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AVERAGING THEORY AT ANY ORDER FOR COMPUTING PERIODIC ORBITS

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ABSTRACT. We provide an explicit expression for the solutions of the perturbed first order differential equations. Using it we give an averaging theory at any order in ε for the following two kinds of analytic differential equations

$$\frac{dx}{d\theta} = \sum_{k\geq 1} \varepsilon^k F_k(\theta, x), \qquad \frac{dx}{d\theta} = \sum_{k\geq 0} \varepsilon^k F_k(\theta, x).$$

We apply these results for studying the limit cycles of planar polynomial differential systems after passing them to polar coordinates.

1. Introduction and statement of the main results

In this work first we deal with the analytic differential equation

$$\frac{dx}{d\theta} = \sum_{k\geq 1} \varepsilon^k F_k(\theta, x), \tag{1}$$

where $x \in \mathbb{R}$, $\theta \in \mathbb{S}^1$ and $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$ with ε_0 a small positive real value, and the functions $F_k(\theta, x)$ are 2π–periodic in the variable θ. So, this differential equation is defined in the cylinder $\mathbb{S}^1 \times \mathbb{R}$. We are interested in the *limit cycles* of the differential equation (1), i.e. in the isolated periodic orbits respect to the set of all periodic orbits of equation (1).

We denote by $x_\varepsilon(\theta, z)$ the solution of equation (1) with initial condition $x_\varepsilon(0, z) = z$. Due to the analyticity of the differential equation (1) and the fact that when $\varepsilon = 0$ we have the trivial equation $dx/d\theta = 0$, the solution can be written as

$$x_\varepsilon(\theta, z) = z + \sum_{j\geq 1} x_j(\theta, z)\varepsilon^j, \tag{2}$$

where $x_j(\theta, z)$ are real analytic functions such that $x_j(0, z) = 0$.

We remark that if the solution $x_e(\theta, z)$ is defined for all $\theta \in \mathbb{S}^1$, then we can consider the Poincaré displacement map associated to the differential equation (1) given by $P_\varepsilon(z) = x_\varepsilon(2\pi, z) - z$. We observe that a limit cycle of equation (1) corresponds to an isolated zero of the Poincaré map. We are interested in the limit cycles which bifurcate from the periodic orbits of

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the unperturbed equation, that is from equation (1) with $\varepsilon = 0$. We say that a limit cycle bifurcates from the periodic orbit at level $z^* \in \mathbb{R}$ if there exists an analytic function $\zeta(\varepsilon)$ defined in a neighborhood of $\varepsilon = 0$ such that $P_\varepsilon(\zeta(\varepsilon)) = 0$ for all ε in this neighborhood, this zero is isolated for each fixed value of $\varepsilon \neq 0$, and $\zeta(0) = z^*$.

A limit cycle is said to be of *multiplicity* m, with $m \geq 1$ an integer, if this is the multiplicity of $\zeta(\varepsilon)$ as a zero of $P_\varepsilon(z)$ in a punctured neighborhood of $\varepsilon = 0$. We note that the expansion of the Poincaré map in powers of ε is $P_\varepsilon(z) = \sum_{j\geq 1} x_j(2\pi, z)\varepsilon^j$. We consider $s \geq 1$ the lowest index such that $x_s(2\pi, z)$ is not identically zero. If there exists a limit cycle of equation (1) bifurcating from z^*, then $x_s(2\pi, z^*) = 0$. Indeed, if there are m limit cycles bifurcating from z^* counted with their multiplicity, then z^* is an isolated zero of $x_s(2\pi, z)$ of multiplicity at least m. On the other hand, for each simple (that is, of multiplicity one) zero z^* of $x_s(2\pi, z)$, there exists a unique limit cycle of equation (1) bifurcating from z^*. For the details about the previous statement, see for instance [8].

The following result provides the explicit expression of the function $x_n(\theta, z)$ for any value of n.

Theorem 1. *The solution* (2) *of equation* (1) *satisfies*

$$
\begin{aligned}
x_1(\theta, z) &= \int_0^\theta F_1(\varphi, z)\, d\varphi, \\
x_n(\theta, z) &= \int_0^\theta \Bigg(F_n(\varphi, z) + \sum_{\ell=1}^{n-1} \sum_{i=1}^{\ell} \frac{1}{i!} \frac{\partial^i F_{n-\ell}}{\partial x^i}(\varphi, z) \\
&\qquad \sum_{j_1+j_2+\cdots+j_i=\ell} x_{j_1}(\varphi, z) x_{j_2}(\varphi, z) \cdots x_{j_i}(\varphi, z) \Bigg) d\varphi,
\end{aligned}
$$

for $n \geq 2$, where $j_m \geq 1$ is an integer for $m = 1, 2, \ldots, i$.

This result is proved in section 2.

Now we shall apply Theorem 1 for obtaining the averaging theory of arbitrarily high order for the differential equations (1), and we develop this theory explicitly until order four. The averaging theory of first order for studying periodic orbits of the differential equations (1) in arbitrary finite dimension is very classical (see for instance [9, 10]), until third order was developed in [1]. More precisely, these authors studied the averaging theory for the differential equations (1) in $\mathbb{R}^n$ up to order 3 in ε. Note that here we are studying the averaging theory for differential equations (1) in $\mathbb{R}$ but at arbitrary order in ε. We are aware that the formula of fourth order was already known by Xiang Zhang, see [11].

We consider the differential system

$$
(3) \quad \frac{dx}{d\theta} = \varepsilon F_1(\theta, x) + \varepsilon^2 F_2(\theta, x) + \varepsilon^3 F_3(\theta, x) + \varepsilon^4 F_4(\theta, x) + \varepsilon^5 R(\theta, x, \varepsilon),
$$

where $F_i(\theta, x)$, $i = 1, 2, 3, 4$, are the functions defined in (1) and

$$R(\theta, x, \varepsilon) = \sum_{k\geq 5} \varepsilon^{k-5} F_k(\theta, x).$$

We define the real functions $F_{i0}(z)$ of real variable z

$$\begin{aligned}
F_{10}(z) &= \int_0^{2\pi} F_1(\varphi, z)\, d\varphi, \\
F_{20}(z) &= \int_0^{2\pi} \left(F_2(\varphi, z) + \frac{\partial F_1}{\partial z}(\varphi, z)\, x_1(\varphi, z) \right) d\varphi, \\
F_{30}(z) &= \int_0^{2\pi} \left(F_3(\varphi, z) + \frac{\partial F_2}{\partial z}(\varphi, z)\, x_1(\varphi, z) + \frac{\partial F_1}{\partial z}(\varphi, z)\, x_2(\varphi, z) \right. \\
&\quad \left. + \frac{1}{2} \frac{\partial^2 F_1}{\partial z^2}(\varphi, z)\, x_1(\varphi, z)^2 \right) d\varphi, \\
F_{40}(z) &= \int_0^{2\pi} \left(F_4(\varphi, z) + \frac{\partial F_3}{\partial z}(\varphi, z)\, x_1(\varphi, z) + \frac{\partial F_2}{\partial z}(\varphi, z)\, x_2(\varphi, z) \right. \\
&\quad + \frac{1}{2} \frac{\partial^2 F_2}{\partial z^2}(\varphi, z)\, x_1(\varphi, z)^2 + \frac{\partial F_1}{\partial z}(\varphi, z)\, x_3(\varphi, z) \\
&\quad \left. + \frac{\partial^2 F_1}{\partial z^2}(\varphi, z)\, x_1(\varphi, z) x_2(\varphi, z) + \frac{1}{6} \frac{\partial^3 F_1}{\partial z^3}(\varphi, z)\, x_1(\varphi, z)^3 \right) d\varphi.
\end{aligned}$$

