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Averaging and integral manifolds

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An integral manifold for a system of differential equations is a manifold such that any solution of the equations which has a point on it is entirely contained on it. The method of averaging establishes the existence of such a manifold for a system which is a perturbation of an autonomous system with a periodic orbit. The existence of the manifold is established here under more general hypotheses, namely for perturbations which are 'integrally small'. The method differs from the original method of Bogolyubov and Mitropolskii and operates directly with the individual solutions. This is made possibly by the use of an appropriate norm, and is equivalent to solving the partial differential equation which occurs in work by Moser and Sacker by the method of characteristics rather than by the introduction of an artificial viscosity term. Moreover, detailed smoothness properties of the manifold are obtained. For periodic perturbations the integral manifold is a torus and these smoothness properties are just sufficient to permit the application of Denjoy's theorem.

1. Introduction

We consider a system of differential equations

$$(1)\qquad \begin{aligned} x' &= f(t, x, y) \\ y' &= A(t)y + g(t, x, y) , \end{aligned}$$

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where $x \in R^m$, $y \in R^n$, $' = d/dt$, and where the linear equation

(2) $$y' = A(t)y$$

has an exponential dichotomy on $(-\infty, \infty)$, so that for any bounded, continuous vector function $f(t)$ the inhomogeneous equation

$$y' = A(t)y + f(t)$$

has a unique solution bounded on the whole real line. For example, $A(t)$ could be a constant matrix all of whose eigenvalues have real part different from zero. Given a real number τ and a vector $\xi \in R^m$, we look for a solution $x(t) = x(t, \xi, \tau)$, $y(t) = y(t, \xi, \tau)$ of (1) such that $x(\tau) = \xi$ and $y(t)$ is bounded. The set of all points $y(\tau, \xi, \tau)$, for varying τ and ξ, is an *integral manifold* of (1), i.e. any solution which has a point on this manifold, is entirely contained in it.

If the system is uncoupled, i.e. if f is independent of y and g is independent of x, then we just have an initial value problem for $x(t)$ and, if g is only weakly nonlinear in y, a boundedness perturbation problem for $y(t)$. Both these problems are of standard types; for the second problem cf. [3], Theorem 5. Our interest here is in coupled systems. Such systems arise in the study of perturbations of an autonomous system with a periodic orbit. In this case x is a scalar $(m = 1)$ and the functions f, g have period 2π in x. In the present paper the problem is studied in its own right.

The functions f and g will be required to be weakly nonlinear. The simplest interpretation of this requirement is that they satisfy Lipschitz conditions in x, y with *small* Lipschitz constants. However, our interpretation may be roughly stated as 'boundedness + integral smallness'. The exact hypotheses on f and g are given at the beginning of Section 3. These hypotheses contain those under which the method of averaging is applied to establish the existence of integral manifolds. However, the method given here is more direct and more general than the original method of Bogolyubov and Mitropolskii [1]. Averaging as such plays no part; what is at issue is a perturbation problem with integrally small perturbations. In contrast to previous treatments of this problem we operate with the individual solutions, rather than with the integral

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manifold. Although we do not consider the case where f and g satisfy Lipschitz conditions with small Lipschitz constants our methods apply also in this case, with substantial simplifications (no integrations by parts!).

Throughout the paper we make systematic use of the norm

$$\|f\| = \sup_{-\infty<t<\infty} \left\{e^{-\beta|t-\tau|} \ |f(t)|\right\}$$

for continuous vector functions $f(t)$. Here τ is a fixed real number and $\beta \geqq 0$. This norm was first used by the second author to prove that for the integral manifold $y = v(t, x)$ whose existence had been established the partial derivative v_x exists and satisfies a Lipschitz condition in x . We have

$$|f(t)| \leqq \|f\|e^{\beta|t-\tau|} \quad \text{for} \quad -\infty < t < \infty .$$

It follows that if

$$g(t) = \int_{\tau}^{t} f(s)ds$$

and if $\beta > 0$, then

$$\|g\| \leqq \beta^{-1}\|f\| .$$

To avoid interrupting the argument some lemmas are collected together in Section 2. The Main Theorem is stated in Section 3, and proved in Sections 3 and 4. The properties of the corresponding integral manifold are summarized at the end of Section 4.

2. Auxiliary results

LEMMA 1. *The function* $H(\beta) = \left(1-e^{-\beta}\right)^{-1}$ *satisfies the following inequalities for* $\beta > 0$ *:*

$$\beta^{-1} < H(\beta) < 1 + \beta^{-1} .$$

Since $H(\beta) > 1$ *we have in addition*

$$\tfrac{1}{2}\left(1+\beta^{-1}\right) < H(\beta) .$$

LEMMA 2. *Let* $A(t)$ *be a continuous matrix function such that*

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$|A(t)| \leqq N$ *for* $-\infty < t < \infty$ *and suppose the linear equation*

$$y' = A(t)y \tag{2}$$

has a fundamental matrix $\tilde{Y}(t)$ *such that*

$$\begin{aligned} |\tilde{Y}(t)P\tilde{Y}^{-1}(s)| &\leqq Ke^{-4\alpha(t-s)} \quad \text{for } t \geqq s \,, \\ |\tilde{Y}(t)(I-P)\tilde{Y}^{-1}(s)| &\leqq Ke^{-4\alpha(s-t)} \quad \text{for } s \geqq t \,, \end{aligned} \tag{3}$$

where P *is a projection matrix and* K, α *are positive constants. Then there exists a positive constant* $\delta = \delta(N, K, \alpha)$ *such that if* $B(t)$ *is a continuous matrix function satisfying*

$$|B(t)-A(t)| \leqq N \quad \text{for } -\infty < t < \infty \,,$$

$$\left|\int_{t_1}^{t_2} \{B(t)-A(t)\}dt\right| \leqq \delta \quad \text{for } |t_2 - t_1| \leqq 1 \,,$$

then the linear equation

$$y' = B(t)y \tag{4}$$

has a fundamental matrix $Y(t)$ *such that*

$$\begin{aligned} |Y(t)PY^{-1}(s)| &\leqq Me^{-2\alpha(t-s)} \quad \text{for } t \geqq s \,, \\ |Y(t)(I-P)Y^{-1}(s)| &\leqq Me^{-2\alpha(s-t)} \quad \text{for } s \geqq t \,, \end{aligned} \tag{5}$$

where $M = M(K) > 1$.

This is a special case of Theorem 2 of [3].

