sum_{k=1}^{N_A^F} e^{\sum_{i=k}^{N_A^F} Z_i^{\{y\}}}\, dF(y)\right\}$$

$$= E_0 N_A^F\left\{\int_0^{1/n} dF(y) + \int_n^{\infty} dF(y)\right\} .$$

Therefore,

$$P_0\left\{\int_0^{1/n} \sum_{k=1}^{N_A^F} e^{\sum_{i=k}^{N_A^F} Z_i^{\{y\}}}\, dF(y) + \int_n^{\infty} \sum_{k=1}^{N_A^F} e^{\sum_{i=k}^{N_A^F} Z_i^{\{y\}}}\, dF(y) \geq \lambda A\right\}$$

$$\leq \frac{\int_0^{1/n} dF(y) + \int_n^{\infty} dF(y)}{\lambda}\; \frac{E_0 N_A^F}{A} ,$$

which, for any $\lambda > 0$, can be made arbitrarily small by taking n to be large enough. Now for λ sufficiently small

$$P(N_A^F \leq N_A^{F_{n,\alpha}}) = P_0\left\{N_A^F \leq N_A^{F_{n,\alpha}};\ \int_0^{1/n} \sum_{k=1}^{N_A^F} e^{\sum_{i=k}^{N_A^F} Z_i^{\{y\}}}\, dF(y)\right.$$

$$\left. + \int_n^{\infty} \sum_{k=1}^{N_A^F} e^{\sum_{i=k}^{N_A^F} Z_i^{\{y\}}}\, dF(y) \geq \lambda A\right\}$$

$$\leq P_0\left\{\int_0^{1/n} \sum_{k=1}^{N_A^F} e^{\sum_{i=k}^{N_A^F} Z_i^{\{y\}}}\, dF(y)\right.$$

$$+ \int_n^\infty \sum_{k=1}^{N_A^F} e^{\sum_{i=k}^{N_A^F} Z_i^{\{y\}}} dF(y) \geq \lambda A\Bigg) \quad .$$

In other words, for arbitrary $\varepsilon > 0$, $P_0(N_A^F \geq N_A^{F^*_{n,\alpha}}) > 1 - \varepsilon$ for large enough n and A. It is easy to see that $E_0(N_A^{F_{n,\alpha}} - N_A^F | N_A^{F_{n,\alpha}} \geq N_A^F) \leq E_0 N_A^{F_{n,\alpha}}$. Hence $E_0 N_A^F \geq E_0 N_A^{F_{n,\alpha}}(1-\varepsilon)$.

Letting $\varepsilon \to 0$, $\alpha \to 0$ completes the proof of Theorem 2(ii) for $(a,b) \subseteq (0,\infty)$.

If $w = \sup\{y | y \in \Omega\} < \infty$ and $I(y) \to \infty$ as $y \to w$, a similar proof is valid, letting b approach w instead of ∞.

The proof for $(a,b) \subseteq (-\infty,\infty)$ is similar.

PROOF OF THEOREM 1(ii). The proof of Theorem 2(ii) can easily be adjusted to be a proof of Theorem 1(ii). In the general non-arithmetic case, Stone's (1965) results can be replaced by the standard renewal theorem. (There is no need for uniformity of the renewal-theoretic results as the support of the mixing measure $F = \delta_{\{\theta\}}$ in this case is made up of one point.) The details are omitted.

IV. THE AVERAGE RUN LENGTH WHEN $\nu = 1$

Define

$$C_2^{y,\theta} = E_1^{(\theta)} \log\left(1 + \sum_{k=1}^{\infty} e^{-\sum_{i=1}^{k} Z_i^{\{y\}}}\right)$$

$$C_3^{y,\theta} = \lim_{B\to\infty} E_1^{(\theta)} \left(\sum_{i=1}^{M_B^y} Z_i^{\{y\}} - B\right)$$

$$C_1^{y,\theta} = C_3^{y,\theta} - C_2^{y,\theta}$$

$$C_2^{\theta,F} = -\frac{1}{2} \log\, [2\pi (F'(\theta))^2/\psi''(\theta)]$$

$$C_4^{\theta} = \frac{1}{2} \log I(\theta) - \frac{1}{2}$$

$$C_1^{\theta,F} = C_2^{\theta,F} + C_3^{\theta,\theta} - C_2^{\theta,\theta} - C_4^{\theta} .$$

The computation of $C_3^{y,\theta}$ is an application of renewal theory. The calculation of $C_2^{y,\theta}$ seems to be feasible only with the aid of Monte Carlo.