Corollary 2 (Averaging theory for periodic orbits up to fourth order for a differential equation (3))**.** *The following statements hold.*

(a) *If $F_{10}(z)$ is not identically zero, then for each simple zero z^* of $F_{10}(z) = 0$ there exists a periodic solution $x_\varepsilon(\theta, z)$ of equation* (3) *such that $x_\varepsilon(0, z) \to z^*$ when $\varepsilon \to 0$.*

(b) *If $F_{10}(z)$ is identically zero and $F_{20}(z)$ is not identically zero, then for each simple zero z^* of $F_{20}(z) = 0$ there exists a periodic solution $x_\varepsilon(\theta, z)$ of equation* (3) *such that $x_\varepsilon(0, z) \to z^*$ when $\varepsilon \to 0$.*

(c) *If $F_{10}(z)$ and $F_{20}(z)$ are identically zero and $F_{30}(z)$ is not identically zero, then for each simple zero z^* of $F_{30}(z) = 0$ there exists a periodic solution $x_\varepsilon(\theta, z)$ of equation* (3) *such that $x_\varepsilon(0, z) \to z^*$ when $\varepsilon \to 0$.*

(d) *If $F_{10}(z)$, $F_{20}(z)$ and $F_{30}(z)$ are identically zero and $F_{40}(z)$ is not identically zero, then for each simple zero z^* of $F_{40}(z) = 0$ there exists a periodic solution $x_\varepsilon(\theta, z)$ of equation* (3) *such that $x_\varepsilon(0, z) \to z^*$ when $\varepsilon \to 0$.*

This corollary of Theorem 1 is proved in section 2.

Remark 3. *Assume that $F_{10}(z)$ is not identically zero. If z^* is a zero of $F_{10}(z)$ with multiplicity larger that* 1*, then considering the zeros of the function $F_{10}(z) + \varepsilon F_{20}(z)$, we can study how many periodic orbits bifurcate from z^*. This observation for statement* (a) *of Corollary* 2 *extends in a natural way to the other statements of that corollary.*

We also deal with the following, more general, analytic differential equation

$$\frac{dx}{d\theta} = G_0(\theta, x) + \sum_{k\geq 1} \varepsilon^k G_k(\theta, x), \tag{4}$$

where $x \in \mathbb{R}$, $\theta \in \mathbb{S}^1$ and $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$ with ε_0 a small positive real value, and the functions $G_k(\theta, x)$ are 2π–periodic in the variable θ. We consider a particular solution $x_0(\theta, z)$ of the unperturbed system, i.e. equation (4) with $\varepsilon = 0$, satisfying that $x_0(0, z) = z$. As before, we denote by $x_\varepsilon(\theta, z)$ the solution of equation (4) with initial condition z when $\theta = 0$, which writes as

$$x_\varepsilon(\theta, z) = x_0(\theta, z) + \sum_{j\geq 1} x_j(\theta, z)\varepsilon^j, \tag{5}$$

where $x_j(\theta, z)$ are real analytic functions with $x_j(0, z) = 0$, for $j \geq 1$.

We consider the unperturbed equation (4) with $\varepsilon = 0$ and we assume that its solution $x_0(\theta, z)$, such that $x_0(0, z) = z$, is 2π–periodic for $z \in \mathcal{I}$ with $\mathcal{I}$ a real open interval. We are interested in the limit cycles of equation (4) which bifurcate from the periodic orbits of the unperturbed equation with initial condition $z \in \mathcal{I}$. We define the Poincaré map in an analogous way as we have done for the differential equation (1).

Let $u = u(\theta, z)$ be the solution of the variational equation

$$\frac{\partial u}{\partial \theta} = \frac{\partial G_0}{\partial x}(\theta, x_0(\theta, z))\, u, \tag{6}$$

satisfying $u(0, z) = 1$. For each $i \geq 1$, we define the functions $u_i(\theta, z)$ as

$$x_i(\theta, z) = u(\theta, z)\, u_i(\theta, z).$$

The following result provides explicit expressions of the function $x_n(\theta, z)$ for any value of n.

Theorem 4. *The solution* (5) *of equation* (4) *satisfies* $x_n(\theta, z) = u(\theta, z)\, u_n(\theta, z)$ *with*

$$u_1(\theta, z) = \int_0^\theta \frac{G_1(\varphi, x_0(\varphi, z))}{u(\varphi, z)}\, d\varphi,$$

$$u_n(\theta, z) = \int_0^\theta \left(\frac{G_n(\varphi, x_0(\varphi, z))}{u(\varphi, z)} + \sum_{\ell=0}^{n-2} \sum_{i=1}^{n-\ell} \frac{1}{i!} \frac{\partial^i G_{n-\ell-i}}{\partial x^i}(\varphi, x_0(\varphi, z)) \right.$$
$$\left. u(\varphi, z)^{i-1} \sum_{j_1+j_2+\cdots+j_i = \ell+i} u_{j_1}(\varphi, z) u_{j_2}(\varphi, z) \cdots u_{j_i}(\varphi, z) \right) d\varphi,$$

for $n \geq 2$*, where* $j_m \geq 1$ *is an integer for* $m = 1, 2, \ldots, i$.

This result is proved in section 3.

We consider the differential equation

$$\frac{dx}{d\theta} = G_0(\theta,x)+\varepsilon G_1(\theta,x)+\varepsilon^2 G_2(\theta,x)+\varepsilon^3 G_3(\theta,x)+\varepsilon^4 R(\theta,x,\varepsilon), \tag{7}$$

where $G_i(\theta,x)$, $i=0,1,2,3$, are the functions defined in (4) and

$$R(\theta,x,\varepsilon) = \sum_{k\geq 4} \varepsilon^{k-4}\, G_k(\theta,x).$$

Now we shall apply Theorem 4 for providing the averaging theory at arbitrarily high order for the differential equation (4), and we develop this theory explicitly until order three. The averaging theory of first order for studying periodic orbits of a differential equation (4) in $\mathbb{R}^n$ is very classical, see for instance Malkin [6], Roseau [7], Buică, Françoise and Llibre [2] where the authors studied the first order averaging theory of equations (7), and Buică, Giné and Llibre [3] where is studied the second order averaging theory of equations (7) in $\mathbb{R}^n$. Here we shall provide the explicit averaging theory of third order with $x\in\mathbb{R}$.

