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H. Tasso, G.N. Throumoulopoulos

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# Axisymmetric ideal magnetohydrodynamic equilibria with incompressible flows

H. Tasso and G. N. Throumoulopoulos[1]

*Max-Planck-Institut für Plasmaphysik, EURATOM Association*

*D-85748 Garching, Germany*

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## Abstract

It is shown that the ideal MHD equilibrium states of an axisymmetric plasma with incompressible flows are governed by an elliptic partial differential equation for the poloidal magnetic flux function $\psi$ containing five surface quantities along with a relation for the pressure. Exact equilibria are constructed including those with non vanishing poloidal and toroidal flows and differentially varying radial electric fields. Unlike the case in cylindrical incompressible equilibria with isothermal magnetic surfaces which should have necessarily circular cross sections [G. N. Throumoulopoulos and H. Tasso, Phys. Plasmas 4, 1492 (1997)], no restriction appears on the shapes of the magnetic surfaces in the corresponding axisymmetric equilibria. The latter equilibria satisfy a set of six ordinary differential equations which for flows parallel to the magnetic field $\mathbf{B}$ can be solved semianalytically. In addition, it is proved the non existence of incompressible axisymmetric equilibria with (a) purely poloidal flows and (b) non-parallel flows with isothermal magnetic surfaces and $|\mathbf{B}| = |\mathbf{B}|(\psi)$ (omnigenous equilibria).

[1]Permanent address: Section of Theoretical Physics, Physics Department, University of Ioannina GR 451 10 Ioannina, Greece

# I. Introduction

Although equilibrium studies of flowing plasmas began in the mid 1950s (e. g. Ref. [1] and references therein) since the early 1970s there has been increasing interest in the investigation of the equilibrium properties of plasmas with mass flow [2]-[13], which was motivated by the observation of plasma rotation in many tokamaks heated by neutral beams [14]-[16]. With the adoption of a specific equation of state, e.g., isentropic magnetic surfaces [4], the symmetric equilibrium states in a two-dimensional geometry obey a partial differential equation for the poloidal magnetic flux function $\psi$, containing five surface quantities, in conjunction with a nonlinear algebraic Bernoulli equation. Unlike the case in static equilibria, the above-mentioned differential equation is not always elliptic: there are three critical values of the poloidal flow at which the type of this equation changes, i.e. it becomes alternatively elliptic and hyperbolic. The existence of hyperbolic regimes may be dangerous for plasma confinement because they are associated with shock waves which can cause equilibrium degradation. In this respect incompressible flows are of particular interest because, as is well known from gas dynamics, it is the compressibility that can give rise to shock waves; thus for incompressible flows the equilibrium equation becomes always elliptic.

In a recent work [12] we found that the equilibrium differential equation of a cylindrical plasma with incompressible flows and arbitrary cross sectional shape is amenable to a variety of analytic solutions. Also, in the case of plasmas with isothermal magnetic surfaces their cross sections are restricted to be circular. The aim of the present report is to extend the study to the most interesting case of axisymmetric plasmas. It should be noted that the particular class of incompressible, axisymmetric equilibria with approximate isobaric magnetic surfaces was investigated by Avinash, Bhattajaryya and Green [9]. Since in flowing plasmas the isobaric surfaces in general depart from the magnetic surfaces (see the discussion after Eq. (19) of Sec. II), in the present work we first consider arbitrary incompressible flows without making any assumption on the pressure. It turns out that, as the case in cylindrical plasmas, the incompressibility condition results in a considerable simplification of the problem, i.e., the equilibrium equations reduce to an elliptic partial differential equation (along with a relation for the pressure) which can be solved analytically when the modulus of the Mach number of the poloidal velocity with respect to the poloidal-magnetic-field Alfvén velocity takes constant values. This is the subject of Sec. II. In Sec. III we construct exact equilibrium solutions for (a) purely toroidal flows, (b) flows parallel to the magnetic field, and (c) non-parallel flows with differentially varying radial electric fields. Incompressible $T = T(\psi)$ equilibria are then examined in Sec. IV. Our conclusions are summarized in Sec. V.