LEMMA 3. *Let* $h(t, x)$ *be a continuous function defined for all* $t \in R^1$ *and all* $x \in R^m$. *Moreover, suppose there exist positive constants* L, q *such that*

$$|h(t, x_1)-h(t, x_2)| \leqq L|x_1 - x_2| \,,$$

$$\left|\int_{t_1}^{t_2} h(t, x)dt\right| \leqq q \quad \text{for all } x \text{ if } |t_2 - t_1| \leqq 1 \,.$$

If $x(t)$ *is any function such that for all* t_1, t_2

$$|x(t_2)-x(t_1)| \leqq N|t_2 - t_1|$$

then

$$\left|\int_{t_1}^{t_2} h[t,\ x(t)]dt\right| \leq p \quad for \quad |t_2 - t_1| \leq 1 ,$$

provided

$$q \leq \tfrac{1}{2}p^2(p+LN)^{-1} .$$

Put

$$s_j = t_1 + j(t_2-t_1)/n \qquad (j = 0,\ \ldots,\ n) .$$

Then for $|t_2 - t_1| \leq 1$

$$\left|\int_{t_1}^{t_2} h[t,\ x(t)]dt\right| \leq \sum_{j=1}^{n} \left|\int_{s_{j-1}}^{s_j} h[t,\ x(s_j)]dt\right| + LN \sum_{j=1}^{n} \int_{s_{j-1}}^{s_j} (s_j-t)dt$$

$$\leq nq + \tfrac{1}{2}LN/n .$$

Choose the positive integer n so that

$$LNp^{-1} \leq n < LNp^{-1} + 1 .$$

Then if $q \leq \frac{1}{2}p^2(p+LN)^{-1}$ we have

$$\left|\int_{t_1}^{t_2} h[t,\ x(t)]dt\right| \leq \tfrac{1}{2}p + \tfrac{1}{2}p = p .$$

LEMMA 4. *Let* $B(t)$ *be a continuous matrix function such that the linear equation*

(4) $$y' = B(t)y$$

has a fundamental matrix $Y(t)$ *satisfying* (5). *If* $0 \leq \beta < 2\alpha$, *and if* $f(t)$ *is a continuous vector function such that* $\|f\| < \infty$, *then the inhomogeneous equation*

(6) $$y' = B(t)y + f(t)$$

has a unique solution $y(t)$ *such that* $\|y\| < \infty$. *Moreover*

$$\|y\| \leq 2(2\alpha-\beta)^{-1}M\|f\| .$$

The uniqueness of the solution is immediate, since the homogeneous equation (4) has no nontrivial solution $y(t)$ such that

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$|y(t)| = O\left(e^{\beta|t|}\right)$ for $t \to \pm\infty$. It is not difficult to see that

$$y(t) = \int_{-\infty}^{t} Y(t)PY^{-1}(s)f(s)ds - \int_{t}^{\infty} Y(t)(I-P)Y^{-1}(s)f(s)ds$$

is a solution of (6). We consider only the estimation of $|y(t)|$ and in fact we restrict attention to the case $t > \tau$. Writing

$$\int_{-\infty}^{t} = \int_{-\infty}^{\tau} + \int_{\tau}^{t}$$

we obtain

$$|y(t)| \leq M\|f\|\left[\int_{-\infty}^{\tau} e^{-2\alpha(t-s)}e^{\beta(\tau-s)}ds + \int_{\tau}^{t} e^{-2\alpha(t-s)}e^{\beta(s-\tau)}ds\right.$$
$$\left. + \int_{t}^{\infty} e^{-2\alpha(s-t)}e^{\beta(s-\tau)}ds\right]$$
$$= M\|f\|\left[(2\alpha-\beta)^{-1}e^{-2\alpha(t-\tau)} + (2\alpha+\beta)^{-1}e^{\beta(t-\tau)}\right.$$
$$\left. - (2\alpha+\beta)^{-1}e^{-2\alpha(t-\tau)} + (2\alpha-\beta)^{-1}e^{\beta(t-\tau)}\right]$$
$$= \left(4\alpha^2-\beta^2\right)^{-1}M\|f\|\left[2\beta e^{-2\alpha(t-\tau)} + 4\alpha e^{\beta(t-\tau)}\right]$$
$$\leq 2(2\alpha-\beta)^{-1}M\|f\|e^{\beta(t-\tau)} .$$

LEMMA 5. *Let* $B(t)$ *be a continuous matrix function such that* $|B(t)| \leq 2N$ *for* $-\infty < t < \infty$, *where* $N \geq 1$, *and suppose the linear equation* (4) *has a fundamental matrix* $Y(t)$ *satisfying* (5). *If* $0 \leq \beta \leq \alpha$, *and if* $f(t)$ *is a continuous vector function such that*

$$\left|\int_{t}^{t+h} f(s)ds\right| \leq re^{\beta|t-\tau|}$$

for $|h| \leq 1$ *and either* $t \geq \tau$, $h > 0$ *or* $\tau \geq t$, $h < 0$, *then the inhomogeneous equation* (6) *has a unique solution* $y(t)$ *such that* $\|y\| < \infty$. *Moreover*

$$\|y\| \leq 8NM\gamma r ,$$

where $\gamma = H(\alpha) = \left(1-e^{-\alpha}\right)^{-1}$.

As in the proof of the previous lemma we consider only the estimation of

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$$y(t) = \int_{-\infty}^{t} Y(t)PY^{-1}(s)f(s)ds - \int_{t}^{\infty} Y(t)(I-P)Y^{-1}(s)f(s)ds$$

for $t > \tau$. We write

$$\int_t^\infty = \sum_{n=0}^{\infty} \int_{t+n}^{t+n+1} .$$

Integrating by parts we get

$$\left|\int_{t+n}^{t+n+1}\right| \leq rMe^{\beta(t-\tau)}e^{-(2\alpha-\beta)n}\left[e^{-2\alpha}+N\alpha^{-1}\left(1-e^{-2\alpha}\right)\right] ,$$

and hence

$$\begin{aligned}\left|\int_t^\infty\right| &\leq rMe^{\beta(t-\tau)}\left[1-e^{-(2\alpha-\beta)}\right]^{-1}\left[e^{-2\alpha}+N\alpha^{-1}\left(1-e^{-2\alpha}\right)\right] \\ &\leq rMe^{\beta(t-\tau)}\gamma\left[1+N\alpha^{-1}\left(1-e^{-2\alpha}\right)\right] \\ &\leq 3NM\gamma re^{\beta(t-\tau)} ,\end{aligned}$$

since $N \geq 1$ and $\alpha^{-1} < \gamma$. Similarly we write

$$\int_{-\infty}^{\tau} = \sum_{n=0}^{\infty} \int_{\tau-n-1}^{\tau-n} .$$

Integrating by parts we get

$$\left|\int_{\tau-n-1}^{\tau-n}\right| \leq rMe^{-2\alpha(t-\tau)}e^{-(2\alpha-\beta)n}\left[e^{-2\alpha}+N\alpha^{-1}\left(1-e^{-2\alpha}\right)\right] ,$$

and hence

$$\left|\int_{-\infty}^{\tau}\right| \leq 3NM\gamma re^{-2\alpha(t-\tau)} .$$

Finally we write

$$\int_\tau^t = \sum_{j=1}^{m} \int_{\tau+j-1}^{\tau+j} + \int_{\tau+m}^{t} ,$$

where m is the unique non-negative integer such that $\tau + m \leq t < \tau + m + 1$, and obtain in the same way

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$$\left|\int_\tau^t\right| \leq rMe^{-2\alpha(t-\tau)}\left[\sum_{j=0}^{m} e^{(2\alpha+\beta)j} + 2N\int_\tau^t e^{(2\alpha+\beta)(s-\tau)}ds\right]$$
$$\leq rMe^{\beta(t-\tau)}\left[e^{2\alpha+\beta}\left(e^{2\alpha+\beta}-1\right)^{-1} + 2N(2\alpha+\beta)^{-1}\right]$$
$$\leq rMe^{\beta(t-\tau)}\left[\left(1-e^{-2\alpha}\right)^{-1} + N\alpha^{-1}\right]$$
$$\leq 2NM\gamma re^{\beta(t-\tau)} .$$

Combining these three estimates, we obtain the lemma.