We define the real functions $G_{i0}(z)$ of real variable z

$$\begin{aligned}
G_{10}(z) &= \int_0^{2\pi} \frac{G_1(\varphi,x_0(\varphi,z))}{u(\varphi,z)}\,d\varphi,\\
G_{20}(z) &= \int_0^{2\pi}\left(\frac{G_2(\varphi,x_0(\varphi,z))}{u(\varphi,z)} + \frac{\partial G_1}{\partial x}(\varphi,x_0(\varphi,z))u_1(\varphi,z)\right.\\
&\quad\left. +\frac{1}{2}\frac{\partial^2 G_0}{\partial x^2}(\varphi,x_0(\varphi,z))u(\varphi,z)u_1(\varphi,z)^2\right)d\varphi,\\
G_{30}(z) &= \int_0^{2\pi}\left(\frac{G_3(\varphi,x_0(\varphi,z))}{u(\varphi,z)} + \frac{\partial G_2}{\partial x}(\varphi,x_0(\varphi,z))u_1(\varphi,z)\right.\\
&\quad +\frac{1}{2}\frac{\partial^2 G_1}{\partial x^2}(\varphi,x_0(\varphi,z))u(\varphi,z)u_1(\varphi,z)^2\\
&\quad +\frac{1}{6}\frac{\partial^3 G_0}{\partial x^3}(\varphi,x_0(\varphi,z))u(\varphi,z)^2u_1(\varphi,z)^3\\
&\quad +\frac{\partial G_1}{\partial x}(\varphi,x_0(\varphi,z))u_2(\varphi,z)\\
&\quad\left. +\frac{\partial^2 G_0}{\partial x^2}(\varphi,x_0(\varphi,z))u(\varphi,z)u_1(\varphi,z)u_2(\varphi,z)\right)d\varphi.
\end{aligned}$$

Corollary 5 (Averaging theory for periodic orbits up to third order for a differential equation (7))**.** *Assume that the solution $x_0(\theta,z)$ of the unperturbed equation (7) such that $x_0(0,z)=z$ is 2π–periodic for $z\in\mathcal{I}$ with $\mathcal{I}$ a real open interval.*

(a) *If $G_{10}(z)$ is not identically zero in $\mathcal{I}$, then for each simple zero $z^*\in\mathcal{I}$ of $G_{10}(z)=0$ there exists a periodic solution $x_\varepsilon(\theta,z)$ of equation (7) such that $x_\varepsilon(0,z)\to z^*$ when $\varepsilon\to 0$.*

(b) *If* $G_{10}(z)$ *is identically zero in* $\mathcal{I}$ *and* $G_{20}(z)$ *is not identically zero in* $\mathcal{I}$*, then for each simple zero* $z^* \in \mathcal{I}$ *of* $G_{20}(z) = 0$ *there exists a periodic solution* $x_\varepsilon(\theta, z)$ *of equation* (7) *such that* $x_\varepsilon(0, z) \to z^*$ *when* $\varepsilon \to 0$*.*

(c) *If* $G_{10}(z)$ *and* $G_{20}(z)$ *are identically zero in* $\mathcal{I}$ *and* $G_{30}(z)$ *is not identically zero in* $\mathcal{I}$*, then for each simple zero* $z^* \in \mathcal{I}$ *of* $G_{30}(z) = 0$ *there exists a periodic solution* $x_\varepsilon(\theta, z)$ *of equation* (7) *such that* $x_\varepsilon(0, z) \to z^*$ *when* $\varepsilon \to 0$*.*

This corollary is proved in section 3.

We consider arbitrary polynomial perturbations

$$\begin{aligned} \dot{x} &= -y + \sum_{j\geq 1} \varepsilon^j f_j(x, y), \\ \dot{y} &= x + \sum_{j\geq 1} \varepsilon^j g_j(x, y), \end{aligned} \tag{8}$$

of the harmonic oscillator, where ε is a small parameter. In this differential equation, the polynomials f_j and g_j are of degree n in the variables x and y and the system is analytic in the variables x, y and ε. We consider the change to polar coordinates $x = r\cos\theta$, $y = r\sin\theta$. In coordinates (r, θ) the differential system (8) becomes

$$\frac{dr}{d\theta} = \frac{\sum_{i\geq 1} \varepsilon^i p_i(\theta, r)}{1 + \sum_{i\geq 1} \varepsilon^i q_i(\theta, r)}, \tag{9}$$

where

$$\begin{aligned} p_i(\theta, r) &= \cos\theta\, f_i(\cos\theta, \sin\theta) + \sin\theta\, g_i(\cos\theta, \sin\theta), \\ q_i(\theta, r) &= \frac{1}{r}\left(\cos\theta\, g_i(\cos\theta, \sin\theta) - \sin\theta\, f_i(\cos\theta, \sin\theta)\right), \end{aligned}$$

and $r \in [0, R]$ with $R > 0$ is arbitrary. We observe that equation (9) is a particular case of equation (1).

We denote by $r_\varepsilon(\theta, z)$ the solution of the differential equation (9) with initial condition $r_\varepsilon(0, z) = z$. Due to the analyticity of this equation in ε, we write

$$r_\varepsilon(\theta, z) = z + \sum_{j\geq 1} r_j(\theta, z)\varepsilon^j, \tag{10}$$

where $r_j(\theta, z)$ are real functions such that $r_j(0, z) = 0$.

In what follows we state the next result due to Iliev [5] and we shall prove it in section 2 in a different way using Theorem 1.

Theorem 6. *Assume that* $r_s(2\pi, z)$ *is the first function in* (10) *which is not identically zero. Then,* $r_s(2\pi, z)$ *is a polynomial and it has no more than* $[s(n-1)/2]$ *positive roots counting their multiplicities.*

Here, $[a]$ denotes the integer part function of a. We note that Theorem 6 was stated by Iliev using Melnikov functions and that in dimension two the zeroes of the s-th Melnikov functions coincide with the zeroes of $r_s(2\pi, z)$, for more details see [4].

Additional applications of Theorem 1 are the following two propositions.

We consider the ordinary analytic differential equation

$$\frac{dx}{d\theta} = a(\theta) + \sum_{k\geq 1} \varepsilon^k F_k(\theta, x), \tag{11}$$

where $x \in \mathbb{R}$, $\theta \in \mathbb{S}^1$ and $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$ with ε_0 a small positive real value, the function $a(\theta)$ is 2π–periodic and the functions $F_k(\theta, x)$ are 2π–periodic in the variable θ. We define $A(\theta) := \int_0^\theta a(\varphi)d\varphi$ and the solution $x_\varepsilon(\theta, z)$ of equation (11) with initial condition $x_\varepsilon(0, z) = z$, writes as

$$x_\varepsilon(\theta, z) = z + A(\theta) + \sum_{j\geq 1} x_j(\theta, z)\varepsilon^j. \tag{12}$$

Proposition 7. *Assume that the functions $F_k(\theta, x)$ in* (11) *are polynomials in x of degree at most n and that $A(2\pi) = 0$. Assume that $x_s(2\pi, z)$ is the first function in* (12) *which is not identically zero. Then $x_s(2\pi, z)$ is a polynomial of degree at most $s(n-1)+1$.*

This proposition is proved in section 2.

