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Averaging Aggregation Functions for Preferences Expressed as Pythagorean Membership Grades and Fuzzy Orthopairs

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***Abstract*—Rather than denoting fuzzy membership with a single value, orthopairs such as Atanassov's intuitionistic membership and non-membership pairs allow the incorporation of uncertainty, as well as positive and negative aspects when providing evaluations in fuzzy decision making problems. Such representations, along with interval-valued fuzzy values and the recently introduced Pythagorean membership grades, present particular challenges when it comes to defining orders and constructing aggregation functions that behave consistently when summarizing evaluations over multiple criteria or experts. In this paper we consider the aggregation of pairwise preferences denoted by membership and non-membership pairs. We look at how mappings from the space of Atanassov orthopairs to more general classes of fuzzy orthopairs can be used to help define averaging aggregation functions in these new settings. In particular, we focus on how the notion of 'averaging' should be treated in the case of Yager's Pythagorean membership grades and how to ensure that such functions produce outputs consistent with the case of ordinary fuzzy membership degrees.**

***Index Terms*—group decision making, preferences aggregation, aggregation functions, Pythagorean fuzzy sets, Atanassov intuitionistic fuzzy sets.**

I. Introduction

In fuzzy modeling and decision making, the need to incorporate uncertainty or multiple aspects of objects simultaneously in many real world problems has given rise to various 'higher order' fuzzy sets, i.e. fuzzy sets where the degree of membership itself can be represented with some degree of fuzziness. Examples include general type-2 fuzzy sets, interval-valued fuzzy sets and Atanassov's intuitionistic fuzzy sets. For Atanassov intuitionistic fuzzy sets, membership is denoted by a pair of values (we will refer to them as Atanassov orthopairs) denoted $\langle \mu(x), \nu(x) \rangle$, often interpreted as giving the degree of membership and non-membership. The membership value $\mu(x)$ tells us the extent to which it is known an object x belongs to the fuzzy set, while $\nu(x)$ indicates how sure we are that x does not belong to the fuzzy set. Such representations can be appealing in fuzzy decision making as membership and non-membership can be used to denote positive and negative aspects of a decision criterion or alternative which needn't be considered as complimentary.

We consider the group decision making setting where an expert provides evaluations x_{ij} expressing his/her preference for alternative i over alternative j. The conditions for these preferences can vary from additive preferences where $a_{ij} + a_{ji} = 1$, to multiplicative where $a_{ij} \cdot a_{ji} = 1$. The use of Atanassov membership and non-membership pairs $\langle \mu(x_{ij}), \nu(x_{ij}) \rangle$ has also been proposed in this context [10], [13]. Suppose we have three decision makers, each providing their preferences in this form as follows.

TABLE I
Decision maker preferences for one alternative over another expressed as Atanassov membership and non-membership pairs.

Decision maker	$x_{ij}^{(1)}$	$x_{ij}^{(2)}$	$x_{ij}^{(3)}$
Preference x_{ij}	$\langle 0.3, 0.3 \rangle$	$\langle 0.5, 0.3 \rangle$	$\langle 0.6, 0.4 \rangle$

Whereas the membership values $\mu(x_{ij})$ are interpreted as the extent to which i is preferred to j, the non-membership $\nu(x_{ij})$ is usually interpreted as to the degree to which i is not preferred to j and consistency might then require that $\mu(x_{ji}) = \nu(x_{ij})$ and $\nu(x_{ji}) = \mu(x_{ij})$, e.g. we would have $x_{ji}^{(2)} = \langle 0.3, 0.5 \rangle$. This kind of representation is hence equivalent to the original fuzzy preferences setting with each preference x_{ij} being denoted by a single value. However the pairs can be interpreted more broadly and this type of consistency may not be necessary, with decision makers allowed to provide contradictory or uncertain preference pairs. We note, in particular that with Atanassov membership pairs, we require

$$\mu(x) \leq N(\nu(x)),$$

with N the standard (or Zadeh) fuzzy negation, and hence $\mu(x) \leq 1 - \nu(x)$. In this decision making setting, we can interpret μ as providing the *direct* expression of preference for i over j, while the negation of non-membership $1-\nu$ provides an *indirect* or inferred expression for the preference of i over j, with the direct preference bounded from above by the indirect preference. For example, when comparing alternatives i and j, $\mu(x_{ij})$ may be used to express the positive aspects of i that make it preferable to j, while $\nu(x_{ij})$ refers to its negative aspects. It would not then be necessary that evaluations *against* choosing i based on these negative qualities are equivalent to arguments *in favor* of choosing j due to its own positive aspects.

The use of alternative fuzzy negations allows the boundary condition relating membership and non-membership, $\mu(x) + \nu(x) \leq 1$ to be relaxed and for decision makers to express membership and non-membership pairs that sum to values greater than 1. One example of such fuzzy membership pairs were introduced in [15], [16] and are referred to as Pythagorean membership grades. Pythagorean membership grades are bounded within the first quadrant of the unit circle, i.e. $\mu(x)^2 + \nu(x)^2 \leq 1$ and can be derived from the Yager family of negations. These pairs then effectively expand the space upon which valid membership pairs can be defined. The idea naturally extends to pairs bounded by $\mu(x)^p + \nu(x)^p \leq 1$ with $p \to \infty$ leading to the limiting case of membership and non-membership contained to the unit interval but unrestricted with respect to each other, meaning that the strength of positive and negative information can be considered independently[1].

