AXISYMMETRIC CREEPING FLOW PAST A POROUS PROLATE SPHEROIDAL PARTICLE USING THE BRINKMAN MODEL

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Summary

A boundary-value solution to axisymmetric creeping flow past and through a porous prolate spheroidal particle is presented. The Brinkman model for the flow inside the porous medium and the Stokes model for the free-flow region in their stream function formulations are used. As boundary conditions, continuity of velocity, pressure and tangential stresses across the interface are employed. A mainly analytical procedure for calculating the required eigenvalues and eigenfunctions for the porous region part of the solution is proposed. The coefficients of the convergent series expansions of the general solutions for the stream functions, and thus for the velocity, pressure, vorticity and stress fields, both for the flow inside and outside the porous particle, can be calculated to any desired degree of accuracy as the solution of a truncated algebraic system of linear equations, once the eigenvalues to the Brinkman equation for a given focal distance and permeability have been computed. The drag force experienced by the porous particle is then given as a function of only one of these coefficients. Streamline-pattern and drag-force dependence on permeability and focal distance are presented and discussed.

1. Introduction

The computational prediction of the relevant hydrodynamic parameters of the flow of a viscous incompressible fluid past a porous particle is of considerable practical and theoretical interest in many physical and engineering applications. The usual macroscopic continuum approach to this complex fluid dynamical problem, as a satisfactory approximation in many real processes, is to neglect inertial and volume forces as well as thermal influences, and to treat it as a *multifield boundary-value problem* governed by the steady-state Stokes equation in the free-flow region and the Darcy or the Brinkman equation in the region occupied by the porous body. In reality, the particle may be arbitrarily shaped and of a fairly odd structure, such as a solid core covered with one or more porous layers, a void one-layer or multi-layered shell of porous material, or only a single porous particle with uniform or varying permeability, exhibiting a uniform slow translational motion relative to the quiescent unbounded liquid space.

Analytical solutions to the creeping-flow problem just stated, despite the inherent as well as practical usefulness that these would have, have not been found except for the geometrically simplest cases of a porous sphere or spherical shell. In the past these simpler cases were investigated by many authors in different ways.

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Ooms *et al.* (**1**) discussed, among others, the reasons which led Brinkman (**2**) and Debye (**3**) independently to suggest a modified Darcy equation which is now commonly known as the Brinkman equation. These authors gave an analytical solution of the problem for the composite porous spherical particle, having in mind its application to a polymer coil in a solvent as a typical example of a porous particle of non-uniform permeability. In passing they also addressed the problem of a porous shell with a solid or cavity core.

Another way to solve the problem under consideration is to use numerical methods, as was done by Youngren and Acrivos (**4**) for creeping flow past a solid body employing the well-known Ladyzhenskaya (**5**) reduction of the Stokes problem to the solution of a system of Cartesian-tensor integral equations for velocity and shear-stress along the boundary. This possibility was discussed by Higdon and Kojima (**6**) for a single porous body of uniform permeability, in which case Howells's (**7**) Green's-function formulation of the Brinkman equation for the porous region has to be used. These authors derived asymptotic results for small and large permeability, which simplifies the analysis for the case of particles with arbitrary geometry. The computer code for the full set of integral equations as well as its numerical implementation turn out to be rather complex even for the simplest case of axisymmetric creeping flow past a uniformly porous sphere. Less complicated solution techniques for specific geometries are needed. Such solutions, whilst being interesting in their own right, can be used as a check for more complicated numerical schemes that solve problems for arbitrary geometries.

Qin and Kaloni (**8**) obtained an exact solution for the axisymmetric creeping flow past a porous one-layer spherical shell immersed in a uniform Newtonian incompressible viscous fluid using Brinkman's equation for the flow inside the porous region, the Stokes equation for the free-flow regions, and continuity conditions for velocity, pressure and tangential stresses across the two permeable interfaces. These authors considered the core to be either a free-flow cavity or a solid region. The practical importance of such a solution is seen, for instance, in the need for understanding the flow past a collection of fine particles. Such collections usually possess low bulk density and very high porosity, so that for the porous-flow region the use of the Brinkman equation rather than Darcy's law is appropriate. These authors, like many others previously, also discussed and emphasized the merits of the Brinkman equation over Darcy's equation, the latter having the drawback of necessitating the use of empirical boundary conditions, and showed how Darcy's solution can be approximated from the Brinkman solution. As for the solution method itself, these authors employed their previously developed Cartesian-tensor solution of the Brinkman equation (Qin and Kaloni (**9**)). Using the stream function formulation, Bahtt and Sacheti (**10**) also solved the same problem of creeping flow past a porous spherical shell in a uniform ambient flow.

Here we are concerned with the analytical solution to the problem of creeping axisymmetric flow past a single porous prolate spheroidal particle. In many applications, porous bodies of spheroidal rather than spherical shape may be encountered, which justifies an investigation of the problem under consideration. As starting equations we also use the Brinkman equation for the flow inside the porous region and the Stokes equation for the free-flow region, but employ the stream function formulation as a boundary-eigenvalue problem to obtain a series expansion of the general solution to the Brinkman equation in modified spheroidal coordinates. A simple but effective and mainly analytical procedure for calculating the eigenvalues for the porous region part of the solution is proposed. The coefficients of the convergent series expansion for the stream function, and thus for the velocity, pressure, vorticity and stress fields, both inside and outside the spheroidal particle, can be calculated to any desired degree of accuracy as the solution

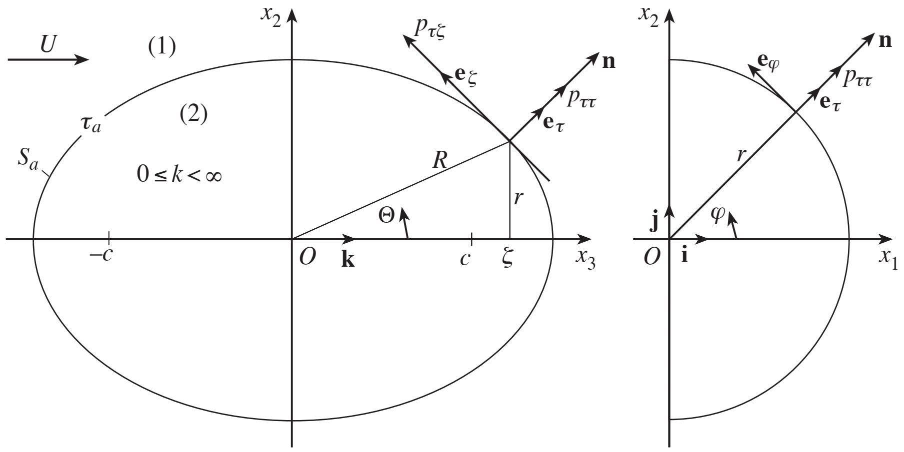

Fig. 1 Porous prolate spheroidal particle in uniform ambient flow, coordinate system and flow regions

of a truncated algebraic system of simultaneous linear equations, once the eigenvalues for a given focal distance and permeability have been computed. The drag force experienced by the spheroidal particle is then given as a function of only one of these coefficients. When the focal distance of the spheroidal particle tends to zero, we obtain the exact solution for the spherical particle.