We consider the ordinary analytic differential equation

$$\frac{dx}{d\theta} = b(\theta)x + \sum_{k\geq 1} \varepsilon^k F_k(\theta, x), \tag{13}$$

where $x \in \mathbb{R}$, $\theta \in \mathbb{S}^1$ and $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$ with ε_0 a small positive real value, the function $b(\theta)$ is 2π–periodic and the functions $F_k(\theta, x)$ are 2π–periodic in the variable θ. We define $B(\theta) := \int_0^\theta b(\varphi)d\varphi$ and the solution $x_\varepsilon(\theta, z)$ of equation (13) with initial condition $x_\varepsilon(0, z) = z$, can be written as

$$x_\varepsilon(\theta, z) = z\, e^{B(\theta)} + \sum_{j\geq 1} x_j(\theta, z)\varepsilon^j. \tag{14}$$

Proposition 8. *Assume that the functions $F_k(\theta, x)$ in* (13) *are polynomials in x of degree at most n and that $B(2\pi) = 0$. Assume that $x_s(2\pi, z)$ is the first function in* (14) *which is not identically zero. Then $x_s(2\pi, z)$ is a polynomial of degree at most $s(n-1)+1$.*

This proposition is proved in section 2.

2. Proof of Theorem 1 and applications

Proof of Theorem 1. By definition we have that the function $x_\varepsilon(\theta, z)$ defined in (2) is the solution of equation (1), for any sufficiently small value of $|\varepsilon|$, that is,

$$\frac{\partial x_\varepsilon(\theta,z)}{\partial\theta} = \sum_{k\geq 1}\varepsilon^k F_k\left(\theta, x_\varepsilon(\theta,z)\right). \tag{15}$$

Since this equality is verified for all sufficiently small value of $|\varepsilon|$, we can equate the coefficients of the same powers of ε in both members of the equality. Taking into account (2), the expansion in powers of ε in the left-hand side of (15) is

$$\frac{\partial x_\varepsilon(\theta,z)}{\partial\theta} = \sum_{n\geq 1}\frac{\partial x_n(\theta,z)}{\partial\theta}\,\varepsilon^n. \tag{16}$$

The expansion in powers of ε in the right-hand side of (15) involves more calculations. First we fix a value of k, $k\geq 1$, and we note that

$$\begin{aligned} F_k\left(\theta, x_\varepsilon(\theta,z)\right) &= F_k\left(\theta, z+\sum_{j\geq 1} x_j(\theta,z)\right) \\ &= F_k(\theta,z) + \sum_{i\geq 1}\frac{1}{i!}\frac{\partial^i F_k}{\partial x^i}(\theta,z)\left(\sum_{j\geq 1} x_j(\theta,z)\varepsilon^j\right)^i, \end{aligned}$$

where we have used the expression of the Taylor expansion of $F_k(\theta,x)$ in a neighborhood of a point $x=z$. We observe that, the expression $\left(\sum_{j\geq 1} x_j(\theta,z)\varepsilon^j\right)^i$ is divisible by ε^i and that given an integer $\ell\geq i$, the coefficient of ε^ℓ in the expression $\left(\sum_{j\geq 1} x_j(\theta,z)\varepsilon^j\right)^i$ corresponds to all the possible ways of obtaining ℓ by adding i indices j_1, j_1, $\ldots$, j_i (with repetition). That is

$$\left(\sum_{j\geq 1} x_j(\theta,z)\varepsilon^j\right)^i = \sum_{\ell\geq i}\varepsilon^\ell \sum_{j_1+j_2+\cdots+j_i=\ell} x_{j_1}(\theta,z)x_{j_2}(\theta,z)\cdots x_{j_i}(\theta,z).$$

In order to simplify notation, we define

$$\Omega_{k,\ell,i}(\theta,z) = \frac{1}{i!}\frac{\partial^i F_k}{\partial x^i}(\theta,z)\sum_{j_1+j_2+\cdots+j_i=\ell} x_{j_1}(\theta,z)x_{j_2}(\theta,z)\cdots x_{j_i}(\theta,z). \tag{17}$$

We can write

$$F_k\left(\theta, x_\varepsilon(\theta,z)\right) = F_k(\theta,z) + \sum_{i\geq 1}\sum_{\ell\geq i}\varepsilon^\ell\,\Omega_{k,\ell,i}(\theta,z).$$

By changing the order of the summation indices we have that

$$F_k\left(\theta, x_\varepsilon(\theta,z)\right) = F_k(\theta,z) + \sum_{\ell\geq 1}\varepsilon^\ell\sum_{i=1}^{\ell}\Omega_{k,\ell,i}(\theta,z).$$

Hence, the expansion in powers of ε in the right-hand side of (15) is

$$\sum_{k\geq 1}\varepsilon^k F_k\left(\theta, x_\varepsilon(\theta,z)\right) = \sum_{k\geq 1}\varepsilon^k F_k(\theta,z) + \sum_{k\geq 1}\sum_{\ell\geq 1}\varepsilon^{k+\ell}\sum_{i=1}^{\ell}\Omega_{k,\ell,i}(\theta,z). \tag{18}$$

We see that the second term of the former expression is divisible by ε^2. Thus, the coefficient of ε^1 in this expression is $F_1(\theta,z)$. Equating the coefficient of ε^1 in (16) we deduce that

$$\frac{\partial x_1(\theta,z)}{\partial\theta} = F_1(\theta,z).$$

We have, by definition, that $x_\varepsilon(0,z) = z$ for any sufficiently small value of $|\varepsilon|$. This fact and (2) imply that

$$x_j(0,z) = 0, \qquad \text{for all } j\geq 1. \tag{19}$$

Hence,

$$x_1(\theta,z) = \int_0^\theta F_1(\varphi,z)\,d\varphi.$$

We fix an integer $n>1$, and the coefficient of ε^n in (18) is $F_n(\theta,z)$ plus all the terms of the second term such that $k\geq 1$, $\ell\geq 1$ and $k+\ell = n$. We see that these terms correspond to substituting k by $n-\ell$ and adding from $\ell=1$ to $\ell=n-1$. Hence, the coefficient of ε^n in (18) is

$$F_n(\theta,z) + \sum_{\ell=1}^{n-1}\sum_{i=1}^{\ell}\Omega_{n-\ell,\ell,i}(\theta,z).$$

By equating the same coefficients of ε^n in (16) and using (19), we get that

$$x_n(\theta,z) = \int_0^\theta\left(F_n(\varphi,z) + \sum_{\ell=1}^{n-1}\sum_{i=1}^{\ell}\Omega_{n-\ell,\ell,i}(\varphi,z)\right)d\varphi.$$

If we substitute the definition of the function $\Omega_{n-\ell,\ell,i}(\theta,z)$ as defined in (17), we are done. $\square$

Proof of Corollary 2. The perturbed differential equation (3) for a sufficiently small value of $|\varepsilon|$ defines a Poincaré map $P_\varepsilon(z)$ for any initial condition $z\in\mathbb{R}$. Since the solution of this differential equation is (2), we note that the Poincaré map reads for $P_\varepsilon(z) = x_\varepsilon(2\pi,z)-z = \sum_{j\geq 1}x_j(2\pi,z)\varepsilon^j$. As usual, the zeroes of the Poincaré map $P_\varepsilon(z)$ correspond to periodic orbits of the differential equation (3).

