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Averaged Equations for an Isothermal, Developing Flow of a Fluid-Solid Mixture

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Abstract

A mathematical description of a flowing fluid with entrained particulate solids is presented within the context of Mixture Theory. The mixture is considered to consist of a linearly viscous fluid and a granular solid. The balance of mass and balance of linear momentum equations for each component are averaged over the cross section of the flow to obtain ordinary differential equations describing developing flow between parallel plates. The resulting coupled equations describe the variation of the average velocities and volume fraction in the direction of flow, and represent a simplified approximate set of equations which are easier to use in engineering applications.

Contents

1. Introduction

Multicomponent flows have become the subject of considerable attention because of their importance in many industrial applications. Flowing mixtures consisting of solid particles entrained in a fluid are relevant to a variety of applications such as fluidized beds and pneumatic transport of solid particles. The importance of these complex flows is discussed, for example, by Soo [1989, 1990] and Marcus et al. [1990] who provide up-to-date accounts of multiphase fluid dynamics and pneumatic conveying of solids. Many of the articles published concerning fluid-solid flows typically employ one of two continuum theories developed to describe such situations; Averaging or Mixture Theory (theory of interacting continua).

In the averaging approach (cf. Anderson and Jackson [1967], Drew and Segel [1971], and Drew [1983]) point-wise equations of motion, valid for a single fluid or a single particle, are modified to account for the presence of the other components and the interactions between components. These equations are then averaged over time, some suitable volume which is large compared with a characteristic dimension (for example, particle spacing or the diameter of solid particles) but small compared to the dimensions of the whole system, or an ensemble. Terms which appear due to the process of averaging, which are not present in the equation being averaged, are usually interpreted as some form of interaction between the constituents. Constitutive relations to represent these interactive forces, as well as the stress tensors for each constituent, are required to make the theory complete. A comparison of recent formulations of multiphase flows is provided in two review articles by Soo [1990b,1991].

Anderson and Jackson [1967], in one of the early applications of this method, obtain a set of equations describing the fluid mechanical aspects of fluidized beds. They propose simple constitutive relations in order to close the system of equations. They further assume that the form of the stress tensor for the granular solid constituent is analogous to that of a Newtonian fluid, with the coefficients of viscosity depending on the local mean voidage. Drew and Segel [1971a,b] use a different averaging technique to obtain equations of motion for a broad class of two-component flows. They consider the application of these equations to the specific case of flow in fluidized beds. The constitutive equations they use for the stress relations are very similar to those used by Anderson and Jackson [1967].

The second method of modeling multi-component systems is mixture theory. This theory, which traces its origins to the work of Fick [1855], was first presented within the framework of continuum mechanics by Truesdell [1957]. It is a means of generalizing the equations and principles of the mechanics of a single continuum to include any number of superimposed continua. The fundamental assumption of the theory is that at any instant of time, every point in space is occupied by one particle from each constituent, in a homogenized sense. The historical development and details of Mixture Theory are given in the review articles by Atkin and Craine [1976], Bedford and Drumheller [1983], Bowen [1976], and several appendices in the recent edition of *Rational Thermodynamics* [Truesdell, 1984]. Like Averaging, Mixture Theory also requires constitutive relations for the stress tensor of each component of the mixture and for momentum exchange between the components.

We recently proposed a mathematical description for a flowing mixture of solid particulates and a fluid within the context of Mixture Theory [Johnson et al., 1990b]. The mixture is modeled as a two-component mixture of a Newtonian fluid and a granular solid, in a manner that the equations reduce to those describing a linearly viscous fluid when the solid volume fraction goes to zero, and to those describing a flowing granular solid when the fluid volume fraction goes to zero. Boundary value problems were solved numerically for steady, fully developed flow of this mixture between parallel plates and through a pipe [Johnson et al., 1990b,c].

In our formulation of the problem we use a constitutive equation of a Newtonian fluid for the fluid consitituent. This could describe either a liquid or a gas. From a practical point of view, a dense suspension of solid particles in a fluid shows different characteristics for different suspending media. For example, Soo [1987] shows that a steady flow in a dense gas-solid suspension is not expected; the flow is often turbulent. Soo [1984] also indicates that "the minimum suspension velocity of the same solid particles and the pressure drop are much lower in a liquid than in a gas at similar temperature and at useful working pressures." Also, gas-solid mixtures have become increasingly important in many of the chemical processes and energy related technologies such as pneumatic transport, flow of pulverized coal in feeder lines to surfaces, and fluidized beds.

In suspensions of gas-solid flows, particle-particle interaction has also received much attention (cf. Soo [1967]).

Cross-sectional or radial variations of flow properties, such as velocity and density, are very difficult to measure in gas-solid flows. In many cases the measurements are restricted to crosssectionally averaged quantities. A review of measurement techniques for two-phase media is given by Hewitt in the book by Hestroni [1982]. The purpose of this report is to apply mixture theory to steady developing flow, and by averaging the equations over the cross section of the flow derive equations governing the average volume fraction and the average velocities of each component. First, we review briefly the basic principles of mixture theory and discuss constitutive equations for the mixture components and for the interactions between components. We then average the balance of mass and balance of linear momentum equations over an appropriate control volume.

2. Theory of a Single Continuous Medium

2.1 Introduction

The mechanics of a single continuous medium is discussed as an introduction to mixture theory. Mixture theory follows conceptually as the superposition of two or more continuous media. The discussion here is restricted to purely mechanical systems; thermal effects and chemical reactions are ignored.

2.2 Kinematics and Notation

Consider a body in an arbitrary fixed reference configuration. Let $\mathbf{X}$ denote a typical particle in the reference configuration. The motion of a particle of the body is given by the one-to-one, invertible mapping:

$$\mathbf{x} = \mathcal{X}(\mathbf{X}, t), \tag{1}$$

where, as noted above $\mathbf{X}$ is the position of a particle in the reference configuration, t the time, and $\mathbf{x}$ the spatial position occupied at time t by the particle that was at position $\mathbf{X}$ in the reference configuration. The invertibility of $\mathcal{X}$ ensures that a single particle cannot occupy two positions at once nor can motion occur such that two discrete particles occupy the same postion at the same time. In general, sufficient smoothness is assumed to make any necessary mathematical operations correct. The velocity vector corresponding to the motion (1) is:

$$\mathbf{v} = \frac{D\mathcal{X}}{Dt}, \tag{2}$$

where $\frac{D}{Dt}$ is the material time derivative (i.e. $\frac{\partial}{\partial t}$ with $\mathbf{X}$ fixed) and the other kinematical quantities associated with the above motion are given by:

$$\mathbf{a} = \frac{D\mathbf{v}}{Dt}, \qquad \mathbf{L} = \frac{\partial \mathbf{v}}{\partial \mathbf{x}}, \tag{3}$$

$$\mathbf{D} = \frac{1}{2}(\mathbf{L} + \mathbf{L}^T), \qquad \mathbf{W} = \frac{1}{2}(\mathbf{L} - \mathbf{L}^T), \tag{4}$$

where $\mathbf{v}$ denotes velocity, $\mathbf{a}$ acceleration, $\mathbf{L}$ is the velocity gradient, $\mathbf{D}$ the stretching tensor, and $\mathbf{W}$ the spin tensor. $\frac{D}{Dt}$ denotes differentiation with respect to t, holding $\mathbf{X}$ fixed.