LEMMA 6. *If* $\beta > 0$, *if* $u(t)$ *is a continuously differentiable vector function such that* $\|u\| + \|u'\| < \infty$, *and if* $G(t)$ *is a continuous matrix function such that*

$$\left|\int_{t_1}^{t_2} G(s)ds\right| \leq p \quad \text{for} \quad |t_2 - t_1| \leq 1 ,$$

then the function

$$v(t) = \int_\tau^t G(s)u(s)ds$$

satisfies

$$\|v\| \leq pH[\|u\| + \|u'\|] ,$$

where $H = H(\beta) = \left(1-e^{-\beta}\right)^{-1}$.

We consider only the case $t > \tau$. Let m be the unique non-negative integer such that $\tau + m \leq t < \tau + m + 1$. Writing

$$\int_\tau^t = \sum_{j=1}^{m} \int_{\tau+j-1}^{\tau+j} + \int_{\tau+m}^{t}$$

and integrating by parts in each subinterval we get

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$$
\begin{aligned}
|v(t)| &= \left|\int_\tau^t G(s)u(s)ds\right| \\
&\leqq p\sum_{j=0}^{m} |u(\tau+j)| + p\int_\tau^t |u'(s)|ds \\
&\leqq p\|u\| \sum_{j=0}^{m} e^{\beta j} + p\beta^{-1}\|u'\|e^{\beta(t-\tau)} \\
&\leqq p\|u\|e^{\beta(t-\tau)}e^{\beta}(e^{\beta}-1)^{-1} + p\beta^{-1}\|u'\|e^{\beta(t-\tau)} \\
&\leqq pH[\|u\| + \|u'\|]e^{\beta(t-\tau)} .
\end{aligned}
$$

By integrating by parts we immediately obtain also

LEMMA 7. *If* $\beta > 0$, *if* $u(t)$ *is a continuously differentiable vector function such that* $\|u\| + \|u'\| < \infty$, *and if* $G(t)$ *is a continuous matrix function such that*

$$\left|\int_{t_1}^{t_2} G(s)ds\right| \leqq p \quad \textit{for} \quad |t_2 - t_1| \leqq 1 ,$$

then

$$\left|\int_t^{t+h} G(s)u(s)ds\right| \leqq p\beta^{-1}(e^{\beta}-1)[\|u\| + \|u'\|]e^{\beta|t-\tau|}$$

for $|h| \leqq 1$ *and either* $t \geqq \tau$, $h > 0$ *or* $\tau \geqq t$, $h < 0$.

3. The main theorem

Suppose $f(t, x, y)$ and $g(t, x, y)$ are bounded, continuous vector functions defined for $-\infty < t < \infty$, $|x| < \infty$, $|y| < \rho$, with bounded, continuous partial derivatives with respect to x and y which satisfy Lipschitz conditions. More precisely, we assume

$$|f(t, x, y)| \leqq N , \quad |g(t, x, y)| \leqq N$$
$$|f_x(t, x, y)| \leqq N ,$$
$$|f_x(t, x_1, y_1) - f_x(t, x_2, y_2)| \leqq L[|x_1 - x_2| + |y_1 - y_2|] ,$$

where $N \geqq 1$, and the same inequalities with f_x replaced by f_y, g_x, g_y . Furthermore we assume

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$$\left|\int_{t_1}^{t_2} g(t,\, x,\, 0)dt\right| \leq q \quad \text{for all} \quad x \quad \text{if} \quad |t_2 - t_1| \leq 1 \, ,$$

and the same inequality with g replaced by f_x, g_x, g_y .

Finally, let $A(t)$ be a bounded, continuous matrix function, with $|A(t)| \leq N$, such that the linear equation

$$y' = A(t)y$$

has a fundamental matrix $\tilde{Y}(t)$ satisfying

$$\left|\tilde{Y}(t)P\tilde{Y}^{-1}(s)\right| \leq Ke^{-4\alpha(t-s)} \quad \text{for} \quad t \geq s \, ,$$

$$\left|\tilde{Y}(t)(I-P)\tilde{Y}^{-1}(s)\right| \leq Ke^{-4\alpha(s-t)} \quad \text{for} \quad s \geq t \, ,$$

where P is a projection matrix and K, α are positive constants.

Under these assumptions we have

THEOREM 1. *For any* β $\left(0 < \beta < \frac{1}{2}\alpha\right)$ *there exists a positive constant* $\mu_0 = \mu_0(N,\, K,\, L,\, \alpha,\, \beta)$ *such that if* $\mu \leq \mu_0$ *and if* $q \leq q_0(N,\, K,\, L,\, \alpha,\, \beta,\, \mu)$ *then the system of differential equations*

$$\text{(1)} \qquad \begin{aligned} x' &= f(t,\, x,\, y) \\ y' &= A(t)y + g(t,\, x,\, y) \end{aligned}$$

has a unique solution $x(t) = x(t,\, \xi,\, \tau)$, $y(t) = y(t,\, \xi,\, \tau)$ *for which*

$$\text{(7)} \qquad x(\tau) = \xi \, , \quad |y(t)| \leq \mu \quad \textit{for} \quad -\infty < t < \infty \, .$$

Moreover the partial derivatives x_ξ, y_ξ *exist and satisfy*

$$|x_\xi(t,\, \xi,\, \tau)| \leq 2e^{\beta|t-\tau|}$$

$$|y_\xi(t,\, \xi,\, \tau)| \leq 2C^{-1}e^{\beta|t-\tau|} \, ,$$

$$|x_\xi(t,\, \xi_1,\, \tau) - x_\xi(t,\, \xi_2,\, \tau)| \leq CD|\xi_1 - \xi_2|e^{2\beta|t-\tau|}$$

$$|y_\xi(t,\, \xi_1,\, \tau) - y_\xi(t,\, \xi_2,\, \tau)| \leq D|\xi_1 - \xi_2|e^{2\beta|t-\tau|} \, ,$$

where $C = 4N\left(1-e^{-\beta}\right)^{-1}$ *and* $D = 4L\left(N^{-1}+4\alpha^{-1}M\right)$. *Here* $\delta = \delta(N,\, K,\, \alpha) > 0$ *and* $M = M(K) > 1$ *are defined as in* Lemma 2.