## II. Equilibrium equations

The ideal MHD equilibrium states of plasma flows are governed by the following set of equations, written in standard notations and convenient units:

$$\nabla \cdot (\rho \mathbf{v}) = 0 \tag{1}$$

$$\rho(\mathbf{v} \cdot \nabla)\mathbf{v} = \mathbf{j} \times \mathbf{B} - \nabla P \tag{2}$$

$$\nabla \times \mathbf{E} = 0 \tag{3}$$

$$\nabla \times \mathbf{B} = \mathbf{j} \tag{4}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{5}$$

$$\mathbf{E} + \mathbf{v} \times \mathbf{B} = 0. \tag{6}$$

It should be pointed out that, unlike the usual procedure followed in equilibrium studies with flow [2]-[9], in the present work an equation of state is not included in the above set of equations from the outset, and therefore equations (15) and (16) below are first derived, independently of the equation of state. This alternative procedure is convenient because the equilibrium problem can then be further reduced for any particular equation of sate. The system under consideration is an axially symmetric magnetically confined plasma with flow. For this configuration the divergence-free fields, i.e. the magnetic field $\mathbf{B}$, the current density $\mathbf{j}$ and the mass flow $\rho\mathbf{v}$ can be expressed in terms of the stream functions $\psi(R, z)$, $I(R, z)$, $F(R, z)$ and $\Theta(R, z)$ as

$$\mathbf{B} = I\nabla\phi + \nabla\phi \times \nabla\psi, \tag{7}$$

$$\mathbf{j} = \Delta^*\psi\nabla\phi - \nabla\phi \times \nabla I \tag{8}$$

and

$$\rho\mathbf{v} = \Theta\nabla\phi + \nabla\phi \times \nabla F. \tag{9}$$

Here, $R, \phi, z$ are cylindrical coordinates with $z$ corresponding to the axis of symmetry, constant $\psi$ surfaces are the magnetic surfaces and $\Delta^*$ is the elliptic operator defined by $\Delta^* = R^2 \nabla \cdot (\nabla / R^2)$.

Eqs. (1)-(6) can be reduced by means of certain integrals of the system, which are shown to be surface quantities. To identify two of these quantities, the time independent electric field is expressed by $\mathbf{E} = -\nabla\Phi$ and the Ohm's law (6) is projected along $\nabla\phi$ and $\mathbf{B}$, respectively, yielding

$$\nabla\phi \cdot (\nabla\phi \times \nabla F) \times (\nabla\phi \times \nabla\psi) = 0 \tag{10}$$

and

$$\mathbf{B} \cdot \nabla\Phi = 0. \tag{11}$$

Eqs. (10) and (11) imply that $F = F(\psi)$ and $\Phi = \Phi(\psi)$, hence the electric field is "radial", i.e., perpendicular to a magnetic surface. Two additional surface quantities are found from the component of Eq. (6) perpendicular to a magnetic surface:

$$\frac{1}{\rho R^2}(IF' - \Theta) = \Phi', \tag{12}$$

and from the component of the momentum conservation equation (2) along $\nabla\phi$:

$$I\left(1 - \frac{(F')^2}{\rho}\right) + R^2 F' \Phi' \equiv X(\psi). \tag{13}$$