THEOREM 3. If $y,\theta \in \Omega$, $0 < y\,\psi'(\theta) - \psi(y) < \infty$, and the $P_1^{\{\theta\}}$-distribution of $\log(f_y(X_1)/f_0(X_1))$ is non-lattice, then

$$E_1^{(\theta)} N_A^{\{y\}} = \frac{1}{y\psi'(\theta) - \psi(y)} [\log A + C_1^{y,\theta} + o(1)]$$

where $o(1) \to 0$ as $A \to \infty$.

THEOREM 4. Suppose $F'(y) = dF(y)/dy$ exists, is positive, and is continuous in an open neighborhood of $\theta \in \Omega$. Then

$$E_1^{(\theta)} N_A^F = \frac{1}{I(\theta)} [\log A + \tfrac{1}{2} \log \log A + C_1^{\theta,F} \pm o(1)]$$

where $o(1) \to 0$ as $A \to \infty$.

PROOF OF THEOREM 4, THEOREM 3. For the proof of Theorem 4, assume (without loss of generality) that $\theta > 0$. Consider first the case where F is concentrated on $[\theta_0,\theta_1]$ where $0 < \theta_0 < \theta < \theta_1 < \infty$ are such that $y\psi'(\theta) - \psi(y) > 0$ for $\theta_0 \leq y \leq \theta_1$ and F has a derivative F' which is positive and continuous on $[\theta_0,\theta_1]$. For $\theta_0 \leq y \leq \theta_1$ denote

$$W^{n,y} = 1 + \sum_{k=2}^{n} e^{-\Sigma_{i=1}^{k-1} Z_i^{\{y\}}} .$$

Note that $W^{n,y}$ converges a.s. $P_1^{(\theta)}$ as $n \to \infty$ to a random variable $W_{y,\theta}$. Since $\Sigma_{n=m}^{\infty} (W^{n+1,y} - W^{n,y}) = \Sigma_{n=m}^{\infty} \exp\{-y \Sigma_{i=1}^{n} X_i - n\,\psi(y)\} \xrightarrow[m \to \infty]{\text{a.s. } P_1^{(\theta)}} 0$ uniformly in $y \in [\theta_0,\theta_1]$, it follows that $W_{y,\theta}$ is a.s. $P_1^{(\theta)}$ continuous in $y \in [\theta_0,\theta_1]$, and $W^{n,y} \xrightarrow[n \to \infty]{\text{a.s. } P_1^{(\theta)}} W_{y,\theta}$ uniformly in $y \in [\theta_0,\theta_1]$. Note that

$$R_n^F = \int_{\theta_0}^{\theta_1} e^{\Sigma_{i=1}^{n} Z_i^{\{y\}}} W^{n,y}\, dF(y) .$$

The proof of Theorem 4 now follows the proof of the asymptotic formula for the expected sample size of power one tests, based on non-linear renewal theory (cf. Lai, Siegmund (1977)). The details presented here follow the proof presented in Woodroofe (1982) Section 6.3. With minor modifications, the proof is the same.

One difference is that Woodroofe's $u_n(\bar{Y}_n)$ now has $\pi(ds)$ replaced by $W^{n,s}\ \pi(ds)$. Note that the upper bound on the newly defined $u_n(\bar{Y}_n)$ is not uniform in $W^{n,s}$, One must show that (13) and (14) of Section 4 of Woodroofe (1982) are nevertheless satisfied. One can dispense with (14) by noting that $W^{n,s} \geq 1$. To show that (13) is satisfied, it more than suffices to prove the existence of a constant $\alpha > 0$ such that

$$(42) \qquad E_1^{(\theta)}\left[\int_{\theta_0}^{\theta_1} W_{y,\theta}\, dF(y)\right]^{\alpha} < \infty \quad .$$