Concerning the basic characteristics of the spheroidal geometry, we refer to Happel and Brenner (**11**) and to the work of Dassios *et al.* (**12**), from which we adopt part of the notation needed. These latter authors derived a semiseparable general solution for the free-flow Stokes operator in modified spheroidal coordinates, which we have used here.

For arbitrary chosen values of the focal distance, the streamline pattern and drag force, which are dependent on the permeability of the single porous prolate spheroid, are presented and discussed.

2. Statement of the problem and the governing equations in the primitive variables

The configuration of a porous prolate spheroid (an 'egg-shaped' ellipsoid) in an unbounded liquid relative to a Cartesian-coordinate system (x_1, x_2, x_3) with the origin at the centre O of the spheroid and unit vectors $(\mathbf{i}, \mathbf{j}, \mathbf{k})$ is shown in Fig. 1. A modified orthogonal prolate spheroidal coordinate system (τ, ζ, φ) with unit vectors $(\mathbf{e}_\tau, \mathbf{e}_\zeta, \mathbf{e}_\varphi)$ is defined through the relations

$$x_1 = c\{\tau^2 - 1\}^{\frac{1}{2}}\{1 - \zeta^2\}^{\frac{1}{2}} \cos\varphi, \quad x_2 = c\{\tau^2 - 1\}^{\frac{1}{2}}\{1 - \zeta^2\}^{\frac{1}{2}} \sin\varphi, \quad x_3 = c\tau\zeta, \tag{1}$$

where $1 \le \tau < \infty$, $-1 \le \zeta \le 1$, $0 \le \varphi < 2\pi$. The positive number $c > 0$ is the semifocal distance of the spheroidal system. The coordinate surfaces $\tau =$ const are a family of confocal prolate spheroids with the centre at the origin. The equation for the surface S_a of a single prolate spheroid in Cartesian coordinates, with the minor semiaxis a_1 and the major semiaxis a_3, is given by

$$\frac{x_3^2}{a_3^2} + \frac{x_1^2 + x_2^2}{a_1^2} = 1, \qquad a_3 = c\tau_a, \qquad a_1 = c\{\tau_a^2 - 1\}^{\frac{1}{2}}. \tag{2}$$

The Lamé coefficients in the prolate spheroidal geometry are

$$\hat{H}_\tau = c\frac{\{\tau^2-\zeta^2\}^{\frac{1}{2}}}{\{\tau^2-1\}^{\frac{1}{2}}}, \qquad \hat{H}_\zeta = c\frac{\{\tau^2-\zeta^2\}^{\frac{1}{2}}}{\{1-\zeta^2\}^{\frac{1}{2}}}, \qquad \hat{H}_\varphi = c\{\tau^2-1\}^{\frac{1}{2}}\{1-\zeta^2\}^{\frac{1}{2}}. \tag{3}$$

We assume that the porous spheroidal particle is stationary and a steady axisymmetric flow has been established around and through it by a uniform far-field flow with velocity of magnitude U directed in the positive orientation of the x_3-axis. The viscous fluid is Newtonian and incompressible. Thermal, inertial and volume forces are neglected. The medium of the porous spheroidal particle has constant permeability k. As reference quantities we use the velocity U, a given pressure p_0, and, as lengthscale L, the spheroid minor semiaxis a_1. The dynamic viscosity and density of the fluid are denoted by μ and ρ respectively.

The governing differential equations for the creeping flow around and through the porous particle must be written for the two regions separated by the interface S_a. For the region outside the porous spheroid, namely (1), we assume the flow to be governed by the Stokes equation

$$\Delta \mathbf{v}^{(1)} = \vartheta_1 \operatorname{grad} p^{(1)}. \tag{4}$$

For the region (2), occupied by the porous particle, we use the modified Darcy equation, that is, the Brinkman equation

$$\Delta \mathbf{v}^{(2)} - K^2\mathbf{v}^{(2)} = \vartheta_2 \operatorname{grad} p^{(2)}. \tag{5}$$

In addition, the continuity equation must be satisfied in both flow regions:

$$\operatorname{div} \mathbf{v}^{(i)} = 0, \qquad i = 1, 2. \tag{6}$$

The dimensionless coefficients appearing in (4) and (5) are defined as $K^2 = \beta L^2/k$ with $\beta = \mu/\hat{\mu}$, and $\vartheta_1 = \mathrm{Eu}\,\mathrm{Re}$, $\vartheta_2 = \beta\,\mathrm{Eu}\,\mathrm{Re}$ with the Euler number $\mathrm{Eu} = p_0/\rho U^2$ and the Reynolds number $\mathrm{Re} = \rho L U/\mu$. The effective viscosity of the fluid saturating the porous medium is denoted by $\hat{\mu}$ (see, for example, (**2**, **1**, **8**) for more details about the so-called effective viscosity). The viscosity coefficients μ and $\hat{\mu}$ are, in general, different. Here, we consider the ratio β to be arbitrary, but constant. The basic assumption pertinent to the Stokes approximation is $\mathrm{Re} < 1$ and $\mathrm{Re}\,\mathrm{Eu} > 1$.

The velocity vector and pressure in the two regions are denoted by $\mathbf{v}^{(i)}$ and $p^{(i)}$, $i = 1, 2$. The porous medium is assumed to be homogeneous and isotropic. It is clear that $\mathbf{v}^{(2)}$ and $p^{(2)}$ represent the macroscopic averaged values of velocity and pressure in the porous medium with respect to an elementary representative volume of the porous medium. This approach is usual for conceptual models of flows through porous media. We assume the velocity $\mathbf{v}^{(2)}$ and the pressure $p^{(2)}$ to be continuous everywhere in the porous medium as though no solid matrix were present. In that way we regard the entire space as a homogeneous isotropic fluid continuum which inside the volume occupied by the porous body, where the fictitious fluid is also assumed to be Stokesian, obeys a different equation of motion from that in the free flow, namely Brinkman's equation (5).

For small permeability of the porous medium, the Brinkman equation is a good approximation of Darcy's law:

$$K^2\mathbf{v}^{(2)} = -\vartheta_2 \operatorname{grad} p^{(2)}. \tag{7}$$

For very high porosity we obtain from (5), but not from (7), the Stokes equation (4) as for the free flow. Darcy's equation is of lower order than the Stokes equation for the free stream, and therefore difficulties are encountered in trying to satisfy physically reasonable conditions of continuity of velocity and pressure across the boundary. This had been the reason for introducing the modified Darcy equation by Brinkman and Debye. However, Darcy's equation is still widely used by many authors to study flow through porous media, together with modified physical boundary conditions supported by experiments (**13**, **14**). In using the Brinkman equation, which is especially relevant for high porosity materials, these difficulties with the boundary conditions are avoided. We can employ realistic dynamical conditions at the interfaces between the free-flow regions and the fictitious dissimilar flow in the porous particle. Following Lamb we can demand that the stress vector at the interface should be continuous, and we could then expect to obtain continuous velocity and pressure distributions across the interface; see (**15**, section G.I.64. Boundary conditions, p. 240). Here we use the same boundary conditions as in (**8**), namely continuity of velocity, pressure and tangential stress across the interface, and known values of velocity and pressure at infinity, as well as regularity of the solution in the entire space.

