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Average Version of Artin's Conjecture in an Algebraic Number Field

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Artin's conjecture on primitive root was proved by Hooley [1] under the generalized Riemann hypothesis for certain family of Dedekind Zeta functions. Its generalization to arbitrary number fields was also obtained by Weinberger [2] and Lenstra [3]. In another direction, Goldfeld [4] obtained a "large sieve type" result for the rational case without Riemann hypothesis. In this paper we shall show that his method can be applied to obtain the similar result for the case treated in [2], [3].

Let K be an algebraic number field of degree n, and q be a fixed integral divisor which contains all the infinite primes of K. For $A>0$ we define a set of non-zero integers of K by

$$B'(A)=\{\alpha;\ |\alpha^{(\nu)}|\leqq A,\ (\nu=1,\ \cdots,\ n)\}\ ,$$

where $\alpha^{(\nu)}$ denotes a conjugate element of α. We write $\alpha\sim\beta$ for integers α, β of K if both α and β generate the same principal ideal. We denote by $B(A)$ a fixed collection of the representatives taken from each equivalence class of $B'(A)/\sim$. Let h be a ray class modulo q, and define

$$(1)\quad N_{\alpha,q,h}(x)=N_{\alpha}(x)$$
$$=\mathrm{card}\,\{\mathfrak{p};\ N\mathfrak{p}\leqq x,\ \mathfrak{p}\in h,\ \alpha \text{ is a primitive root modulo } \mathfrak{p}.\}\ ,$$

and

$$g(K)=\sup_{1\leqq x<\infty}\sum_{N\mathfrak{a}\leqq x}1/x$$

where $\mathfrak{a}$ denotes an integral ideal of K. We denote

$$(2)\qquad \mathrm{Li}\,(x)=\int_2^x\frac{dt}{\log t}\ .$$

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In the following p always denotes a rational prime, and $\mathfrak{p}$, $\mathfrak{p}_1$, $\mathfrak{p}_2$ prime ideals of K. We denote by K_q the ray class field modulo q over K, and by $Q(\zeta_k)$ the kth cyclotomic field. Then our result is:

Theorem. Let $1 \leqq A^n \leqq x$. Then for each $D \geqq 1$

$$N_\alpha(x) = C(K, q, h)\,\mathrm{Li}\,(x) + O\left(\frac{x}{\log^D x}\right) \tag{3}$$

for all $\alpha \in B(A)$ with at most

$$c_1 A^{(9/10)n}(g(K)(5\log x+1))^{(\log x)/(n\log A)+D+2} \tag{4}$$

exceptions, where c_1 and the constant represented by O-symbol are positive and depend at most on K, q, h and D. And

$$C(K, q, h) = \frac{1}{[K_q; K]} \prod_p \left(1 - \frac{F(p)\cdot[Q(\zeta_p)\cap K_q : Q]}{p(p-1)}\right), \tag{5}$$

where $F(\ )$ is defined by

$$F(k) = \begin{cases} 1 & \text{if there exists an ideal } \mathfrak{b} \text{ of } K \text{ such} \\ & \text{that } N\mathfrak{b} \equiv 1 (\mathrm{mod}\, k),\ \mathfrak{b} \in h\,, \\ 0 & \text{otherwise}\,. \end{cases} \tag{6}$$

We note that $B(A)$ has at least $c_2 A^n$ elements as will be proved in Lemma 1, so that (3) is satisfied actually for "almost all" $\alpha \in B(A)$. Further we note that, by putting $K=Q$, $q=(p_\infty)$ our result coincides with Goldfeld's one since $g(K)=1$, $K_q=Q$, and $C(K, q, h)=\prod_p (1-(1/p(p-1)))$ in this case.

In the followings $c_2, c_3, \cdots$ are positive constants depending only on K, O-symbol and $\ll$ may contain constants depending on K, q, h and D.

§1. Lemmas.

LEMMA 1. *There exist positive constants c_2, c_3, and c_4 depending only on K such that*

$$c_3 A^n \leqq |B(A)| \leqq c_4 A^n \tag{7}$$

for all $A \geqq c_2$.