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We first announce our determination of the constants μ_0, q_0. Set

$$\nu = (8N)^{-1}, \quad R = 8NM\alpha^{-1}e^{\alpha},$$

and choose $\mu_0 > 0$ so that

$$\mu_0 L \leqq \tfrac{1}{2}\delta, \quad 32\mu_0 LNC(2\nu+R) \leqq 1.$$

Next, for any μ $(0 < \mu \leqq \mu_0)$, choose $p_0 > 0$ so that

$$p_0 \leqq \tfrac{1}{2}\delta, \quad 16NM\gamma p_0 \leqq \mu, \quad 32p_0 NC(2\nu+R) \leqq 1,$$

where $\gamma = H(\alpha) = \left(1-e^{-\alpha}\right)^{-1}$. Then we take

$$q_0 = \tfrac{1}{2}p_0{}^2\min\left[\left(p_0+LN\right)^{-1}, \left(p_0+N^2\right)^{-1}\right].$$

If $x(t)$ is a continuously differentiable function and $y(t)$ a continuous function such that

$$(8) \qquad |x'(t)| \leqq N, \quad |y(t)| \leqq \mu \quad \text{for} \quad -\infty < t < \infty$$

then for $|t_2 - t_1| \leqq 1$ we have

$$\left|\int_{t_1}^{t_2} g_y[t, x(t), y(t)]dt\right| \leqq \mu L + \left|\int_{t_1}^{t_2} g_y[t, x(t), 0]dt\right|$$
$$\leqq \mu L + p_0,$$

by Lemma 3 and the inequality $q \leqq q_0$. Since $\mu L + p_0 \leqq \delta$ it follows that

$$B(t) = A(t) + g_y[t, x(t), y(t)]$$

satisfies the conditions of Lemma 2.

We can write

$$(9) \qquad \begin{aligned} f(t, x, y) &= f(t, x, 0) + f_y(t, x, 0)y + F(t, x, y) \\ g(t, x, y) &= g(t, x, 0) + g_y(t, x, 0)y + G(t, x, y), \end{aligned}$$

where $F(t, x, 0) = 0$, $G(t, x, 0) = 0$, and if $|y_1| \leqq \mu$, $|y_2| \leqq \mu$ then

$$|F(t, x_1, y_1) - F(t, x_2, y_2)| \leq 2\mu L[|x_1 - x_2| + |y_1 - y_2|] ,$$
$$|G(t, x_1, y_1) - G(t, x_2, y_2)| \leq 2\mu L[|x_1 - x_2| + |y_1 - y_2|] .$$

Let $x(t)$ be a continuously differentiable function and $y(t)$ a continuous function satisfying (8). Put

$$\hat{x}(t) = \xi + \int_\tau^t f[s, x(s), y(s)]ds$$

and let $\hat{y}(t)$ denote the unique bounded solution of the equation

$$y' = \left\{A(t) + g_y[t, x(t), 0]\right\}y + h(t) ,$$

where

$$h(t) = g[t, x(t), y(t)] - g_y[t, x(t), 0]y(t)$$
$$= g[t, x(t), 0] + G[t, x(t), y(t)] .$$

Since

$$|G(t, x, y)| = \left|\int_0^1 \left\{g_y(t, x, \theta y) - g_y(t, x, 0)\right\}yd\theta\right|$$
$$\leq L|y|^2 \int_0^1 \theta d\theta$$
$$\leq \tfrac{1}{2}\mu^2 L$$

and

$$\left|\int_{t_1}^{t_2} g[t, x(t), 0]dt\right| \leq p_0 \quad \text{for} \quad |t_2 - t_1| \leq 1$$

it follows from Lemmas 4 and 5, with $\beta = 0$, and from the superposition principle that for all t

$$|\hat{y}(t)| \leq \tfrac{1}{2}\mu^2 LM\alpha^{-1} + 8NM\gamma p_0$$
$$\leq \tfrac{1}{2}\mu + \tfrac{1}{2}\mu = \mu .$$

Moreover, for all t ,

$$|\hat{x}'(t)| = |f[t, x(t), y(t)]| \leq N .$$

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For a fixed β such that $0 < \beta \leq \frac{1}{2}\alpha$ let B denote the set of all pairs (x, y), where $x = x(t)$ is a continuously differentiable function, $y = y(t)$ is a continuous function and $\|x\| + \|x'\| + \|y\| < \infty$. The set B becomes a Banach space if we define

$$(x_1, y_1) + (x_2, y_2) = (x_1+x_2, y_1+y_2)$$
$$\lambda(x, y) = (\lambda x, \lambda y)$$
$$|(x, y)| = \|x\| + \nu\|x'\| + C\|y\| ,$$

where ν and C are the positive constants defined above. Thus $\nu \leq \frac{1}{8}$ and $C \geq 4$. Let S denote the set of all pairs (x, y) in B satisfying (8). Then S is a closed subset of B, since convergence in norm of a sequence $\{(x_n, y_n)\}$ implies pointwise convergence of the coordinates $x_n(t)$, $y_n(t)$ and the derivatives $x_n'(t)$. By what we have just proved the transformation $T : (x, y) \to (\hat{x}, \hat{y})$ maps S into itself. Clearly any solution of the problem (1) - (7) belongs to S and is a fixed point of T. Thus to prove the existence of a unique solution of (1) - (7) we need only show that T is a contraction on S.

Let (x_1, y_1) and (x_2, y_2) be any two points in S and set

(10) $$z(t) = x_1(t) - x_2(t) , \quad w(t) = y_1(t) - y_2(t) .$$

Similarly, if $(\hat{x}_1, \hat{y}_1) = T(x_1, y_1)$ and $(\hat{x}_2, \hat{y}_2) = T(x_2, y_2)$, we set

$$\hat{z}(t) = \hat{x}_1(t) - \hat{x}_2(t) , \quad \hat{w}(t) = \hat{y}_1(t) - \hat{y}_2(t) .$$

Then

$$\hat{z}(t) = \int_\tau^t \{f[s, x_1(s), y_1(s)] - f[s, x_2(s), y_2(s)]\}ds$$
$$= J_1 + J_2 + J_3 ,$$

where the three integrals J_1, J_2, J_3 correspond to the three terms in the decomposition (9) of $f(t, x, y)$. Thus

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$$|J_3| = \left|\int_\tau^t \{F[s,\, x_1(s),\, y_1(s)] - F[s,\, x_2(s),\, y_2(s)]\}ds\right|$$
$$\leqq 2\mu L \int_\tau^t \{|z(s)| + |w(s)|\}ds$$
$$\leqq 2\mu L\beta^{-1}[\|z\| + \|w\|]e^{\beta|t-\tau|} .$$