3. The stream function formulation and solution of the problem

Because the flow is axisymmetric ($v_\varphi = 0$, $\partial/\partial\varphi = 0$) we can introduce a stream function $\Psi^{(i)}(\tau,\zeta)$ which has to satisfy the continuity equation (6), that is, we may write

$$\mathbf{v}^{(i)} = v_\tau^{(i)}\mathbf{e}_\tau + v_\zeta^{(i)}\mathbf{e}_\zeta = \operatorname{curl}\left(\frac{1}{\hat{H}_\varphi}\Psi^{(i)}\mathbf{e}_\varphi\right) \tag{8}$$

and obtain the two velocity components of the flow in the two regions $i = 1, 2$ as

$$v_\tau^{(i)} = \frac{1}{\hat{H}_\zeta \hat{H}_\varphi}\frac{\partial \Psi^{(i)}}{\partial \zeta} = \frac{1}{c^2\{\tau^2-\zeta^2\}^{\frac{1}{2}}\{\tau^2-1\}^{\frac{1}{2}}}\frac{\partial \Psi^{(i)}}{\partial \zeta}, \tag{9a}$$

$$v_\zeta^{(i)} = -\frac{1}{\hat{H}_\tau \hat{H}_\varphi}\frac{\partial \Psi^{(i)}}{\partial \tau} = \frac{-1}{c^2\{\tau^2-\zeta^2\}^{\frac{1}{2}}\{1-\zeta^2\}^{\frac{1}{2}}}\frac{\partial \Psi^{(i)}}{\partial \tau}. \tag{9b}$$

For the vorticity field we have

$$\boldsymbol{\omega}^{(i)} = \operatorname{curl}\mathbf{v}^{(i)} = \operatorname{curl}\operatorname{curl}\left(\frac{1}{\hat{H}_\varphi}\Psi^{(i)}\mathbf{e}_\varphi\right) = \omega_\varphi^{(i)}\mathbf{e}_\varphi = -\frac{1}{\hat{H}_\varphi}E^2\Psi^{(i)}\mathbf{e}_\varphi, \tag{10}$$

where the Stokes operator E^2 in prolate spheroidal coordinates is given by

$$E^2 = \frac{1}{c^2(\tau^2-\zeta^2)}\left[(\tau^2-1)\frac{\partial^2}{\partial\tau^2} + (1-\zeta^2)\frac{\partial^2}{\partial\zeta^2}\right]. \tag{11}$$

The pressure can be eliminated from (4) and (5) by using $\Delta\mathbf{v}^{(i)} = \operatorname{grad}\operatorname{div}\mathbf{v}^{(i)} - \operatorname{curl}\operatorname{curl}\mathbf{v}^{(i)}$, and applying the curl operator on both sides of these equations. This gives the following differential equations to be solved for the stream function:

$$E^4\Psi^{(1)} = 0, \qquad E^4\Psi^{(2)} - K^2E^2\Psi^{(2)} = 0. \tag{12a, b}$$

The normal and tangential stresses of interest are

$$p_{\tau\tau}^{(i)} = -p^{(i)} + \frac{1}{\vartheta_i} \frac{2}{c\{\tau^2 - \zeta^2\}^{\frac{1}{2}}} \left[\{\tau^2 - 1\}^{\frac{1}{2}} \frac{\partial v_\tau^{(i)}}{\partial \tau} - \frac{\zeta\{1 - \zeta^2\}^{\frac{1}{2}}}{\tau^2 - \zeta^2} v_\zeta^{(i)} \right], \tag{13a}$$

$$\begin{aligned} p_{\tau\zeta}^{(i)} = {} & \frac{1}{\vartheta_i} \frac{1}{c(\tau^2 - \zeta^2)\{\tau^2 - \zeta^2\}^{\frac{1}{2}}} \left\{ \{1 - \zeta^2\}^{\frac{1}{2}} \left[(\tau^2 - \zeta^2) \frac{\partial v_\tau^{(i)}}{\partial \zeta} + \zeta v_\tau^{(i)} \right] \right. \\ & \left. + \{\tau^2 - 1\}^{\frac{1}{2}} \left[(\tau^2 - \zeta^2) \frac{\partial v_\zeta^{(i)}}{\partial \tau} - \tau v_\zeta^{(i)} \right] \right\}, \qquad i = 1, 2. \end{aligned} \tag{13b}$$

The boundary conditions which comprise continuity of velocity, pressure and tangential stresses across the interface S_a are

$$\begin{aligned} v_\tau^{(1)}(\tau_a, \zeta) = v_\tau^{(2)}(\tau_a, \zeta), \qquad & v_\zeta^{(1)}(\tau_a, \zeta) = v_\zeta^{(2)}(\tau_a, \zeta), \\ p_{\tau\zeta}^{(1)}(\tau_a, \zeta) = p_{\tau\zeta}^{(2)}(\tau_a, \zeta), \qquad & p^{(1)}(\tau_a, \zeta) = p^{(2)}(\tau_a, \zeta). \end{aligned} \tag{14}$$

Additionally, we have the velocity conditions at infinity:

$$\lim_{\tau \to \infty} v_\tau^{(1)} = \zeta, \qquad \lim_{\tau \to \infty} v_\zeta^{(1)} = -\{1 - \zeta^2\}^{\frac{1}{2}} \tag{15}$$

and the condition that velocity and pressure must be non-singular everywhere in the flow field.

3.1 *The spherical particle as a limiting case of the spheroidal particle*

Before considering the solution of the problem for arbitrary semifocal distance $0 < c < \infty$ we briefly recall the solution for the limiting case of the same problem when the semifocal distance c tends to zero. For this limiting process we have $\lim_{c \to 0} c\tau = R = \{x_1^2 + x_2^2 + x_3^2\}^{\frac{1}{2}}$, whereas the other two coordinates remain unchanged, that is, the spheroidal coordinate system reduces to the spherical one (R, ζ, φ), $\zeta = \cos\theta$. Thus the relations (1) change to

$$x_1 = R\{1 - \zeta^2\}^{\frac{1}{2}} \cos\varphi, \qquad x_2 = R\{1 - \zeta^2\}^{\frac{1}{2}} \sin\varphi, \qquad x_3 = R\zeta \tag{16}$$

and consequently all the equations (8) to (15) transform into the corresponding equations of the spherical geometry. In particular, the Stokes operator E^2 reduces to the known Stokes operator D^2 in spherical coordinates:

$$D^2 = \frac{\partial^2}{\partial R^2} + \frac{1 - \zeta^2}{R^2} \frac{\partial^2}{\partial \zeta^2} \tag{17}$$

and the equation system (12a, b) becomes

$$D^4 \Psi^{(1)} = 0, \qquad D^4 \Psi^{(2)} - K^2 D^2 \Psi^{(2)} = 0. \tag{18a, b}$$

Of course, we would have obtained the same result if we had started with spherical coordinates from the beginning.