Given an index k with $k\in\{1,2,3,4\}$, we note that the expression of the function $F_{k0}(z)$ correspond to $x_k(2\pi,z)$ as a direct consequence of Theorem 1. Assume that k is the lowest index such that $F_{k0}(z)$ is not identically zero. We want to show that if z^* is a simple zero of $F_{k0}(z)$, then there exists a periodic solution of equation (3) whose initial condition at $\theta = 0$ tends to z^* when $\varepsilon \to 0$. We define the auxiliary function

$$\phi(\varepsilon, z) = \frac{x_\varepsilon(2\pi, z) - z}{\varepsilon^k} = \frac{P_\varepsilon(z)}{\varepsilon^k}.$$

We remark that this function is analytic for $z \in \mathbb{R}$ and for ε in a neighborhood of $\varepsilon = 0$. Indeed, its Taylor expansion around the point $\varepsilon = 0$ is $\phi(\varepsilon, z) = F_{k0}(z) + \varepsilon\psi(\varepsilon, z)$, with $\psi(\varepsilon, z)$ an analytic function for $z \in \mathbb{R}$ and for ε in a neighborhood of $\varepsilon = 0$. We observe that the function $\phi(\varepsilon, z)$ satisfies the hypothesis of the Implicit Function Theorem in a neighborhood of the point $(\varepsilon, z) = (0, z^*)$, that is

$$\phi(0, z^*) = F_{k0}(z^*) = 0 \qquad \text{and} \qquad \left.\frac{\partial\phi(\varepsilon, z)}{\partial z}\right|_{(0,z^*)} = F'_{k0}(z^*) \neq 0.$$

Therefore, there exists a unique analytic function $\zeta(\varepsilon)$ defined in a neighborhood of $\varepsilon = 0$ such that

$$\phi(\varepsilon, \zeta(\varepsilon)) \equiv 0 \qquad \text{and} \qquad \zeta(0) = z^*.$$

We remark that this function also verifies that $P_\varepsilon(\zeta(\varepsilon)) \equiv 0$. Therefore, the solution $x_\varepsilon(\theta, \zeta(\varepsilon))$, with initial condition $z = \zeta(\varepsilon)$, is a periodic solution of the perturbed differential equation (3), such that $x_\varepsilon(\theta, \zeta(\varepsilon)) \to z^*$ when $\varepsilon \to 0$. ☐

The proofs of Propositions 7 and 8 will use the following result.

Lemma 9. *Assume that the functions $F_k(\theta, x)$ in (1) are polynomials in x of degree at most n. Assume that $x_s(2\pi, z)$ is the first function in (2) which is not identically zero. Then, the function $x_s(2\pi, z)$ is a polynomial in z of degree at most $s(n-1)+1$.*

Proof. We remark that the functions $x_k(\theta, z)$ in (2) verify the integral expressions given in Theorem 1. We will show by induction on k that the function $x_k(\theta, z)$ is a polynomial in z of degree at most $k(n-1)+1$.

Case $k = 1$. Since $F_1(\theta, z)$ is a polynomial in z of degree at most n and $x_1(\theta, z) = \int_0^\theta F_1(\varphi, z) d\varphi$, we have that $x_1(\theta, z)$ is a polynomial in z of degree at most $n = 1(n-1)+1$.

We assume, by induction hypothesis, that $x_j(\theta, z)$ is a polynomial in z of degree at most $j(n-1)+1$, for $1 \leq j \leq k$, and we want to show that $x_{k+1}(\theta, z)$ is a polynomial in z of degree at most $(k+1)(n-1)+1$. In the integral expression of $x_{k+1}(\theta, z)$ given in Theorem 1, there only appear the previous functions $x_j(\theta, z)$, for $1 \leq j \leq k$. Hence, given ℓ with $1 \leq \ell \leq k$ and given i with $1 \leq i \leq \ell$, the expression

$$\sum_{j_1+j_2+\cdots+j_i=\ell} x_{j_1}(\varphi, z) x_{j_2}(\varphi, z) \cdots x_{j_i}(\varphi, z)$$

is a polynomial in z of degree at most

$$(j_1(n-1)+1)+(j_2(n-1)+1)+\ldots+(j_i(n-1)+1) = \\ = (j_1+j_2+\cdots+j_i)\,(n-1) + i = \ell\,(n-1) + i.$$

On the other hand, the function $(\partial^i F_{k+1-\ell}/\partial x^i)(\varphi, z)$ is a polynomial in z of degree at most $n-i$. Hence, the expression

$$\frac{1}{i!}\frac{\partial^i F_{k+1-\ell}}{\partial x^i}(\varphi, z) \sum_{j_1+j_2+\cdots+j_i=\ell} x_{j_1}(\varphi, z)x_{j_2}(\varphi, z)\cdots x_{j_i}(\varphi, z),$$

is a polynomial in z of degree at most

$$\ell(n-1)+i+n-i = \ell(n-1)+n.$$

Since this degree increases with ℓ and the maximum value of ℓ is k, we have that this expression is a polynomial in z of degree at most $k(n-1)+n = (k+1)(n-1)+1$. Since $k \geq 1$, this degree is greater than or equal to n and, therefore, we have that the integrand of the expression of $x_{k+1}(\theta, z)$ in Theorem 1 is a polynomial in z of degree at most $(k+1)(n-1)+1$ and, thus, $x_{k+1}(\theta, z)$ is a polynomial in z of degree at most $(k+1)(n-1)+1$.