Let $\varepsilon > 0$, $\Lambda = \min\{n \mid |\bar{X}_m - \psi'(\theta)| \leq \varepsilon$ for all $m \geq n\}$. Suppose that ε is small enough so that there exists $\beta > 0$ such that $\Sigma_{i=1}^{n} Z_i^{\{y\}} \geq \beta n$ if $n \geq \Lambda$ for all $\theta_0 \leq y \leq \theta_1$. There exists $\gamma > 0$ such that $|\psi(\theta-y) + \psi(y) - \psi(\theta)| < \gamma$ for all $\theta_0 \leq y \leq \theta_1$. There exists a constant $\delta > 0$ such that $P_1^{(\theta)}(\Lambda=\lambda) \leq \exp\{-\delta\lambda\}$. Choose $1 > \alpha > 0$ such that $\alpha\gamma - \delta(1-\alpha) < 0$. Now

$$\int_{\theta_0}^{\theta_1} W_{y,\theta} dF(y) = \int_{\theta_0}^{\theta_1}\left[1 + \sum_{k=2}^{\Lambda} e^{-\Sigma_{i=1}^{k-1} Z_i^{\{y\}}} + \sum_{k=\Lambda+1}^{\infty} e^{-\Sigma_{i=1}^{k-1} Z_i^{\{y\}}}\right] dF(y)$$

$$\leq \int_{\theta_0}^{\theta_1}\left[1 + \sum_{k=2}^{\Lambda} e^{-\Sigma_{i=1}^{k-1} Z_i^{\{y\}}} + \frac{1}{1-e^{-\beta}}\right] dF(y) \quad ,$$

$$E_1^{(\theta)}\left[\int \sum_{k=2}^{\Lambda} e^{-\Sigma_{i=1}^{k-1} Z_i^{\{y\}}} dF(y) \middle| \Lambda = \lambda\right] \leq \frac{E_1^{(\theta)} \int_{\theta_0}^{\theta_1} \sum_{k=2}^{\lambda} e^{-\Sigma_{i=1}^{k-1} Z_i^{\{y\}}} dF(y)}{P_1^{(\theta)}(\Lambda = \lambda)}$$

$$= \frac{1}{P_1^{(\theta)}(\Lambda = \lambda)} \int_{\theta_0}^{\theta_1} \sum_{k=2}^{\lambda} e^{[\psi(\theta-y)+\psi(y)-\psi(\theta)](k-1)} dF(y)$$

$$\leq \frac{1}{P_1^{(\theta)}(\Lambda = \lambda)} \frac{1}{\gamma} e^{\gamma\lambda} \quad .$$

By Jensen's inequality,

$$E_1^{(\theta)}\left(\int_{\theta_0}^{\theta_1} W_{y,\theta} dF(y)\right)^{\alpha} = E_1^{(\theta)} E_1^{(\theta)}\left[\left(\int_{\theta_0}^{\theta_1} W_{y,\theta} dF(y)\right)^{\alpha} \middle| \Lambda\right]$$

$$\leq \sum_{\lambda=1}^{\infty} \left\{\frac{1}{P_1^{(\theta)}(\Lambda=\lambda)} \frac{1}{\gamma} e^{\gamma\lambda} + \frac{2 - e^{-\beta}}{1 - e^{-\beta}}\right\}^{\alpha} P_1^{(\theta)}(\Lambda = \lambda) \quad .$$

The inequality (42) now follows because

$$\sum_{\lambda=1}^{\infty} \left\{\frac{1}{P_1^{(\theta)}(\Lambda = \lambda)} \frac{1}{\gamma} e^{\gamma\lambda}\right\}^{\alpha} P_1^{(\theta)}(\Lambda = \lambda) = \frac{1}{\gamma^{\alpha}} \sum_{\lambda=1}^{\infty} e^{\alpha\gamma\lambda}(P_1^{(\theta)}(\Lambda = \lambda))^{1-\alpha}$$

$$\leq \frac{1}{\gamma^{\alpha}} \sum_{\lambda=1}^{\infty} e^{\lambda(\alpha\gamma-(1-\alpha)\delta)}$$

$$< \infty \quad .$$

To complete the proof of Theorem 4 for the case that F is concentrated on $[\theta_0, \theta_1]$ as above, one need only show that (16) of Woodroofe (1982), Section 4, holds. For this, following Woodroofe's (1982 Section 6.3) proof, it suffices to note that

$$P_0(N_A^F \le (\log A)/(2I(\theta))) = \sum_{i=1}^{(\log A)/(2I(\theta))} P_0(R_i^F \not\le A)$$

$$\le \sum_{i=1}^{(\log A)/(2I(\theta))} \frac{i}{A}$$

$$\le \frac{1}{(I(\theta))^2} \frac{(\log A)^2}{A}$$

and hence

$$P_1^{(y)}(N_A^F \le (\log A)/(2I(\theta))) \le \frac{\exp\{(3/4)A\}}{(I(\theta))^2} \frac{(\log A)^2}{A} + o\left(\frac{1}{\log A}\right) = o\left(\frac{1}{\log A}\right)$$

which is equivalent to (16) of Woodroofe (1982), Section 4.