As is known, both equations (18a, b) are completely separable. Their complete general solutions are easily obtained and can be written as

$$\begin{aligned}\Psi^{(1)} &= \sum_{m=0}^{\infty}\big\{\big[A_m^{(1)}R^{1-m} + B_m^{(1)}R^m + C_m^{(1)}R^{3-m} + D_m^{(1)}R^{2+m}\big]G_m(\zeta)\\ &\quad + \big[\tilde{A}_m^{(1)}R^{1-m} + \tilde{B}_m^{(1)}R^m + \tilde{C}_m^{(1)}R^{3-m} + \tilde{D}_m^{(1)}R^{2+m}\big]H_m(\zeta)\big\}, \qquad (19a)\\ \Psi^{(2)} &= \sum_{m=0}^{\infty}\big\{\big[A_m^{(2)}R^{1-m} + B_m^{(2)}R^m + C_m^{(2)}\sqrt{R}I_{-\nu}(KR) + D_m^{(2)}\sqrt{R}I_{\nu}(KR)\big]G_m(\zeta)\\ &\quad + \big[\tilde{A}_m^{(2)}R^{1-m} + \tilde{B}_m^{(2)}R^m + \tilde{C}_m^{(2)}\sqrt{R}I_{-\nu}(KR) + \tilde{D}_m^{(2)}\sqrt{R}I_{\nu}(KR)\big]H_m(\zeta)\big\}, \qquad (19b)\end{aligned}$$

where $I_{\pm\nu}(KR)$ denote the modified Bessel functions of the first kind and of non-integer index $v = \{m(m-1) + \frac{1}{4}\}^{\frac{1}{2}}$; $G_m(\zeta)$, $H_m(\zeta)$ are the Gegenbauer functions of the first and second kind respectively, of order m, and of degree $-\frac{1}{2}$.

The nice feature of the general solutions (19a, b) in spherical coordinates, apart from separability and closed-form eigenfunctions, is that the ζ-dependent functions in (19a, b) are the same for both flow regions, that is, they do not contain K as a parameter. This allows us to equate only the R-dependent factors of every order m by satisfying the spherical boundary conditions at the interface corresponding to (14), whatever the value of the factor K. In that way we obtain a set of linear algebraic systems for determining the unknown constants in (19a, b). Before doing this we note that here we seek solutions which are regular on the x_3-axis. If we retain the terms of the general solutions (19a, b) which are multiplied by $G_0(\zeta)$ and $G_1(\zeta)$, the velocities will be rendered irregular at the x_3-axis, whereas $H_m(\zeta)$ are irregular on the x_3-axis for every m. In order to satisfy the spherical far-field conditions for the free flow corresponding to (15), we are forced to retain only the term of order $m = 2$ of the general solution for the free-flow stream function and to additionally take $D_2^{(1)} = 0$. Furthermore, the modified Bessel functions $I_{-\nu}(KR)$ are irregular at $R = 0$ for $m \geq 2$, hence we have to take $C_m^{(2)} = 0$ for $m \geq 2$. The same is true for the terms multiplied by $A_m^{(2)}$, $m \geq 2$. Thus (19a) and (19b) reduce only to terms multiplying $G_2(\zeta)$. Using relations between the modified Bessel functions and the hyperbolic functions, we finally obtain

$$\Psi^{(1)}(R,\zeta) = \left[A_2^{(1)}\frac{1}{R} + B_2^{(1)}R^2 + C_2^{(1)}R\right]G_2(\zeta), \qquad (20a)$$

$$\Psi^{(2)}(R,\zeta) = \left\{B_2^{(2)}R^2 + D_2^{(2)}\left[K\cosh(KR) - \frac{1}{R}\sinh(KR)\right]\right\}G_2(\zeta). \qquad (20b)$$

Thus, we have obtained closed-form expressions for the stream functions $\Psi^{(i)}(R,\zeta)$, $i = 1, 2$, and can also easily derive the closed-form expressions for velocity, pressure, viscous stresses, vorticity and the drag force exerted on the porous spherical particle in terms of the constants $A_2^{(1)}, B_2^{(1)}, C_2^{(1)}, B_2^{(2)}, D_2^{(2)}$. These constants are obtained as solutions of the linear algebraic system which results from using the spherical boundary conditions corresponding to (14) and (15). For the drag force exerted on a porous spherical particle we obtain

$$F_D = -4\pi C_2^{(1)}. \qquad (21)$$

More explicit expressions for the drag force are known for the solid sphere (Stokes formula), for the porous sphere and for the porous one-layer spherical shell with cavity or solid core.

3.2 *The generalized eigenfunctions of E^2 in spheroidal coordinates*

The equations (12a, b) in spheroidal coordinates do not permit such nice separable general solutions as in the limiting case when the focal distance $c \to 0$, where the spheroidal coordinate system becomes spherical. For the equation (12a), Dassios *et al.* (**12**) gave a complete *semiseparable* general solution in the form

$$\Psi^{(1)}(\tau, \zeta) = g_0(\tau)\, G_0(\zeta) + g_1(\tau)\, G_1(\zeta) + \sum_{m=2}^{\infty} [g_m(\tau)\, G_m(\zeta) + h_m(\tau)\, H_m(\zeta)], \qquad (22)$$

where the functions $g_m(\tau)$ and $h_m(\tau)$ are given in (**12**) as linear finite combinations of the Gegenbauer functions $G_m(\tau)$ and $H_m(\tau)$. In contrast to the usual form of separable general solutions of partial differential equations, two examples of which are the solutions (19a, b) to the equations (18a, b), the individual terms of (22) are not solutions of the equation (12a). However, the full expansion is a solution and this kind of general solution is therefore referred to as a *semiseparable* general solution by Dassios *et al.* (**12**).