2.3 Basic Equations

Since we are interested in a purely mechanical problem, the appropriate balance laws are conservation of mass, balance of linear momentum, and balance of angular momentum. We will consider an arbitrary fixed region R in three-dimensional Euclidean space of volume V bounded by a surface ∂R of area a. All equations are postulated at the current time t and all field quantities are functions of $\mathbf{x}$ and t. Let Ω_0 denote the reference configuration of the body and Ω_t denote the configuration of the body at time t. Conservation of mass is given in its Lagrangian form as:

$$\int_{P_0} \rho_0 dV = \int_{P_0} \rho \det \mathbf{F} dV \qquad \forall P_0 \subseteq \Omega_0, \tag{5}$$

where ρ_0 is the reference density of the material, ρ is the current density, and $\mathbf{F}$ is the deformation gradient given by:

$$\mathbf{F} = \frac{\partial \mathcal{X}}{\partial \mathbf{X}}. \tag{6}$$

If the integrand is sufficiently smooth, we immediately get the local form for the balance of mass:

$$\rho_0 = \rho \det \mathbf{F}. \tag{7}$$

The conservation of mass is given in Eulerian form by:

$$\frac{D}{Dt} \int_{P_t} \rho dV = 0 \qquad \forall P_t \subseteq \Omega_t, \tag{8}$$

where, again, ρ is the current density of the material, from which we immediately get the local Eulerian form for the balance of mass:

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \mathbf{v}) = 0, \tag{9}$$

The balance of linear momentum is:

$$\frac{D}{Dt} \int_{P_t} \rho \mathbf{v} dV = \int_{\partial P_t} \mathbf{T}\mathbf{n} da + \int_{P_t} \rho \mathbf{b} dV \qquad \forall P_t \subseteq \Omega_t, \tag{10}$$

where $\mathbf{b}$ is the body force vector, $\mathbf{n}$ is the unit normal to the surface, and $\mathbf{T}$ is the Cauchy stress tensor, which leads to the local form:

$$\rho \frac{D\mathbf{v}}{Dt} = \operatorname{div} \mathbf{T} + \rho \mathbf{b}, \tag{11}$$

The balance of angular momentum (in the absence of couple stresses) yields the result that the Cauchy stress is symmetric:

$$\mathbf{T} = \mathbf{T}^T \tag{12}$$

3. Mixture Theory

3.1 Introduction

Materials such as steel, water, or rubber are usually regarded as a single continuum. In many applications such as a fluid containing particulate solid, however, it is useful to describe the two components as separate, interacting continua. A general mixture theory, or theory of interacting continua, can be used to derive balance equations for any number of continuous bodies occupying the same space. The details and historical development of mixture theory are found in review articles by Atkin and Craine[4], Bedford and Drumheller[5], Bowen[6], and several appendices in the recent edition of *Rational Thermodynamics*[40].

3.2 Kinematics and Notation

The underlying assumption of mixture theory is that the mixture may be regarded as n superimposed continua, each having its own motion. At any time, t, each position in the mixture is occupied by one particle from each constituent of the mixture. As in the case of a single continuum, each constituent of the mixture is assigned an arbitrary fixed reference configuration. The motion of a particle of constituent α is a one-to-one, invertible mapping denoted by:

$$\mathbf{x}_\alpha = \mathcal{X}_\alpha(\mathbf{X}_\alpha, t), \tag{13}$$

where $\mathbf{X}_\alpha$ is the position of a particle of the α^{th} body or constituent in its reference configuration, t the time, and $\mathbf{x}_\alpha$ the spatial position occupied at time t by the particle that was at $\mathbf{X}_\alpha$ in the reference configuration. In general, sufficient smoothness is assumed in order to make any needed mathematical operations correct. The velocity vectors corresponding to the motions are:

$$\mathbf{v}_\alpha = \frac{D_\alpha \mathcal{X}_\alpha}{Dt}, \tag{14}$$

$\frac{D_\alpha}{Dt}$ denotes differentiation with respect to t, holding $\mathbf{X}_\alpha$ fixed. Note that there is no sum on α. The densities of each component of the mixture, measured per unit volume of the mixture, are written ρ_α. The mean velocity of the mixture, $\mathbf{V}$, is defined through:

$$\rho \mathbf{V} = \sum_{\alpha=1}^{n} \rho_\alpha \mathbf{v}_\alpha, \tag{15}$$

where ρ is the mixture density, defined by:

$$\rho = \sum_{\alpha=1}^{n} \rho_\alpha. \tag{16}$$

Consider the special case of a two component mixture consisting of a Newtonian fluid and a granular material. The fluid in the mixture will be represented by S_1 and the granular solid by

S_2. Let $\mathbf{X}_1$ and $\mathbf{X}_2$ denote the positions of particles of S_1 and S_2 in the reference configuration. The motion of the constituents is represented by the mappings:

$$\mathbf{x}_1 = \mathcal{X}_1(\mathbf{X}_1, t), \quad \text{and} \quad \mathbf{x}_2 = \mathcal{X}_2(\mathbf{X}_2, t). \tag{17}$$

where the subscripts 1 and 2 refer to the fluid and granular solid, respectively. The kinematical quantities associated with these motions are:

$$\mathbf{v}_1 = \frac{D_1 \mathcal{X}_1}{Dt}, \qquad \mathbf{v}_2 = \frac{D_2 \mathcal{X}_2}{Dt}, \tag{18}$$

$$\mathbf{a}_1 = \frac{D_1 \mathbf{v}_1}{Dt}, \qquad \mathbf{a}_2 = \frac{D_2 \mathbf{v}_2}{Dt}, \tag{19}$$

$$\mathbf{L}_1 = \frac{\partial \mathbf{v}_1}{\partial \mathbf{x}_1}, \qquad \mathbf{L}_2 = \frac{\partial \mathbf{v}_2}{\partial \mathbf{x}_2}, \tag{20}$$

$$\mathbf{D}_1 = \frac{1}{2}(\mathbf{L}_1 + \mathbf{L}_1^T), \qquad \mathbf{D}_2 = \frac{1}{2}(\mathbf{L}_2 + \mathbf{L}_2^T), \tag{21}$$

$$\mathbf{W}_1 = \frac{1}{2}(\mathbf{L}_1 - \mathbf{L}_1^T), \qquad \mathbf{W}_2 = \frac{1}{2}(\mathbf{L}_2 - \mathbf{L}_2^T), \tag{22}$$

where $\mathbf{v}$ denotes velocity, $\mathbf{a}$ acceleration, $\mathbf{L}$ is the velocity gradient, $\mathbf{D}$ the stretching tensor, and $\mathbf{W}$ the spin tensor.

