Commun. Korean Math. Soc. **27** (2012), No. 2, pp. 217–222
http://dx.doi.org/10.4134/CKMS.2012.27.2.217

β-ALGEBRAS AND RELATED TOPICS

Young Hee Kim and Keum Sook So

Abstract. In this note we investigate some properties of β-algebras and further relations with B-algebras. Especially, we show that if $(X, -, +, 0)$ is a B^*-algebra, then $(X, +)$ is a semigroup with identity 0. We discuss some constructions of linear β-algebras in a field K.

1. Introduction

Y. Imai and K. Iséki introduced two classes of abstract algebras: BCK-algebras and BCI-algebras ([3, 4]). We refer useful textbooks for BCK/BCI-algebra to [2, 6, 9]. J. Neggers and H. S. Kim ([7]) introduced another class related to some of the previous ones, viz., B-algebras and studied some of its properties. They also introduced the notion of β-algebra ([8]) where two operations are coupled in such a way as to reflect the natural coupling which exists between the usual group operation and its associated B-algebra which is naturally defined by it. P. J. Allen et al. ([1]) gave another proof of the close relationship of B-algebras with groups using the observation that the zero adjoint mapping is surjective. H. S. Kim and H. G. Park ([5]) showed that if X is a 0-commutative B-algebra, then $(x * a) * (y * b) = (b * a) * (y * x)$. Using this property they showed that the class of p-semisimple BCI-algebras is equivalent to the class of 0-commutative B-algebras.

In this note we investigate some properties of β-algebras and further relations with B-algebras. Especially, we show that if $(X, -, +, 0)$ is a B^*-algebra, then $(X, +)$ is a semigroup with identity 0. Finally we discuss some constructions of linear β-algebras in a field K.

2. Preliminaries

A *β-algebra* ([8]) is a non-empty set X with a constant 0 and two binary operations "+" and "−" satisfying the following axioms: for any $x, y, z \in X$,

(I) $x - 0 = x$,
(II) $(0 - x) + x = 0$,

Received October 12, 2010.
2010 *Mathematics Subject Classification.* 06F35.
Key words and phrases. β-algebra, B-algebra, B^*-algebra, linear.

(III) $(x - y) - x = x - (z + y)$.

Example 2.1 ([8])**.** Let $X := \{0, 1, 2, 3\}$ be a set with the following tables:

$+$	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

$-$	0	1	2	3
0	0	3	2	1
1	1	0	3	2
2	2	1	0	3
3	3	2	1	0

Then $(X, +, -, 0)$ is a β-algebra.

Proposition 2.2 ([8])**.** *Let $(G, \cdot, e)$ be a group. If we define $x+y := x\cdot y, x-y := x \cdot y^{-1}, 0 := e$ for any $x, y \in G$, then $(G, +, -, 0)$ is a β-algebra, called a group-derived β-algebra and denoted by $A(G)$.*

Proposition 2.3 ([8])**.** *Let S be a set. If we define $x + y := x, x - y := x$ and $0 \in S$, then $(S, +, -, 0)$ is a β-algebra, called a left β-algebra and denoted by A_S.*

It is known that the Cartesian product $X \times Y$ of a group-derived β-algebra X and a left β-algebra Y is a β-algebra which is neither group-derived nor a left β-algebra, and denoted by $A(G) \times A_S$.

We note that if a β-algebra is either $A(G)$ or A_S, then it is also the case that

(IV) $x + y = x - (0 - y)$.

Hence the condition (IV) holds for β-algebras of the type $A(G) \times A_S$ as well.

Group-derived and left β-algebras part ways via the following conditions:

(V_a) $x - x = 0$ (group derived),
(V_b) $x - x = x$ (left).

We list two classes of β-algebras of special interest. A β-algebra X is said to be a *B^*-algebra* if (IV) and (V_b) hold. J. Neggers and H. S. Kim introduced the notion of B-algebra, and obtained various properties. An algebra $(X, -, 0)$ is said to be a *B-algebra* ([7]) if it satisfies $(I), (V_a)$ and

(VI) $(x - y) - z = x - (z - (0 - y))$

for any $x, y, z \in X$.

3. β-algebras and related topics

Given a β-algebra X, we denote $x^* := 0 - x$ for any $x \in X$.

Proposition 3.1. *Let $(X, +, -, 0)$ be a β-algebra with condition* (IV)*. Then the following holds: for any $x, y, z \in X$,*

(1) $x^* + y = x^* - y^*$,
(2) $x^* + x = 0$,
(3) $x^* - x^* = 0$,
(4) $x^* - y^* = (y^* - x^*)^*$,

(5) $x + y = x - y^*$,
(6) $x = (x - y) + y = (x - y) - y^*$,
(7) $y - x = y' - x$ *implies* $y = y'$.