(The prime denotes differentiation with respect to $\psi$). From Eq. (13) it follows that, unlike the case in static equilibria, $I$ is not a surface quantity. On the basis of Eq. (12) the velocity (Eq. (9)) can be written in the form

$$\mathbf{v} = \frac{F'}{\rho}\mathbf{B} - R^2 \Phi' \nabla\phi. \tag{14}$$

With the aid of Eqs. (10)-(14), the components of Eq. (2) along $\mathbf{B}$ and perpendicular to a magnetic surface are put in the respective forms

$$\mathbf{B} \cdot \left[\nabla\left(\frac{v^2}{2} + \frac{\Theta}{\rho}\Phi'\right) + \frac{\nabla P}{\rho}\right] = 0 \tag{15}$$

and

$$\left\{\nabla \cdot \left[\left(1 - \frac{(F')^2}{\rho}\right)\frac{\nabla\psi}{R^2}\right] + \frac{F''F'}{\rho}\frac{|\nabla\psi|^2}{R^2}\right\}|\nabla\psi|^2$$
$$+ \left[\frac{\rho}{2}\left(\nabla v^2 - \frac{\nabla(\Theta/\rho)^2}{R^2}\right) + \frac{\nabla(I^2)}{2R^2} + \nabla P\right] \cdot \nabla\psi = 0 \tag{16}$$

In order to reduce the equilibrium equations further, we employ the incompressibility condition

$$\nabla \cdot \mathbf{v} = 0. \tag{17}$$

Then Eq. (1) implies that the density is a surface quantity,

$$\rho = \rho(\psi). \tag{18}$$

Consequently, from Eqs. (12) and (13) it follows that, unlike the case in cylindrical plasmas [11], axisymmetric incompressible equilibria with purely poloidal flows ($\Theta = 0$) can not exist; the only possible equilibria of this kind are of cylindrical shape. It may be noted that, as proved in Ref. [20], this holds also for resistive plasmas in the particular case of $\beta_p = 1$ equilibria.

With the aid of Eq. (18), Eq. (15) can be integrated to yield an expression for the pressure, i.e.

$$P = P_s(\psi) - \rho\left(\frac{v^2}{2} + \frac{\Phi'\Theta}{\rho}\right). \tag{19}$$

We note here that, unlike in static equilibria, in the presence of flow magnetic surfaces in general do not coincide with isobaric surfaces because Eq. (2) implies that $\mathbf{B} \cdot \nabla P$ in general differs from zero. In this respect, the term $P_s(\psi)$ is the static part of the pressure which does not vanish when $\mathbf{v} = \mathbf{0}$.

Eq. (16) has a singularity when

$$\frac{(F')^2}{\rho} = 1. \tag{20}$$

On the basis of Eq. (9) for $\rho\mathbf{v}$ and the definitions $v_{Ap}^2 \equiv \frac{|\nabla\psi|^2}{\rho}$ for the Alfvén velocity associated with the poloidal magnetic field and the Mach number

$$M^2 \equiv \frac{v_p^2}{v_{Ap}^2} = \frac{(F')^2}{\rho}, \tag{21}$$

Eq. (20) can be written as $M^2 = 1$. If it is now assumed that $\frac{(F')^2}{\rho} \neq 1$ and Eq. (19) is inserted into Eq. (16), the latter reduces to the *elliptic* differential equation

$$(1 - M^2)\Delta^\star\psi - \frac{1}{2}(M^2)'|\nabla\psi|^2$$
$$+\frac{1}{2}\left(\frac{X^2}{1-M^2}\right)' + R^2\left(P_s - \frac{XF'\Phi'}{1-M^2}\right)' + \frac{R^4}{2}\left(\frac{\rho(\Phi')^2}{1-M^2}\right)' = 0. \tag{22}$$

This is the equilibrium equation for an axisymmetric plasma with incompressible flows. Once its solutions are known, the pressure can be determined from Eq. (19). Eq. (22) contains the arbitrary surface quantities $F(\psi)$, $\Phi(\psi)$, $X(\psi)$, $\rho(\psi)$ and $P_s(\psi)$ which must be found from other physical considerations. As shown in next section for appropriate physically reasonable choices of the surfaces functions, it can be linearized and solved analytically.