Also, ρ_1 and ρ_2 are the densities of the mixture components in the current configuration given by:

$$\rho_1 = \phi \rho_f, \qquad \rho_2 = \nu \rho_s, \tag{23}$$

where ρ_f is the density of the pure fluid, ρ_s is the density of the solid grains, and ν is the volume fraction of the solid component and ϕ is the volume fraction of the fluid. For a saturated mixture $\phi = 1 - \nu$. The mixture density, ρ_m is given by:

$$\rho_m = \rho_1 + \rho_2, \tag{24}$$

and the mean velocity $\mathbf{v}$ of the mixture is defined by:

$$\rho_m \mathbf{v} = \rho_1 \mathbf{v}_1 + \rho_2 \mathbf{v}_2. \tag{25}$$

3.3 Basic Equations

Balance equations for the mixture, and its constituents, may again be in either integral form or in differential form. Conservation of mass for the fluid and granular material is:

$$\frac{D}{Dt} \int_{P_t} \rho_1 dV = \int_{P_t} c_1 dV \qquad \forall P_t \subseteq \Omega_t, \tag{26}$$

and:

$$\frac{D}{Dt} \int_{P_t} \rho_2 dV = \int_{P_t} c_2 dV \qquad \forall P_t \subseteq \Omega_t, \tag{27}$$

where $c_1$and c_2 are the mass supplies to the first and second constituents, respectively. These equations take the local form:

$$\frac{\partial \rho_1}{\partial t} + \mathrm{div}\,(\rho_1 \mathbf{v}_1) = c_1, \tag{28}$$

and:

$$\frac{\partial \rho_2}{\partial t} + \mathrm{div}\,(\rho_2 \mathbf{v}_2) = c_2. \tag{29}$$

Let $\mathbf{T}_1$ and $\mathbf{T}_2$ denote the partial stress tensors of the fluid S_1 and the solid S_2, respectively. Then the balance of linear momentum for the fluid and solid are given by:

$$\frac{D}{Dt}\int_{P_t} \rho_1 \mathbf{v}_1 dV = \int_{\partial P_t} \mathbf{T}_1 \mathbf{n}_1 da + \int_{P_t} (\rho_1 \mathbf{b}_1 + \mathbf{f}_I + c_1 \mathbf{v}_1)\, dV \qquad \forall P_t \subseteq \Omega_t, \tag{30}$$

and:

$$\frac{D}{Dt}\int_{P_t} \rho_2 \mathbf{v}_2 dV = \int_{\partial P_t} \mathbf{T}_2 \mathbf{n}_2 da + \int_{P_t} (\rho_2 \mathbf{b}_2 - \mathbf{f}_I + c_2 \mathbf{v}_2)\, dV \qquad \forall P_t \subseteq \Omega_t, \tag{31}$$

or:

$$\rho_1 \frac{D_1 \mathbf{v}_1}{Dt} = \mathrm{div}\,\mathbf{T}_1 + \rho_1 \mathbf{b}_1 + \mathbf{f}_I + c_1 \mathbf{v}_1 \tag{32}$$

and:

$$\rho_2 \frac{D_2 \mathbf{v}_2}{Dt} = \mathrm{div}\,\mathbf{T}_2 + \rho_2 \mathbf{b}_2 - \mathbf{f}_I + c_2 \mathbf{v}_2 \tag{33}$$

where $\mathbf{b}$ represents the external body force, and $\mathbf{f}_I$ represents the mechanical interaction (local exchange of momentum) between the components.

The balance of moment of momentum implies that:

$$\mathbf{T}_1 + \mathbf{T}_2 = \mathbf{T}_1^T + \mathbf{T}_2^T. \tag{34}$$

The partial stresses need not be symmetric, however.

3.4 Boundary Conditions

One problem in using mixture theory is specifying the boundary conditions. Boundary conditions can be prescribed based on the tractions acting on the boundary, known displacements (or velocities) on the boundary, or some combination of the two.

The difficulty in specifying tractions is that one must ultimately determine how much of the total traction is supported by each constituent. Rajagopal et al.[25] have addressed this issue for a certain class of boundary value problems. The problems considered here belong to the second class in which the velocities are specified at the boundaries, for instance the adherence boundary condition or a slip condition that is specified on the basis of experiments.

4. Constitutive Equations

We shall assume that the fluid and solid phases are dense enough to be modeled as homogeneous continuous media so that we may exploit the theory of interacting continua. Based on our knowledge of modeling in the theory of granular materials and a linearly viscous fluid, it would be natural to assume all the constitutive functions depend on [cf. Rajagopal et al., 1990]:

$$\rho_1, \rho_2, \nabla\rho_1, \nabla\rho_2, \nabla\nabla\rho_1, \nabla\nabla\rho_2, \mathbf{v}_1 - \mathbf{v}_2, \mathbf{D}_1, \mathbf{D}_2, \tag{35}$$

and possibly other vectors and tensors. Then, using methods that are now standard in continuum mechanics (cf. Shi et al. [1981] and Atkin and Craine [1976]), we can obtain restrictions and forms for such constitutive expressions. Here, we discuss an alternative approach, which is to postulate the constitutive expressions by simply generalizing the structure of the constitutive relations from a single constituent theory. In general, the constitutive expressions for $\mathbf{T}_f$ and $\mathbf{T}_s$ depend on the kinematical quantities associated with both the constituents. However, we assume that $\mathbf{T}_s$ and $\mathbf{T}_f$ depend only on the kinematical quantities associated with the solid and fluid, respectively. This assumption is sometimes called "the principle of phase separation" and was first used in mixture theory by Adkins [1963a,b].

In the majority of fluid-solid mixtures, the fluid is either a gas or water. Therefore, it is appropriate to assume that the fluid behaves as a linearly viscous fluid, whose constitutive equation is:

$$\mathbf{T}_f = [-p(\rho_1) + \lambda_f(\rho_1)\mathrm{tr}\,\mathbf{D}_1]\mathbf{I} + 2\mu_f(\rho_1)\mathbf{D}_1, \tag{36}$$

where p is the fluid pressure, λ_f and μ_f are the viscosities, $\mathbf{D}_1$ is the stretching tensor for the fluid defined in equation (21), and $\mathbf{I}$ is the identity tensor. If the fluid is incompressible, then p is one of the unknown quantities in the problem that would have to be calculated. If the fluid is compressible, an equation of state is needed. In general, p, λ_f, and μ_f are functions of ρ_1.