In our case, we will use the restricted form of the complete semiseparable general solution of (12a), given in (**12**, equation (30)). This takes into account that the solution of the flow problem has to be regular on the axis and at infinity, and is, additionally, even in the ζ-coordinate, because of the symmetry of the Ψ-field with respect to the equatorial plane ($\zeta = 0$), as in the case of the porous spherical particle. We give here this part of the general solution to (12a) in the following form:

$$\Psi^{(1)}(\tau, \zeta) = 2c^2 G_2(\tau)\, G_2(\zeta) + \sum_{m=2,4,\ldots}^{\infty} A_m^{(1)} H_m(\tau)\, G_m(\zeta) + \sum_{m=2,4,\ldots}^{\infty} C_m^{(1)}\, \Omega_m^{(3)}(\tau, \zeta), \qquad (23)$$

where $\Omega_m^{(3)}(\tau, \zeta)$ (note that here the upper index (3) has nothing to do with the upper index for designating the flow regions) is the third of the four (**12**) generalized eigenfunctions of (12a):

$$\Omega_2^{(3)}(\tau, \zeta) = \tfrac{2}{25} H_2(\tau) G_4(\zeta) + \tfrac{2}{25} H_4(\tau) G_2(\zeta) + \tfrac{1}{6} G_1(\tau) G_2(\zeta), \qquad (24a)$$

$$\begin{aligned} \Omega_m^{(3)}(\tau, \zeta) = & -\frac{\alpha_m}{2(2m-3)} [H_{m-2}(\tau) G_m(\zeta) + H_m(\tau) G_{m-2}(\zeta)] \\ & + \frac{\beta_m}{2(2m+1)} [H_{m+2}(\tau) G_m(\zeta) + H_m(\tau) G_{m+2}(\zeta)], \qquad m = 4, 6, \ldots, \end{aligned} \qquad (24b)$$

where the coefficients α_m, β_m and γ_m (the last of these will be used later) are given by

$$\begin{aligned} \alpha_m &= \frac{(m-3)(m-2)}{(2m-3)(2m-1)}, \\ \beta_m &= \frac{(m+1)(m+2)}{(2m-1)(2m+1)}, \\ \gamma_m &= \frac{2m^2 - 2m - 3}{(2m+1)(2m-3)}. \end{aligned} \qquad (25)$$

3.2.1 Axisymmetric Stokes flow around a solid spheroid in uniform ambient flow. Nevertheless, some boundary-value problems in spheroidal coordinates associated with equation (12a) can still be solved by the method of separation of variables without resorting to the above general semiseparable solution. This is the case, for example, for the Stokes flow problem for a solid spheroid immersed in a homogenous flow parallel to its main axis. As is known and more closely elaborated in (**12**), this problem was treated and solved by several authors in the past. We briefly give here two possible ways of investigating the solution to this problem. Making an *ad hoc* ansatz $\Psi^{(1)}(\tau,\zeta) = F(\tau)\,G_2(\zeta)$, equation (12a) gives

$$\left[\tau^2(1-\tau^2)F'''' - 4\tau F''' + 4(1+\tau^2)F'' - 8\tau F' + 8F\right] - \left[(1-\tau^2)F'''' - 4\tau F'''\right]\zeta^2 = 0. \tag{26}$$

Equating the expressions in brackets to zero and eliminating $F''''(\tau)$ gives a homogenous third-order linear differential equation

$$\tau(\tau^2-1)F''' + (\tau^2+1)F'' - 2\tau F' + 2F = 0, \tag{27}$$

the solution of which is easily found to be

$$\Psi^{(1)}(\tau,\zeta) = \left[A_2^{(1)}H_2(\tau) + B_2^{(1)}G_2(\tau) + C_2^{(1)}G_1(\tau)\right]G_2(\zeta). \tag{28}$$

Using the far-field velocity condition (15) and the no-slip boundary condition at the surface of the solid prolate spheroid ($\tau = \tau_a = \text{const}$) we find the values for the constants $A_2^{(1)}$, $B_2^{(1)} = 2c^2$, and $C_2^{(1)}$, and thus the complete solution of the problem.

We must arrive at the same result by using the semiseparable general solution of Dassios *et al.* (**12**) as given by equation (23). This solution meets all the above requirements for regularity, evenness and satisfaction of the far-field condition. We also necessarily have to take $A_m^{(1)} = 0$, $m = 4, 6, \ldots$. However, we still have infinitely many unknown constants $C_m^{(1)}$, $m = 2, 4, \ldots$, and only two boundary conditions for the velocity on the surface of the spheroid. It is also easily seen that the no-slip boundary condition on the spheroidal particle surface cannot be satisfied by taking any finite combination of the generalized eigenfunctions $\Omega_m^{(3)}(\tau,\zeta)$, $m = 2, 4, \ldots$. The problem can only be solved by taking the following additional relations between the constants $C_m^{(1)}$:

$$\alpha_m C_m^{(1)} - \beta_{m-2}C_{m-2}^{(1)} = 0, \qquad m = 4, 6, \ldots. \tag{29}$$

In this way we arrive at the solution (28) which we have obtained by the method of an *ad hoc* variable separation ansatz.

3.3 *Axisymmetric creeping flow past a porous prolate spheroidal particle*

In order to solve the problem of axisymmetric creeping flow past a single porous prolate spheroidal particle using the stream function formulation of the Brinkman equation, we must find a general solution of equation (12b). For this purpose, we write the equation (12b) in the following two equivalent forms:

$$(E^2 - K^2)E^2\Psi^{(2)} = 0, \qquad E^2\left(E^2\Psi^{(2)} - K^2\Psi^{(2)}\right) = 0, \tag{30a, b}$$

and seek the solution in the form of a linear sum of two functions, $\Psi^{(2)} = \Psi_1^{(2)} + \Psi_2^{(2)}$. The functions $\Psi_1^{(2)}$ and $\Psi_2^{(2)}$ are obtained by solving the following two second-order partial differential equations:

$$E^2\Psi_1^{(2)} = 0, \qquad E^2\Psi_2^{(2)} - K^2\Psi_2^{(2)} = 0. \tag{31a, b}$$

We use the method of separation of variables to find general solutions of (31a, b). The complete separable general solution to (31a) is obtained as

$$\Psi_1^{(2)} = \sum_{m=0}^{\infty} \big\{\big[A_m^{(2)} H_2(\tau) + B_m^{(2)} G_m(\tau)\big] G_m(\zeta) + \big[\tilde{A}_m^{(2)} H_2(\tau) + \tilde{B}_m^{(2)} G_m(\tau)\big] H_m(\zeta)\big\}. \tag{32}$$

Again, because of reasons to do with regularity and symmetry, as for the spherical geometry, we retain, in this case also, only the following part of the above general solution:

$$\Psi_1^{(2)} = \sum_{m=2,4,\ldots}^{\infty} B_m^{(2)}\, G_m(\tau)\, G_m(\zeta). \tag{33}$$

To find general solutions to (31b) we make the ansatz $\Psi_2^{(2)} = T(\tau)\, Z(\zeta)$ and obtain the following system of two second-order ordinary differential equations for $Z(\zeta)$ and $T(\tau)$:

$$\big(1-\zeta^2\big)Z'' + \big(q^2\zeta^2 + \lambda\big)Z = 0, \qquad \big(1-\tau^2\big)T'' + \big(q^2\tau^2 + \lambda\big)T = 0, \tag{34a, b}$$

where $q = cK$ has been introduced. We note that the equations (34a, b) are identical. They are therefore both satisfied by the same general solutions. The difference is that in satisfying the ζ-dependent equation (34a) we need here only a general solution that is regular at least in the domain $-1 \le \zeta \le 1$. In satisfying the τ-dependent equation (34b), however, we need a general solution which is regular at least for $1 \le \tau \le \tau_a$.