## III. Analytic equilibrium solutions

With the ansatz

$$\frac{(F')^2}{\rho} \equiv M_c^2 = \text{const.}, \tag{23}$$

Eq. (22) reduces to

$$\Delta^\star\psi + \frac{1}{(1-M_c^2)^2}\left[XX' + R^2\left((1-M_c^2)P_s - XF'\Phi'\right)' + \frac{R^4}{2}(\rho(\Phi')^2)'\right] = 0. \tag{24}$$

The singularity $M_c^2 = 1$ is the limit at which the confinement can be assured by the toroidal current $\Delta^\star\psi/R$ alone. For $M_c^2 > 1$ the derivative of $X^2/2$, related to the derivative of $(RB_\phi)^2/2$ by Eq. (13), partly compensates for the pressure gradient and inertial flow forces.

Three classes of exact equilibria of Eq. (24) can be constructed as follows.

### *(a) Purely toroidal flows*

This kind of equilibria correspond to $M_c^2 = F' = 0$. From Eqs. (9), (12) and (13) it then follows the relation $I = X(\psi)$ and that the angular frequency of the toroidal flow becomes a surface quantity:

$$\omega(\psi) \equiv \frac{v_\phi}{R} = \frac{\Theta}{\rho R^2} = -\Phi'(\psi). \tag{25}$$

Consequently, Eq. (24) becomes

$$\Delta^\star\psi + II' + R^2P_s' + \frac{R^4}{2}(\rho\omega^2)' = 0. \tag{26}$$

With the ansatz $II' = \text{const.}$ and $P_s' = \text{const.}$ Eq. (26) can be solved analytically for (a) $\rho\omega^2 = \text{const.}$ and (b) $\rho\omega^2 \propto \psi$. In the latter case the simplest solution corresponding to $I = \text{const.}$ is given by

$$\psi = \frac{\psi_c}{5}\frac{R^2}{R_c^6}(9R_c^4 - 3R_c^2R^2 - R^4 - d^2R_c^2z^2), \tag{27}$$

where $\psi_c$ is the value of the flux function at the position of the magnetic axis ($z = 0$, $R = R_c$) and $d^2$ is a parameter related to the shape of the flux surfaces.

### *(b) Flows parallel to* B

Equilibria with **B**-aligned flows correspond to $\Phi' = 0$. Eq. (24) then reduces to

$$\Delta^\star\psi + \hat{X}\hat{X}' + R^2\hat{P}_s' = 0 \tag{28}$$

where $\hat{X} \equiv X(\psi)/(1 - M_c^2) = I$ and $\hat{P}_s = P_s(\psi)/(1 - M_c^2)$. Eq. (28) is identical in form to the equation governing static equilibria; the only reminiscence of the flow is the presence of $M_c$ in $\hat{X}$ and $\hat{P}_s$. It can be linearized and solved for (a) $\hat{X}\hat{X}' =$ const., and $\hat{P}_s' =$ const., and (b) $\hat{X}\hat{X}' \propto \psi$ and $\hat{P}_s' \propto \psi$. In case (a) the simplest solution, corresponding to $\hat{X} =$ const., is given by

$$\psi = \psi_c \frac{R^2}{R_c^4}\left(2R_c^2 - R^2 - 4d^2z^2\right). \tag{29}$$

Eq. (29) describes the Hill's vortex configuration [17]. Also, one can derive more general solutions by adding higher order polynomials in $(R, z)$ which satisfy $\Delta^\star\psi = 0$ to Hill's vortex solution. If $\hat{P}_s = b_0\psi$, the general form for these solutions is given by

$$\psi = -\frac{b_0}{8}R^4 + \sum_{m=0}\frac{a_m R^2}{2}\sum_{l=0}^{m}\frac{(-1)^l (R/2)^{2(m-l)} z^{2l}}{(m-l)!(m-l+1)!(2l)!}, \tag{30}$$

where $b_0$ and $a_m$ are constant quantities.