There are basically two different ways of deriving a constitutive relation for the stress tensor of granular materials - the continuum approach and the statistical approach. We use the continuum approach in our analysis. In this study, we assume that the stress tensor for a granular material is given by [cf. Goodman and Cowin, 1971,1972; Savage, 1979; Rajagopal and Massoudi, 1990]:

$$\mathbf{T}_s = [\hat{\beta}_0(\rho_2) + \hat{\beta}_1(\rho_2)\mathrm{grad}\,\rho_2 \bullet \mathrm{grad}\,\rho_2 + \hat{\beta}_2(\rho_2)\mathrm{tr}\,\mathbf{D}_2]\mathbf{I}$$

$$+ \hat{\beta}_3(\rho_2)\mathbf{D}_2 + \hat{\beta}_4(\rho_2)\mathrm{grad}\,\rho_2 \otimes \mathrm{grad}\,\rho_2, \tag{37}$$

where $\bullet$ denotes the scalar product of two vectors and $\otimes$ denotes the outer, or tensor, product of two vectors. The spherical part of the stress in equation (37) can be interpreted as the solid pressure p_s. The material moduli $\hat{\beta}_1$ and $\hat{\beta}_4$ are material parameters that reflect the distribution of the granular particles, and $\hat{\beta}_0$ plays a role akin to pressure in a compressible fluid and is given by an equation of state. The material modulus $\hat{\beta}_2$ is a viscosity akin to the second coefficient of viscosity in a compressible fluid and $\hat{\beta}_3$ denotes the viscosity (i.e., the resistance of the material to flow) of the granular solids. Recently, Rajagopal and Massoudi [1990] have outlined an experimental/theoretical approach to determine these material moduli. Based on the available experimental measurements of Savage [1979], Savage and Sayed [1984], and Hanes and Inman [1985] and the computer simulations of Walton and Braun [1986a,b], it is clear that granular materials exhibit normal stress effects. The above model (equation 37) is a simplified version of the model proposed by Rajagopal and Massoudi [1990] which predicts the possibility of both the normal stress differences. Furthermore, Boyle and Massoudi [1990], using Enskog's dense gas theory, have obtained explicit expressions for the material moduli $\hat{\beta}_0$ through $\hat{\beta}_4$.

A mixture stress tensor is defined as (cf. Green and Naghdi [1969]):

$$\mathbf{T}_m = \mathbf{T}_1 + \mathbf{T}_2, \tag{38}$$

where:

$$\mathbf{T}_1 = (1-\nu)\mathbf{T}_f, \quad \text{and} \quad \mathbf{T}_2 = \mathbf{T}_s, \tag{39}$$

so that the mixture stress tensor reduces to that of a pure fluid as $\nu \to 0$ and to that of a granular material as $\phi \to 0$.[1] $\mathbf{T}_2$ may also be written as $\mathbf{T}_2 = \nu\hat{\mathbf{T}}_s$, where $\hat{\mathbf{T}}_s$ may be thought of as representing the stress tensor for some (quite densely packed) reference configuration of the granular material.

The mechanical interaction between the mixture components, $\mathbf{f}_I$, is written as [Johnson et al., 1990]:

$$\mathbf{f}_I = A_1 \operatorname{grad}\nu + A_2\mathcal{F}(\nu)(\mathbf{v}_2 - \mathbf{v}_1) + A_3\nu(2\operatorname{tr}\mathbf{D}_1^2)^{-\frac{1}{4}}\mathbf{D}_1(\mathbf{v}_2 - \mathbf{v}_1)$$
$$+ A_4\nu(\mathbf{W}_2 - \mathbf{W}_1)(\mathbf{v}_2 - \mathbf{v}_1) + A_5\mathbf{a}_{vm}, \tag{41}$$

where $\mathbf{a}_{vm}$ is a properly frame invariant measure of the relative acceleration between the mixture components and $\mathcal{F}(\nu)$ represents the dependence of the drag coefficient on the volume fraction. The terms in equation (41) reflect the presence of density gradients[2], drag, 'slip-shear' lift, 'spin' lift, and virtual mass, respectively. Müller's [1968] work indicates that a term of the form $A_1 \operatorname{grad}\nu$ must be included in the interactions in order to get well-posed problems. The term multiplying A_3 is a generalization of Saffman's [1965,1968] single particle result first proposed in this form by McTigue et al. [1986]. One of the earliest studies examining the effect of lift force is discussed by Soo [1969] and Soo and Tung [1972].

[1]Note that $\phi \to 0$ is equivalent to $\nu \to 1$ only in a saturated mixture. Thus the theory allows for the case of the mixture tending to a pure granular material without $\nu \to 1$ but to some value ν_m, strictly less than unity, usually referred to as the maximum packing fraction. We are interested here, however, in the case when there is a sufficient amount of both the constituents and hence we are not close to either of the limiting cases. Further, in keeping with the usual weighting procedures in multiphase flow we shall represent $\mathbf{T}_m$ as:

$$\mathbf{T}_m = (1-\nu)\mathbf{T}_f + \nu\hat{\mathbf{T}}_s, \tag{40}$$

where $\hat{\mathbf{T}}_s$ is discussed above.

[2]The actual form of this interaction should include the terms $\alpha_1 \operatorname{grad}\rho_1 + \alpha_2 \operatorname{grad}\rho_2$ where α_1 and α_2 are constants. If we assume that the system is a saturated mixture with incompressible components, this expression simplifies to $A_1 \operatorname{grad}\nu$ where $A_1 = \alpha_2 - \alpha_1$. Since no information concerning the coefficients α_1 and α_2 is available and a term of the same form arises in the balance of linear momentum from the granular solid stress tensor, this term will be neglected in the present work.

5. Averaging

5.1 Conservation of Mass

Consider a box shaped fixed region R in three-dimensional Euclidean space of volume V bounded by a surface ∂R of area a. The box has unit depth in the z direction and surfaces ∂R_t, ∂R_b, ∂R_1, and ∂R_2 as indicated in Figure 1. All equations are postulated at the current time t and all field quantities are functions of $\mathbf{x}$ and t.

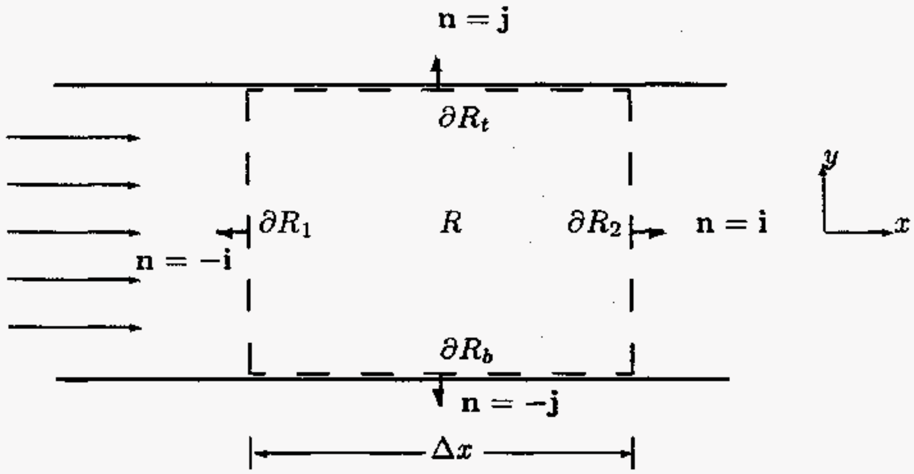

Figure 1. Control Volume

The appropriate balance of mass for a control volume is given by:

$$\frac{\partial}{\partial t}\int_R \rho_\alpha dV = -\int_{\partial R}\rho_\alpha(\mathbf{v}_\alpha \bullet \mathbf{n})da + \int_R c_\alpha dV \quad , \tag{42}$$

where ρ_α is the density of the α^{th} constituent, $\mathbf{v}_\alpha$ is the velocity, $\mathbf{n}$ is the unit normal to the surface, and c_α is the mass supply. Suppose there is no chemical reaction, i.e. $c_\alpha \equiv 0$, and that the flow is steady and unidirectional, i.e. $\mathbf{v}_\alpha = v_\alpha(x, y, z)\mathbf{i}$. Equation (42) implies that:

$$\int_{\partial R}\rho_\alpha(\mathbf{v}_\alpha \bullet \mathbf{n})da = 0. \tag{43}$$