For the general proof of Theorem 4, let F be a measure on the real line. There exist constants $0 < \xi < I(\theta)/2$, $\omega > 0$, and $0 < \theta_0 < \theta < \theta_1 < \infty$ such that $y\psi'(\theta) - \psi(y) > 0$ for $y \in [\theta_0,\theta_1]$, $\max\{y\psi'(\theta-\omega) - \psi(y),\ y\psi'(\theta+\omega) - \psi(y)\} < \xi$ for $y \notin [\theta_0,\theta_1]$, and F(y) has a derivative F'(y) for $\theta_0 \le y \le \theta_1$ which is positive and continuous for $\theta_0 \le y \le \theta_1$. Since $P_1^{(\theta)}(N_A^F \ge (2 \log A)/I(\theta))$ is arbitrarily small when A is large enough, and since for all $x > 0$ $E_1^{(\theta)}(N_A^F | N_A^F > x) \le x + (2 \log A)/I(\theta)$ for large enough A, it suffices to show that

$$(44) \qquad (\log A)P_1^{(\theta)}\left\{\max_{n=1,\ldots,(2\log A)/I(\theta)} \int_{R-[\theta_0,\theta_1]} \sum_{k=1}^{n} e^{y\Sigma_{i=k}^{n} X_i - (n-k+1)\psi(y)} dF(y) \ge \frac{3A}{\log A}\right\} \xrightarrow[A\to\infty]{} 0 .$$

The remainder of the proof is therefore an analysis of this expression.

Let $-\infty < \theta_0^* < 0 < \theta_1 < \theta_1^* < \infty$ be such that

$$-\zeta = \max\{y\psi'(\theta) - \psi(y) \,|\, y \in (\theta_0^*, \theta_1^*)\} < 0 \quad .$$

In the same manner which lead to (42) above, it can be shown that there exists a constant $\alpha > 0$ such that

$$\Gamma = E_1^{(\theta)} \left(\int_{\mathbb{R}-[\theta_0^*,\theta_1^*]} \sum_{k=1}^{\infty} e^{y\Sigma_{i=1}^k X_i - k\psi(y)} dF(y) \right)^{\alpha} < \infty \quad ,$$

and hence by Jensen's inequality

$$(45) \quad (\log A)\, P_1^{(\theta)} \left\{ \max_{n=1,\ldots,(2\log A)/I(\theta)} \int_{\mathbb{R}-[\theta_0^*,\theta_1^*]} \sum_{k=1}^{n} e^{y\Sigma_{i=k}^n X_i - (n-k+1)\psi(y)} dF(y) \geq \frac{A}{\log A} \right\}$$

$$\leq (\log A) \sum_{n=1}^{(2\log A)/I(\theta)} P_1^{(\theta)} \left(\left(\int \sum_{k=1}^{n} e^{y\Sigma_{i=k}^n X_i - (n-k+1)\psi(y)} dF(y) \right)^{\alpha} \geq \left(\frac{A}{\log A}\right)^{\alpha} \right)$$

$$\leq \frac{2\log A}{I(\theta)} \; \frac{\Gamma}{(A/\log A)^{\alpha}}$$

$$\xrightarrow[A\to\infty]{} 0 \; .$$

For large enough A

(46) $$\max_{n=1,\dots,(2\log A)/I(\theta)} \int_{[\theta_0^*,\theta_1^*]-[\theta_0,\theta_1]} \sum_{k=1}^{n} e^{y\Sigma_{i=k}^{n} X_i-(n-k+1)\psi(y)}$$

$$\cdot 1\left(\frac{1}{n-k+1}\sum_{i=k}^{n} X_i \in (\psi'(\theta-\omega),\psi'(\theta+\omega))\right) dF(y)$$

$$\leq \max_{n=1,\dots,(2\log A)/I(\theta)} \int \sum_{k=1}^{n} e^{\xi(n-k+1)}\, dF(y)$$