Proof. (1) By (IV), $x^* + y = (0 - x) + y = (0 - x) - (0 - y) = x^* - y^*$. (2) From (II) $0 = (0 - x) + x = x^* + x$. (3) If we let $y := x$ in (1), then $x^* - x^* = x^* + x = 0$ by (2). (4) It follows from (III) and (1) that $x^* - y^* = (0 - x) - (0 - y) = 0 - ((0 - y) + x) = 0 - (y^* + x) = 0 - (y^* - x^*) = (y^* - x^*)^*$. (5) It follows from (IV) immediately. (6) $x = x - 0 = x - ((0 - y) + y) = (x - y) - y^* = (x - y) + y$. (7) Suppose that $y - x = y^* - x$. Then $y = (y - x) + x = (y^* - x) + x = y^*$, proving the proposition. □

Let $(X, +, -, 0)$ be a β-algebra with condition (IV) and let $x \in X$. We denote sum of x as follows:

$$0x = 0, \quad 1x = x,$$
$$2x = x - (0 - x) = x + x, \quad 3x = 2x + x = (x + x) + x,$$
$$nx = (n - 1)x + x \quad \text{where } n \text{ is a natural number} \geq 2.$$

Proposition 3.2. *Let* $(X, +, -, 0)$ *be a* β*-algebra with condition* (IV)*. Then*

$$(x - ny) + y = x - (n - 1)y$$

for any $x, y \in X$ *where* n *is a natural number.*

Proof. For any $x, y \in X$, $(x - 2y) + y = (x - (y + y)) + y = ((x - y) - y) + y = x - y$ by Proposition 3.1(6).

$$\begin{aligned}(x - 3y) + y &= (x - (2y + y)) + y \\ &= ((x - y) - 2y) + y \\ &= [(x - y) - (y + y)] + y \\ &= [((x - y) - y) - y] + y && \text{[by (III)]} \\ &= (x - y) - y && \text{[by Proposition 3.1(6)]} \\ &= x - 2y.\end{aligned}$$

Using mathematical induction on n, we obtain $(x - ny) + y = x - (n - 1)y$ for any natural number n. □

Proposition 3.3. *Let* $(X, -, 0)$ *be a* B*-algebra with* (IV)*. Then* $(X, -, +, 0)$ *is a* β*-algebra.*

Proof. (II) By applying (IV) and (V_a), we obtain $(0 - x) + x = (0 - x) - (0 - x) = 0$. (III) By applying (VI) and (IV), we obtain $(x - y) - z = x - (z - (0 - y)) = x - (z + y)$. Hence $(X, -, +, 0)$ is a β-algebra. □

Proposition 3.4. *Let* $(X, -, +, 0)$ *be a* β*-algebra with condition* (IV)*. Then it satisfies the condition* (VI)*.*

Proof. Given $x, y, z \in X$, we have

$$x - (z - (0 - y)) = (x - (z + y) \qquad \text{[by (IV)]}$$
$$= (x - y) - z, \qquad \text{[by (III)]}$$

proving the proposition. □

Lemma 3.5. *Let $(X, -, +, 0)$ be a B^*-algebra. Then for any $x \in X$, we have*

$$x = 0 - (0 - x).$$

Proof. For any $x \in X$, we have

$$\begin{aligned} x &= x - 0 && \text{[by (I)]} \\ &= x - [(0 - x) + x] && \text{[by (II)]} \\ &= (x - x) - (0 - x) && \text{[by (III)]} \\ &= 0 - (0 - x), && \text{[by (V}_\text{a}\text{)]} \end{aligned}$$

proving the lemma. □

Theorem 3.6. *If $(X, -, +, 0)$ is a B^*-algebra, then $(X, +)$ is a semigroup with identity 0.*

Proof. We claim that $(0-z)+(0-y) = 0-(y+z)$. By applying (IV), Lemma 3.5 and (III), we obtain $(0-z)+(0-y) = (0-z)-(0-(0-y)) = (0-z)-y = 0-(y+z)$. For any $x, y, z \in X$, we have

$$\begin{aligned} (x + y) + z &= (x + y) - (0 - z) \\ &= (x - (0 - y)) - (0 - z) \\ &= x - [(0 - z) + (0 - y)] \\ &= x - [0 - (y + z)] && \text{[by claim]} \\ &= x + (y + z). \end{aligned}$$

Hence $(X, +)$ is a semigroup. Since $x + 0 = x - (0 - 0) = x - 0 = x$ and $0 + x = 0 - (0 - x) = x$, 0 acts as an identity. □

Corollary 3.7. *Let $(X, -, +, 0)$ be a B^*-algebra. If $0-x = 0-y$, then $x+y = 0$.*

Proof. Suppose that $0 - x = 0 - y$. Then $0 = (0 - x) + x = (0 - y) + x = (0 - y) - (0 - x) = 0 - (x + y)$ by applying the claim in the proof of Theorem 3.6. Since $0 - 0 = 0$, by applying Proposition 3.1(7), we obtain $x + y = 0$. □