Denoting the entrance boundary as ∂R_1 and the exit as ∂R_2, we find:

$$-\int_{\partial R_1}\rho_\alpha \mathbf{v}_\alpha da + \int_{\partial R_2}\rho_\alpha \mathbf{v}_\alpha da = 0. \tag{44}$$

Let ∂R_1 be located at x and ∂R_2 be located at $x + \Delta x$. Also let us suppose that the cross-sectional area is constant. Then (by the mean value theorem[3]):

$$\int_{\partial R_1}\rho_\alpha \mathbf{v}_\alpha da = \rho_\alpha^* \mathbf{v}_\alpha^* A, \tag{45}$$

[3] Mean Value Theorem for Integrals. If f is continuous on $[a, b]$, then there is a number c in $[a, b]$ such that:

$$\int_a^b f(x)dx = (b-a)f(c)$$

where the asterisks represent some average value on ∂R_1. Using Taylor's expansion:

$$\int_{\partial R_2} \rho_\alpha \mathbf{v}_\alpha da \approx \rho_\alpha^* \mathbf{v}_\alpha^* A + \frac{d}{dx}(\rho_\alpha^* \mathbf{v}_\alpha^* A)\Delta x, \tag{46}$$

Thus equation (44) implies:

$$\frac{d}{dx}(\rho_\alpha^* \mathbf{v}_\alpha^* A) = 0. \tag{47}$$

5.2 Balance of Linear Momentum for the Fluid

The balance of linear momentum for the fluid is:

$$\frac{\partial}{\partial t}\int_R \rho_1 \mathbf{v}_1 \, dV = -\int_{\partial R} \rho_1 \mathbf{v}_1 (\mathbf{v}_1 \bullet \mathbf{n}) \, da + \int_{\partial R} \mathbf{T}_1 \mathbf{n} \, da + \int_R (\rho_1 \mathbf{b}_1 + \mathbf{f}_I + c_1 \mathbf{v}_1) \, dV \tag{48}$$

where R represents a control volume, ∂R represents the surface of that control volume, ρ is density, $\mathbf{v}_1$ is velocity, $\mathbf{b}_1$ is the body force, $\mathbf{f}_I$ is the interaction between the components, $\mathbf{T}_1$ is the stress tensor, c_1 is the mass supply, and $\mathbf{n}$ is the unit normal to the surface of the control volume. The subscript 1 refers to the fluid component. With the assumptions of steady flow and no chemical reaction the above equation reduces to:

$$\int_{\partial R} \rho_1 \mathbf{v}_1 (\mathbf{v}_1 \bullet \mathbf{n}) \, da = \int_{\partial R} \mathbf{T}_1 \mathbf{n} \, da + \int_R (\rho_1 \mathbf{b} + \mathbf{f}_I) \, dV. \tag{49}$$

We now derive the averaged form of equation (49) by considering each term individually.

Convective Term The velocity fields of the fluid and solid are assumed to have the form:

$$\mathbf{v}_1 = v(x,y)\mathbf{i}, \qquad \text{and} \qquad \mathbf{v}_2 = u(x,y)\mathbf{i}, \tag{50}$$

then:

$$\int_{\partial R} \rho_1 \mathbf{v}_1 (\mathbf{v}_1 \bullet \mathbf{n}) \, da = -\int_{\partial R_1} \rho_1 v^2 \mathbf{i} \, da + \int_{\partial R_2} \rho_1 v^2 \mathbf{i} \, da, \tag{51}$$

The Mean Value Theorem implies that:

$$\int_{\partial R_1} \rho_1 v^2 \mathbf{i} \, da = \rho_1^* (v^*)^2 A \, \mathbf{i}, \tag{52}$$

where the asterisks denote average values (i.e. averaged over the cross-section) and A is the cross sectional area of the control volume. Assuming that Δx is small and applying Taylor's expansion yields:

$$\int_{\partial R_2} \rho_1 v^2 \mathbf{i} \, da \approx \left[\rho_1^* (v^*)^2 A + \frac{d}{dx}\left[\rho_1^* (v^*)^2 A\right] \Delta x\right] \mathbf{i}, \tag{53}$$

Combining equations 51, 52 and 53 gives the following result:

$$\int_{\partial R} \rho \mathbf{v}_1 (\mathbf{v}_1 \bullet \mathbf{n}) \, da = \frac{d}{dx}\left[\rho_1^* (v^*)^2 A\right] \Delta x \, \mathbf{i}. \tag{54}$$

Body Forces The effect of the body force can be averaged over the control volume to yield:

$$\int_R \rho_1 \mathbf{b}_1 \, dV = \rho_1^* \mathbf{b}_1 A \, \Delta x \,, \tag{55}$$

Interactions In general, the interactions between the mixture components will include density gradients, drag, lift, and virtual mass. Because we are considering the interactions to be averaged in the y direction, lift and density gradient interactions can be neglected. Drag and virtual mass interactions will be considered here so that the interaction term has the form:

$$\mathbf{f}_I = \alpha_3(\mathbf{u} - \mathbf{v}) + \alpha_6 \mathbf{a}_{vm}, \tag{56}$$

where $\mathbf{a}_{vm}$ is a frame-indifferent relative acceleration given by:

$$\mathbf{a}_{vm} = \left[\frac{D\mathbf{v}_2}{Dt} - (\operatorname{grad} \mathbf{v}_2)(\mathbf{v}_2 - \mathbf{v}_1)\right] - \left[\frac{D\mathbf{v}_1}{Dt} - (\operatorname{grad} \mathbf{v}_1)(\mathbf{v}_1 - \mathbf{v}_2)\right]. \tag{57}$$

With our assumptions about the velocity fields the interaction force vector becomes:

$$\mathbf{f}_I = \begin{pmatrix} \alpha_3(u - v) + \alpha_6 \left(v\frac{\partial u}{\partial x} - u\frac{\partial v}{\partial x}\right) \\ 0 \\ 0 \end{pmatrix}, \tag{58}$$

and:

$$\int_R \mathbf{f}_I dV = \left[\alpha_3^*(u^* - v^*) + \alpha_6^* \left(v^*\frac{\partial u^*}{\partial x} - u^*\frac{\partial v^*}{\partial x}\right)\right] A \, \Delta x \, \mathbf{i}. \tag{59}$$