We have proved that $x_s(2\pi, z)$ is a polynomial in z of degree at most $s(n-1)+1$. □

Proof of Proposition 7. We consider the change of the dependent variable x by y in the differential equation (11) given by $x = y + A(\theta)$. We observe that with this change, the differential equation (11) becomes

$$\frac{dy}{d\theta} = \sum_{k\geq 1} \varepsilon^k F_k(\theta, y + A(\theta)), \tag{20}$$

where the functions $F_k(\theta, y + A(\theta))$ are polynomial in y of degree at most n. We remark that, since $A(\theta)$ is 2π-periodic in θ, the number of limit cycles of the differential equation (11) coincides with the number of limit cycles of equation (20), counted with multiplicities, by the change $x = y + A(\theta)$. On the other hand, the solution $x_\varepsilon(\theta, z)$ given in (12) verifies the equality $x_\varepsilon(\theta, z) = A(\theta)+y_\varepsilon(\theta, z)$, where $y_\varepsilon(\theta, z)$ is the solution of the above differential equation (20) with initial condition $y_\varepsilon(0, z) = z$. Therefore, if we write

$$y_\varepsilon(\theta, z) = z + \sum_{j\geq 1} y_j(\theta, z)\varepsilon^j,$$

we have that the function $x_j(\theta, z)$ coincides with the function $y_j(\theta, z)$ for any value of $(\theta, z) \in \mathbb{S}^1 \times \mathbb{R}$ and for any $j \geq 1$. Hence, assume that $x_s(2\pi, z)$ is the first function in (12) which is not identically zero, then $y_s(2\pi, z)$ is the first function in the former expansion which is not identically zero and we are under the hypothesis of Lemma 9. □

Proof of Proposition 8. We consider the change of the dependent variable x by y in the differential equation (13) given by $x = y\, e^{B(\theta)}$. We observe that with this change, the differential equation is

$$\frac{dy}{d\theta} = \sum_{k\geq 1} \varepsilon^k F_k(\theta, y\, e^{B(\theta)}), \tag{21}$$

where the functions $F_k(\theta, y\, e^{B(\theta)})$ are polynomial in y of degree at most n. We remark that, since $B(\theta)$ is 2π-periodic in θ, the number of limit cycles of the differential equation (13) coincides with the number of limit cycles of equation (21), counted with multiplicities, by the change $x = y\, e^{B(\theta)}$. On the other hand, the solution $x_\varepsilon(\theta, z)$ given in (14) verifies the equality $x_\varepsilon(\theta, z) = e^{B(\theta)}\, y_\varepsilon(\theta, z)$, where $y_\varepsilon(\theta, z)$ is the solution of the above differential equation (21) with initial condition $y_\varepsilon(0, z) = z$. Therefore, if we write

$$y_\varepsilon(\theta, z) = z + \sum_{j\geq 1} y_j(\theta, z)\varepsilon^j,$$

we have that the function $x_j(\theta, z) = e^{B(\theta)}\, y_j(\theta, z)$ for any value of $(\theta, z) \in \mathbb{S}^1 \times \mathbb{R}$ and for any $j \geq 1$. Hence, assume that $x_s(2\pi, z)$ is the first function in (14) which is not identically zero, then $y_s(2\pi, z)$ is the first function in the former expansion which is not identically zero and we are under the hypothesis of Lemma 9. □

3. Proof of Theorem 4 and applications

Lemma 10. *Assume that the solution $x_0(\theta, z)$ of the unperturbed differential equation* (4) *such that $x_0(0, z) = z$ is 2π–periodic for $z \in \mathcal{I}$ with $\mathcal{I}$ a real open interval. Then the function $u(\theta, z)$ defined in* (6) *is 2π–periodic in θ when $z \in \mathcal{I}$.*

Proof. We claim that $u = \partial x_0/\partial z$. Indeed, since $x_0(\theta, z)$ satisfies

$$\frac{\partial x_0}{\partial \theta} = G_0(\theta, x_0(\theta, z)),$$

derivating the equality with respect to z we get that

$$\frac{\partial^2 x_0}{\partial \theta \partial z} = \frac{\partial G_0}{\partial x}(\theta, x_0(\theta, z))\, \frac{\partial x_0}{\partial z},$$

or equivalently

$$\frac{\partial u}{\partial \theta} = \frac{\partial G_0}{\partial x}(\theta, x_0(\theta, z))\, u,$$

which is (6). From $x_0(0, z) = z$, it follows that $u(0, z) = 1$. So the claim is proved.

Due to the fact that the solution $x_0(\theta, z)$ of the unperturbed equation (4) is 2π–periodic for $z \in \mathcal{I}$, that is $x_0(2\pi, z) = z$, it follows $(\partial x_0/\partial z)(2\pi, z) = 1$. Hence, $u(2\pi, z) = 1$ when $z \in \mathcal{I}$. □

Proof of Theorem 4. We have defined the function $x_\varepsilon(\theta, z)$ given in (5) as the solution of equation (4). Thus, for any sufficiently small value of $|\varepsilon|$, the following equality holds

$$\frac{\partial x_\varepsilon(\theta, z)}{\partial \theta} = \sum_{k\geq 0} \varepsilon^k G_k\left(\theta, x_\varepsilon(\theta, z)\right). \tag{22}$$

As we have argued in the proof of Theorem 1, we can equate the coefficients of the same powers of ε in both members of this equality. Taking into account (5), the expansion in powers of ε in the left-hand side of (22) is

$$\frac{\partial x_\varepsilon(\theta, z)}{\partial \theta} = \sum_{n\geq 0} \frac{\partial x_n(\theta, z)}{\partial \theta}\, \varepsilon^n. \tag{23}$$

The expansion in powers of ε in the right-hand side of (22) is more involved. We fix an index k, with $k \geq 0$, and by analogous reasonings as the ones given in the proof of Theorem 1, we have that

$$G_k\left(\theta, x_\varepsilon(\theta, z)\right) = G_k(\theta, x_0(\theta, z)) + \sum_{\ell\geq 1} \varepsilon^\ell \sum_{i=1}^{\ell} \Theta_{k,\ell,i}(\theta, x_0(\theta, z), z),$$

where

$$\Theta_{k,\ell,i}(\theta, x_0(\theta, z), z) = \\ \frac{1}{i!}\frac{\partial^i G_k}{\partial x^i}(\theta, x_0(\theta, z)) \sum_{j_1+j_2+\cdots+j_i=\ell} x_{j_1}(\theta, z) x_{j_2}(\theta, z) \cdots x_{j_i}(\theta, z).$$

We consider the function $u(\theta, z)$ defined in (6), which is 2π-periodic in θ as it has been proved in Lemma 10. For each $i \geq 1$, we consider the defined functions $u_i(\theta, z)$ such that $x_i(\theta, z) = u(\theta, z)\, u_i(\theta, z)$. Therefore, the function $\Theta_{k,\ell,i}(\theta, x_0(\theta, z), z)$ becomes

$$\begin{aligned} \Theta_{k,\ell,i}(\theta, x_0(\theta, z), z) &= \frac{1}{i!}\frac{\partial^i G_k}{\partial x^i}(\theta, x_0(\theta, z)) \cdot \\ & u(\theta, z)^i \sum_{j_1+j_2+\cdots+j_i=\ell} u_{j_1}(\theta, z) u_{j_2}(\theta, z) \cdots u_{j_i}(\theta, z). \end{aligned} \tag{24}$$

So, the expansion in powers of ε of the right-hand side of equality (22) is

$$\sum_{k\geq 0} \varepsilon^k G_k(\theta, x_0(\theta, z)) + \sum_{k\geq 0}\sum_{\ell\geq 1} \varepsilon^{k+\ell} \sum_{i=1}^{\ell} \Theta_{k,\ell,i}(\theta, x_0(\theta, z), z). \tag{25}$$

From equality (22) we can equate the coefficients of the same powers of ε of expressions (23) and (25).

