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AXISYMMETRIC STAGNATION FLOW ON A CYLINDER*

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Due to the inherent nonlinearity of the Navier–Stokes equations, there exist only three exact solutions of stagnation flows: Hiemenz [1] found a solution to the two-dimensional stagnation flow against a plate, Homann [2] investigated the axisymmetric stagnation flow, also against a plate, and Howarth [3] and Davey [4] extended the results to unsymmetric cases.

The present note presents a new exact solution, namely, axisymmetric stagnation flow on an infinite circular cylinder. Fig. 1 shows a cylinder described by $r = a$ in the cylindrical polar coordinates. The flow is axisymmetric about the z axis and also symmetric to the $z = 0$ plane. The stagnation "line" is at $z = 0$, $r = a$. This flow may be useful in certain cooling processes.

Let u and w be the velocities in the directions r and z respectively. If the flow is inviscid, the potential velocity and pressure distribution in the neighborhood of the stagnation line are

$$u = -k(r - a^2/r), \tag{1}$$

$$w = 2kz,$$

$$p = p_0 - \rho k^2[2z^2 + \tfrac{1}{2}(r - a^2/r)^2], \tag{3}$$

where k is a given constant of dimensions $[1/T]$, p_0 is the stagnation pressure, and ρ is the density. We expect the viscous flow to approach the potential solution as $r \to \infty$.

The constant-density Navier–Stokes equations in cylindrical coordinates are

$$uu_r + wu_z = -\frac{1}{\rho}p_r + \nu\left(u_{rr} + \frac{1}{r}u_r + u_{zz} - \frac{u}{r^2}\right), \tag{4}$$

$$uw_r + ww_z = -\frac{1}{\rho}p_z + \nu\left(w_{rr} + \frac{1}{r}w_r + w_{zz}\right), \tag{5}$$

$$rw_z + (ru)_r = 0. \tag{6}$$

Let

$$u = -ka\eta^{-1/2}f(\eta), \tag{7}$$

$$w = 2kf'(\eta)z, \tag{8}$$

where $\eta = (r/a)^2$. After some algebra, Eqs. (4) and (5) reduce to

$$\eta f''' + f'' + R(1 + ff'' - f'^2) = 0, \tag{9}$$

$$p = p_0 - \rho\left(\frac{k^2a^2}{2}\frac{f^2}{\eta} + 2\nu kf' + 2k^2z^2\right), \tag{10}$$

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where $R \equiv ka^2/2\nu$ is a Reynolds number. The boundary conditions are

$$f(1) = 0, \quad f'(1) = 0, \quad f'(\infty) = 1. \tag{11}$$

Eq. (9) is integrated numerically by the Runge–Kutta method. The accuracy is determined by varying the step size. Figs. 2 and 3 shows $f(\eta)$ and $f'(\eta)$ respectively for several values of R. Due to the natural length scale a which enters in the parameter