The eigenvalue problem (34a) cannot be solved in closed form in terms of known functions. We therefore solve the problem by means of appropriate series expansions of the unknown functions, taking into account that the solution to (34a) has to be regular on the axis and even because of the symmetry of the Ψ-field on either side of the equatorial plane ($\zeta = 0$). Because of the rotational symmetry about the x_3-axis we can declare the x_3-axis to be the zero-streamline, that is, $Z(\pm 1) = 0$. As the flow rate through the porous region must be finite, the stream function $\Psi^{(2)}(\tau, \zeta)$ must be bounded in the domain $-1 \le \zeta \le 1$, $1 \le \tau \le \tau_a < \infty$. The eigenvalue problem can now be stated as follows: find the set of eigenvalues $\{\lambda\}$ for which (34a) has eigensolutions bounded in the interval $0 \le \zeta \le 1$ and, additionally, $Z(\pm 1) = 0$. The Dassios *et al.* (**12**) general solutions to (12a) are given in terms of Gegenbauer functions. To achieve compatibility when satisfying the boundary conditions at the interface we must also seek eigenfunctions of (34a) represented as convergent series expansions in terms of Gegenbauer functions. Therefore, for $Z(\zeta)$ we make a series ansatz in terms of even Gegenbauer polynomials:

$$Z(\zeta) = \sum_{m=2,4,\ldots}^{\infty} a_m G_m(\zeta), \tag{35}$$

where $a_m = a_m\big(\lambda, q^2\big)$ are unknown coefficients to be determined. The coefficient $a_2 \neq 0$ can be taken as arbitrary. Substituting the series (35) into the differential equation (34a) and using the relation stemming from the differential equation of the Gegenbauer functions

$$G''_m(\zeta) = -\frac{m(m-1)}{1-\zeta^2} G_m(\zeta), \tag{36}$$

we arrive at the relation

$$\sum_{m=2,4,\ldots}^{\infty} \big[m(m-1) - \lambda - q^2\zeta^2\big] a_m G_m(\zeta) = 0, \tag{37}$$

which is just another expression for the differential equation (34a) and has to be satisfied for every complex ζ and $0 < q < \infty$. Using further the relations

$$\zeta^2 G_m(\zeta) = \alpha_m G_{m-2}(\zeta) + \gamma_m G_m(\zeta) + \beta_m G_{m+2}(\zeta), \qquad m = 2, 4, \ldots, \tag{38}$$

where the coefficients α_m, β_m and γ_m are given by (25), and considering the fact that the system $G_m(\zeta)$, $m = 2, 4, \ldots$, is complete and its functions are linearly independent, we obtain the following recursive formula for determining the coefficients a_m:

$$\begin{aligned} a_2 &= 1, \qquad a_4 = \frac{2 - \lambda - \frac{1}{5}q^2}{q^2\alpha_4}, \\ a_m &= \frac{\big[(m-2)(m-3) - \lambda - q^2\gamma_{m-2}\big]a_{m-2} - q^2\beta_{m-4}a_{m-4}}{q^2\alpha_m}, \qquad m = 6, 8, \ldots . \end{aligned} \tag{39}$$

From the requirement that the relation (37) also has to be satisfied at the point $\zeta = 0$ we find a relation which can be used for determining the set of eigenvalues $\{\lambda\}$. Putting $\zeta = 0$ in (37) we obtain a characteristic equation for the eigenvalue problem (34a):

$$P(\lambda) = \sum_{m=2,4,\ldots}^{\infty} [m(m-1) - \lambda] a_m G_m(0) = 0. \tag{40}$$

The function $P(\lambda)$ can be rearranged into an infinite power series in terms of λ, whose coefficients depend only on the parameter q^2. The infinitely many zeros of the entire function $P(\lambda)$ are all real and distinct and provide the exact eigenvalues to the eigenvalue problem (34a). Approximate values of the first $N/2$ eigenvalues λ_n, $n = 2, 4, \ldots, N$ (N is an arbitrary even positive integer) are obtained by truncating the infinite power series $P(\lambda)$ at $m = N$ and solving (factoring) the resulting polynomial equation $P_{N/2}(\lambda) = 0$ of degree $N/2$ numerically. As N becomes larger, this numerical calculation renders the $N/2$ approximate eigenvalues closer to their exact values. For $q = 0$ we obtain in this way the $N/2$ exact eigenvalues $\lambda_n = n(n-1)$, $n = 2, 4, \ldots, N$. For the same purpose, instead of (37) we can also use the equation $a_m(\lambda, q^2) = 0$. We have found recurrent formulae for determining the coefficients of the characteristic polynomial $P_{N/2}(\lambda) = 0$ although these are not presented here. For determining the eigenvalues, instead of (35) we can also use other series expansions as well as pure numerical methods, which, however, are more cumbersome and less accurate.

With the sequence of eigenvalues $\lambda_n(q^2)$, $n = 2, 4, \ldots$, the coefficients in (39) become a corresponding sequence of coefficients $a_{n,m}$, and thus the expression (35) gives a corresponding sequence of eigenfunctions $Z_n(\zeta)$, which satisfy the differential equation (34a) and converge in the entire complex plane. The accuracy of the eigenvalues influences significantly the convergence rate of the eigenfunction series solution (35). Also, for q large, more terms in the series expansion (35) should be taken, and $\lambda_n\left(q^2\right)$ should be calculated more accurately.

The odd eigenvalues and the corresponding odd eigenfunctions to (34a) can be found in a similar way, but we do not use them in this paper. The second-kind solutions of (34b), uniformly convergent when τ lies in any closed domain of the complex τ-plane supposed cut along the real axis from -1 to $+1$, can be presented in an integral form. However, no use is made of that solution in this paper either.

Having found the way of calculating the required eigenvalue spectrum and the infinite sequence of everywhere-convergent eigenfunctions to the eigenvalue problem (34a), we could take the same sequence of functions as the one required solution of the τ-dependent differential equation (34b) for the domain $1 \leq \tau \leq \tau_a$. However, we found it more convenient to derive the required solution of (34b), which we designate here as $f_n^{(2)}(\tau)$, for every eigenvalue λ_n, $n = 2, 4, \ldots$ in the form of a convergent (in the domain $1 \leq \tau \leq \tau_a$) power series expansion in terms of $(\tau - \tau_M)$, where $\tau_M = \frac{1}{2}(1 + \tau_a)$:

$$f_n^{(2)}(\tau) = \sum_{l=0}^{\infty} \hat{a}_{n,l}(\tau - \tau_M)^l. \tag{41}$$