To estimate J_1 we use Lemma 6. We have

$$f[s,\, x_1(s),\, 0] - f[s,\, x_2(s),\, 0] = \int_0^1 f_x[s,\, x_2(s) + \theta z(s),\, 0]z(s)d\theta$$

and hence

$$|J_1| \leqq \int_0^1 \left|\int_\tau^t f_x[s,\, x_2(s) + \theta z(s),\, 0]z(s)ds\right|d\theta$$
$$\leqq p_0 H[\|z\| + \|z'\|]e^{\beta|t-\tau|} ,$$

where $H = H(\beta) = \left(1-e^{-\beta}\right)^{-1}$. To estimate J_2 we write

$$f_y[s,\, x_1(s),\, 0]y_1(s) - f_y[s,\, x_2(s),\, 0]y_2(s)$$
$$= f_y[s,\, x_1(s),\, 0]w(s) + \left\{f_y[s,\, x_1(s),\, 0] - f_y[s,\, x_2(s),\, 0]\right\}y_2(s) .$$

Then

$$|J_2| \leqq \int_\tau^t \left\{N|w(s)| + \mu L|z(s)|\right\}ds$$
$$\leqq \beta^{-1}[N\|w\| + \mu L\|z\|]e^{\beta|t-\tau|} .$$

Collecting together these three estimates we get, since $\beta^{-1} < H$,

$$\text{(11)} \qquad \|\hat{z}\| \leqq H\left\{3\mu L[\|z\| + \|w\|] + N\|w\| + p_0[\|z\| + \|z'\|]\right\} .$$

Since

$$\hat{z}'(t) = f[t,\, x_1(t),\, y_1(t)] - f[t,\, x_2(t),\, y_2(t)]$$

we also have

$$\text{(12)} \qquad \|\hat{z}'\| \leqq N[\|z\| + \|w\|] .$$

The difference $\hat{w}(t) = \hat{y}_1(t) - \hat{y}_2(t)$ is a bounded solution of the

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equation

$$(13) \qquad w' = \left\{A(t) + g_y[t, x_1(t), 0]\right\}w + \phi(t) + \psi(t) ,$$

where

$$\phi(t) = g[t, x_1(t), 0] - g[t, x_2(t), 0] ,$$

$$\psi(t) = \left\{g_y[t, x_1(t), 0] - g_y[t, x_2(t), 0]\right\}\hat{y}_2(t) + G[t, x_1(t), y_1(t)] - G[t, x_2(t), y_2(t)] .$$

Thus

$$|\psi(t)| \leqq 3\mu L[|z(t)| + |w(t)|]$$

and hence

$$\|\psi\| \leqq 3\mu L[\|z\| + \|w\|] .$$

Also

$$\int_t^{t+h} \phi(s)ds = \int_0^1 \int_t^{t+h} g_x[s, x_2(s) + \theta z(s), 0]z(s)dsd\theta$$

and hence by Lemma 7,

$$\left|\int_t^{t+h} \phi(s)ds\right| \leqq p_0\beta^{-1}\left(e^{\beta}-1\right)[\|z\| + \|z'\|]e^{\beta|t-\tau|}$$

for $|h| \leqq 1$ and either $t \geqq \tau$, $h > 0$ or $\tau \geqq t$, $h < 0$. It follows from Lemmas 4 and 5 and from the superposition principle that

$$\|\hat{\omega}\| \leqq 2(2\alpha-\beta)^{-1}M \cdot 3\mu L[\|z\| + \|w\|] + 8NM\gamma p_0\beta^{-1}\left(e^{\beta}-1\right)[\|z\| + \|z'\|] .$$

Therefore, since $(2\alpha-\beta)^{-1} \leqq \alpha^{-1}$ and $\gamma\beta^{-1}\left(e^{\beta}-1\right) \leqq \alpha^{-1}e^{\alpha}$,

$$(14) \qquad \|\hat{\omega}\| \leqq 6\alpha^{-1}\mu LM[\|z\| + \|w\|] + p_0R[\|z\| + \|z'\|] .$$

From (11), (12) and (14) we obtain

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$$|(\hat{z},\ \hat{w})| \leq 3\mu L\left(H+2\alpha^{-1}MC\right)[\|z\| + \|w\|] + NH\|w\| + \nu N[\|z\| + \|w\|] + p_0(H+RC)[\|z\| + \|z'\|]$$
$$\leq \frac{1}{4}[\|z\| + \|w\|] + \frac{1}{4}C\|w\| + \frac{1}{4}\nu[\|z\| + \|z'\|]$$
$$\leq \frac{1}{2}|(z,\ w)| \ .$$

Thus T really is a contraction.

Now let $(x_1,\ y_1)$ and $(x_2,\ y_2)$ denote the fixed points of T when ξ in (7) is replaced by ξ_1 and ξ_2. If we again set

$$z(t) = x_1(t) - x_2(t)\ ,\quad w(t) = y_1(t) - y_2(t) \tag{10}$$

then

$$z(t) = \xi_1 - \xi_2 + \int_\tau^t \left\{f[s,\ x_1(s),\ y_1(s)] - f[s,\ x_2(s),\ y_2(s)]\right\}ds$$

and $w(t)$ is the unique bounded solution of the equation (13), with $\hat{y}_2(t) = y_2(t)$. Therefore, by the same argument as before,

$$|(z,\ w)| \leq |\xi_1 - \xi_2| + \frac{1}{2}|(z,\ w)| \ .$$

Hence

$$|(z,\ w)| \leq 2|\xi_1 - \xi_2|\ ,$$

and in particular

$$\begin{aligned} |x(t,\ \xi_1,\ \tau) - x(t,\ \xi_2,\ \tau)| &\leq 2|\xi_1 - \xi_2|e^{\beta|t-\tau|} \\ |y(t,\ \xi_1,\ \tau) - y(t,\ \xi_2,\ \tau)| &\leq 2C^{-1}|\xi_1 - \xi_2|e^{\beta|t-\tau|}\ . \end{aligned} \tag{15}$$

Similarly, let $(x_1,\ y_1)$ and $(x_2,\ y_2)$ denote the fixed points of T when τ in (7) is replaced by $\tau + h$ and τ, and again define $z(t)$, $w(t)$ by (10). Then

$$z(t) = -\int_\tau^{\tau+h} f[s,\ x_1(s),\ y_1(s)]ds + \int_\tau^t \left\{f[s,\ x_1(s),\ y_1(s)] - f[s,\ x_2(s),\ y_2(s)]\right\}ds$$

and $w(t)$ is the unique bounded solution of the equation (13), with

$\hat{y}_2(t) = y_2(t)$, from which it follows again that

$$|(z, w)| \leqq 2N|h| .$$

Thus

(16)
$$|x(t, \xi, \tau+h) - x(t, \xi, \tau)| \leqq 2N|h|e^{\beta|t-\tau|}$$
$$|y(t, \xi, \tau+h) - y(t, \xi, \tau)| \leqq \frac{1}{2}(1-e^{-\beta})|h|e^{\beta|t-\tau|} .$$

It remains to prove the part of the theorem concerning the partial derivatives x_ξ, y_ξ .