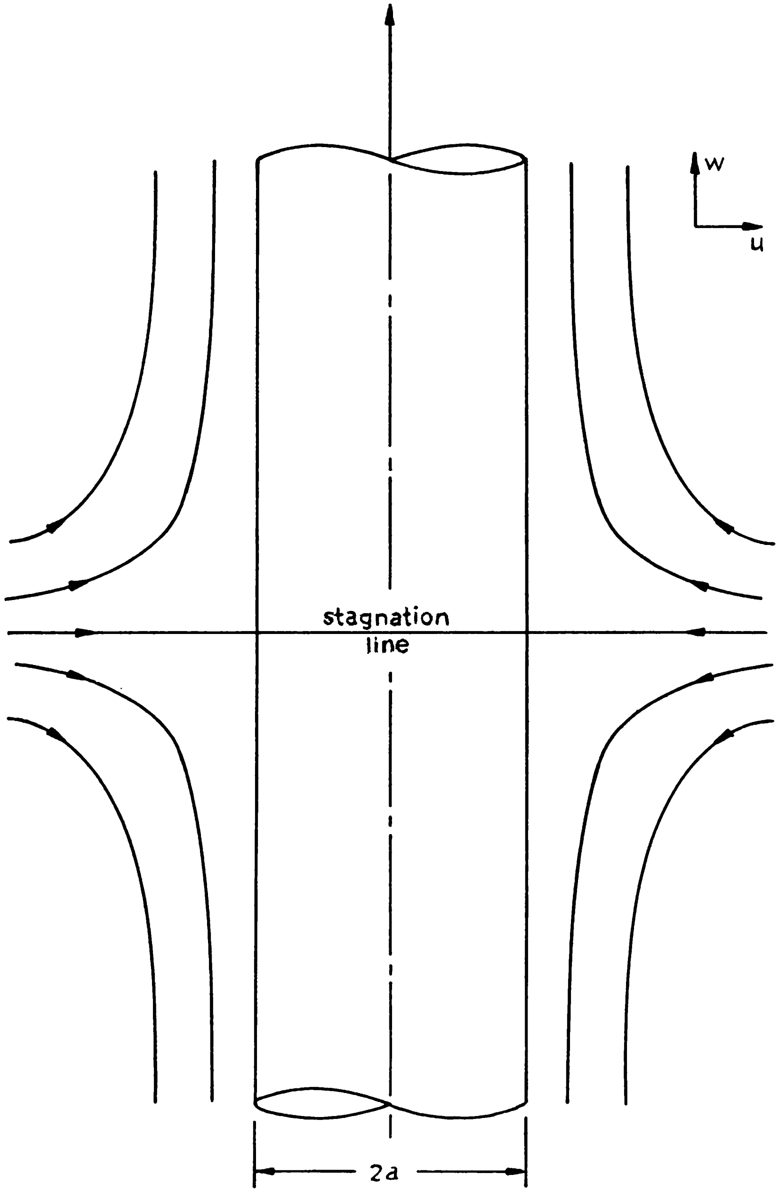

FIG. 1. The coordinate axis.

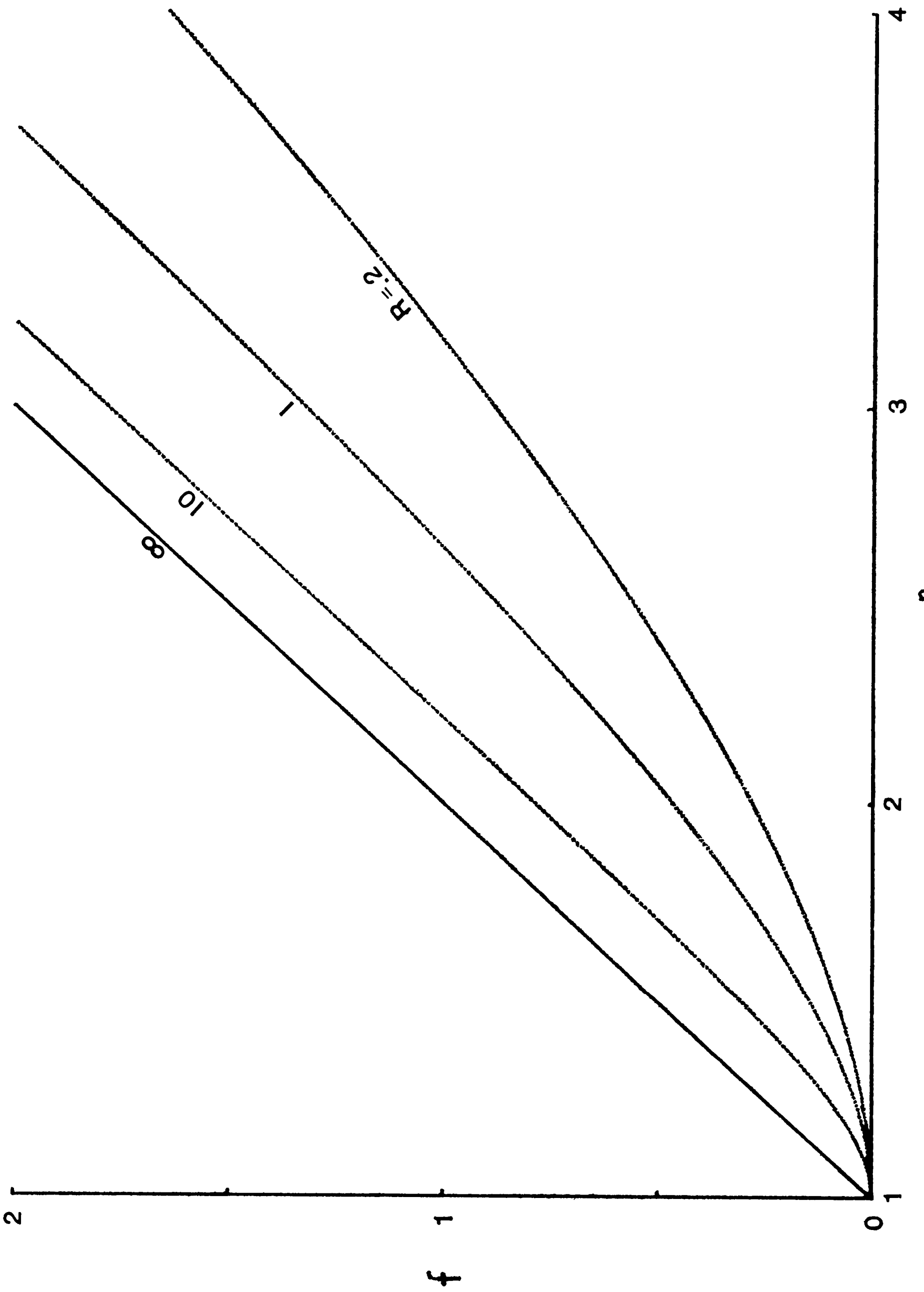

FIG. 2. Numerical solution for $f(\eta)$.