The understanding of the role of negations in the fuzzy evaluation process is hence crucial if such membership pairs are to be used for decision making, however we can also consider relationships and mappings between the resulting spaces for understanding and interpreting the behavior of operators defined in these settings. As well as for preferences aggregation, in other contexts such as the aggregation of multiple sensor readings, it may not necessarily follow that the bounds of membership and non-membership should be related by the standard negation.

Furthermore, Yager's Pythagorean membership grades have given rise to the consideration of aggregating values over the *strength of commitment* and *direction of commitment* rather than the membership and non-membership pairs. In turn, this representation can be related to the space of complex numbers, which have also been gaining interest in terms of their relationship to Type-2 fuzzy sets [9].

In this paper, we explore some alternative ways to map from the space of Atanassov orthopairs to the space of Pythagorean membership grades and other membership pairs related by fuzzy negations. We then look at ways for defining aggregation functions in these settings with the requirement that the functions generalize the case of ordinary fuzzy sets, and investigate their resulting behavior.

The article will be set out as follows. In the Preliminaries section, we give an overview of Atanassov orthopairs and Pythagorean fuzzy membership grades. We also give the definitions required from the study of aggregation functions. In Section III, we present mappings from the space of Atanassov orthopairs to fuzzy orthopairs obtained through alternative fuzzy negations. In Section IV, we consider the problem of defining averaging aggregation functions in this setting, while in Section V we consider how such functions can be defined so that they generalize ordinary fuzzy sets in a manner consistent with the mappings given in the preceding sections. We make some concluding remarks in Section VI.

[1]Such membership pairs have been studied in [17] under the name of conflicting bifuzzy evaluations.

II. Preliminaries

Here we give an overview of fuzzy sets and extensions relating to our investigations of membership and non-membership pairs. We then consider aggregation functions for these cases.

A. Fuzzy sets and negations

Whereas for the case of crisp sets, elements are either in the set or not, fuzzy sets allow membership to graduate from 0 to 1. For a given fuzzy set $\mathcal{A}$, $\mu_{\mathcal{A}}(x) \in [0, 1]$ indicates the degree to which x belongs to $\mathcal{A}$. Fuzzy sets are hence very useful for modeling concepts where the boundary between belonging or not belonging to a set may be vague or imprecise. In fuzzy decision making, membership is often used to indicate the strength of preference for an alternative with respect to a given criterion or expert. In this setting, not all operations defined for fuzzy sets in general will necessarily be meaningful.

An operation of fundamental importance to us is the fuzzy negation [4].

Definition 1: A fuzzy negation is a decreasing function $N : [0, 1] \to [0, 1]$ such that $N(0) = 1$ and $N(1) = 0$. A negation is called *strict* if N is continuous and strictly decreasing. A negation is *strong* if N is involutive, i.e., $N(N(x)) = x$ for every $x \in [0, 1]$.

In particular, we have the standard negation (sometimes called the Zadeh negation) where $N(x) = 1 - x$.

Negations can be used to build fuzzy implications for use in fuzzy logic, and define fuzzy complements in fuzzy set theory [14]. For a given fuzzy set $\mathcal{A}$, the membership degrees for its fuzzy complement $\mathcal{A}^c$ can be determined by $\mu_{\mathcal{A}^c}(x) = 1 - \mu_{\mathcal{A}}(x)$. The negation of membership hence can be seen to indicate the degree to which x is in the complement of $\mathcal{A}$, not necessarily the degree to which x is not in $\mathcal{A}$. In the context of decision making, negations can be used to transform evaluations pertaining to negative attributes of an alternative, e.g. in determining the desirability of a car according to the extent of its membership to the fuzzy sets *expensive* and *quality*, we could take the negation of *expensive* before aggregating the membership values into an overall score.

Although the standard negation is by far the most commonly used, other families include the Yager negations, determined with respect to a paramenter p,

$$N_p^Y(x) = (1 - x^p)^{\frac{1}{p}},$$

and the Sugeno family of negations,

$$N_p^S(x) = \frac{1-x}{1+px} \ ,$$

where it is required that $p > -1$ for N_p^S to be order reversing.

Both families are involutive and hence provide us with *strong* negations[2]. The parameter choices $p = 1$ for N_p^Y and $p = 0$ for N_p^S return the standard negation.

B. Atanassov fuzzy orthopairs

Atanassov's extension of fuzzy sets [2] (originally referred to as intuitionistic fuzzy sets) allows both the degree of membership $\mu_{\mathcal{A}}(x)$ and the degree of non-membership $\nu_{\mathcal{A}}(x)$ to the set $\mathcal{A}$ to be specified.

For clarity and to simplify our notation, we will focus on the pairs of fuzzy membership and non-membership relating to a single object, which we will denote by

$$a = \langle \mu, \nu \rangle.$$

These values should satisfy

$$\mu \le N(\nu), \tag{1}$$

with N the standard fuzzy negation, leading to the restriction,

$$\mu + \nu \le 1.$$

For an ordinary fuzzy set, non-membership is assumed to be the negation of membership and $\mu = \langle \mu, N(\mu) \rangle$.

We also note that interval-valued fuzzy sets were shown to be mathematically equivalent to AIFS in [1] (and later in [5]–[7]).