The series coefficients appearing in (41) are obtained recursively as follows:

$$\begin{aligned}
\hat{a}_{n,0} &= 1, \qquad \hat{a}_{n,1} = 0, \qquad \hat{a}_{n,2} = \frac{q^2\tau_M^2 + \lambda_n}{2(\tau_M^2 - 1)}, \qquad \hat{a}_{n,3} = \frac{\tau_M\left(q^2 - 2\hat{a}_{n,2}\right)}{3(\tau_M^2 - 1)}, \\
\hat{a}_{n,4} &= \frac{q^2 - \left(2 - \lambda_n - q^2\tau_M^2\right)\hat{a}_{n,2} - 12\tau_M\hat{a}_{n,3}}{12\left(\tau_M^2 - 1\right)}, \\
\hat{a}_{n,5} &= \frac{2\tau_M q^2\hat{a}_{n,2} - \left(6 - \lambda_n - q^2\tau_M^2\right)\hat{a}_{n,3} - 24\tau_M\hat{a}_{n,4}}{20\left(\tau_M^2 - 1\right)}, \\
\hat{a}_{n,6} &= \frac{q^2\hat{a}_{n,2} + 2\tau_M q^2\hat{a}_{n,3} - \left(12 - \lambda_n - q^2\tau_M^2\right)\hat{a}_{n,4} - 40\tau_M\hat{a}_{n,5}}{30\left(\tau_M^2 - 1\right)}, \\
\hat{a}_{n,l} &= \left\{q^2\hat{a}_{n,l-4} + 2\tau_M q^2\hat{a}_{n,l-3} - \left[(l-2)(l-3) - \lambda_n - q^2\tau_M^2\right]\hat{a}_{n,l-2}\right. \\
&\quad \left. -2(l-2)(l-1)\tau_M\hat{a}_{n,l-1}\right\}\left\{l(l-1)\left(\tau_M^2 - 1\right)\right\}^{-1},
\end{aligned} \tag{42}$$

where $l = 7, 8, \ldots, n = 2, 4, \ldots$.

Thus, finally, the required general solution representation of $\Psi^{(2)}(\tau, \zeta)$ is

$$\Psi^{(2)}(\tau, \zeta) = \sum_{m=2,4,\ldots}^{\infty} \left\{B_m^{(2)} G_m(\tau) + \sum_{n=2,4,\ldots}^{\infty} D_n^{(2)} f_n^{(2)}(\tau)\, a_{n,m}\right\} G_m(\zeta). \tag{43}$$

For the free-flow region (1) we use the Dassios *et al.* (**12**) general solution (23). The corresponding series expansion of (23) in terms of ζ-dependent Gegenbauer polynomials is

$$\Psi^{(1)}(\tau, \zeta) = \sum_{m=2,4,\ldots}^{\infty} f_m^{(1)}(\tau)\, G_m(\zeta), \tag{44}$$

where the τ-dependent coefficients are given by

$$f_2^{(1)}(\tau) = 2c^2 G_2(\tau) + A_2^{(1)} H_2(\tau) + \tfrac{1}{6} C_2^{(1)} G_1(\tau) + \left(\tfrac{2}{25} C_2^{(1)} + \alpha_4^* C_4^{(1)}\right) H_4(\tau),$$
$$f_m^{(1)}(\tau) = A_m^{(1)} H_m(\tau) + \left(\beta_{m-2}^* C_{m-2}^{(1)} + \alpha_m^* C_m^{(1)}\right) H_{m-2}(\tau) \quad (45)$$
$$+ \left(\beta_m^* C_m^{(1)} + \alpha_{m+2}^* C_{m+2}^{(1)}\right) H_{m+2}(\tau), \qquad m = 4, 6, \ldots.$$

In (45) we have used

$$\alpha_m^* = -\frac{\alpha_m}{2(2m-3)}, \qquad \beta_m^* = \frac{\beta_m}{2(2m+1)}. \quad (46)$$

If we truncate the series expansion (44) at an arbitrary $m = N$, then, as can be seen from (24b), only the term $\beta_N^* C_N^{(1)} H_N(\tau)\, G_{N+2}(\zeta)$ will be missing for these truncated series to be exact solutions of (12a). We can use this fact for checking the accuracy of the numerical solution of the problem.

We further need general expressions for the pressure for both flow regions. We can obtain these expressions by either using the easily verifiable fact that

$$\Delta p^{(i)} = 0, \qquad i = 1, 2 \quad (47)$$

and then finding the general solutions of (47) by the method of separation of variables, or by using the Navier–Stokes equations (4) and (5), in which the velocities are expressed using the general solutions for the respective stream functions. We suppress here the full presentation of these procedures and give only the final expressions for the pressure as follows:

$$p^{(1)}(\tau, \zeta) = \frac{1}{c^3 \vartheta_1} \sum_{m=2,4,\ldots}^{\infty} C_m^{(1)} \frac{1}{m(m-1)} Q_{m-1}(\tau) P_{m-1}(\zeta), \quad (48)$$

$$p^{(2)}(\tau, \zeta) = -\frac{q^2}{c^3 \vartheta_2} \sum_{m=2,4,\ldots}^{\infty} B_m^{(2)} \frac{1}{m(m-1)} P_{m-1}(\tau) P_{m-1}(\zeta), \quad (49)$$

where $P_m(x)$, $Q_m(x)$ are the Legendre functions of the first and second kinds respectively, of order m.

It can be shown that the boundary conditions (14), which comprise continuity of velocity, pressure and tangential stress across the interface S_a, can also be written as

$$\Psi^{(1)}(\tau_a, \zeta) = \Psi^{(2)}(\tau_a, \zeta), \qquad \Psi_\tau^{(1)}(\tau_a, \zeta) = \Psi_\tau^{(2)}(\tau_a, \zeta), \quad (50a, b)$$

$$\Psi_{\tau\tau}^{(1)}(\tau_a, \zeta) = \Psi_{\tau\tau}^{(2)}(\tau_a, \zeta), \qquad p^{(1)}(\tau_a, \zeta) = p^{(2)}(\tau_a, \zeta). \quad (51a, b)$$

As the sets of functions $G_m(\zeta)$ and $P_{m-1}(\zeta)$, $m = 2, 4, \ldots$, are complete and linearly independent, in (50), (51) we can equate the corresponding coefficients. We truncate the series expansions belonging to equations (50), (51) at an appropriate order $N = 2, 4, \ldots$, depending on the accuracy we wish to achieve, so that the indices m and n run over the values $m = 2, 4, \ldots, N$ and $n = 2, 4, \ldots, N$. Thereby

$$f_N^{(1)}(\tau) = A_N^{(1)} H_N(\tau) + \left(\beta_{N-2}^* C_{N-2}^{(1)} + \alpha_N^* C_N^{(1)}\right) H_{N-2}(\tau) + \beta_N^* C_N^{(1)} H_{N+2}(\tau) \quad (52)$$

is to be taken, that is, we reject the term $\alpha^*_{N+2} C^{(1)}_{N+2} H_{N+2}(\tau)\, G_N(\zeta)$ from (45). In this way, we obtain a system of $2N$ linear algebraic equations in as many unknowns:

$$A^{(1)}_m,\, C^{(1)}_m,\, B^{(2)}_m,\, D^{(2)}_m, \qquad m = 2, 4, \ldots, N, \tag{53}$$

that can be uniquely solved. When performing numerical calculations, the power series expansions (41) must also be truncated at an order L depending on the accuracy we wish to achieve.