R, these curves are not similar to each other. The numerical results are given in Tables I, II, III.

Of some interest is the asymptotic behavior for large η. Let $f(\eta) = \eta + c + g(\eta)$ where c is a constant and $g(\eta)$ is small. Then Eq. (9) linearizes to

$$\xi g'''(\xi) + (1 + \xi)g'' - 2g' = 0 \tag{12}$$

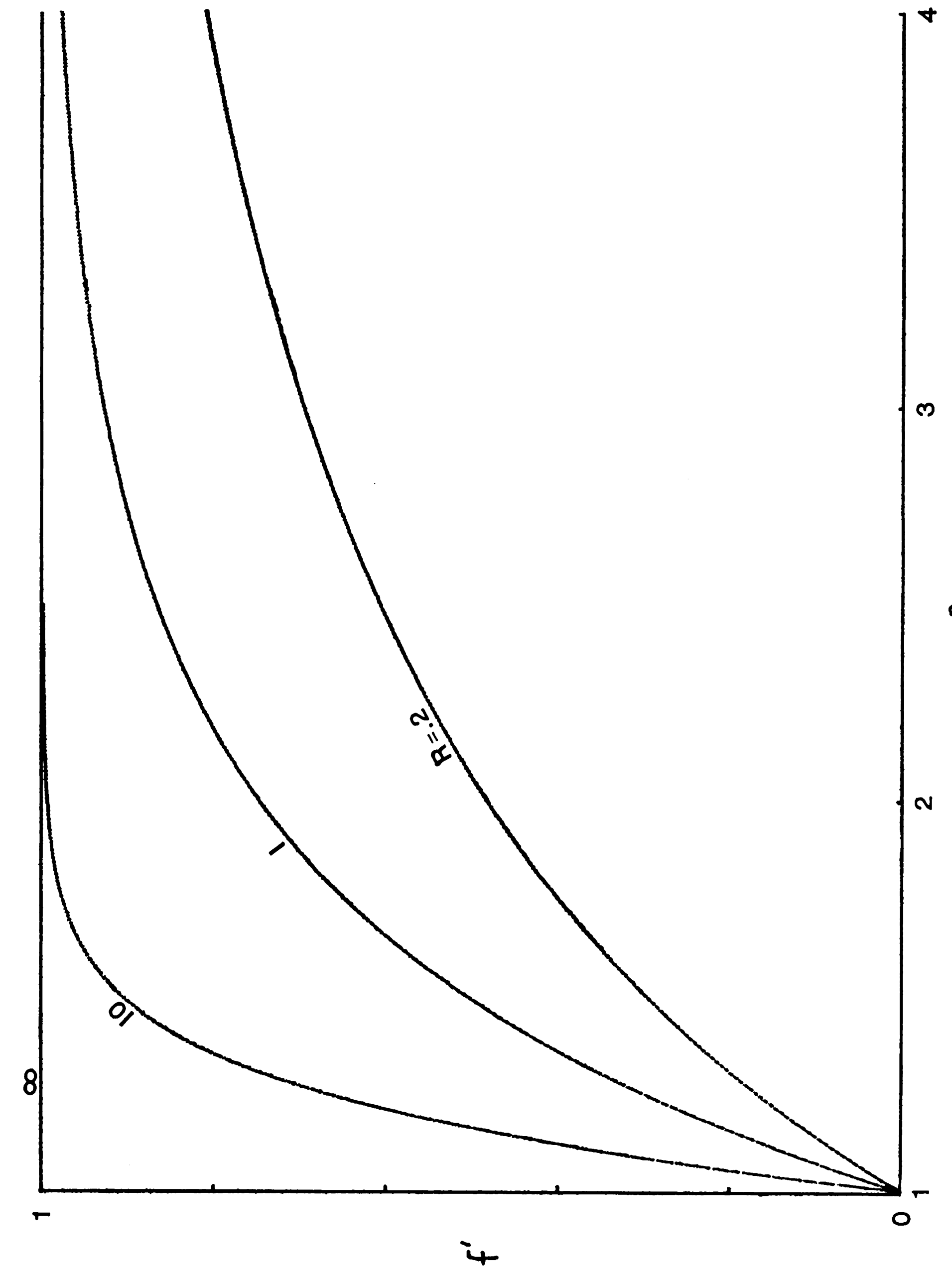

FIG. 3. The axial velocity $f'(\eta)$.