C. Pythagorean fuzzy sets

The Pythagorean fuzzy sets proposed in [15], [16] are inspired by the Atanassov fuzzy orthopairs and essentially replace the negation in Eq.(1) with a Yager negation with $p = 2$, i.e. we have,

$$\mu \le (1-\nu^2)^{\frac{1}{2}},$$

which leads to the restriction,

$$\mu^2 + \nu^2 \le 1.$$

They can be seen to extend the space of the Atanassov orthopairs with the boundary corresponding with ordinary fuzzy sets now denoted by the curve $\nu = \sqrt{1-\mu^2}$.

To distinguish between Atanassov orthopairs and Pythagorean membership grades, we will usually employ the notation

$$a_{(2)} = \langle \mu_{(2)}, \nu_{(2)} \rangle_{(2)} \quad \text{where} \quad \mu_{(2)}^2 + \nu_{(2)}^2 \le 1,$$

to denote the latter. Although we are interested in fuzzy preferences where x_{ij} gives the preference for i over j, we will avoid the use of these sub-indices to avoid confusion with powers and multiple inputs. We will instead use the notation $a_{(2)_i}$ to indicate the evaluation of the i-th expert or i-th criterion.

Whereas with Atanassov orthopairs the degree of certainty with which the membership grades are expressed can be found by summing the membership and non-membership components, Yager noted that in the case of Pythagorean membership grades it is convenient to consider their polar co-ordinates, with the r value (the Euclidean distance from $\langle 0, 0 \rangle$ to $\langle \mu_{(2)}, \nu_{(2)} \rangle_{(2)}$ providing us with the *strength of commitment*. The angle θ this ray makes with the membership axis can then be seen as the *direction of commitment*, with $\theta = 0$ corresponding with the direction towards membership, $\theta = \frac{\pi}{2}$ representing the direction of non-membership, and $\theta = \frac{\pi}{4}$ indicating that the degree of membership and non-membership are equal. For such inputs, we will employ the notation,

$$a_{(2)} = \langle r(\mu_{(2)}, \nu_{(2)}), \theta((\mu_{(2)}, \nu_{(2)})) \rangle_{r,\theta}$$

where r is the Euclidean distance from $\langle 0, 0 \rangle$ to $\langle \mu_{(2)}, \nu_{(2)} \rangle_{(2)}$ and θ is the angle made with the membership axis. We hence have

$$r(\mu_{(2)}, \nu_{(2)}) = \sqrt{\mu_{(2)}^2 + \nu_{(2)}^2} \ ,$$

and

$$\theta(\mu_{(2)}, \nu_{(2)}) = \tan^{-1}\left(\frac{\nu_{(2)}}{\mu_{(2)}}\right).$$

For the case of more general fuzzy orthopairs with respect to the Yager negation and power p, we will write,

$$a_{(p)} = \langle \mu_{(p)}, \nu_{(p)} \rangle_{(p)} \quad \text{where} \quad \mu_{(p)}^p + \nu_{(p)}^p \le 1.$$

D. Aggregation functions

When Atanassov orthopairs, Pythagorean membership grades and other fuzzy membership representations are used in decision making to evaluate alternatives, it may be useful to combine them into a single overall evaluation or pair. For this, we need well-defined aggregation functions. Overviews of aggregation functions can be found in [3], [8], [11]. We begin by stating their definition where the inputs are given over the unit interval.

Definition 2: An aggregation function $f : [0,1]^n \to [0,1]$ is a function non-decreasing in each argument and satisfying $f(0, \ldots, 0) = 0$ and $f(1, \ldots, 1) = 1$.

An aggregation function is considered:

- *averaging* where the output is bounded by the minimum and maximum input, i.e. $\min(\mathbf{x}) \le f(\mathbf{x}) \le \max(\mathbf{x})$,
- *conjunctive* where the output is bounded from above by the minimum input, i.e. $f(\mathbf{x}) \le \min(\mathbf{x})$,

[2]They can be obtained from the standard negation using automorphisms [12].

- *disjunctive* where the output is bounded from below by the maximum input, i.e. $f(\mathbf{x}) \geq \max(\mathbf{x})$,
- *mixed* otherwise.

Due to the monotonicity of aggregation functions, averaging behavior is equivalent to idempotency, i.e. $f(t, t, ..., t) = t$.

While conjunctive and disjunctive functions have been used in fuzzy set theory to model the AND and OR operations of traditional logic, it is clear that if degrees of membership denote strength of preference or some kind of evaluation, we may need averaging functions in order to obtain an overall output from individual scores. A particularly expressive family of averaging aggregation functions is the weighted power means,

$$M^q(\mathbf{x}) = \left(\sum_{i=1}^{n} w_i x_i^q \right)^{\frac{1}{q}}, \tag{2}$$

where the weight w_i usually denotes the relative importance of the i-th input and $\sum_{i=1}^{n} w_i = 1$.

When $q = 1$ we obtain the weighted arithmetic mean, $q = 0$ corresponds with the weighted geometric mean, while $q = -\infty$ and $q = \infty$ correspond with the minimum and maximum functions respectively.

A number of aggregation functions have been proposed for Atanassov orthopairs, in most cases using pairs of dual aggregation functions defined with respect to the standard negation.