3.4 *Drag force acting on the porous spheroidal particle*

Once the coefficients appearing in the general solutions of the problem have been computed, we can then compute all the important physical quantities: the stream function, velocity components, pressure, vorticity and stress fields. It remains to derive the expression for the main integral quantity of the problem, the drag force acting on the porous particle. Due to the symmetry of our problem there is only a net force per unit area in the x_3-direction:

$$dF_D = p^{(1)}_{\tau\tau}(\tau_a, \zeta)\mathbf{e}_\tau \mathbf{k} + p^{(1)}_{\tau\zeta}(\tau_a, \zeta)\mathbf{e}_\zeta \mathbf{k}. \tag{54}$$

With

$$\mathbf{e}_\tau \mathbf{k} = \frac{\left\{\tau_a^2 - 1\right\}^{\frac{1}{2}}}{\left\{\tau_a^2 - \zeta^2\right\}^{\frac{1}{2}}}\zeta, \qquad \mathbf{e}_\zeta \mathbf{k} = -\tau_a \frac{\left\{1 - \zeta^2\right\}^{\frac{1}{2}}}{\left\{\tau_a^2 - \zeta^2\right\}^{\frac{1}{2}}}, \qquad dA_\tau = \hat{H}_\varphi \hat{H}_\zeta \, d\varphi \, d\zeta, \tag{55}$$

the total drag force is given by

$$F_D = 2\pi c^2 \int_{-1}^{1} \left[\zeta \left\{\tau_a^2 - 1\right\}^{\frac{1}{2}} p^{(1)}_{\tau\tau}(\tau_a, \zeta) + \left\{1 - \zeta^2\right\}^{\frac{1}{2}} \tau_a p^{(1)}_{\tau\zeta}(\tau_a, \zeta)\right] \{\tau_a^2 - 1\}^{\frac{1}{2}} d\zeta. \tag{56}$$

Upon substituting the expressions (13a, b) for the stresses $p^{(1)}_{\tau\tau}$ and $p^{(1)}_{\tau\zeta}$ into (56), wherein the velocity and its partial derivatives are expressed via the corresponding derivatives of the stream function $\Psi^{(1)}(\tau, \zeta)$, and then carrying out the integration making use of symmetry properties, we obtain the final expression for the total drag force as

$$F_D = -\frac{2\pi}{3c} C^{(1)}_2. \tag{57}$$

In the limiting case $c \to 0$ we obtain from (57) the drag force formula (21) for the porous spherical particle.

4. Presentation of some of the computed results and discussion

To obtain numerical results for given input parameter values, the computer code for the solution of the problem was written in MATHEMATICA and successfully tested. As illustration we present here only some of the computed results, showing the dependence of the streamline pattern and drag force on the permeability and focal distance. These are the two fluid-flow properties that are usually needed in practice. The streamline pattern is needed to find the flow-rate through the internal part of the porous particle, and the drag force is needed for calculating, for instance, the settling time of the porous particle in a fluid.

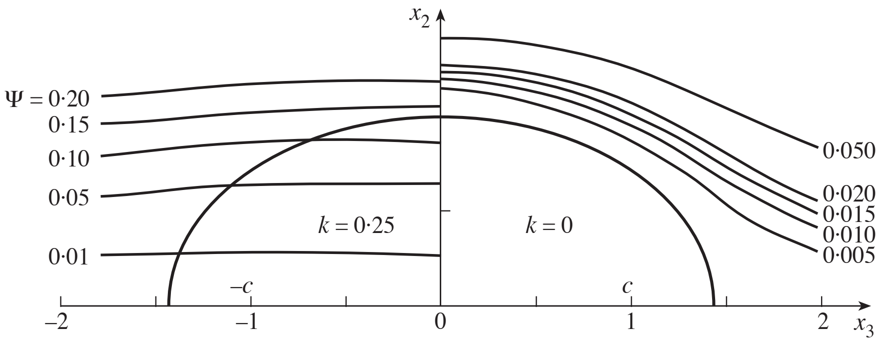

Fig. 2 Streamline patterns for a set of arbitrary chosen values of permeability and focal distance

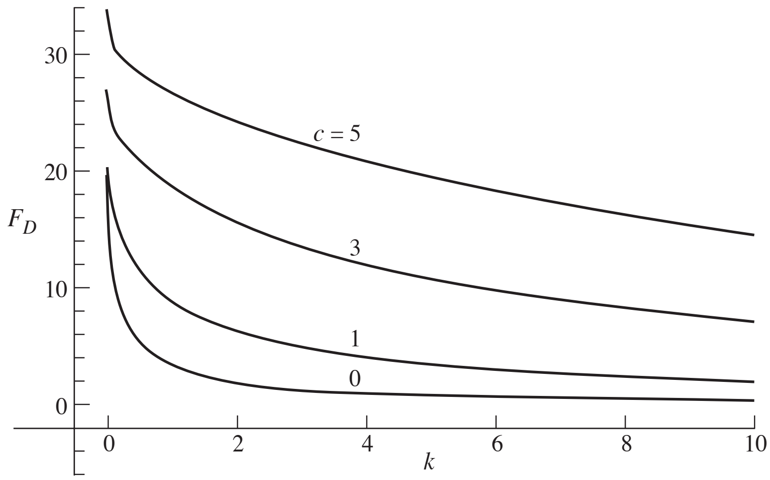

Fig. 3 Drag force F_D against permeability k; varying parameter is the semifocal distance c; $\beta = 1$

In Fig. 2 the streamline patterns for arbitrary chosen values of permeability, effective viscosity and focal distance ($k = 0{\cdot}25$, $\beta = 1$, $c = 1$) for the porous and (comparatively) solid prolate spheroid are presented. Due to the flow symmetry it suffices to show the streamlines only in one quadrant of the plane. As can be seen from the figure, and as is to be expected, increasing permeability generally flattens the streamlines.

The dependence of the drag force F_D on permeability k for the porous prolate spheroid is shown in Fig. 3. The semifocal distance c is varied as a parameter. It is seen that the drag force decreases with increasing permeability k, and increases with increasing semifocal distance c. The values of the drag force for the case of a porous sphere (curve $c = 0$ in the figure) have been computed by using the known exact drag force formula for that case.

The accuracy of the computed results generally depends on the chosen eigenvalue order N and on the series truncation index L of the τ-dependent series expansion (41) and its derivatives.

A measure of the overall accuracy of the computed field quantities is provided by the accuracy of the computed drag force. The presented results were calculated to at least 5-digit accuracy in the drag force. The corresponding eigenvalues have been calculated to an accuracy of at least 5 digits, too. The choice of N for achieving a given accuracy depends in the first place on the value of the semifocal distance c. For large values of c, N should be chosen larger too. The left-hand starting points of the drag force curves ($k = 0$) in Fig. 3 have been computed by using the known exact drag-force formula for the solid spheroid. It is also seen from Fig. 3 that for $k \to \infty$ all the drag force curves tend, as is to be expected, asymptotically to zero. We finally remark that for approximately $c \leq 0{\cdot}005$, when using normal computer precision, the solution for the spherical geometry should be used.

The solutions of the problem for a multi-layered porous prolate spheroidal shell with porous, cavity or solid core and for the corresponding problems in the oblate spheroidal geometry will be presented in a forthcoming paper.

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