4. Completion of the proof.

We consider first a linear system of differential equations

(17)
$$x' = F_1(t)x + F_2(t)y + \eta(t)$$
$$y' = G_1(t)x + G_2(t)y + \zeta(t) ,$$

where the matrix functions F_k, G_k $(k = 1, 2)$ are continuous and bounded by N , except G_2 which is bounded by $2N$, and the vector functions η, ζ are continuous with $\|\eta\| < \infty$., $\|\zeta\| < \infty$. We assume also that for $|t_2 - t_1| \leqq 1$

$$\left|\int_{t_1}^{t_2} F_1(s)ds\right| \leqq r , \quad \left|\int_{t_1}^{t_2} G_1(s)ds\right| \leqq r ,$$

and that the linear equation

(18)
$$y' = G_2(t)y$$

has a fundamental matrix $Y(t)$ satisfying (5). We wish to show that if r is so small that

(19)
$$16NC(2\nu+R)r \leqq 1 ,$$

then the system (17) has a unique solution $x(t)$, $y(t)$ in B such that $x(\tau) = \xi$.

For any (x, y) in B set

$$\hat{x}(t) = \xi + \int_\tau^t \left\{F_1(s)x(s) + F_2(s)y(s) + \eta(s)\right\}ds$$

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and let $\hat{y}(t)$ denote the unique solution with $\|y\| < \infty$ of the equation

$$y' = G_2(t)y + G_1(t)x(t) + \zeta(t) .$$

Then, using the same notation as in the previous Section, we have

$$\hat{z}(t) = \int_\tau^t \left\{F_1(s)z(s) + F_2(s)w(s)\right\}ds$$

and $\hat{w}(t)$ is the unique solution with $\|w\| < \infty$ of the equation

$$w' = G_2(t)w + G_1(t)z(t) .$$

By Lemma 6 we have

$$\|\hat{z}\| \leq rH[\|z\| + \|z'\|] + N\beta^{-1}\|w\| ,$$

while

$$\|\hat{z}'\| \leq N[\|z\| + \|w\|] .$$

By Lemma 7

$$\left|\int_t^{t+h} G_1(s)z(s)ds\right| \leq r\beta^{-1}\left(e^\beta-1\right)[\|z\| + \|z'\|]e^{\beta|t-\tau|}$$

for $|h| \leq 1$ and either $t \geqq \tau$, $h > 0$ or $\tau \geqq t$, $h < 0$, and hence by Lemma 5

$$\|\hat{w}\| \leq rR[\|z\| + \|z'\|] .$$

It follows that

$$\begin{aligned}|(\hat{z}, \hat{w})| &\leq N\beta^{-1}\|w\| + \nu N[\|z\| + \|w\|] + r(H+RC)[\|z\| + \|z'\|] \\ &\leq \tfrac{1}{4}C\|w\| + \tfrac{1}{8}[\|z\| + \|w\|] + \tfrac{1}{2}\nu[\|z\| + \|z'\|] \\ &\leq \tfrac{1}{2}|(z, w)| .\end{aligned}$$

Therefore the mapping $(x, y) \to (\hat{x}, \hat{y})$ has a unique fixed point in $\mathcal{B}$. If we denote this fixed point by $\left(x_0, y_0\right)$ then

$$x_0(t) = \xi + \int_\tau^t \left\{F_1(s)x_0(s) + F_2(s)y_0(s) + \eta(s)\right\}ds$$

and $y_0(t)$ is the unique solution with $\|y\| < \infty$ of the equation

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$$y' = G_2(t)y + G_1(t)x_0(t) + \zeta(t) ,$$

from which it follows in the same way that

$$|(x_0, y_0)| \leq \tfrac{1}{2}|(x_0, y_0)| + |\xi| + 2H\|\eta\| + 2\alpha^{-1}MC\|\zeta\| .$$

Therefore

(20) $$|(x_0, y_0)| \leq 2|\xi| + C\left[N^{-1}\|\eta\| + 4\alpha^{-1}M\|\zeta\|\right] .$$

If

(21) $$\begin{aligned} F_1(t) &= f_x[t, x(t, \xi, \tau), y(t, \xi, \tau)] , \\ F_2(t) &= f_y[t, x(t, \xi, \tau), y(t, \xi, \tau)] , \\ G_1(t) &= g_x[t, x(t, \xi, \tau), y(t, \xi, \tau)] , \\ G_2(t) &= A(t) + g_y[t, x(t, \xi, \tau), y(t, \xi, \tau)] , \end{aligned}$$

then we can take $r = p_0 + \mu L$ and the inequality (19) is satisfied. Moreover (18) has a fundamental matrix $Y(t)$ satisfying (5). Let $X_0(t)$, $Y_0(t)$ denote the corresponding solution in B of the matrix system

$$\begin{aligned} X' &= F_1(t)X + F_2(t)Y \\ Y' &= G_1(t)X + G_2(t)Y \end{aligned}$$

with $X_0(\tau) = I$. By (20) we have

(22) $$\|X_0\| + C\|Y_0\| \leq 2 .$$

Put

$$\begin{aligned} x_1(t) &= x(t, \tilde{\xi}, \tau) , \quad x_2(t) = x(t, \xi, \tau) \\ y_1(t) &= y(t, \tilde{\xi}, \tau) , \quad y_2(t) = y(t, \xi, \tau) \end{aligned}$$

and with $z(t)$, $w(t)$ defined by (10) set

$$\phi(t) = z(t) - X_0(t)(\tilde{\xi}-\xi) , \quad \psi(t) = w(t) - Y_0(t)(\tilde{\xi}-\xi) .$$

Then

$$\phi' = F_1(t)\phi + F_2(t)\psi + \eta(t) ,$$

where

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$$\eta(t) = f[t, x_1(t), y_1(t)] - f[t, x_2(t), y_2(t)] - F_1(t)z(t) - F_2(t)w(t)$$

$$= \int_0^1 \Big\{f_x[t, x_2(t) + \theta z(t), y_2(t) + \theta w(t)] - f_x[t, x_2(t), y_2(t)]\Big\}z(t)d\theta$$

$$+ \int_0^1 \Big\{f_y[t, x_2(t) + \theta z(t), y_2(t) + \theta w(t)] - f_y[t, x_2(t), y_2(t)]\Big\}w(t)d\theta .$$

Hence

$$|\eta(t)| \leqq L[|z(t)| + |w(t)|]^2 .$$

Also

$$\psi' = G_1(t)\phi + G_2(t)\psi + \zeta(t) ,$$

where

$$\zeta(t) = A(t)w(t) + g[t, x_1(t), y_1(t)] - g[t, x_2(t), y_2(t)] - G_1(t)z(t) - G_2(t)w(t)$$

$$= \int_0^1 \Big\{g_x[t, x_2(t) + \theta z(t), y_2(t) + \theta w(t)] - g_x[t, x_2(t), y_2(t)]\Big\}z(t)d\theta$$

$$+ \int_0^1 \Big\{g_y[t, x_2(t) + \theta z(t), y_2(t) + \theta w(t)] - g_y[t, x_2(t), y_2(t)]\Big\}w(t)d\theta ,$$

and hence

$$|\zeta(t)| \leqq L[|z(t)| + |w(t)|]^2 .$$

Thus with the norm

$$\|f\|_2 = \sup_{-\infty<t<\infty} \Big\{e^{-2\beta|t-\tau|}|f(t)|\Big\}$$

we have

$$\|\eta\|_2,\ \|\zeta\|_2 \leqq L(\|z\| + \|w\|)^2 \leqq 4L|\tilde{\xi} - \xi|^2$$

by the formula line preceding (15). If we write

$$H_2 = H(2\beta) \leqq H , \quad C_2 = 4NH_2 \leqq C ,$$

then, since $\phi(\tau) = 0$, it follows from (20) that

$$\|\phi\|_2 + C_2\|\psi\|_2 \leqq 4LC_2\left(N^{-1}+4\alpha^{-1}M\right)|\tilde{\xi} - \xi|^2 .$$