Definition 3: For an aggregation function f, a corresponding function for Atanassov orthopairs can be given by

$$f_A(\langle \mu_1, \nu_1, \rangle, \ldots, \langle \mu_n, \nu_n, \rangle) = \langle f(\mu_1, \ldots, \mu_n), f_d(\nu_1, \ldots, \nu_n) \rangle, \tag{3}$$

where f_d is the dual aggregation function $f_d(x_1, \ldots, x_n) = N(f(N(x_1), \ldots, N(x_n)))$ with N the standard negation.

Aggregation functions for Pythagorean fuzzy sets can be defined similarly using the negation $N_2^Y(t) = \sqrt{1 - t^2}$.

Inspired by the product operation for complex sets, Yager also presented an aggregation function defined for the polar representation of Pythagorean fuzzy sets. We will use the notation $G_{r,\theta}$.

$$G_{r,\theta}((r_1, \theta_1), \ldots, (r_n, \theta_n)) = \left\langle \prod_{i=1}^{n} r_i^{w_i} \ , \ \sum_{i=1}^{n} w_i \theta_i \right\rangle. \tag{4}$$

We note that whilst the aggregation in this representation of the Pythagorean fuzzy sets is appealing, such a function will not coincide with the case of ordinary fuzzy sets. In the following, we focus on different ways of expressing ordinary fuzzy sets or Atanassov orthopairs in the extended space.

III. Mapping Atanassov membership pairs to alternative representations

If we consider the relationship between Atanassov orthopairs and Pythagorean membership grades, we note that the specification of membership is assumed to be equivalent, i.e. $\mu_{(2)} = \mu$, while non-membership with respect to the negations used can be related according to

$$\nu_{(2)} = \sqrt{1 - (1 - \nu)^2}.$$

As such, we can think of there being a dilation in the non-membership dimension only. The equivalent pair of $\langle 0.5, 0.5 \rangle$ (which could be interpreted as an equilibrium point) would be $\langle 0.5, 0.866 \rangle_{(2)}$ and so interpretations might need to bear this in mind.

However there are other ways to map from the space of Atanassov orthopairs to fuzzy membership pairs defined according to different negations. One alternative would be to keep the non-membership values the same and consider transformations of the membership values, while another is to maintain the ratio of membership to non-membershp. In the latter case we require a multiplier depending on the position of $\langle \mu, \nu \rangle$ in the 2-dimensional plane. Since we need to map points along the boundary $\nu = 1 - \mu$ to $\nu = N(\mu)$, we consider each value as a proportion of the Manhattan distance from the origin (this first projects the point to the boundary since it will always hold that $\frac{\mu}{\mu+\nu} + \frac{\nu}{\mu+\nu} = 1$). The multiplier required is hence k such that

$$k \frac{\mu}{\mu + \nu} = N\left(k \frac{\nu}{\mu + \nu} \right).$$

For the case of Yager negations $N_p^Y(x) = (1 - x^p)^{\frac{1}{p}}$, this leads us to

$$\begin{aligned} k^p \left(\frac{\mu}{\mu + \nu} \right)^p + k^p \left(\frac{\nu}{\mu + \nu} \right)^p &= 1 \\ k^p &= \frac{(\mu + \nu)^p}{\mu^p + \nu^p} \\ k &= \frac{(\mu + \nu)}{(\mu^p + \nu^p)^{\frac{1}{p}}}. \end{aligned}$$

We then have the three cases as follows.

Definition 4: Mappings from $\langle \mu, \nu \rangle \to \langle \mu_{(p)}, \nu_{(p)} \rangle_{(p)}$:

Case 1: $\mu_{(p)} = \mu$, and $\nu_{(p)} = (1 - (1 - \nu)^p)^{\frac{1}{p}}$;

Case 2: $\mu_{(p)} = (1 - (1 - \mu)^p)^{\frac{1}{p}}$ and $\nu_{(p)} = \nu$;

Case 3: $\mu_{(p)} = \frac{\mu(\mu+\nu)}{(\mu^p+\nu^p)^{\frac{1}{p}}}$ and $\nu_{(p)} = \frac{\nu(\mu+\nu)}{(\mu^p+\nu^p)^{\frac{1}{p}}}$.

A visual depiction of these three mappings is shown in Fig. 1. The three cases correspond with dilating the space vertically, horizontally, and from the origin respectively, between the two boundaries. In the figure, $p = 4$ is used and the Atanassov orthopair $\langle 0.3, 0.4 \rangle$ is mapped to

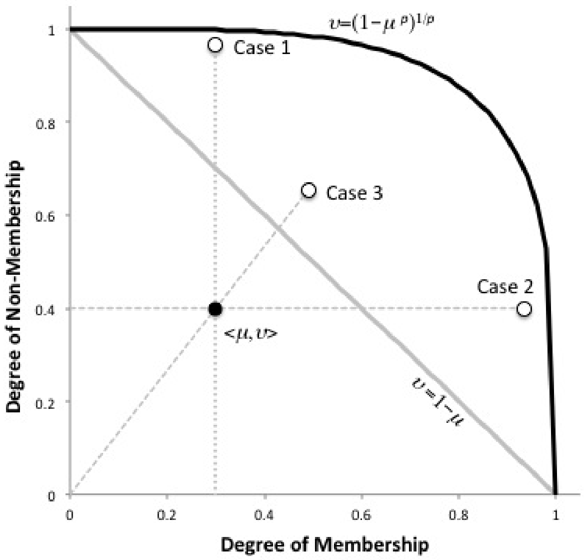

Fig. 1. Cases 1, 2 and 3 (Def. 4) mapping an Atanassov orthopair $\langle \mu, \nu \rangle$ to fuzzy membership pairs according to Yager negations. Each method maps every point in the space bounded by $\mu = 1 - \nu$ to a point in the larger space bounded by $\mu = (1 - \nu^p)^{\frac{1}{p}}$.