### *(c) Non-parallel flows*

Equilibria of this kind are of particular interest because non-parallel flows with non-vanishing poloidal components are associated with radial electric fields which play a role in the transitions to improved confinement regimes [18]. They can be derived with the ansatz $\Phi' \propto \psi^{-k/2}$ and $\rho \propto \psi^k$, where $k$ is a parameter. The electric field is then of the form $\mathbf{E} \propto -\psi^{-k/2}\nabla\psi$ and Eq. (24) becomes

$$\Delta^\star\psi + \hat{X}\hat{X}' + R^2[\hat{P}_s - d_0\hat{X}]' = 0, \tag{31}$$

where $d_0 =$ const. As the case in equilibria with parallel flows, the simplest solution of Eq. (31) ($\hat{X} = \hat{X}_0 =$ const.) is given by Eq. (29) and, e.g., for a plasma with constant density ($k = 0$) the $|\mathbf{E}|$-profile at the midplane $z = 0$ is hollow. Also, according to Eq. (13), the toroidal magnetic field, in addition to the usual $1/R$ component, contains a flow term linear in $R$:

$$B_\phi = \frac{I}{R} = \frac{\hat{X}_0}{R} - d_0 R. \tag{32}$$

Thus, the modification of $B_\phi$ may affect the shape of the safety factor profile. This indicates that the flow along with the associated radial electric field may contribute to the creation of improved confinement regimes related to appropriate shaping of the safety factor profiles, e.g. inverse-magnetic-shear profiles [18].

## III. Equilibria with isothermal magnetic surfaces

For fusion plasmas the thermal conduction along $\mathbf{B}$ is fast compared to the heat transport perpendicular to a magnetic surface and therefore equilibria with isothermal magnetic surfaces are of particular interest. It is noted that for cylindrical plasmas the relation $T = T(\psi)$ imposes a limitation on the possible incompressible equilibria, i.e., the cross sections of the magnetic surfaces must be circular [12]. In the following we show that axisymmetric incompressible $T = T(\psi)$ equilibria are free of such a restriction except near to the magnetic axis.

Under the assumption that the plasma obeys to the ideal gas low $P = \hat{R}\rho T$, Eqs. (19), (12) and (14) lead to the following expression for the magnetic field modulus:

$$|\mathbf{B}|^2 = \Xi(\psi) + R^2 H(\psi), \tag{33}$$

where $\Xi(\psi) \equiv 2(P_s - P)\rho/(F')^2$ and $H(\psi) \equiv (\rho\Phi'/F')^2$. Consequently, apart from the case of field aligned flows ($H = 0$), omnigenous equilibria, viz. equilibria with $|\mathbf{B}|$ being a surface quantity, are not possible.

Solving the set of equations (16) and (33) for $|\nabla\psi|^2$ and $\Delta^*\psi$ one obtains

$$\left(\frac{\partial\psi}{\partial R}\right)^2 + \left(\frac{\partial\psi}{\partial z}\right)^2 = 2(i(\psi) + R^2 j(\psi) + R^4 k(\psi)) \tag{34}$$

and

$$\frac{\partial^2\psi}{\partial^2 R^2} - \frac{1}{R}\frac{\partial\psi}{\partial R} + \frac{\partial^2\psi}{\partial^2 z^2} = -f(\psi) - R^2 g(\psi) - R^4 h(\psi), \tag{35}$$

where

$$i(\psi) \equiv -\frac{1}{2}\left(\frac{X}{1-M^2}\right)^2,$$

$$j(\psi) \equiv \frac{P_s - P}{M^2} - \frac{X\Phi' F'}{(1-M^2)^2},$$

$$k(\psi) \equiv \frac{1}{2}\left[\rho(\Phi')^2 - \frac{(F'\Phi')^2}{(1-M^2)^2}\right],$$

$$f(\psi) \equiv \frac{1}{2}\left[\left(\frac{X}{1-M^2}\right)^2\right]',$$

$$g(\psi) \equiv \frac{M^2}{1-M^2}\left(\frac{P_s}{M^2}\right)' + \left[\frac{X\Phi' F'}{(1-M^2)^2}\right]' - \frac{(M^2)'}{M^2(1-M^2)}P,$$

and

$$h(\psi) \equiv \frac{1}{2(1-M^2)}\left[\left(\frac{\rho(\Phi')^2}{1-M^2}\right)' + \left(\frac{F'\Phi'}{1-M^2}\right)^2 - \frac{\rho(\Phi')^2}{M^2}\right].$$

