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AXIOMATIZATIONS OF INTUITIONISTIC DOUBLE NEGATION

We investigate intuitionistic propositional modal logics in which a modal operator $\Box$ is equivalent to intuitionistic double negation. Whereas $\neg\neg$ is divisible into two negations, $\Box$ is a single indivisible operator. We shall first consider an axiomatization of the Heyting propositional calculus H, with the connectives $\rightarrow, \wedge, \vee$ and $\neg$, extended with $\Box$. This system will be called Hdn ("dn" stands for "double negation"). Next, we shall consider an axiomatization of the fragment of H without $\neg$ extended with $\Box$. This system will be called Hdn^+. We shall show that these systems are sound and complete with respect to specific classes of Kripke-style models with two accessibility relations, one intuitionistic and the other modal. This type of models is investigated in [2] and [3], and here we try to apply the techniques of these papers to an intuitionistic modal operator with a natural interpretation. The full results of our investigation will be published in [4] and [1].

The system Hdn. The language L is the language of propositional modal logic with the propositional variables $p, q, \ldots$ and the connectives $\rightarrow, \wedge, \vee, \neg$ and $\Box$ ($\leftrightarrow$ is defined as usual is usual in terms of $\rightarrow$ and $\wedge$, and in formulae bind more strongly than $\rightarrow$). As schemata for formulae we use $A, B, C, \ldots$. The system Hdn is axiomatized with *modus ponens* and the following axiom-schemata:

$H1$. $A \rightarrow (B \rightarrow A)$;
$H2$. $(A \rightarrow (B \rightarrow C)) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C))$;
$H3$. $(C \rightarrow A) \rightarrow ((C \rightarrow B) \rightarrow (C \rightarrow A \wedge B))$;
$H4$. $A \wedge B \rightarrow A$;

$H5.$ $A \wedge B \to B$;
$H6.$ $A \to A \vee B$;
$H7.$ $B \to A \vee B$;
$H8.$ $(A \to C) \to ((B \to C) \to (A \vee B \to C))$;
$H9.$ $(A \to \neg B) \to (B \to \neg A)$;
$H10.$ $\neg A \to (A \to B)$;
$dn1.$ $\Box(A \to B) \to (\Box A \to \Box B)$;
$dn2.$ $A \to \Box A$;
$dn3.$ $\Box(((A \to B) \to A) \to A)$;
$dn4.$ $\neg\Box\neg(A \to A)$.

It is easy to show that the system obtained by replacing $dn1 - dn4$ by

$dn0.$ $A \leftrightarrow \neg\neg A$

has the same theorems as Hdn. Using $dn1 - dn4$ is, however, more suitable when one wants to connect Hdn with the models given below and to compare Hdn with Hdn^+. Since Hdn is closed under replacement of equivalent formulae, $dn0$ guarantees that $\Box$ in Hdn stands for intuitionistic double negation.

An Hdn *frame* is $\langle X, R_I, R_M \rangle$ where $X \neq \emptyset$, $R_I \subseteq X^2$ is reflexive and transitive, $R_M \subseteq X^2$ and

(1) $R_I \circ R_M \subseteq R_M \circ R_I$,
(2) $R_M \subseteq R_I$,
(3) $\forall x, y(xR_My \Rightarrow \forall z(yR_Iz \Rightarrow zR_Iy))$,
(4) $\forall x \exists y xR_My$; the variables $x, y, z, \ldots$ range over X.

An Hdn *model* is $\langle X, R_I, R_M, V \rangle$ where $\langle X, R_I, R_M \rangle$ is an Hdn frame and the valuation V is a mapping from the set of propositional variables of L to the power set of X such that for every p, $\forall x, y(xR_Iy \Rightarrow (x \in V(p) \Rightarrow y \in V(p)))$. The relation $\models$ in $x \models A$ is defined as usual for $\to, \wedge, \vee$ and $\neg$, using R_I for $\to$ and $\neg$, whereas $x \models \Box A \Leftrightarrow_{df} \forall y(xR_My \Rightarrow y \models A)$. A formula A holds in a frame Fr iff A holds in every model with the frame Fr; and A is valid iff A holds in every frame. An Hdn frame (model) is *condensed* iff $R_I \circ R_M = R_M$, and it is *strictly condensed* iff $R_I \circ R_M = R_M \circ R_I = R_M$. Strictly condensed Hdn frames from a proper subclass of condensed Hdn frames, with form a proper subclass of the class of all Hdn frames.