Case 1: $\langle 0.3, 0.966 \rangle_{(4)}$, Case 2: $\langle 0.934, 0.4 \rangle_{(4)}$ and Case 3: $\langle 0.490, 0.654 \rangle_{(4)}$.

Interestingly with Case 3, we can interpret the multiplier as ensuring that the Manhattan distance is equal to the p-norm distance in the corresponding orthopair space of order p. This means that for Pythagorean membership grades in polar representation, r will simply be equivalent to the addition of the components μ and ν, i.e. we have $p = 2$ and

$$\begin{aligned} r &= \sqrt{\mu_{(2)}^2 + \nu_{(2)}^2} \\ &= \sqrt{\left(\frac{\mu(\mu+\nu)}{(\mu^2+\nu^2)^{\frac{1}{2}}}\right)^2 + \left(\frac{\nu(\mu+\nu)}{(\mu^2+\nu^2)^{\frac{1}{2}}}\right)^2} \\ &= \sqrt{\frac{\mu^2(\mu+\nu)^2}{(\mu^2+\nu^2)} + \frac{\nu^2(\mu+\nu)^2}{(\mu^2+\nu^2)}} \\ &= \sqrt{\frac{(\mu^2+\nu^2)(\mu+\nu)^2}{(\mu^2+\nu^2)}} \\ &= \mu + \nu. \end{aligned}$$

As we will see later on, this can help us when defining aggregation functions to generalize the case of ordinary fuzzy sets.

For the Sugeno family of negations, using the negation N_p^S requires that $\mu_{(p)} \leq \frac{1-\nu_{(p)}}{1+p\nu_{(p)}}$ and hence membership and non-membership are restricted such that,

$$\mu_{(p)} + \nu_{(p)} + p\mu_{(p)}\nu_{(p)} \leq 1.$$

For Cases 1 and 2, we can use this negation in the same way as we did in Def. 4, i.e. $\nu_{(p)} = N_p^S(1-\nu)$ for Case 1. Case 3 on the other hand requires finding a k such that,

$$k\left(\frac{\mu}{\mu+\nu} + \frac{\nu}{\mu+\nu}\right) + k^2 p \frac{\mu}{\mu+\nu}\frac{\nu}{\mu+\nu} = 1.$$

We hence solve for

$$k^2 \frac{p\mu\nu}{(\mu+\nu)^2} + k - 1 = 0,$$

which has solutions,

$$k = \frac{-1 \pm \sqrt{1 + \frac{4p\mu\nu}{(\mu+\nu)^2}}}{\frac{2p\mu\nu}{(\mu+\nu)^2}}.$$

Provided $\mu, \nu > 0$ and since $p > -1$ and $\mu\nu < (\mu+\nu)^2$, k can always be determined. In fact, we can contain ourselves to the solution obtained by adding the square root since subtracting results in a k that is too large.

IV. Defining averaging aggregation functions for Pythagorean fuzzy sets and alternative membership pairs

Here we consider the problem of defining averaging aggregation functions for fuzzy membership pairs in light of the mappings presented in the previous section.

We firstly note that Atanassov orthopairs can be considered to be defined over a lattice with the partial order $\leq$ defined such that for $a_i = \langle \mu_i, \nu_i \rangle$,

$$a_1 \leq a_2 \iff \mu_1 \leq \mu_2 \text{ and } \nu_1 \geq \nu_2.$$

The partial order on the lattice is important for considering monotonicity and averaging behavior. An aggregation function f_A for Atanassov orthopairs can hence be said to be averaging if it is bounded by the infimum and supremum, i.e.,

$$\begin{aligned} \langle \min(\mu_1, \ldots, \mu_n), \max(\nu_1, \ldots, \nu_n) \rangle &\leq f_A(a_1, \ldots, a_n) \\ &\leq \langle \max(\mu_1, \ldots, \mu_n), \min(\nu_1, \ldots, \nu_n) \rangle. \end{aligned}$$

Fig. 2 helps illustrate the averaging bounds with a graphical interpretation. The averaging rectangle denotes the boundary for which outputs are comparable with the supremum and infimum of the inputs. Also shown in the diagram are outputs for the power mean of the two inputs, shown to graduate along a curve from the infimum to the supremum.

There have also been attempts to define a total order on the set of Atanassov orthopairs. A score function $s(\mu, \nu) = \mu - \nu$ and accuracy $z(\mu, \nu) = \mu + \nu$ can be used where a_1 is ordered lower than a_2 if and only if either,

1) $s(\mu_1, \nu_1) < s(\mu_2, \nu_2)$; or,
2) $s(\mu_1, \nu_1) = s(\mu_2, \nu_2)$ and $z(\mu_1, \nu_1) < z(\mu_2, \nu_2)$.