The coefficient of ε^0 in (25) is $G_0(\theta, x_0(\theta, z))$ and equating with the corresponding one in (23) we get that

$$\frac{\partial x_0(\theta, z)}{\partial \theta} = G_0(\theta, x_0(\theta, z)).$$

This is the definition of the function $x_0(\theta, z)$ as a particular solution of the unperturbed equation, that is, equation (4) with $\varepsilon = 0$.

The coefficient of ε^1 in (23) is $\partial x_1/\partial\theta$, where we have avoided to write the dependence in (θ, z) in order to simplify the notation. Since $x_1 = u\, u_1$, we have that

$$\frac{\partial x_1}{\partial \theta} = \frac{\partial u}{\partial \theta} u_1 + u \frac{\partial u_1}{\partial \theta}.$$

The definition (6) of u gives

$$\frac{\partial x_1(\theta, z)}{\partial \theta} = \left(\frac{\partial G_0}{\partial x}(\theta, x_0(\theta, z))\, u_1(\theta, z) + \frac{\partial u_1(\theta, z)}{\partial \theta} \right) u(\theta, z).$$

The coefficient of ε^1 in (25) is

$$\begin{aligned} & G_1(\theta, x_0(\theta, z)) + \Theta_{0,1,1}(\theta, x_0(\theta, z), z) = \\ & \quad = G_1(\theta, x_0(\theta, z)) + \frac{\partial G_0}{\partial x}(\theta, x_0(\theta, z))\, u(\theta, z)\, u_1(\theta, z). \end{aligned}$$

If we equate the two corresponding coefficients, we get that

$$\frac{\partial u_1(\theta, z)}{\partial \theta} u(\theta, z) = G_1(\theta, x_0(\theta, z)). \tag{26}$$

We remark that, given $i \geq 1$, since $x_i(0, z) = 0$ and $u(0, z) = 1$ (see (5) and (6)), we have that

$$u_i(0, z) = 0 \qquad \text{for all } i \geq 1. \tag{27}$$

From (26) and condition (27) with $i = 1$, we deduce that

$$u_1(\theta, z) = \int_0^\theta \frac{G_1(\varphi, x_0(\varphi, z))}{u(\varphi, z)}\, d\varphi.$$

Given an integer $n \geq 2$ and following similar arguments, we have that the coefficient of ε^n in (23) is

$$\frac{\partial x_n(\theta, z)}{\partial \theta} = \left(\frac{\partial G_0}{\partial x}(\theta, x_0(\theta, z))\, u_n(\theta, z) + \frac{\partial u_n(\theta, z)}{\partial \theta} \right) u(\theta, z). \tag{28}$$

On the other hand, the coefficient of ε^n in (25) is

$$G_n(\theta, x_0(\theta, z)) + \sum_{\ell=1}^{n} \sum_{i=1}^{\ell} \Theta_{n-\ell,\ell,i}(\theta, x_0(\theta, z), z),$$

where we have written $k = n - \ell$ and we have considered that $k \geq 0$ and $\ell \geq 1$. We change the order of summation in the indices ℓ and i and we do the change ℓ by s with $s = \ell - i$. We get that the previous expression is

$$G_n(\theta, x_0(\theta, z)) + \sum_{s=0}^{n-1} \sum_{i=1}^{n-s} \Theta_{n-s-i,s+i,i}(\theta, x_0(\theta, z), z). \tag{29}$$

We will write separately the term corresponding to $s = n-1$ (and, thus, $i = 1$) which is

$$\Theta_{0,n,1}(\theta, x_0(\theta, z), z) \;=\; \frac{\partial G_0}{\partial x}(\theta, x_0(\theta, z))\, u(\theta, z)\, u_n(\theta, z),$$

where we have used the definition (24). Hence, equating (28) with (29) we obtain

$$\frac{\partial u_n(\theta, z)}{\partial \theta} u(\theta, z) \;=\; G_n(\theta, x_0(\theta, z)) \;+\; \sum_{s=0}^{n-2}\sum_{i=1}^{n-s} \Theta_{n-s-i,s+i,i}(\theta, x_0(\theta, z), z).$$

From condition (27) and the definition (24) we directly get the integral expression written in the statement of Theorem 4. □

Proof of Corollary 5. This proof is a verbatim expression of the proof of Corollary 2 with the obvious changes of notation. □

Proof of Theorem 6. We recall that given any real value $|z| < 1$, the following expansion holds

$$\frac{1}{1+z} \;=\; \sum_{j\geq 0} (-1)^j z^j.$$

Thus, equation (9) can be written

$$\frac{dr}{d\theta} \;=\; \left(\sum_{i\geq 1} \varepsilon^i p_i(\theta, r)\right) \left[1 + \sum_{j\geq 1} (-1)^j \left(\sum_{i\geq 1} \varepsilon^i q_i(\theta, r)\right)^j\right].$$

Given $\ell \geq j$, we join together the coefficient of ε^ℓ in $\left(\sum_{i\geq 1} \varepsilon^i q_i(\theta, r)\right)^j$ which corresponds to summing up to ℓ with j indices $i_1, i_2, \ldots, i_j \geq 1$, that is

$$\left(\sum_{i\geq 1} \varepsilon^i q_i(\theta, r)\right)^j \;=\; \sum_{\ell\geq j} \varepsilon^\ell \left(\sum_{i_1+i_2+\ldots+i_j=\ell} q_{i_1}(\theta, r) q_{i_2}(\theta, r) \cdots q_{i_j}(\theta, r)\right).$$

Hence,

$$\begin{aligned} \sum_{j\geq 1} (-1)^j \left(\sum_{i\geq 1} \varepsilon^i q_i\right)^j \;&=\; \sum_{j\geq 1}\sum_{\ell\geq j} \varepsilon^\ell (-1)^j \left(\sum_{i_1+i_2+\ldots+i_j=\ell} q_{i_1} q_{i_2} \cdots q_{i_j}\right) \\ &=\; \sum_{\ell\geq 1}\sum_{j=1}^{\ell} \varepsilon^\ell (-1)^j \left(\sum_{i_1+i_2+\ldots+i_j=\ell} q_{i_1} q_{i_2} \cdots q_{i_j}\right), \end{aligned}$$

where we have avoided the dependence on (θ, r) for simplifying the notation, and where we have changed the order of summation in the indices. In order to simplify notation, we define the auxiliary function

$$\Lambda_\ell(\theta, r) \;:=\; \sum_{j=1}^{\ell} (-1)^j \sum_{i_1+i_2+\ldots+i_j=\ell} q_{i_1}(\theta, r) q_{i_2}(\theta, r) \cdots q_{i_j}(\theta, r). \tag{30}$$

Substituting it in the differential equation we have

$$\frac{dr}{d\theta} \;=\; \left(\sum_{i\geq 1} \varepsilon^i p_i(\theta, r) \right) \left[1 + \sum_{\ell \geq 1} \varepsilon^\ell \Lambda_\ell(\theta, r) \right].$$