With the introduction of the quantities $x \equiv R^2$, $p = \partial\psi/\partial x$, $q = \partial\psi/\partial z$, $r = \partial^2\psi/\partial x^2$ and $t = \partial^2\psi/\partial z^2$ Eqs. (34) and (35) are written in the respective forms

$$4xp^2 + q^2 = 2(i + xj + x^2 k) \tag{36}$$

and

$$4xr + t = -f - xg - x^2 h. \tag{37}$$

To integrate Eqs. (36) and (37) we apply a procedure suggested by Palumbo [19]. Accordingly, employing $R$ and $\psi$ as independent coordinates instead of $R$ and $z$ (then $z = z(x,\psi)$), we have

$$r = \left.\frac{\partial p}{\partial x}\right|_z = \left.\frac{\partial p}{\partial x}\right|_\psi + p\left.\frac{\partial p}{\partial \psi}\right|_x \tag{38}$$

and

$$t = \left.\frac{\partial q}{\partial z}\right|_x = q\left.\frac{\partial q}{\partial \psi}\right|_x. \tag{39}$$

With the aid of Eqs. (36), (38), (39) and $f + i' = 0$ Eq. (37) reduces to

$$4\left.\frac{\partial p}{\partial x}\right|_\psi = -x(g + j') - x^2(h + k')$$

and consequently

$$p = -\frac{1}{4}(g + j')x - \frac{1}{8}(h + k')x^2 + \frac{d(\psi)}{4}. \tag{40}$$

Since z is a function of $x$ and $\psi$, solutions of equation

$$dz = -\frac{p}{q}dx + \frac{1}{q}d\psi \tag{41}$$

exist provided

$$\frac{\partial}{\partial\psi}\left(-\frac{p}{q}\right) = \frac{\partial}{\partial\psi}\left(\frac{1}{q}\right). \tag{42}$$

Eq. (42) leads to the solvability condition

$$-q^2\frac{\partial p}{\partial\psi} + \frac{1}{2}p\frac{\partial q^2}{\partial\psi} + \frac{1}{2}\frac{\partial q^2}{\partial x} = 0. \tag{43}$$

Substituting $q^2$ and $p$ from Eqs. (36) and (40) in Eq. (43) yields a relation of the form $\sum_{j=0}^{4} a_j(\psi)x^j = 0$; hence, since $x$ and $\psi$ are independent variables, the equations $a_j = 0$ should be satisfied for all $j$. They are explicitly given by

$$-\frac{1}{8}d^2 + j - \frac{1}{2}id' + \frac{1}{4}di' = 0, \tag{44}$$

$$\frac{1}{2}dg + 2k - \frac{1}{2}jd' + \frac{1}{2}ig' - \frac{1}{4}gi' + \frac{3}{4}dj' - \frac{1}{4}i'j' + \frac{1}{2}ij'' = 0, \tag{45}$$

$$-\frac{3}{8}g^2+\frac{3}{8}dh-\frac{1}{2}kd'+\frac{1}{2}jg'+\frac{1}{4}ih'-\frac{1}{8}hi'-gj'-\frac{5}{8}(j')^2+\frac{5}{8}dk'-\frac{1}{8}i'k'+\frac{1}{2}jj''+\frac{1}{4}ik'' = 0, \tag{46}$$

$$-\frac{1}{2}gh + \frac{1}{2}kg' + \frac{1}{4}jh' - \frac{5}{8}hj' - \frac{3}{4}gk' - \frac{7}{8}j'k' + \frac{1}{2}kj'' + \frac{1}{4}jk'' = 0, \tag{47}$$

and

$$-\frac{5}{32}h^2 + \frac{1}{4}kh' - \frac{7}{16}hk' - \frac{9}{32}(k')^2 + \frac{1}{4}kk'' = 0. \tag{48}$$