A number of averaging aggregation functions, however will not be monotone with respect to this total order. We provide the following example.

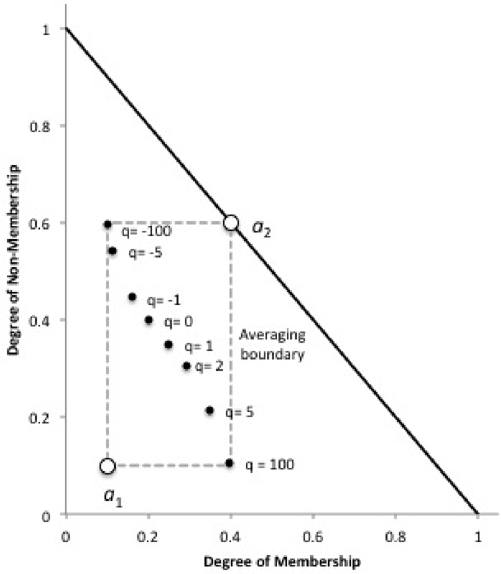

Fig. 2. Aggregation of two orthopairs a_1, a_2 using a power mean to power q for membership and the dual function for non-membership. The function output graduates between the minimum and maximum and stays bound within the averaging window.

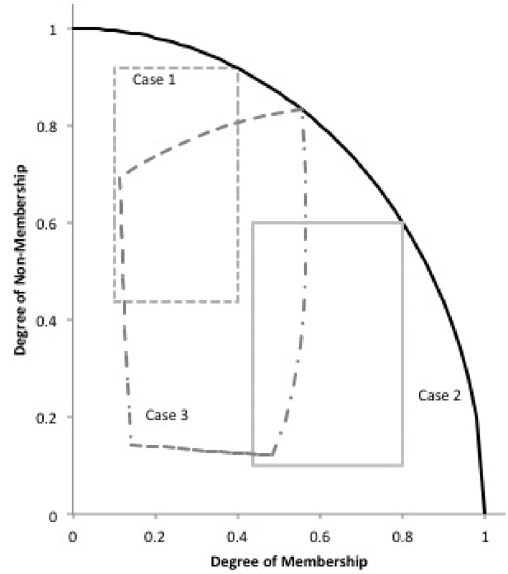

Fig. 3. Equivalent averaging windows for data values in Fig. 2 for Pythagorean orthopairs depending on each case. Note that the non-linear borders for Case 3 would make it difficult to define an equivalent partial order.

Example 1: Consider the evaluations provided previously in Table I. The score and accuracy values for each of the decision makers' preferences are

$$s(x_{ij}^{(1)}) = 0 \quad s(x_{ij}^{(2)}) = 0.2 \quad s(x_{ij}^{(3)}) = 0.2$$
$$z(x_{ij}^{(1)}) = 0.4 \quad z(x_{ij}^{(2)}) = 0.8 \quad z(x_{ij}^{(3)}) = 1.$$

and we have $x_{ij}^{(1)} < x_{ij}^{(2)} < x_{ij}^{(3)}$. If we take a geometric mean (according to Def.3) of decision makers 1 and 2, we have $G(x_{ij}^{(1)}, x_{ij}^{(2)}) = \langle 0.387, 0.3 \rangle$ which has a score 0.087. However if we aggregate the preferences of decision makers 1 and 3, we have $G(x_{ij}^{(1)}, x_{ij}^{(3)}) = \langle 0.424, 0.352 \rangle$ which has a score of 0.072. So even though $x_{ij}^{(3)} > x_{ij}^{(2)}$, we have $G(x_{ij}^{(1)}, x_{ij}^{(3)}) < G(x_{ij}^{(1)}, x_{ij}^{(2)})$ and hence the geometric mean would not be considered monotone with respect to this ordering.

An explanation for this is that the geometric mean increases more with respect to increases in lower inputs than higher inputs. Alternative score functions that reflect the behavior of the geometric mean could overcome this problem.

We can now turn to how averaging behavior is carried through when inputs are mapped from the space of Atanassov orthopairs to Pythagorean membership grades. Fig. 3 illustrates this for the three different cases. We note that for Cases 1 and 2, since the dilation is only in one dimension, the averaging bounds remain rectangular when represented in this way. These would then be consistent with a partial order defined according to,

$$a_{(2)_1} \leq a_{(2)_2} \iff \mu_{(2)_1} \leq \mu_{(2)_2} \text{ and } \nu_{(2)_1} \geq \nu_{(2)_2}.$$

For Case 3, however, we see that the corresponding averaging region is not rectangular. This means that if corresponding inputs were used, averaging behavior for Atanassov orthopairs inputs may not correspond with averaging behavior for Pythagorean membership grades when these partial orderings are used. The following example helps illustrate this.