Therefore, by the definitions of ϕ and ψ,

$$|x(t, \tilde{\xi}, \tau) - x(t, \xi, \tau) - X_0(t)(\tilde{\xi}-\xi)| \leq CD|\tilde{\xi} - \xi|^2 e^{2\beta|t-\tau|}$$

$$|y(t, \tilde{\xi}, \tau) - y(t, \xi, \tau) - Y_0(t)(\tilde{\xi}-\xi)| \leq D|\tilde{\xi} - \xi|^2 e^{2\beta|t-\tau|}$$

where $D = 4L(N^{-1}+4\alpha^{-1}M)$ as in the statement of Theorem 1. Thus the partial derivatives $x_\xi(t, \xi, \tau)$, $y_\xi(t, \xi, \tau)$ exist and equal $X_0(t)$, $Y_0(t)$ respectively.

We show next that the partial derivatives satisfy a Lipschitz condition. If

$$\tilde{X}_0(t) = x_\xi(t, \tilde{\xi}, \tau) , \quad \tilde{Y}_0(t) = y_\xi(t, \tilde{\xi}, \tau)$$

then $\tilde{X}_0(t)$, $\tilde{Y}_0(t)$ is the solution in $\mathcal{B}$ of a system

$$X' = \tilde{F}_1(t)X + \tilde{F}_2(t)Y$$
$$Y' = \tilde{G}_1(t)X + \tilde{G}_2(t)Y ,$$

corresponding to that satisfied by $X_0(t)$, $Y_0(t)$, with $\tilde{X}_0(\tau) = I$. Thus

$$Z(t) = X_0(t) - \tilde{X}_0(t) , \quad W(t) = Y_0(t) - \tilde{Y}_0(t)$$

is a solution of the system

$$Z' = F_1(t)Z + F_2(t)W + \eta(t)$$
$$W' = G_1(t)Z + G_2(t)W + \zeta(t) ,$$

where

$$\eta(t) = [F_1(t) - \tilde{F}_1(t)]\tilde{X}_0(t) + [F_2(t) - \tilde{F}_2(t)]\tilde{Y}_0(t) ,$$

$$\zeta(t) = [G_1(t) - \tilde{G}_1(t)]\tilde{X}_0(t) + [G_2(t) - \tilde{G}_2(t)]\tilde{Y}_0(t) .$$

Thus we have

$$|\eta(t)|, \ |\zeta(t)| \leq L[|z(t)| + |w(t)|][|\tilde{X}_0(t)| + |\tilde{Y}_0(t)|]$$

and hence, by (22) and the formula line preceding (15),

$$\|\eta\|_2, \ \|\zeta\|_2 \leq 4L|\tilde{\xi} - \xi| .$$

Since $Z(\tau) = 0$ it follows from (20) that

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$$\|Z\|_2 + C_2\|W\|_2 \leq 4LC_2\left(N^{-1}+4\alpha^{-1}M\right)|\tilde{\xi} - \xi| = C_2D|\tilde{\xi} - \xi| .$$

Therefore

(23) $$|x_\xi(t, \tilde{\xi}, \tau) - x_\xi(t, \xi, \tau)| \leq CD|\tilde{\xi} - \xi|e^{2\beta|t-\tau|}$$
$$|y_\xi(t, \tilde{\xi}, \tau) - y_\xi(t, \xi, \tau)| \leq D|\tilde{\xi} - \xi|e^{2\beta|t-\tau|} .$$

Similarly, if we put

$$Z(t) = x_\xi(t, \xi, \tau+h) - x_\xi(t, \xi, \tau)$$
$$W(t) = y_\xi(t, \xi, \tau+h) - y_\xi(t, \xi, \tau)$$

then $Z(t)$, $W(t)$ are the solutions of a linear system of the same form as before where, by (22) and the formula line preceding (16),

$$\|\eta\|_2, \|\zeta\|_2 \leq 4LN|h| .$$

Since

$$Z(\tau) = \tilde{X}(\tau) - I = -\int_\tau^{\tau+h} (\tilde{F}_1\tilde{X}+\tilde{F}_2\tilde{Y})dt$$

we have, for $0 \leq h \leq 1$,

$$|Z(\tau)| \leq N\int_\tau^{\tau+h} [|\tilde{X}(t)| + |\tilde{Y}(t)|]dt$$
$$\leq 2N\int_\tau^{\tau+h} e^{\beta(\tau+h-t)}dt$$
$$= 2N\left(e^{\beta h}-1\right)/\beta$$
$$\leq 2Ne^{\beta}h .$$

Therefore, by (20),

$$\|Z\|_2 + C_2\|W\|_2 \leq 4Ne^{\beta}h + C_2NDh .$$

Since $C_2 \leq C$ it follows that for $0 \leq h \leq 1$

$$|x_\xi(t, \xi, \tau+h) - x_\xi(t, \xi, \tau)| \leq [2\left(e^{\beta}-1\right) + ND]Che^{2\beta|t-\tau|}$$
$$|y_\xi(t, \xi, \tau+h) - y_\xi(t, \xi, \tau)| \leq [2\left(e^{\beta}-1\right) + ND]he^{2\beta|t-\tau|} .$$

Finally we consider the properties of the corresponding integral manifold. Put

$$v(\tau, \xi) = y(\tau, \xi, \tau) \ . \tag{24}$$

Then

$$|v(\tau, \xi)| \leq \mu \ , \tag{25}$$

the partial derivative v_ξ exists and

$$|v_\xi(\tau, \xi)| \leq 2C^{-1} \tag{26}$$

$$|v_\xi(\tau, \tilde{\xi}) - v_\xi(\tau, \xi)| \leq D|\tilde{\xi} - \xi| \ . \tag{27}$$

Since the solutions guaranteed by Theorem 1 are unique we have

$$x[t, x(s, \xi, \tau), s] = x(t, \xi, \tau)$$
$$y[t, x(s, \xi, \tau), s] = y(t, \xi, \tau) \ .$$