4. Linear β-algebras

Let $(K, +, \cdot, e)$ be a field (sufficiently large) and let $x, y \in K$. Define two binary operations "$\ominus, \oplus$" on K as follows:

$$x \ominus y := \alpha + \beta x + \gamma y,$$
$$x \oplus y := A + Bx + Cy,$$

where $\alpha, \beta, \gamma, A, B, C \in K$ (fixed). Assume that $(K, \ominus, \oplus, e)$ is a β-algebra. It is necessary to find proper solutions for two equations. Since $x = x \ominus e = \alpha + \beta x + \gamma e$, we obtain $(\beta - 1)x + (\alpha + \gamma e) = 0$, and hence $\beta = 1$ and $\alpha = -\gamma e$. It follows that

$$x \ominus y = x + \gamma(y - e). \tag{1}$$

Since $(e \ominus x) \oplus x = e$, we obtain

$$[A - Be(1 - \gamma) - e] + (B\gamma + C)x = 0, \forall x \in K. \tag{2}$$

It follows from (2) that $C = -B\gamma, A = [1 + B(1 - \gamma)]e$. Hence we have

$$x \oplus y = [1 + B(1 - \gamma)]e + B(x - \gamma y). \tag{3}$$

Using (1) we obtain

$$(x \ominus y) \ominus z = x + \gamma(y + z - 2e) \tag{4}$$

and

$$x \ominus (z \oplus y) = x + \gamma(z \oplus y - e). \tag{5}$$

To satisfy condition (III), if $\gamma \neq 0$, then

$$z \oplus y - e = y + z - 2e,$$

i.e., $z \oplus y = z + y - e$. Hence $x \oplus y = x + y - e$ and $B = C = 1$. Since $C = -B\gamma$, $\gamma = -1$, and hence $x \ominus y = x - y + e$. In the case of $\gamma = 0$, we obtain from (1) and (3) that $x \ominus y = x$ and $x \oplus y = (1 + B)e + Bx$, which leads to a contradiction, since $(e \ominus x) \oplus x = 1 + 2Be \neq e$. We summarize:

Theorem 4.1. *Let $(K, +, \cdot, e)$ be a field (sufficiently large) and let $x, y \in K$. Then $(K, \ominus, \oplus, e)$ is a β-algebra, where $x \ominus y = x - y + e$ and $x \oplus y = x + y - e$ for any $x, y \in K$.*

We call such a β-algebra described in Theorem 4.1 a *linear β-algebra*.

If we let $\varphi : X \to X$ be a map defined by $\varphi(x) = e + bx$ for some $b \in K$. Then we have

$$\begin{aligned}\varphi(x + y) &= e + b(x + y)\\ &= (e + bx) + (e + by) - e\\ &= \varphi(x) \oplus \varphi(y)\end{aligned}$$

and

$$\begin{aligned}\varphi(x - y) &= e + b(x - y)\\ &= (e + bx) - (e + by) + e\\ &= \varphi(x) \ominus \varphi(y),\end{aligned}$$

so that $\varphi(0) = e$ implies $\varphi : (K, -, +, 0) \to (K, \ominus, \oplus, e)$ is a homomorphism of β-algebras, where "$-$" is usual subtraction in the field K. If $b \neq 0$, then $\psi : (K, \ominus, \oplus, e) \to (K, -, +, 0)$ defined by $\psi(x) := (x - e)/b$ is a homomorphism of

β-algebras and the inverse mapping of the mapping φ, so that $(K, \ominus, \oplus, e)$ and $(K, -, +, 0)$ are isomorphic as β-algebras, i.e., there is only one isomorphism type in this case. We summarize:

Proposition 4.2. *The β-algebra $(K, \ominus, \oplus, e)$ discussed in Theorem 4.1 is unique up to isomorphism.*

References

[1] P. J. Allen, J. Neggers, and H. S. Kim, *B-algebras and groups*, Sci. Math. Jpn. **59** (2004), no. 1, 23–29.
[2] A. Iorgulescu, *Algebras of Logic as BCK-Algebras*, Editura ASE, Bucharest, 2008.
[3] K. Iséki, *On BCI-algebras*, Math. Sem. Notes Kobe Univ. **8** (1980), no. 1, 125–130.
[4] K. Iséki and S. Tanaka, *An introduction to theory of BCK-algebras*, Math. Japon. **23** (1978), no. 1, 1–26.
[5] H. S. Kim and H. G. Park, *On 0-commutative B-algebras*, Sci. Math. Jpn. **62** (2005), no. 1, 7–12.
[6] J. Meng and Y. B. Jun, *BCK-Algebras*, Kyungmoon Sa, Seoul, 1994.
[7] J. Neggers and H. S. Kim, *On B-algebras*, Mat. Vesnik **54** (2002), no. 1-2, 21–29.
[8] ———, *On β-algebras*, Math. Slovaca **52** (2002), no. 5, 517–530.
[9] H. Yisheng, *BCI-Algebras*, Science Press, Beijing, 2006.

Young Hee Kim
Department of Mathematics
Chungbuk National University
Cheongju 361-791, Korea
E-mail address: yhkim@chungbuk.ac.kr

Keum Sook So
Department of Mathematics
Hallym University
Chuncheon 200-702, Korea
E-mail address: ksso@hallym.ac.kr