We write the product of the two summation signs joining together the terms whose coefficient is ε^k and we have

$$\begin{aligned} \frac{dr}{d\theta} &= \sum_{i\geq 1} \varepsilon^i p_i(\theta, r) \,+\, \sum_{k\geq 2} \left(\sum_{\ell=1}^{k-1} p_{k-\ell}(\theta, r)\Lambda_\ell(\theta, r) \right) \varepsilon^k \\ &= p_1(\theta, r)\varepsilon \,+\, \sum_{k\geq 2} \left(p_k(\theta, r) \,+\, \sum_{\ell=1}^{k-1} p_{k-\ell}(\theta, r)\Lambda_\ell(\theta, r) \right) \varepsilon^k. \end{aligned}$$

Hence, if we define the functions $F_1(\theta, r) \,:=\, p_1(\theta, r)$ and

$$F_k(\theta, r) \,:=\, p_k(\theta, r) \,+\, \sum_{\ell=1}^{k-1} p_{k-\ell}(\theta, r)\Lambda_\ell(\theta, r),$$

for $k \geq 2$, we have that equation (9) becomes

$$\frac{dr}{d\theta} = \sum_{k\geq 1} F_k(\theta, r)\, \varepsilon^k. \tag{31}$$

By assumption, the polynomials $f_j(x, y)$ and $g_j(x, y)$ of system (8) are of degree at most n in x and y, which implies that the functions $p_i(\theta, r)$ and $q_i(\theta, r)$ are polynomials in r of degrees at most n and $n - 1$, respectively. From the definition (30) of the function $\Lambda_\ell(\theta, r)$, we have that it is a polynomial in r of degree at most $\ell(n-1)$. We see that $F_1(\theta, r)$ is a polynomial in r of degree at most n and $F_k(\theta, r)$ is a polynomial in r of degree at most $(k-1)(n-1) + n \;=\; k(n-1) + 1$. In summary, the functions $F_k(\theta, r)$ in equation (31) are polynomials in r of degree at most $k(n-1)+1$, for $k \geq 1$.

We note that equation (31) is a particular case of equation (1) and, thus, the functions $r_j(\theta, z)$ defined in (10) verify the integral expressions given in Theorem 1. We are going to show, by induction in j, that the function $r_j(\theta, z)$ is a polynomial in z of degree at most $j(n-1)+1$.

Case $j = 1$. Since $F_1(\theta, r)$ is a polynomial in r of degree at most n, we have by Theorem 1 that $r_1(\theta, z)$ is a polynomial in z of degree at most n.

We assume, by induction hypothesis, that $r_i(\theta, z)$ is a polynomial in z of degree at most $i(n-1)+1$, for $1 \leq i \leq j$. In the expression of $r_{j+1}(\theta, z)$ given in Theorem 1, there only appear the previous functions $r_i(\theta, z)$, for $1 \leq i \leq j$. We have that, given an integer ℓ with $1 \leq \ell \leq j$ and an integer i with $1 \leq i \leq \ell$, the summation function

$$\sum_{j_1+j_2+\cdots+j_i=\ell} r_{j_1}(\varphi, z) r_{j_2}(\varphi, z) \cdots r_{j_i}(\varphi, z)$$

is a polynomial in z of degree at most

$$(j_1(n-1)+1)+(j_2(n-1)+1)+\ldots+(j_i(n-1)+1) = \\ = (j_1+j_2+\cdots+j_i)(n-1)+i = \ell(n-1)+i.$$

On the other hand, the function $(\partial^i F_{j+1-\ell}/\partial x^i)(\varphi,z)$ is a polynomial in z of degree at most $(j+1-\ell)(n-1)+1-i$. Hence, the expression

$$\frac{1}{i!}\frac{\partial^i F_{j+1-\ell}}{\partial x^i}(\varphi,z) \sum_{j_1+j_2+\cdots+j_i=\ell} r_{j_1}(\varphi,z)r_{j_2}(\varphi,z)\cdots r_{j_i}(\varphi,z),$$

is a polynomial in z of degree at most

$$\ell(n-1)+i+(j+1-\ell)(n-1)+1-i = jn+n-j = (j+1)(n-1)+1.$$

Since $j \geq 1$, this degree is greater than or equal to n and, therefore, we have that the integrand of the expression of $r_{j+1}(\theta,z)$ in Theorem 1 is a polynomial in z of degree at most $(j+1)(n-1)+1$ and, thus, $r_{j+1}(\theta,z)$ is a polynomial in z of degree at most $(j+1)(n-1)+1$.

We recall that since the differential equation (9) is the transformation to polar coordinates of the planar differential system (8), we have that its flow $r_\varepsilon(\theta,z)$ satisfies that $r_\varepsilon(\theta+\pi,z) = -r_\varepsilon(\theta,-z)$ for any real value of θ and z, see for instance [12]. This symmetry implies that $r_s(2\pi,z)$ has $z=0$ as zero and that if z^* is one of its zeroes, then $-z^*$ is also a zero.

We have proved that $r_s(2\pi,z)$ is a polynomial in z of degree $s(n-1)+1$. And we conclude that it can have at most $[s(n-1)/2]$ positive zeroes. □

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References

[1] A. Buică and J. Llibre, *Averaging methods for finding periodic orbits via Brouwer degree*, Bull. Sci. Math. **128** (2004), 7–22.

[2] A. Buică, J.P. Françoise and J. Llibre, *Periodic solutions of nonlinear periodic differential systems with a small parameter*, Comm. on Pure and Appl. Anal. **6** (2007), 103–111.

[3] A. Buică, J. Giné and J. Llibre, *A second order analysis of the periodic solutions for nonlinear periodic differential systems with a small parameter*, Physica D **241** (2012), 528–533.

[4] J. Giné and J. Llibre, *Limit cycles of cubic polynomial vector fields via the averaging theory*, Nonlinear Analysis **66** (2007), 1707–1721.

[5] I. Iliev, *The number of limit cycles due to polynomial perturbations of the harmonic oscillator*, Math. Proc. Cambridge Philos. Soc. **127** (1999), no. 2, 317–322.

[6] I.G. Malkin, Some problems of the theory of nonlinear oscillations, (Russian) Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1956.

[7] M. Roseau, Vibrations non linéaires et théorie de la stabilité, (French) Springer Tracts in Natural Philosophy, Vol.8, Springer-Verlag, Berlin-New York. 1966.
[8] R. Roussarie, Bifurcation of planar vector fields and Hilbert's sixteenth problem. Progress in Mathematics, 164. Birkhäuser Verlag, Basel, 1998.
[9] J.A. Sanders, F. Verhulst and J. Murdock, Averaging methods in nonlinear dynamical systems. Second edition. Applied Mathematical Sciences, **59**. Springer, New York, 2007.
[10] F. Verhulst, Nonlinear Differential Equations and Dynamical Systems, second edition, Universitext, Springer-Verlag, Berlin. 1996.
[11] Xiang Zhang, private communication.
[12] Zhang Zhi Fen *et al.* Qualitative theory of differential equations. Translations of Mathematical Monographs, **101**. American Mathematical Society, Providence, RI, 1992.

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