PROOF. Note first that

$$|B(A)| \leqq \sum_{N\mathfrak{A} \leqq A^n} 1 \leqq c_4 A^n\,.$$

Let r_1, r_2 be the numbers of the real or the imaginary infinite primes of K respectively. Let $\{\varepsilon_1, \cdots, \varepsilon_{r_1+r_2-1}\}$ be a system of the fundamental units of K. We denote for $\alpha \in K$

$$l^{(i)}\alpha = \begin{cases} \log |\alpha^{(i)}| , & i=1, \cdots, r_1 \\ 2\log |\alpha^{(i)}| , & i=r_1+1, \cdots, r_1+r_2 \end{cases}$$

where the indices i are chosen so that $\{1, \cdots, r_1\}$, $\{r_1+1, \cdots, r_1+r_2\}$ corresponds to the complete set of real or imaginary primes of K respectively. Set

$$c_2 = \exp\left(\frac{1}{n}\sum_{i=1}^{r}\sum_{j=1}^{r-1}|l^{(i)}\varepsilon_j|\right),$$

where $r=r_1+r_2$.

Let α be an integer of K satisfying $|N\alpha| \leqq (c_2^{-1}A)^n$. Put

$$x_i = l^{(i)}\alpha , \qquad i=1, \cdots, r ,$$

$$y_i = \begin{cases} \log A - \sum_{j=1}^{r-1}|l^{(i)}\varepsilon_j| & i=1, \cdots, r_1 , \\ 2\log A - \sum_{j=1}^{r-1}|l^{(i)}\varepsilon_j| & i=r_1+1, \cdots, r , \end{cases}$$

and

$$u_{ij} = \begin{cases} l^{(i)}\varepsilon_j & i=1, \cdots, r;\ j=1, \cdots, r \\ 0 & i=1, \cdots, r-1;\ j=r \\ 1 & i=r;\ j=r . \end{cases}$$

Since the matrix $U=\{u_{ij}\}$ is regular, there exist real numbers $z_1, \cdots, z_r$ satisfying

$$U\cdot\begin{bmatrix} z_1 \\ \vdots \\ z_r \end{bmatrix} = \begin{bmatrix} y_1-x_1 \\ \vdots \\ y_r-x_r \end{bmatrix} .$$

By putting $n_j=[z_j]$ (the integer part) we have

$$y_i - x_i = \sum_{j=1}^{r-1} u_{ij}z_j \geqq \sum_{j=1}^{r-1} u_{ij}n_j - \sum_{j=1}^{r-1}|u_{ij}|$$

for $i=1, \cdots, r-1$. This inequality is also valid for $i=r$, since

$$y_r - x_r = \sum_{j=1}^{r-1} u_{rj}z_j + z_r$$

and

$$z_r=\sum_{i=1}^{r} y_i-x_i\geqq 0\ .$$

Thus we have

$$x_i+\sum_{j=1}^{r-1} u_{ij}n_j\leqq y_i+\sum_{j=1}^{r-1}|u_{ij}|$$

for $i=1, \cdots, r$, which yields $\alpha\cdot\varepsilon_1^{n_1}\cdots\varepsilon_r^{n_r}\in B(A)$. Hence we have

$$|B(A)|\geqq \sum_{\substack{N\mathfrak{a}\leqq(c_2^{-1}A)^n \\ \mathfrak{a}\ \text{principal}}} 1\geqq c_3A^n$$

if $A\geqq c_2$.

Let k be a positive integer and set $f=kq$. We denote by I_f the group of all the fractional ideals of K having no common divisor to f, and set

$$(8)\qquad H_f=\{(\alpha)\in I_f;\ N(\alpha)\equiv 1(\mathrm{mod}\ k),\ \alpha\equiv 1(\mathrm{mod}\ q)\}\ ,$$

where (α) denotes the principal ideal of K generated by α.