Stress Tensor: Isotropic Part The fluid stress tensor is given by:

$$\mathbf{T}_1 = [-p_f + \lambda \operatorname{tr} \mathbf{D}_1]\mathbf{I} + 2\mu \mathbf{D}_1, \tag{60}$$

where $\mathbf{D}$ is the stretching tensor, $\mathbf{I}$ is the identity tensor, p_f is the fluid pressure, μ is the first coefficient of viscosity of the fluid, and λ is the second coefficient of viscosity. Following the same procedure as above, we have:

$$\begin{aligned} \int_{\partial R} [-p_f + \lambda \operatorname{tr} \mathbf{D}_1] \mathbf{I} \, \mathbf{n} \, da = &- \int_{\partial R_1} \left[-p_f + \lambda \frac{\partial v}{\partial x}\right] \mathbf{i} \, da + \int_{\partial R_2} \left[-p_f + \lambda \frac{\partial v}{\partial x}\right] \mathbf{i} \, da \\ &+ \int_{\partial R_t} \left[-p_f + \lambda \frac{\partial v}{\partial x}\right] \mathbf{j} \, da - \int_{\partial R_b} \left[-p_f + \lambda \frac{\partial v}{\partial x}\right] \mathbf{j} \, da, \end{aligned} \tag{61}$$

where:

$$\int_{\partial R_1} \left[-p_f + \lambda \frac{\partial v}{\partial x}\right] da = \left[-p_f^* + \lambda^* \frac{\partial v^*}{\partial x}\right] A, \tag{62}$$

and:

$$\int_{\partial R_2} \left[-p_f + \lambda \frac{\partial v}{\partial x}\right] da = \left[-p_f^* + \lambda^* \frac{\partial v^*}{\partial x}\right] A + \frac{d}{dx}\left[-p_f^* + \lambda^* \frac{\partial v^*}{\partial x}\right] A \, \Delta x, \tag{63}$$

so that:

$$\int_{\partial R}\left[-p_f+\lambda\frac{\partial v}{\partial x}\right]\mathbf{I}\,\mathbf{n}\,da=\frac{d}{dx}\left[-p_f^*+\lambda^*\frac{\partial v^*}{\partial x}\right]A\,\Delta x\,\mathbf{i}$$

$$+\int_{\partial R_t}\left[-p_f+\lambda\frac{\partial v}{\partial x}\right]\mathbf{j}\,da-\int_{\partial R_b}\left[-p_f+\lambda\frac{\partial v}{\partial x}\right]\mathbf{j}\,da. \quad (64)$$

Stress Tensor: Viscous Part

$$\int_{\partial R}2\mu\mathbf{D}_1\,\mathbf{n}\,da=-\int_{\partial R_1}2\mu\frac{\partial v}{\partial x}\,\mathbf{i}\,da+\int_{\partial R_2}2\mu\frac{\partial v}{\partial x}\,\mathbf{i}\,da-\int_{\partial R_b}\mu\frac{\partial v}{\partial y}\,\mathbf{i}\,da+\int_{\partial R_t}\mu\frac{\partial v}{\partial y}\,\mathbf{i}\,da$$

$$-\int_{\partial R_1}\mu\frac{\partial v}{\partial y}\,\mathbf{j}\,da+\int_{\partial R_2}\mu\frac{\partial v}{\partial y}\,\mathbf{j}\,da, \quad (65)$$

$$\int_{\partial R_1}2\mu\frac{\partial v}{\partial x}\,da=2\mu^*\frac{\partial v^*}{\partial x}\,A, \quad (66)$$

$$\int_{\partial R_2}2\mu\frac{\partial v}{\partial x}\,da=2\mu^*\frac{\partial v^*}{\partial x}\,A+\frac{d}{dx}\left[2\mu^*\frac{du^*}{dx}A\right]\,\Delta x. \quad (67)$$

The shear force of the fluid on the upper and lower surfaces of the control volume is defined as:

$$F_w=\tau_f\,\Delta x=\int_{\partial R_t}\mu\frac{\partial v}{\partial y}\,da-\int_{\partial R_b}\mu\frac{\partial v}{\partial y}\,da, \quad (68)$$

and:

$$\int_{\partial R}2\mu\mathbf{D}_1\,\mathbf{n}\,da=\frac{d}{dx}\left[2\mu^*\frac{dv^*}{dx}A\right]\,\Delta x\,\mathbf{i}-\tau_f\,\Delta x\,\mathbf{i}-\int_{\partial R_1}\mu\frac{\partial v}{\partial y}\,\mathbf{j}\,da+\int_{\partial R_2}\mu\frac{\partial v}{\partial y}\,\mathbf{j}\,da. \quad (69)$$

Thus the balance of linear momentum for the fluid phase in the x-direction becomes:

$$\frac{d}{dx}\left[\rho_1^*(v^*)^2A\right]\,\Delta x=\frac{d}{dx}\left[-p_f^*+\lambda^*\frac{\partial v^*}{\partial x}\right]A\,\Delta x+\frac{d}{dx}\left[2\mu^*\frac{dv^*}{dx}A\right]\,\Delta x$$

$$-\tau_f\Delta x+\rho_1^*(b_1)_xA\Delta x+\left[\alpha_3^*(u^*-v^*)+\alpha_6^*\left(v^*\frac{\partial u^*}{\partial x}-u^*\frac{\partial v^*}{\partial x}\right)\right]A\Delta x. \quad (70)$$

and in the y-direction, we have:

$$\int_{\partial R_t}\left[-p_f+\lambda\frac{\partial v}{\partial x}\right]da-\int_{\partial R_b}\left[-p_f+\lambda\frac{\partial v}{\partial x}\right]da-\int_{\partial R_1}\mu\frac{\partial v}{\partial y}da$$

$$+\int_{\partial R_2}\mu\frac{\partial v}{\partial y}da+\rho_1^*(b_1)_yA\Delta x=0 \quad (71)$$

5.3 Balance of Linear Momentum for the Granular Solid

The balance of linear momentum for the granular solid is:

$$\frac{\partial}{\partial t}\int_R \rho_2 \mathbf{v}_2\, dV = -\int_{\partial R} \rho_2 \mathbf{v}_2(\mathbf{v}_2 \bullet \mathbf{n})\, da + \int_{\partial R} \mathbf{T}_2 \mathbf{n}\, da + \int_R (\rho_2 \mathbf{b}_2 - \mathbf{f}_I + c_2 \mathbf{v}_2)\, dV, \tag{72}$$

where R represents a control volume, ∂R represents the surface of that control volume, ρ_2 is density, $\mathbf{v}_2$ is velocity, $\mathbf{b}_2$ is the body force, $\mathbf{f}_I$ is the interaction between the components, $\mathbf{T}_2$ is the stress tensor, c_2 is the mass supply, and $\mathbf{n}$ is the unit normal to the surface of the control volume. The subscript 2 refers to the solid component. With the assumptions of steady flow and no chemical reaction the above equation reduces to:

$$\int_{\partial R} \rho_2 \mathbf{v}_2(\mathbf{v}_2 \bullet \mathbf{n})\, da = \int_{\partial R} \mathbf{T}_2 \mathbf{n}\, da + \int_R (\rho_2 \mathbf{b}_2 - \mathbf{f}_I)\, dV. \tag{73}$$