$$< \max_{n=1,\dots,(2\log A)/I(\theta)} \frac{e^{\xi(n+1)}}{\xi}$$

$$= \frac{e^{\xi}}{\xi} A^{2\xi/I(\theta)}$$

$$< \frac{A}{\log A} .$$

Let $\eta > 0$ be such that $P_1^{(\theta)}(\Sigma_{i=1}^{k} X_i/k \in (\psi'(\theta-\omega),\psi'(\theta+\omega)))$ $\leq \exp\{-\eta k\}$ for all k. Let $\lambda > 0$ be such that $E_1^{(\theta)}[\Sigma_{i=1}^{k}(X_i-\psi'(\theta))]^4$ $\leq \lambda k^2$ for all k. For large enough A and for $n \leq (2\log A)/I(\theta)$

$$P_1^{(\theta)}\left\{\int_{\theta_1}^{\theta_1^*} \sum_{k=1}^{n} e^{y\Sigma_{i=k}^{n} X_i-(n-k+1)\psi(y)}\right.$$

$$\left.\cdot 1\left(\frac{1}{n-k+1}\sum_{i=k}^{n} X_i \in (\psi'(\theta-\omega),\psi'(\theta+\omega))\right) > \frac{A}{\log a}\right\}$$

$$\le P_1^{(\theta)}\left\{\sum_{k=1}^{n} e^{\theta_1^* \Sigma_{i=k}^n X_i}\right.$$

$$\left.\cdot 1\left\{\frac{1}{n-k+1}\sum_{i=k}^{n} X_i \in (\psi'(\theta-\omega),\psi'(\theta+\omega))\right\} > \frac{A}{\log A}\right\}$$

$$\le P_1^{(\theta)}\left\{\max_{k=1,\ldots,n} e^{\theta_1^* \Sigma_{i=1}^k X_i}\right.$$

$$\left.\cdot 1\left\{\frac{1}{k}\sum_{i=1}^{k} X_i \in (\psi'(\theta-\omega),\psi'(\theta+\omega))\right\} > \frac{A}{\log A}\right\}$$

$$\le \sum_{k=1}^{n} P_1^{(\theta)}\left\{\sum_{i=1}^{k}(X_i-\psi'(\theta))\right.$$

$$\cdot 1\left\{\frac{1}{k}\sum_{i=1}^{k} X_i \notin (\psi'(\theta-\omega),\psi'(\theta+\omega))\right\} > \frac{1}{\theta_1^*}\log A$$

$$\left.-\frac{2}{\theta_1^*}\log\log A - k\psi'(\theta) + \log(I(\theta)/2)\right\}$$

$$\le \sum_{k=1}^{n\wedge(\log A)^{1/4}} \frac{\lambda k^2}{\left\{\frac{1}{\theta_1^*}\log A - \frac{2}{\theta_1^*}\log\log A - k\psi'(\theta) + \log(I(\theta)/2)\right\}^2}$$

$$+\sum_{k=(\log A)^{1/4}}^{n} P_1^{(\theta)}\left\{\frac{1}{k}\sum_{i=1}^{k} X_i \in (\psi'(\theta-\omega),\psi'(\theta+\omega))\right\}$$

$$\le \frac{\lambda(\log A)^{3/4}}{\left\{\frac{1}{\theta_1^*}\log A - \frac{2}{\theta_1^*}\log\log A - (\log A)^{3/4}\psi'(\theta) + \log(I(\theta)/2)\right\}^4}$$

$$+ e^{-\eta(\log A)^{3/4}}\frac{1}{1-e^{-\eta}} .$$

It follows that

$$(47)\qquad (\log A)\, P_1^{(\theta)}\left\{\max_{k=1,\ldots,(2\log A)/I(\theta)} \int_{\theta_1}^{\theta_1^*} \sum_{k=1}^{n} e^{y\Sigma_{i=k}^{n} X_i-(n-k+1)\psi(y)} \cdot 1\left[\frac{1}{n-k+1}\sum_{i=k}^{n} X_i \notin (\psi'(\theta-\omega),\psi'(\theta+\omega))\right] dF(y) > \frac{A}{\log A}\right\} \xrightarrow[A\to\infty]{} 0 \ .$$

In a similar fashion one gets that

$$(48)\qquad (\log A)\, P_1^{(\theta)}\left\{\max_{k=1,\ldots,(2\log A)/I(\theta)} \int_{\theta_0^*}^{\theta_0} \sum_{k=1}^{n} e^{y\Sigma_{i=k}^{n} X_i-(n-k+1)\psi(y)} \cdot 1\left[\frac{1}{n-k+1}\sum_{i=k}^{n} X_i \notin (\psi'(\theta-\omega),\psi'(\theta+\omega))\right] dF(y) > \frac{A}{\log A}\right\} \xrightarrow[A\to\infty]{} 0 \ .$$

Formulas (45)-(48) account for (44) and so the proof of Theorem 4 is complete.