Equations (44)-(48) contain the surface functions $P(\psi)$, $P_s(\psi)$, $F(\psi)$, $\Phi(\psi)$, $X(\psi)$, $\rho(\psi)$ and $d(\psi)$ two of which remain free. If the free functions are assigned along with boundary conditions the set of Eqs. (44)-(48) can be solved numerically. Furthermore, to completely solve the equilibrium problem, one should determine the function $z(x,\psi)$ which by Eq. (41) satisfies the equation

$$\left.\frac{\partial z}{\partial x}\right|_{\psi} = -\frac{p}{q} = \frac{\frac{1}{4}\left[(g+j')x + \frac{1}{2}(h+k')x^2 - d\right]}{\left\{2(i+xj+x^2k) - \frac{x}{4}\left[(g+j')x + \frac{1}{2}(h+k')x^2 - d\right]^2\right\}^{1/2}}. \tag{49}$$

Once the set of equations (44)-(48) is solved, the $\psi$ dependence on the rhs of Eq. (49) is known and the function $z(x,\psi)$ can be expressed in terms of hyperelliptic integrals [21]. For the particular case of field aligned flows (h=k=0) the hyperelliptic integrals reduce to elliptic ones, as suggested by Ref. [22].

## IV. Conclusions

For an axisymmetric plasma with incompressible flows we found that the ideal MHD equilibrium equations are reduced to a second-order elliptic partial differential equation for the poloidal magnetic flux function $\psi$ (Eq. (22) containing the density $\rho(\psi)$, the electrostatic potential $\Phi(\psi)$, the static equilibrium pressure $P_s(\psi)$, the function $F(\psi)$ associated with the poloidal flow and the function $X(\psi)$ related to the toroidal magnetic field) in conjunction with a relation for the pressure (Eq. (19)). When the Mach number of the poloidal velocity with respect to the poloidal-magnetic-field Alfvén velocity takes constant values, the equilibrium differential equation can be solved analytically. Exact equilibria were obtained for (a) purely toroidal flows, (b) flows parallel to the magnetic field, and (c) non-parallel flows with differentially varying electric fields.

Unlike the case of cylindrical incompressible equilibria with isothermal magnetic surfaces, which should have necessarily circular cross section, no restriction appears on the shape of magnetic surfaces of the corresponding axisymmetric equilibria, though in analogy with the cylindrical case it can be conjectured that only the near magnetic axis surfaces have to be circular. In fact, the equilibrium problem reduces to a set a five nonlinear ordinary differential equations with respect to $\psi$, containing the above mentioned surface quantities and the pressure $P(\psi)$, along with an ordinary differential equation for the function $z(R, \psi)$, where $z$ pertains to the axis of symmetry and $R$ to the direction perpendicular to the axis of symmetry. Once the solution of the former set of five equations is numerically found, the function $z(R, \psi)$ can be expressed in terms of hyperelliptic integrals which for field aligned flows reduce to elliptic ones. In addition, it was proved the non existence of axisymmetric incompressible equilibria with (a) purely poloidal flows and (b) non-parallel flows with isothermal magnetic surfaces and $|\mathbf{B}| = |\mathbf{B}|(\psi)$ (omnigenous equilibria).

The equilibrium equations for incompressible axisymmetric flows and the analytic solutions derived in the present work can be employed for stability and transport investigations, which would be of relevance to magnetic confinement systems. In particular, they may help in understanding the physics of the transitions to the improved confinement regimes, which are related to differential flows and radial electric fields. In addition, they can be used in benchmarking relevant equilibrium codes. Finally, let us note that it is interesting to extend the study to the more general case of helically symmetric equilibria.


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