The solid stress tensor is given by:

$$\mathbf{T}_2 = [-p_s + \hat{\beta}_2 \operatorname{tr} \mathbf{D}_2]\mathbf{I} + \hat{\beta}_3 \mathbf{D}_2 + \hat{\beta}_4 \operatorname{grad} \nu \otimes \operatorname{grad} \nu, \tag{74}$$

where:

$$p_s = -\hat{\beta}_o - \hat{\beta}_1 \operatorname{grad} \nu \bullet \operatorname{grad} \nu, \tag{75}$$

also, $\mathbf{D}$ is the stretching tensor, $\mathbf{I}$ is the identity tensor, the $\hat{\beta}'s$ are the material coefficients of the granular material, and $\otimes$ denotes the outer or tensor product of two vectors. The following results are analogous to those for the fluid:

$$\int_{\partial R} \rho \mathbf{v}_2(\mathbf{v}_2 \bullet \mathbf{n})\, da = \frac{d}{dx}\left[\rho_2^*(u^*)^2 A\right] \Delta x\, \mathbf{i}, \tag{76}$$

$$\int_R \rho_2 \mathbf{b}\, dV = \rho_2^* \mathbf{b} A\, \Delta x\,, \tag{77}$$

$$\int_R \mathbf{f}_I dV = \left[\alpha_3^*(u^* - v^*) + \alpha_6^*\left(v^*\frac{\partial u^*}{\partial x} - u^*\frac{\partial v^*}{\partial x}\right)\right] A\, \Delta x\, \mathbf{i}, \tag{78}$$

$$\begin{aligned}\int_{\partial R}\left[-p_s + \hat{\beta}_2\frac{\partial u}{\partial x}\right]\mathbf{I}\,\mathbf{n}\, da &= \frac{d}{dx}\left[-p_s^* + \hat{\beta}_2^*\frac{\partial u^*}{\partial x}\right] A\, \Delta x\, \mathbf{i} \\ &\quad + \int_{\partial R_t}\left[-p_s + \hat{\beta}_2\frac{\partial u}{\partial x}\right]\mathbf{j}\, da - \int_{\partial R_b}\left[-p_s + \hat{\beta}_2\frac{\partial u}{\partial x}\right]\mathbf{j}\, da,\end{aligned} \tag{79}$$

$$\int_{\partial R} \hat{\beta}_3 \mathbf{D}_2\, \mathbf{n}\, da = \frac{d}{dx}\left[\hat{\beta}_3^*\frac{du^*}{dx}A\right] \Delta x\, \mathbf{i} - \tau_s\, \Delta x\, \mathbf{i} - \frac{1}{2}\int_{\partial R_1} \hat{\beta}_3\frac{\partial u}{\partial y}\mathbf{j}\, da + \frac{1}{2}\int_{\partial R_2} \hat{\beta}_3\frac{\partial u}{\partial y}\mathbf{j}\, da, \tag{80}$$

where τ_s is defined in an identical fashion to τ_f in equation (68) through:

$$\tau_s\, \Delta x = \int_{\partial R_t} \hat{\beta}_3\frac{\partial u}{\partial y}\, da - \int_{\partial R_b} \hat{\beta}_3\frac{\partial u}{\partial y}\, da. \tag{81}$$

The solid stress tensor has an additional term not found in the fluid equation. To simplify the following calculations, we define:

$$\mathbf{M} = \operatorname{grad}\nu \otimes \operatorname{grad}\nu, \tag{82}$$

which for our assumed form of the density field becomes:

$$\mathbf{M} = \begin{pmatrix} \left(\frac{\partial\nu}{\partial x}\right)^2 & \frac{\partial\nu}{\partial x}\frac{\partial\nu}{\partial y} & 0 \\ \frac{\partial\nu}{\partial x}\frac{\partial\nu}{\partial y} & \left(\frac{\partial\nu}{\partial y}\right)^2 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{83}$$

The additional term in the solid balance of linear momentum is:

$$\begin{aligned}\int_{\partial R}\hat{\beta}_4\mathbf{M}\,\mathbf{n}\,da = &-\int_{\partial R_1}\hat{\beta}_4\left(\frac{\partial\nu}{\partial x}\right)^2\mathbf{i}\,da + \int_{\partial R_2}\hat{\beta}_4\left(\frac{\partial\nu}{\partial x}\right)^2\mathbf{i}\,da + \int_{\partial R_t}\hat{\beta}_4\frac{\partial\nu}{\partial x}\frac{\partial\nu}{\partial y}\mathbf{i}\,da \\ &-\int_{\partial R_b}\hat{\beta}_4\frac{\partial\nu}{\partial x}\frac{\partial\nu}{\partial y}\mathbf{i}\,da + \int_{\partial R_2}\hat{\beta}_4\frac{\partial\nu}{\partial x}\frac{\partial\nu}{\partial y}\mathbf{j}\,da \\ &-\int_{\partial R_1}\hat{\beta}_4\frac{\partial\nu}{\partial x}\frac{\partial\nu}{\partial y}\mathbf{j}\,da + \int_{\partial R_t}\hat{\beta}_4\left(\frac{\partial\nu}{\partial y}\right)^2\mathbf{j}\,da - \int_{\partial R_b}\hat{\beta}_4\left(\frac{\partial\nu}{\partial y}\right)^2\mathbf{j}\,da.\end{aligned} \tag{84}$$

Proceeding as before, we have:

$$\int_{\partial R_1}\hat{\beta}_4\left(\frac{\partial\nu}{\partial x}\right)^2 da = \hat{\beta}_4^*\left(\frac{\partial\nu^*}{\partial x}\right)^2 A, \tag{85}$$

$$\int_{\partial R_2}\hat{\beta}_4\left(\frac{\partial\nu}{\partial x}\right)^2 da = \hat{\beta}_4^*\left(\frac{\partial\nu^*}{\partial x}\right)^2 A + \frac{d}{dx}\left[\hat{\beta}_4^*\left(\frac{\partial\nu^*}{\partial x}\right)^2 A\right]\Delta x. \tag{86}$$