LEMMA 2.

$$(9)\qquad [I_f\colon H_f]=[K_q\colon K]\cdot\frac{\varphi(k)}{[Q(\zeta_k)\cap K_q\colon Q]}\ ,$$

where $\varphi(k)$ is Euler function. If k is square free, we have

$$(10)\qquad [I_f\colon H_f]=[K_q\colon K]\prod_{p\mid k}\frac{p-1}{[Q(\zeta_p)\cap K_q\colon Q]}\ .$$

PROOF. (10) is a simple consequence of (9) since $Q(\zeta_p)\cap Q(\zeta_{p'})=Q$ for different primes p and p'. According to class field theory $[I_f\colon H_f]=[K_q(\zeta_k)\colon K]$. Since $K_q(\zeta_k)=K_q\cdot Q(\zeta_k)$ we have

$$\begin{aligned}[K_q(\zeta_k)\colon K]&=[K_q(\zeta_k)\colon K_q]\cdot[K_q\colon K]\\&=[Q(\zeta_k)\colon Q(\zeta_k)\cap K_q]\cdot[K_q\colon K]=\frac{[Q(\zeta_k)\colon Q]}{[Q(\zeta_k)\cap K_q\colon Q]}\cdot[K_q\colon K]\\&=[K_q\colon K]\frac{\varphi(k)}{[Q(\zeta_k)K_q\colon Q]}\ ,\end{aligned}$$

which yields Lemma 2.

Set

$$\pi(x; k)=\sum_{\substack{N\mathfrak{p}\leqq x,\mathfrak{p}\in h\\ N\mathfrak{p}\equiv 1(\mathrm{mod}\, k)}} 1 .$$

The following lemma is an easy consequence of the theorem of Goldstein [6].

LEMMA 3. *For arbitrary $D\geqq 1$ we have*

$$\pi(x; k)=\frac{F(k)}{[I_f: H_f]}\left(\mathrm{Li}\,(x)+O\left(\frac{x}{\log^{D+1} x}\right)\right),$$

if $k\leqq\log^D x$.

Next we quote a large sieve inequality of Huxley ([7]; Theorem 4) in a slightly modified form which is appropriate for our purpose. Let μ be a positive integer and set

$$B(A)^\mu=\{\alpha_1\cdots\alpha_\mu;\ \alpha_i\in B(A)\ \ (i=1, \cdots, \mu)\} .$$

LEMMA 4. *Let $a(\alpha)$ be a complex valued function defined on $B(A)^\mu$, then for $x\geqq 0$*

$$\sum_{N\mathfrak{a}\leqq x}\sum_{\chi \bmod \mathfrak{a}}{}^{*}\Big|\sum_{\alpha\in B(A)^\mu} a(\alpha)\chi(\alpha)\Big|^2 \leqq c_5(A^{n\mu}+x^2)\sum_{\alpha\in B(A)^\mu}|a(\alpha)|^2 ,$$

where the sum $\sum^{*}_{\chi \bmod \mathfrak{a}}$ is taken over all the primitive characters modulo $\mathfrak{a}$.

Let

$$\tau_\mu(\mathfrak{a})=\sum_{\substack{(\mathfrak{a}_1,\cdots,\mathfrak{a}_\mu)\\ \mathfrak{a}=\mathfrak{a}_1\cdots\mathfrak{a}_\mu}} 1 .$$

Similarly to the rational case (cf. Hua [8]), we get the following estimate.

LEMMA 5. *For $y\geqq 0$ and nonnegative integer l,*

$$\sum_{N\mathfrak{a}\leqq y}\tau_\mu(\mathfrak{a})^l\leqq y\cdot g(K)^{\mu^l}(\log y+1)^{\mu^l-1} .$$

LEMMA 6. *$F(k)$ is a multiplicative function.*

PROOF. Let k and k' are relatively prime. If $F(kk')=1$, clearly $F(k)=1$ and $F(k')=1$ by the definition. Suppose $F(k)=F(k')=1$, then there exist integral ideals $\mathfrak{b}$, $\mathfrak{b}'$ of K such that

(11) $N\mathfrak{b}\equiv 1(\mathrm{mod}\, k)$, $\mathfrak{b}\in h$; $N\mathfrak{b}'\equiv 1(\mathrm{mod}\, k')$, $\mathfrak{b}'\in h$.

let $\mathfrak{c}$ be an integral ideal in h^{-1}. Then there exist integers α, α' of K such that

(12) $\mathfrak{b}\mathfrak{c}=(\alpha)$, $\alpha\equiv 1(\mathrm{mod}\, q)$; $\mathfrak{b}'\mathfrak{c}=(\alpha')$, $\alpha'\equiv 1(\mathrm{mod}\, q)$.