In particular

$$v[t, x(t, \xi, \tau)] = y(t, \xi, \tau) \ , \tag{28}$$

which shows that the equation $y = v(t, x)$ defines an integral manifold of (1). For fixed τ, ξ and small h we have, by (16),

$$\begin{aligned} x(\tau, \xi, \tau+h) - \xi &= -\int_\tau^{\tau+h} f[s, x(s, \xi, \tau+h), y(s, \xi, \tau+h)]ds \\ &= -\int_\tau^{\tau+h} f[s, x(s, \xi, \tau), y(s, \xi, \tau)]ds + O(h^2) \\ &= -\int_\tau^{\tau+h} f[s, x(\tau, \xi, \tau), y(\tau, \xi, \tau)]ds + O(h^2) \\ &= -f[\tau, \xi, v(\tau, \xi)]h + o(|h|) \ . \end{aligned}$$

Also, for fixed t ,

$$x(t, \tilde{\xi}, \tau) = x(t, \xi, \tau) + x_\xi(t, \xi, \tau)(\tilde{\xi}-\xi) + O(|\tilde{\xi}-\xi|^2) \ .$$

Therefore, taking $\tilde{\xi} = x(\tau, \xi, \tau+h)$,

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$$
\begin{aligned}
x(t, \xi, \tau+h) &= x(t, \tilde{\xi}, \tau) \\
&= x(t, \xi, \tau) + x_\xi(t, \xi, \tau)(\tilde{\xi}-\xi) + O(h^2) \\
&= x(t, \xi, \tau) - x_\xi(t, \xi, \tau)f[\tau, \xi, v(\tau, \xi)]h + o(|h|) .
\end{aligned}
$$

Thus $x_\tau(t, \xi, \tau)$ exists and is equal to

$$
- x_\xi(t, \xi, \tau)f[\tau, \xi, v(\tau, \xi)] .
$$

Therefore, by (28), $y_\tau(t, \xi, \tau)$ also exists and is equal to

$$
v_\xi[t, x(t, \xi, \tau)]x_\tau(t, \xi, \tau) .
$$

Since $v(\tau, \xi) = y(\tau, \xi, \tau)$ it follows that $v_\tau(\tau, \xi)$ exists and is equal to

$$
\begin{aligned}
&A(\tau)y(\tau, \xi, \tau) + g[\tau, x(\tau, \xi, \tau), y(\tau, \xi, \tau)] \\
&\qquad\qquad + v_\xi[\tau, x(\tau, \xi, \tau)]x_\tau(\tau, \xi, \tau) \\
&= A(\tau)v(\tau, \xi) + g[\tau, \xi, v(\tau, \xi)] - v_\xi(\tau, \xi)f[\tau, \xi, v(\tau, \xi)] .
\end{aligned}
$$

Thus $v(\tau, \xi)$ is a solution of the partial differential equation

$$
v_\tau + v_\xi f(\tau, \xi, v) = A(\tau)v + g(\tau, \xi, v) . \tag{29}
$$

Since $C \geqq 4$ it now follows from (25), (26) and (27) that

$$
|v_\tau(\tau, \xi)| \leqq (\mu+2)N , \tag{30}
$$

$$
|v_\tau(\tau, \tilde{\xi}) - v_\tau(\tau, \xi)| \leqq (D+3)N|\tilde{\xi} - \xi| . \tag{31}
$$

Also

$$
\begin{aligned}
v_\xi(\tau+h, \xi) - v_\xi(\tau, \xi) &= y_\xi(\tau+h, \xi, \tau+h) - y_\xi(\tau, \xi, \tau+h) \\
&\qquad + y_\xi(\tau, \xi, \tau+h) - y_\xi(\tau, \xi, \tau) .
\end{aligned}
$$

From (22) and the differential equation satisfied by y_ξ we have, for $0 \leqq h \leqq 1$,

$$
\begin{aligned}
|y_\xi(\tau+h,\ \xi,\ \tau+h) - y_\xi(\tau,\ \xi,\ \tau+h)| & \\
&\leq N \int_\tau^{\tau+h} \left[|x_\xi(t,\ \xi,\ \tau+h)| + 2|y_\xi(t,\ \xi,\ \tau+h)|\right] dt \\
&\leq 2N \int_\tau^{\tau+h} e^{\beta(\tau+h-t)} dt \\
&= 2N\left(e^{\beta h}-1\right)/\beta \\
&\leq 2Ne^{\beta}h \ .
\end{aligned}
$$

Therefore

$$
\begin{aligned}
|v_\xi(\tau+h,\ \xi) - v_\xi(\tau,\ \xi)| &\leq \left[2Ne^{\beta} + 2\left(e^{\beta}-1\right) + ND\right]|h| \\
&\leq \left(D+4e^{\beta}\right)N|h| \ .
\end{aligned}
$$

This inequality has been established for $0 \leq h \leq 1$, but extends at once to arbitrary real h . The two Lipschitz conditions show that v_ξ is a continuous function of $(\tau,\ \xi)$, and hence v_τ is also by the partial differential equation (29). Moreover from the fact that v_τ is continuous in τ for fixed ξ we can show in general that v_ξ satisfies a Lipschitz condition in τ with the *same* constant as in the Lipschitz condition which v_τ satisfies in ξ .

Summing up, we have proved

THEOREM 2. *Under the hypotheses of* Theorem 1 *the system* (1) *has an integral manifold* $y = v(t,\ x)$ *where* v *is a continuous function with continuous partial derivatives* v_t, v_x *such that*

$$
\begin{gathered}
|v(t,\ x)| \leq \mu \ , \quad |v_x(t,\ x)| \leq 2C^{-1} \ , \quad |v_t(t,\ x)| \leq (\mu+2)N \ , \\
|v_x(t,\ x_1) - v_x(t,\ x_2)| \leq D|x_1 - x_2| \ , \\
|v_t(t,\ x_1) - v_t(t,\ x_2)| \leq (D+3)N|x_1 - x_2| \ , \\
|v_x(t_1,\ x) - v_x(t_2,\ x)| \leq (D+3)N|t_1 - t_2| \ .
\end{gathered}
$$

The behaviour of the solutions on the integral manifold $y = v(t,\ x)$ is determined by the differential equation

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(32) $$x' = k(t, x) ,$$

where $k(t, x) = f[t, x, v(t, x)]$. Thus the partial derivative k_x exists and is equal to $f_x + f_y v_x$. Hence

(33) $$|k_x(t, x)| \leq 2N$$

and

(34) $$|k_x(t, x_1) - k_x(t, x_2)| \leq \left(DN+\tfrac{9}{4}L\right)|x_1 - x_2| .$$

In the case referred to in the Introduction, where the system (1) arises in the study of a *periodic* perturbation of an autonomous system with a periodic orbit, x is a scalar and the functions f, g, v, k are periodic in both t and x . Thus (32) describes a flow without stationary points on a torus. The smoothness properties which have been established for the function $k(t, x)$ are just sufficient to exclude the singular case, by virtue of Denjoy's Theorem (see [2]). Consequently the solutions of (32) are asymptotically periodic if the rotation number is rational, and quasiperiodic with two basic frequencies if the rotation number is irrational.

References

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