Example 2: We define an aggregation function for Atanassov orthopairs as follows. Let $\max(\mu_1, \mu_2) + \max(\nu_1 + \nu_2) = \omega$,

$$f_A(\langle \mu_1, \nu_1 \rangle, \ldots, \langle \mu_n, \nu_n \rangle)$$
$$= \begin{cases} (\langle \max(\mu_1, \ldots, \mu_n), 1 - \max(\mu_1, \ldots, \mu_n) \rangle, & \omega \geq 1, \\ \langle \max(\mu_1, \ldots, \mu_n), \max(\nu_1, \ldots, \nu_n) \rangle, & \omega < 1. \end{cases}$$

For the inputs $a_1 = \langle 0.1, 0.6 \rangle$ and $a_2 = \langle 0.4, 0.1 \rangle$, we have $f_A(a_1, a_2) = \langle 0.4, 0.6 \rangle$. Mapping these to the Pythagorean fuzzy sets space gives $a_{(2)_1} = \langle 0.115, 0.690 \rangle_{(2)}$, and $a_{(2)_2} = \langle 0.485, 0.121 \rangle_{(2)}$ and $f_A(a_{(2)_1}, a_{(2)_2}) = \langle 0.555, 0.832 \rangle_{(2)}$. We hence see that the output membership value is not bounded from above by $\mu_{(2)_2} = 0.485$, nor is the non-membership bounded from above by $\nu_{(2)_1} = 0.690$.

We note that this does not violate the partial order as such, rather the output is *incomparable* with the infimum and supremum. In practice it is worth considering whether certain functions will remain averaging regardless of the negation by which membership and non-membership are related.

The polar representation of the Pythagorean membership grades space also prompts us to consider alternative orders according to the value of r and θ. In a decision-making setting, r can be viewed analogously to the accuracy function. Indeed, while the accuracy function is equivalent to the Manhattan distance from $\langle 0, 0\rangle$, r is equal to the Euclidean distance. These values represent how close the membership values are to the boundary of the space coinciding with the case of ordinary fuzzy sets. When $r = 1$, the membership and non-membership are known completely, while $r = 0$ is usually interpreted as complete ignorance or nothing being known about the degree of membership or non-membership. This value is referred to as the *strength of commitment* in [15]. On the other hand, θ can be interpreted similarly to the score function and is related to the *direction of commitment* value in [15][3]. If r is fixed, as θ decreases the values increase in their membership and simultaneously decrease in their non-membership. A resulting partial ordering can hence be considered,

$$a_{(2)_1} \le a_{(2)_2} \iff r_1 \le r_2 \text{ and } \theta_1 \ge \theta_2.$$

Of course, this partial order emphasizes different aspects of the decision maker evaluations than focusing on the membership and non-membership pairs. We could therefore consider averaging aggregation functions defined using functions and their duals for r and θ respectively, however, while in the case of Atanassov fuzzy orthopairs, the dual relationship for aggregating membership and non-membership ensures that the output is a permissible orthopair, that need not be the case here. We merely require r and θ to be bounded by their respective arguments from all the inputs. Fig. 4 gives an example of two inputs with their averaging boundary according to this partial order. The outputs obtained for various power means are also shown, calculated according to,

$$f_q(a_{(2)_1}, a_{(2)_2}) = \langle M^q(r_1, r_2), M_d^q(\theta_1, \theta_2)\rangle_{r,\theta}.$$

We note that the averaging boundary is quite different to any of those produced in Fig. 3.

There has also been a total order for the strength/direction of commitment representation proposed in [16]. A score $s_r(r, \theta)$ is calculated according to

$$s_r(r, \theta) = \frac{1}{2} + r\left(\frac{1}{2} - \frac{2\theta}{\pi}\right).$$

As with the total order based on score and accuracy for Atanassov orthopairs, some of the aggregation functions defined for the Pythagorean membership grades may not be monotone with respect to this ordering. We can show this with the following example.

Example 3: If we have the inputs $a_{(2)_1} = \langle 0.35, 0.644\rangle_{r,\theta}$, and $a_{(2)_2} = \langle 1, 1.460\rangle_{r,\theta}$ with scores of $s_r(a_{(2)_1}) = 0.532$ and $s_r(a_{(2)_2}) = 0.070$, the aggregation using $G_{r,\theta}$ gives $\langle 0.592, 1.052\rangle_{r,\theta}$ which has $s_r = 0.399$. However when we double the value of r in $a_{(2)_1}$ to give $a'_{(2)_1} = \langle 0.7, 0.644\rangle_{r,\theta}$, its corresponding score will increase to 0.563 but the aggregated value $G_{r,\theta}(a'_{(2)_1}, a_{(2)_2}) = \langle 0.837, 1.051\rangle_{r,\theta}$ has the lower score $s_r = 0.358$. Therefore according to this ordering, the operator $G_{r,\theta}$ would not be considered monotone.

[3]The direction of commitment was a value between 0 and 1 calculated from θ using $d = 1 - \frac{2\theta}{\pi}$.

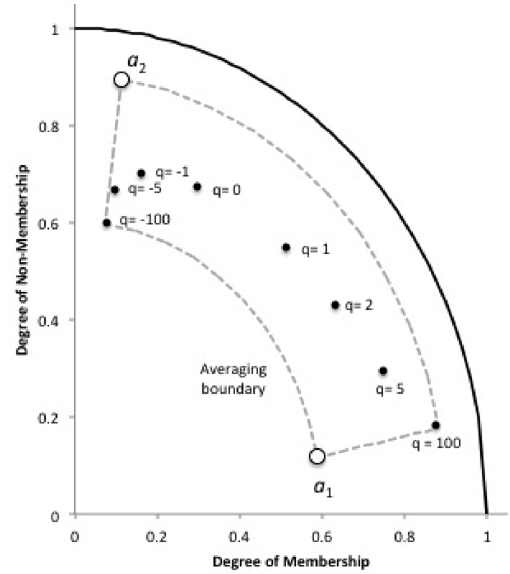

Fig. 4. Averaging window for Pythagorean pairs expressed in polar coordinates. The r values are aggregated by a power mean with power q and the θ values are averaged with the dual function. A larger r represents a degree of certainty and proximity to the boundary, while smaller θ increases the tendency toward degree of membership.