Since $\alpha\equiv\alpha'(\mathrm{mod}\, q)$ and $(k, k')=1$, there exists an integer β of K satisfying

$$\beta\equiv\alpha(\mathrm{mod}\, qk) , \qquad \beta\equiv\alpha'(\mathrm{mod}\, qk') .$$

Now set $\mathfrak{b}^*=(\beta)\mathfrak{c}^{-1}$, then clearly $\mathfrak{b}^*\in h$. Furthermore we have

(13) $$N\beta\equiv N\alpha(\mathrm{mod}\, k) , \qquad N\beta\equiv N\alpha'(\mathrm{mod}\, k') .$$

But $N(\alpha)=N\alpha$ and $N(\beta)=N\beta$ since both α and β are totally real. Therefore we have $N\mathfrak{b}^*\equiv 1(\mathrm{mod}\, kk')$ from (11), (12), and (13), which proves Lemma 6.

§2. Proof of Theorem.

Let α be an integer of K. For a prime ideal $\mathfrak{p}\nmid\alpha$ we denote by $e_\alpha(\mathfrak{p})$ the least positive integer m satisfying $\alpha^m\equiv 1(\mathrm{mod}\, \mathfrak{p})$, and put $f_\alpha(\mathfrak{p})=(N\mathfrak{p}-1)/e_\alpha(\mathfrak{p})$. Then we have

$$N_\alpha(x)=\sum_{k\leq x}\mu(k)\cdot P_\alpha(x, k) ,$$

where

$$P_\alpha(x, k)=\sum_{\substack{N\mathfrak{p}\leq x,\ \mathfrak{p}\in h\\ k\mid f_\alpha(\mathfrak{p})}} 1 ,$$

and $\mu(k)$ is the Mobius function.

We show first

(14) $$\sum_{k>x^{3/4}} P_\alpha(x, k)=O(x^{1/2}\log x) ,$$

for $\alpha\in B(A)$. Since

$$\prod_{k>x^{3/4}}\prod_{\substack{N\mathfrak{p}\leq x\\ k\mid f_\alpha(\mathfrak{p})}}\mathfrak{p}\Big|\prod_{m\leq x^{1/4}}(\alpha^m-1)$$

and $N\mathfrak{p}\geqq 2$, we have

(15) $$\Big(\sum_{k>x^{3/4}} P_\alpha(x, k)\Big)\log 2\leqq\sum_{m<x^{1/4}}\log|N(\alpha^m-1)| .$$

But by $A^n\leqq x$,

$$\log|N(\alpha^m-1)| = \sum_{\nu=1}^{n} \log|\alpha^{(\nu)m}-1| \leqq \sum_{\nu=1}^{n} \log(1+|\alpha^{(\nu)}|^m) \leqq c_5 m \log x \ . \tag{16}$$

Therefore we obtain (14) from (15) and (16). We have thus proved

$$N_\alpha(x) = \sum_{k \leqq x^{3/4}} \mu(k) P_\alpha(x, k) + O(x^{1/2} \log x) \ . \tag{17}$$

Let $\chi_{\mathfrak{p},k}$ be a character modulo $\mathfrak{p}$ of order k. Since

$$\sum_{\nu=1}^{k} \chi_{\mathfrak{p},k}^{\nu}(\alpha) = \begin{cases} k & \text{if } \alpha \text{ is a } k\text{th power residue modulo } \mathfrak{p} \ , \\ 0 & \text{otherwise} \ , \end{cases}$$

we obtain

$$P_\alpha(x, k) = \sum_{\substack{N\mathfrak{p} \leqq x, \mathfrak{p} \in h \\ N\mathfrak{p} \equiv 1 \pmod{k}}} \frac{1}{k} \sum_{\nu=1}^{k} \chi_{\mathfrak{p},k}^{\nu}(\alpha) = \frac{1}{k}\{\pi(x; k) + S_\alpha(x, k) + O(\log|N\alpha|)\} \ , \tag{18}$$

where

$$S_\alpha(x, k) = \sum_{\substack{N\mathfrak{p} \leqq x, \mathfrak{p} \in h \\ N\mathfrak{p} \equiv 1 \pmod{k}}} \sum_{\nu=1}^{k-1} \chi_{\mathfrak{p},k}^{\nu}(\alpha) \ .$$