Let Fr be a frame which satisfies only (1), and not necessarily also (2)-(4). Then it is possible to show that: $dn2$ holds in Fr iff (2) holds for

Fr; $dn3$ holds in Fr iff (3) holds for Fr; and $dn4$ holds in Fr iff (4) holds for Fr.

By a fairly standard proof with a canonical model it is possible to show that Hdn is sound and complete with respect to the class of all (all condensed, all strictly condensed) Hdn frames.

In the definition of strictly condensed Hdn frames (1)-(3) and the condition $R_I \circ R_M = R_M \circ R_I = R_M$ can all be replaced by the condition

$$\forall x, y(xR_My \Leftrightarrow (xR_Iy \textit{ and } \forall z(yR_Iz \Rightarrow zR_Iy)))$$

yielding the same class of frames. So in these frames R_M is definable in terms of R_I. Now, if in the definition of Hdn frames we require that R_I is not only reflexive and transitive, but a partial ordering, our soundness and completeness results still hold. However, in that case all Hdn frames are strictly condensed (just show $R_M \circ R_I \subseteq R_M$). Hence, we have shown Hdn sound and complete with respect to partially ordered frames where for any x there is a maximal element y above x, xR_My means that y is one of these maximal elements, and $x \models \Box A$ means that A holds in all these maximal elements.

The system Hdn^+. The system Hdn^+ will be formulated in the language L^+ which is L without $\neg$, and in addition to *modus ponens* and the axiom-schemata $H1 - H8$, $dn1 - dn3$ it will have the axiom-schema

$dn5$. $\Box(\Box A \rightarrow A)$.

This system axiomatizes Heyting's positive propositional logic extended with intuitionistic double negation, but not with negation. To show that we proceed as follows.

An Hdn^+ *frame* differs from an Hdn frame in having

(5) $\forall x, y(xR_M \circ R_Iy \Rightarrow yR_M \circ R_Iy)$

instead of (4). It is easy to show that Hdn frames form a proper subclass of Hdn^+ frames. It is also possible to show that for any frame Fr which satisfies only (1), $dn5$ holds in Fr iff (5) holds for Fr.

Again by a standard proof with a canonical model shows that Hdn^+ is sound and complete with respect to the class of all Hdn^+ frames.

In order to prove that Hdn^+ captures all the theorems of Hdn without $\neg$ we proceed as follows. Suppose a formula A from L^+ is not a theorem of Hdn^+; hence, it is falsified in an Hdn^+ model $\langle X, R_I, R_M, V\rangle$. The *closure* of this model will be $\langle \overline{X}, \overline{R}_I, \overline{R}_M, \overline{V} \rangle$ where $\overline{X} = X \cup \{1\}$, $x\overline{R}_I y \Leftrightarrow (xR_I y$ *or* $(y = 1$ *and* $\exists z(xR_I z$ *and not* $\exists tzR_M t))$ *or* $x = y = 1)$, $x\overline{R}_M y \Leftrightarrow (xR_M y$ *or* $(x\overline{R}_I y$ and $y = 1))$, and $\overline{V}(p) = V(p) \cup \{1\}$. Since it is possible to show that the closure of an Hdn^+ model is an Hdn model, and that in these two models the same formulae for L^+ holds in the members of X, it follows that A is falsified in Hdn model, and hence A is not a theorem of Hdn.

The system Hdn^+ extended with $H9$ and $H10$ is weaker than Hdn, since $dn4$ and $\Box A \to \neg\neg A$ are not provable in it. Alternatively, it is also possible to axiomatize Hdn^+ using $\Box\Box A \to \Box A$ instead of $dn5$.

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