The role of partial and total orders can be important for ensuring the functions provide results that are consistent with our intuitions about changes in the input. Another key thing to consider when defining functions on these extended spaces is whether they will produce the results we expect when the inputs are equivalent to ordinary fuzzy set membership degrees.

V. Aggregation Functions that Generalize Functions on Ordinary Fuzzy Sets

For Cases 1 and 2, defining aggregation functions such that they generalize Atanassov orthopairs and, in turn, ordinary fuzzy sets can be achieved using the dual aggregation functions in a straightforward way, i.e. as specified in Def. 3. The transformations we have proposed for Case 3 are somewhat more complicated. The value of the multiplier k changes depending on the location of the membership pair. This means that to aggregate in the Pythagorean membership grade space in a way that is consistent with aggregating Atanassov orthopairs would effectively require first transforming inputs from Pythagorean pairs to Atanassov orthopairs, aggregating, and then finally transforming back to Pythagorean membership grades.

It then clearly becomes questionable as to whether we gain any advantage by working with Pythagorean membership grades rather than pre-processing the fuzzy preference evaluations (that is, if indeed we want to have coincidence with the case of ordinary membership values).

On the other hand, the polar representation and relationship between the different types of distances that can be used allows us to consider developing new functions in this framework which would then be equivalent in all membership pair spaces defined according to the Yager negations.

We consider $r_{(p)}$ to represent the p-norm distance in the associated space from the origin $\langle 0,0\rangle$ to the pair, i.e. $r_{(p)} = (\mu_{(p)}^p + \nu_{(p)}^p)^{\frac{1}{p}}$. As in [16], we let $d_{(p)} = 1 - \frac{2\theta}{\pi}$, which indicates how close the ray traced to the pair is to the membership axis. Aggregation functions defined according to the following definition will be equivalent in all spaces.

Definition 5: Given two aggregation functions $f_1 : [0,1]^n \to [0,1]$ and $f_2 : [0,1] \to [0,1]$, the following will be an aggregation function over the $\langle r_{(p)}, d_{(p)}\rangle_{r,d}$ pairs. $f_{r,d}(a_{(p)_1},\ldots,a_{(p)_n}) =$

$$\left\langle f_1(r_{(p)_1},\ldots,r_{(p)_n}),\ f_2(d_{(p)_1},\ldots,d_{(p)_n})\right\rangle_{r,d}\ .$$

If we can define a function for Atanassov orthopairs that holds in this representation, we hence automatically establish the equivalent functions for all pairs related by the Yager negations.

For example, we can consider the case of the weighted arithmetic mean. The weighted arithmetic mean is self dual by the standard negation and hence the output value of r, which in this space is the sum of the membership and non-membership components, can be calculated as the corresponding weighted arithmetic mean of the input $r_{(p)}$ values.

To calculate the angle made with the membership axis, we note that θ is calculated from the inverse tangent of the ratio of non-membership to membership. The weighted arithmetic mean in this polar coordinates setting will therefore be given by,

$$WAM_{r,\theta}(\langle r_{(p)_1}, d_{(p)_1}\rangle,\ldots,\langle r_{(p)_n}, d_{(p)_n}\rangle) =$$

$$\left\langle \sum_{i=1}^{n} w_i(r_{(p)_i})\ ,\ 1 - \frac{2}{\pi}\tan^{-1}\left(\frac{\sum_{i=1}^{n} w_i(r_{(p)_i})\sin\theta_i}{\sum_{i=1}^{n} w_i(r_{(p)_i})\cos\theta_i}\right)\right\rangle, \quad (5)$$

where $\theta = \frac{\pi}{2}(1 - d_{(p)})$.

This does not mean we have to restrict ourselves to operators which are equivalent in all spaces, however operators such as $WAM_{r,\theta}$ can be used to help us understand the behavior of operators defined solely in the new setting.

VI. Conclusion

We have considered mappings from the space of Atanassov orthopairs to other membership and non-membership pairs restricted according the relationship $\mu \leq N(\nu)$ where N is a negation. Such representations can be useful in fuzzy decision making for expressing preferences, however need not always be related by the standard negation.

We have focused on partial and total orders defined over these alternative spaces and whether these are consistent with orders for Atanassov orthopairs. These results have implications when it comes to defining averaging aggregation functions and ensuring monotone behavior that matches our intuitions that extend from ordinary fuzzy membership values. The orders in the extended spaces could also potentially be used to induce orderings for standard Atanassov orthopairs.

Finally we looked at aggregation functions which are equivalent across families of membership pairs defined by Yager negations. We saw that the polar representation with respect to the p-norm distance allows us a simple way to ensure that aggregation functions behave consistently regardless of the p chosen.

Having a better understanding of how to construct and interpret operators for these fuzzy membership pairs is appealing for fuzzy decision making as the restriction according to the standard negation may not always be meaningful for the problem at hand.

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