Thus we have from (17) and (18) that

$$N_\alpha(x) = \sum_{k \leqq x^{3/4}} \frac{\mu(k)}{k} \pi(x; k) + \sum_{k \leqq x^{3/4}} \frac{\mu(k)}{k} S_\alpha(x, k) + O(x^{1/2} \log x) \ .$$

We denote the first sum by $\sum_1$ and the second by $\sum_2$. It follows from Lemma 3 that

$$\begin{aligned} \sum\nolimits_1 &= \sum_{k \leqq \log^D x} \frac{\mu(k) \cdot F(k)}{k[I_f : H_f]} \operatorname{Li}(x) + O\Big(\frac{x}{\log^{D+1} x} \sum_{k \leqq \log^D x} \frac{1}{k}\Big) \\ &\quad + \sum_{\log^D x < k \leqq x^{3/4}} \frac{\mu(k)}{k} \cdot \pi(x; k) \\ &= \operatorname{Li}(x) \sum_{k \leqq \log^D x} \frac{\mu(k) F(k)}{k[I_f : H_f]} + O\Big(\frac{x}{\log^D x}\Big) \end{aligned} \tag{19}$$

using

$$\pi(x; k) \ll \sum_{m \leq x,\, m \equiv 1 (\mathrm{mod} k)} 1 \ll \frac{x}{k}$$

for $k > \log^D x$. Furthermore, since

$$[I_f : H_f] \gg \varphi(k) \geqq c_6 \frac{k}{\log \log k}$$

we have for any $\varepsilon > 0$

$$\sum_{k=1}^{\infty} \frac{\mu(k)F(k)}{k[I_f : H_f]} = \sum_{k \leq \log^D x} \frac{\mu(k)F(k)}{k[I_f : H_f]} + O\left(\frac{x}{(\log^D x)^{1-\varepsilon}}\right).$$

Combinning this with (19) we obtain,

$$\begin{aligned} \Sigma_1 &= \mathrm{Li}(x) \sum_{k=1}^{\infty} \frac{\mu(k)F(k)}{k[I_f : H_f]} + O\left(\frac{x}{\log^D x}\right) \\ &= C(K, q, h)\mathrm{Li}(x) + O\left(\frac{x}{\log^D x}\right) \end{aligned}$$

using Lemma 2 and Lemma 6.

To complete the proof it is sufficient to show

$$\Sigma_2 = O\left(\frac{x}{\log^D x}\right)$$

except some $\alpha \in B(A)$ indicated in (4). The number of α's such that $\Sigma_2 \geqq x/\log^D x$ does not exceed

$$\sum_{\substack{\alpha \in B(A) \\ S_\alpha(x) \geqq x/\log^D x}} 1 \leqq \frac{\log^D x}{x} \sum_{\alpha \in B(A)} S_\alpha(x), \tag{20}$$

where

$$S_\alpha(x) = \sum_{k \leq x^{3/4}} \frac{1}{k} |S_\alpha(x, k)|. \tag{21}$$

It follows from Cauchy-Schwarz's inequality that

$$\sum_{\alpha \in B(A)} |S_\alpha(x, k)| \leqq |B(A)|^{1/2} \left(\sum_{\alpha \in B(A)} |S_\alpha(x, k)|^2 \right)^{1/2}.$$