The proof of Theorem 3 follows along similar lines. The details are omitted.

V. MONTE CARLO

A Monte Carlo study was made for the normal model with unit variance. Letting f_θ denote the density of the $N(\theta,1)$ distribution, simulations of $N_A^{\{\theta\}}$, $R^{\{\theta\}}_{N_A^{\{\theta\}}}$ were made for θ = .4, .8, 1.0, 1.2, 1.6, 2.0, 2.5, 3.0, 4.0 and A = 10, 20, 30, 100 using $X_i \sim N(0,1)$ random numbers. For each of the 36 combinations of and A, 10,000 realizations were obtained. The results show the asymptotic formulae (derived in the previous sections) to give a very good picture of $E_0 N_A^{\{\theta\}}$ even for surprisingly low values of A.

As expected, the Monte Carlo estimate of $E_0(R^{\{\theta\}}_{N_A^{\{\theta\}}} - N_A^{\{\theta\}})$ was zero: in only one of the 36 cases did $(R^{\{\theta\}}_{N_A^{\{\theta\}}} - N_A^{\{\theta\}})$ exceed two (Monte Carlo) standard deviations of $R^{\{\theta\}}_{N_A^{\{\theta\}}} - N_A^{\{\theta\}}$. Results of Lai and Siegmund (1977) lead one to conjecture that the linear correlation coefficient between $N_A^{\{\theta\}}$ and $R^{\{\theta\}}_{N_A^{\{\theta\}}}$ is asymptotically $(A \to \infty)$ zero. The Monte Carlo results support this conjecture - the highest Monte Carlo correlation between $N_A^{\{\theta\}}$ and $R^{\{\theta\}}_{N_A^{\{\theta\}}}$ was .0234. (In 28 of the 36 cases the correlation between $N_A^{\{\theta\}}$ and $R^{\{\theta\}}_{N_A^{\{\theta\}}}$ was not significantly different from zero at a 5% level of significance, and in all of the 36 cases this correlation was not significantly different from zero at a 1% level of significance.) Therefore, estimates of $E_0 N_A^\theta$ were made using a linear combination $\alpha_{A,\theta} N_A^{\{\theta\}} + (1 - \alpha_{A,\theta}) R^{\{\theta\}}_{N_A^{\{\theta\}}}$, where $\alpha_{A,\theta}$ was chosen to minimize $\alpha^2 \text{ Var } N_A^{\{\theta\}} + (1-\alpha)^2 \text{ Var } R^{\{\theta\}}_{N_A^{\{\theta\}}}$ (the variances being Monte Carlo variances). The results are presented in Table 1.

TABLE 1: Values of $E_0N_A^{\{\theta\}}$ predicted by asymptotic theory (TH) and estimated by Monte Carlo (MC)

θ \ A		10		20		30		100	
		$E_0N_A^{\{\theta\}}$	S.D. of MC	$E_0N_A^{\{\theta\}}$	S.D. of MC	$E_0N_A^{\{\theta\}}$	S.D. of MC	$E_0N_A^{\{\theta\}}$	S.D. of MC
.4	TH	12.62		25.24		37.86		126.21	*
	MC	13.01	.03	25.57	.05	38.20	.08	126.44	.27
.8	TH	15.91		31.82		47.73		159.09	*
	MC	16.51	.07	32.32	.14	48.58	.22	159.61	.68
1.0	TH	17.85		35.69		53.54		178.45	*
	MC	18.44	.09	36.23	.21	54.53	.30	178.25	.95
1.2	TH	20.00		40.00		60.00	*	200.01	*
	MC	20.98	.13	40.59	.27	60.56	.40	200.71	.40
1.6	TH	25.05		50.09		75.14	*	250.47	*
	MC	26.62	.20	52.65	.42	76.00	.60	248.27	1.91
2.0	TH	31.21		62.42		93.62	*	312.08	*
	MC	34.54	.32	65.52	.58	93.93	.84	315.47	2.74
2.5	TH	40.72		81.44		122.16		407.20	*
	MC	48.27	.46	89.65	.87	128.39	1.22	406.78	3.91
3.0	TH	52.52		105.04		157.56		525.21	*
	MC	72.75	.71	127.08	1.25	180.15	1.80	533.15	5.23
4.0	TH	83.93		167.87		251.80		839.35	
	MC	189.58	1.86	315.25	3.10	428.10	4.30	1099.02	10.96

(In Table 1, TH represents the theoretical value one would expect for $E_0 N_A^{\theta}$ using Theorem 1(ii); MC represents the estimated based on the Monte Carlo trials. The (Monte Carlo) standard deviation of this estimate is given under the heading of "S.D. of MC." The starred cells in Table 1 are those where TH - MC did not exceed 2 (Monte Carlo) standard deviations of TH.)