where $\xi = R\eta$. This shows the solutions are at least similar for large η. The nontrivial solution which decays to zero at infinity is found to be

$$g(\xi) = \text{const} \int_{\infty}^{\xi} \left[(\lambda^2 + 4\lambda + 2) \int_{\infty}^{\lambda} \beta^{-1}(\beta^2 + 4\beta + 2)^{-2} e^{-\beta}\, d\beta \right] d\lambda. \tag{13}$$

Eq. (13) can be rewritten in terms of one integral

$$g(\xi) = \text{const} \int_{\infty}^{\xi} \left(\frac{\xi^3}{3} + 2\xi^2 + 2\xi - \frac{t^3}{3} - 2t^2 - 2t\right) t^{-1}(t^2 + 4t + 2)^{-2} e^{-t} \, dt. \tag{14}$$

The decay is proportional to $\xi^{-3}e^{-\xi}$.

For large R the solution is closely related to the two-dimensional stagnation flow against a flat plate. The transformation

$$f(\eta) = R^{1/2}\varphi(\xi) \tag{15}$$

where $\xi = R^{1/2}(\eta - 1)$ yields Hiemenz's equation

$$\begin{gathered} \varphi''' + \varphi\varphi'' - \varphi'^2 + 1 = 0, \\ \varphi(0) = \varphi'(0) = 0, \qquad \varphi'(\infty) = 1 \end{gathered} \tag{16}$$

as a first approximation. The error is of order $R^{-1/2}$. Table IV gives a comparison to Hiemenz's value.

If the Reynolds number were small, Eq. (9) yields as first approximation

$$f = \text{const } (\eta \ln \eta - \eta + 1) \tag{17}$$

which is singular at infinity. The solution breaks down at a distance of $\eta = 0(1/R)$,

TABLE I (R = .2)

η	f	f′	f″
1.0	0	0	.78605
1.5	.08170	.29998	.45864
2.0	.28134	.48534	.29900
2.5	.55708	.61005	.20737
3.0	.88537	.69829	.14970
3.5	1.2514	.76290	.11117
4.0	1.6456	.81139	.08432
4.5	2.0609	.84848	.06503
5.0	2.4926	.87726	.05083
6.0	3.3917	.91786	.03205
7.0	4.3235	.94386	.02087
8.0	5.2766	.96100	.01392
9.0	6.2437	.97254	.00947
10.0	7.2204	.98045	.00654
12.0	9.1918	.98982	.00324
14.0	11.176	.99455	.00167
16.0	13.168	.99702	.00088
18.0	15.164	.99834	.00048
20.0	17.161	.99906	.00026
25.0	22.159	.99977	.00006
30.0	27.158	.99995	.00001
35.0	32.158	1.00000	.00000

TABLE II (R = 1.)

η	f	f′	f″
1	0	0	1.484185
1.2	.02667	.25302	1.07223
1.4	.09665	.43724	.78662
1.6	.19836	.57315	.58369
1.8	.32361	.67444	.43697
2.0	.46647	.75054	.32949
2.5	.87488	.86968	.16664
3.0	1.3266	.93068	.08647
3.5	1.8008	.96261	.04572
4.0	2.2867	.97961	.02453
4.5	2.7791	.98878	.01331
5.0	3.2748	.99378	.00729
6.0	4.2712	.99805	.00224
7.0	5.2700	.99938	.00070
8.0	6.2697	.99980	.00022
9.0	7.2695	.99993	.00007
10.0	8.2695	.99998	.00002
11.0	9.2695	1.00000	.00000

TABLE III (R = 10.)

η	f	f′	f″
1.0	0	0	4.16292
1.1	.01857	.35045	2.89754
1.2	.06638	.58982	1.93962
1.3	.13382	.74753	1.25571
1.4	.21400	.84821	.78923
1.5	.30220	.91071	.48298
1.6	.39532	.94852	.28845
1.7	.49139	.97087	.16847
1.8	.58919	.98380	.09640
1.9	.68797	.99114	.05412
2.0	.78731	.99522	.02986
2.2	.98677	.99867	.00866
2.4	1.1866	.99965	.00238
2.6	1.3865	.99991	.00062
2.8	1.5865	.99998	.00015
3.0	1.7865	.99999	.00004
3.5	2.2865	1.00000	.00000

TABLE IV

R	$R^{-1/2}f''(1)$	
.2	1.7577	
1.	1.484185	
10.	1.31643	
∞	1.232588	(Hiemenz flow, $\varphi''(0)$)

where, in order to bring in the nonlinear terms, we set $f = (1/R)h(\xi)$, $\xi = R\eta$. One obtains

$$\xi h''' + h'' + 1 + hh'' - h'^2 = 0$$

which is an equation as difficult as the original one. As η is increased further, linearization is possible and we obtain Eq. (12) by setting $h(\xi) = \xi + c + g(\xi)$.

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