Since $\chi_{\mathfrak{p}_1, k}^{\nu_1} \cdot \chi_{\mathfrak{p}_2, k}^{-\nu_2}$ is principal only if $\mathfrak{p}_1 = \mathfrak{p}_2$, we get using Lemma 1,

$$\begin{aligned} \sum_{\alpha \in B(A)} |S_\alpha(x, k)| &\ll A^n (xk)^{1/2} \\ &+ A^{n/2} \Bigg| \sum_{\substack{N\mathfrak{p}_1 \leq x,\, \mathfrak{p}_1 \in h \\ N\mathfrak{p}_1 \equiv 1 (\mathrm{mod}\, k) \\ \mathfrak{p}_1 \neq \mathfrak{p}_2}} \sum_{\substack{N\mathfrak{p}_2 \leq x,\, \mathfrak{p}_2 \in h \\ N\mathfrak{p}_2 \equiv 1 (\mathrm{mod}\, k)}} \sum_{\chi}{}' \sum_{\alpha \in B(A)} \chi(\alpha) \Bigg|^{1/2}, \end{aligned} \tag{22}$$

where the sum $\sum'_\chi$ is taken over all the primitive characters modulo $\mathfrak{p}_1\mathfrak{p}_2$ of order k. Let

$$T=\Big|\sum_{\substack{N\mathfrak{p}_1\leqq x,\mathfrak{p}_1\in h\\ N\mathfrak{p}_1\equiv 1(\mathrm{mod}\,k)\\ \mathfrak{p}_1\neq\mathfrak{p}_2}}\sum_{\substack{N\mathfrak{p}_2\leqq x,\mathfrak{p}_2\in h\\ N\mathfrak{p}_2\equiv 1(\mathrm{mod}\,k)}}\sum'_\chi\sum_{\alpha\in B(A)}\chi(\alpha)\Big| .$$

Applying Hölder's inequality we have for each positive integer μ

$$T^{1/2}\ll x^{1-1/2\mu}\Big(\sum_{\substack{N\mathfrak{p}_1\leqq x,\mathfrak{p}_1\in h\\ N\mathfrak{p}_1\equiv 1(\mathrm{mod}\,k)\\ \mathfrak{p}_1\neq\mathfrak{p}_2}}\sum_{\substack{N\mathfrak{p}_2\leqq x,\mathfrak{p}_2\in h\\ N\mathfrak{p}_2\equiv 1(\mathrm{mod}\,k)}}\sum'_\chi\Big|\sum_{\alpha\in B(A)}\chi(\alpha)\Big|^{2\mu}\Big)^{1/4\mu} .$$

Let $a_\mu(\alpha)$ denote the number of ways α can be written as a product of μ elements of $B(A)$. Then we have

$$\Big(\sum_{\alpha\in B(A)}\chi(\alpha)\Big)^\mu=\sum_{\alpha\in B(A)^\mu}a_\mu(\alpha)\chi(\alpha) .$$

By Lemma 4 we obtain

$$T^{1/2}\ll x^{1-1/2\mu}\Big((A^{n\mu}+x^4)\sum_{\alpha\in B(A)^\mu}a_\mu(\alpha)^2\Big)^{1/4\mu} .$$

Note that $a_\mu(\alpha)\leqq\tau_\mu((\alpha))$ and $N((\alpha))\leqq A^{n\mu}$ for $\alpha\in B(A)^\mu$. Therefore we obtain

$$T^{1/2}\ll x^{1-1/2\mu}\{(A^{n\mu}+x^4)A^{n\mu}((\log A^{n\mu}+1)g(K))^{\mu^2}\}^{1/4\mu} . \tag{23}$$

Now we take μ as an integer satisfying

$$\frac{4\log x}{n\log A}\leqq\mu<\frac{4\log x}{n\log A}+1 . \tag{24}$$

It follows from (22), (23), and (24) that

$$\sum_{\alpha\in B(A)}|S_\alpha(x,k)|$$
$$\ll A^n(xk)^{1/2}+A^{(9/10)n}x(g(K)(5\log x+1))\frac{\log x}{n\log A}+1 .$$

Combining this with (20) and (21), we see that

$$\sum_{\substack{\alpha\in B(A)\\ \Sigma_2>x/\log^D x}}1\ll A^{(9/10)n}(g(K)(5\log x+1))^{(\log x/n\log A)+D+2} ,$$

which completes the proof of Theorem.

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