TABLE 2: Ratios of asymptotic theory predictions of $E_0 N_A^{\{\theta\}}$ to Monte Carlo estimates (TH/MC)

θ \ A	10	20	30	100
.4	.97	.99	.99	1.00
.8	.96	.98	.98	1.00
1.0	.97	.99	.98	1.00
1.2	.95	.99	.98	1.00
1.6	.94	.94	.99	1.01
2.0	.90	.95	1.00	.99
2.5	.84	.91	.95	1.00
3.0	.72	.83	.87	.99
4.0	.44	.53	.59	.76

The results show a surprisingly good fit, even for low values of A (as long as θ is not too large). (Table 2 presents the ratio between the theoretical value of TH and the Monte Carlo estimate MC.) It seems clear that for most practical purposes the asymptotic formula could be safely applied. (Shewhart control charts using "3σ limits" - often used in practice - have a P_0-expected stopping time of 741.)

For an indication of how well one may expect the formula of Theorem 4 to fit, see Pollak and Siegmund (1975). One would expect the formula presented there to hold as well as the formulae presented here, provided that $E_1^{(\theta)} N_A^F$ is large enough for the distribution of $\log[1 + {}_{k=1}\ \exp\{-\Sigma_{i=1}^{k} Z_i^{\{\theta\}}\}]$ to have approximately reached its limiting distribution.

VI. REMARKS

1. In Theorems 1, 3, 4 if $I(\theta) = \infty$, it is possible to show that $E_0 N_A^{\{\theta\}}/A \to \infty$ as $A \to \infty$ and $E_1^{(\theta)} N_A^F/\log A \to 0$ as $A \to \infty$.

2. Using the method involved in showing the validity of Remark 1, one can show that Theorem 2 remains valid with $F(\{y | I(y) < \infty\}) > 0$.

3. It seems reasonable to conjecture that Theorem 2 remains valid if the $P_1^{(y)}$-distribution of X_1 is just assumed to be non-lattice. The proof given above for Theorem 2 breaks down because the uniformity of a renewal-theoretic convergence used in the proof of Lemma 1 need not exist if the strongly non-lattice assumption is dropped.

4. In the lattice case, even a version of Theorem 1 seems to be difficult to formulate. Despite X_1's being lattice, R_n is not, and the proof presented here - which conditions on $N_{A/C}^{\{\theta\}}$ - does not yield an expression for the non-lattice part of the asymptotic P_0-distribution of $\log R_{N_A^{\{\theta\}}}^{\{\theta\}} - \log A$.

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Suppose one is able to observe sequentially a series of independent observations $X_1, X_2, \ldots,$ such that $X_1, X_2, \ldots, X_{\nu-1}$ are i.i.d. with known density f_0 and $X_\nu, X_{\nu+1}, \ldots,$ are i.i.d. with density f_θ where ν is unknown. Define

$$R_n^{\{\theta\}} = \sum_{k=1}^{n} \prod_{i=k}^{n} \frac{f_\theta(X_i)}{f_0(X_i)} .$$

It is known that rules which call for stopping and raising an alarm the first time n that $R_n^{\{\theta\}}$ or a mixture thereof exceeds a prespecified level A are optimal methods of detecting that the density of the observations is not f_0 any more.

Practical applications of such stopping rules require knowledge of their operating characteristics, whose exact evaluation is difficult. Here are presented asymptotic $(A \to \infty)$ expressions for the expected stopping times of such stopping rules (a) when $\nu = \infty$ and (b) when $\nu = 1$. We assume that the densities f_θ form an exponential family and that the distribution of $\log(f_\theta(X_1)/f_0(X_1))$ is (strongly) non-lattice.

Monte Carlo studies indicate that the asymptotic expressions are very good approximations even when the expected sample sizes are small.

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