The shear force of the solid on the upper and lower surfaces of the control volume due to the normal stresses is defined as:

$$\tau_n\,\Delta x = \int_{\partial R_t}\hat{\beta}_4\frac{\partial\nu}{\partial y}\frac{\partial\nu}{\partial x}da - \int_{\partial R_b}\hat{\beta}_4\frac{\partial\nu}{\partial y}\frac{\partial\nu}{\partial x}da, \tag{87}$$

and:

$$\begin{aligned}\int_{\partial R}\hat{\beta}_4\mathbf{M}\,\mathbf{n}\,da = &\frac{d}{dx}\left[\hat{\beta}_4^*\left(\frac{\partial\nu^*}{\partial x}\right)^2 A\right]\Delta x\,\mathbf{i} - \tau_n\,\Delta x\,\mathbf{i} + \int_{\partial R_2}\hat{\beta}_4\frac{\partial\nu}{\partial x}\frac{\partial\nu}{\partial y}\mathbf{j}\,da \\ &-\int_{\partial R_1}\hat{\beta}_4\frac{\partial\nu}{\partial x}\frac{\partial\nu}{\partial y}\mathbf{j}\,da + \int_{\partial R_t}\hat{\beta}_4\left(\frac{\partial\nu}{\partial y}\right)^2\mathbf{j}\,da - \int_{\partial R_b}\hat{\beta}_4\left(\frac{\partial\nu}{\partial y}\right)^2\mathbf{j}\,da.\end{aligned} \tag{88}$$

Thus the balance of linear momentum in the x-direction becomes:

$$\frac{d}{dx}\left[\rho_2^*(u^*)^2A\right]\Delta x = \frac{d}{dx}\left[-\rho_s^* + \hat{\beta}_2^*\frac{\partial u^*}{\partial x}\right]A\,\Delta x + \frac{d}{dx}\left[\hat{\beta}_3^*\frac{\partial u^*}{\partial x}A\right]\Delta x - \tau_s\,\Delta x$$

$$+\frac{d}{dx}\left[\hat{\beta}_4^*\left(\frac{\partial\nu^*}{\partial x}\right)^2 A\right]\Delta x-\tau_n\Delta x+\rho_2^* b_x A\Delta x-\left[\alpha_3^*(u^*-v^*)+\alpha_6^*\left(v^*\frac{\partial u^*}{\partial x}-u^*\frac{\partial v^*}{\partial x}\right)\right]A\Delta x\,. \tag{89}$$

and in the y-direction, we have:

$$\int_{\partial R_t}\left[-\rho_s+\hat{\beta}_2\frac{\partial u}{\partial x}\right]da-\int_{\partial R_b}\left[-\rho_s+\hat{\beta}_2\frac{\partial u}{\partial x}\right]da-\frac{1}{2}\int_{\partial R_1}\hat{\beta}_3\frac{\partial u}{\partial y}da+\frac{1}{2}\int_{\partial R_2}\hat{\beta}_3\frac{\partial u}{\partial y}da-\int_{\partial R_1}\hat{\beta}_4\frac{\partial\nu}{\partial x}\frac{\partial\nu}{\partial y}da$$

$$+\int_{\partial R_2}\hat{\beta}_4\frac{\partial\nu}{\partial x}\frac{\partial\nu}{\partial y}da+\int_{\partial R_t}\hat{\beta}_4\left(\frac{\partial\nu}{\partial y}\right)^2 da-\int_{\partial R_b}\hat{\beta}_4\left(\frac{\partial\nu}{\partial y}\right)^2 da+\rho_2^* b_y A\Delta x=0 \tag{90}$$

6. Summary and Conclusions

Combining equations 54, 55, 59, 64, 69 with 49 and 76, 77, 78, 79, 80, and 88 with 73, and treating the averaged values for the variable as the variable, yields (in the x-direction):

$$\frac{d}{dx}\left(\rho_1^* v^{*2}\right)=-\frac{dp_f^*}{dx}+(\lambda^*+2\mu^*)\frac{d^2v^*}{dx^2}-\frac{1}{A}\tau_f+\rho_1^* b_x$$

$$+\alpha_3^*(u^*-v^*)+\alpha_6^*\left[v^*\frac{du^*}{dx}-u^*\frac{dv^*}{dx}\right], \tag{91}$$

$$\frac{d}{dx}\left(\rho_2^* u^{*2}\right)=-\frac{dp_s^*}{dx}+(\hat{\beta}_2^*+\hat{\beta}_3^*)\frac{d^2u^*}{dx^2}-\frac{1}{A}\tau_s+\rho_2^* b_x$$

$$+2\hat{\beta}_4^*\frac{d\nu^*}{dx}\frac{d^2\nu^*}{dx^2}-\frac{1}{A}\tau_n-\alpha_3^*(u^*-v^*)-\alpha_6^*\left[v^*\frac{du^*}{dx}-u^*\frac{dv^*}{dx}\right], \tag{92}$$

where b_x is the x-component of the body force and:

$$\tau_f=\frac{2\mu}{\Delta x}\int_{\partial R_t}\frac{\partial v}{\partial y}da, \tag{93}$$

$$\tau_s=\frac{\hat{\beta}_3}{\Delta x}\int_{\partial R_t}\frac{\partial u}{\partial y}da, \tag{94}$$

$$\tau_n=\frac{2\hat{\beta}_4}{\Delta x}\int_{\partial R_t}\frac{\partial\nu}{\partial y}\frac{\partial\nu}{\partial x}da. \tag{95}$$

In many applications (cf. Soo [1967]), the equation of interest is the fluid-particle (or mixture) momentum equation. This equation is obtained by adding equations (91) and (92):

$$\frac{d}{dx}\left(\rho_1^* v^{*2}\right)+\frac{d}{dx}\left(\rho_2^* u^{*2}\right)=-\frac{dp_f^*}{dx}-\frac{dp_s^*}{dx}+(\lambda^*+2\mu^*)\frac{d^2v^*}{dx^2}$$

$$+ (\hat{\beta}_2^* + \hat{\beta}_3^*)\frac{d^2u^*}{dx^2} - \frac{1}{A}\tau_f - \frac{1}{A}\tau_s + \rho_1^* b_x + \rho_2^* b_x + 2\hat{\beta}_4^* \frac{d\nu^*}{dx}\frac{d^2\nu^*}{dx^2} - \frac{1}{A}\tau_n, \tag{96}$$

which can be rewritten as:

$$\frac{d}{dx}\left(\rho_1^* v^{*2}\right) + \frac{d}{dx}\left(\rho_2^* u^{*2}\right) = -\frac{d}{dx}\left(p_f^* + p_s^*\right) + (\lambda^* + 2\mu^*)\frac{d^2v^*}{dx^2}$$

$$+ (\hat{\beta}_2^* + \hat{\beta}_3^*)\frac{d^2u^*}{dx^2} - \frac{1}{A}(\tau_f + \tau_s) + (\rho_1^* + \rho_2^*)b_x + 2\hat{\beta}_4^* \frac{d\nu^*}{dx}\frac{d^2\nu^*}{dx^2} - \frac{1}{A}\tau_n. \tag{97}$$

In the spirit of previous work in this area, the balance of momentum in the y-direction is not included in the averaged equations. Of course, the terms in the y-direction are such that the pressure field adjusts itelf to satisfy the governing equations.

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