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B-branes and supersymmetric quivers in 2d

Cyril Closset,[a] **Jirui Guo**[b] **and Eric Sharpe**[c]

[a] *Theory Department, CERN, Geneva 23, CH-1211 Switzerland*
[b] *Department of Physics, Fudan University, 220 Handan Road, Shanghai, 200433 China*
[c] *Department of Physics MC 0435, 850 West Campus Drive, Virginia Tech, Blacksburg, VA, 24061 U.S.A.*

E-mail: cyril.closset@cern.ch, jrguo@fudan.edu.cn, ersharpe@vt.edu

Abstract: We study 2d $\mathcal{N} = (0, 2)$ supersymmetric quiver gauge theories that describe the low-energy dynamics of D1-branes at Calabi-Yau fourfold (CY_4) singularities. On general grounds, the holomorphic sector of these theories — matter content and (classical) superpotential interactions — should be fully captured by the topological B-model on the CY_4. By studying a number of examples, we confirm this expectation and flesh out the dictionary between B-brane category and supersymmetric quiver: the matter content of the supersymmetric quiver is encoded in morphisms between B-branes (that is, Ext groups of coherent sheaves), while the superpotential interactions are encoded in the A_∞ algebra satisfied by the morphisms. This provides us with a derivation of the supersymmetric quiver directly from the CY_4 geometry. We also suggest a relation between triality of $\mathcal{N} = (0, 2)$ gauge theories and certain mutations of exceptional collections of sheaves. 0d $\mathcal{N} = 1$ supersymmetric quivers, corresponding to D-instantons probing CY_5 singularities, can be discussed similarly.

Keywords: D-branes, Field Theories in Lower Dimensions, Supersymmetry and Duality, Topological Strings

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1 Introduction

Many supersymmetric quantum field theories can be engineered on systems of branes in string theory. The string theory embedding often provides us with an elegant geometric understanding of field theory phenomena. In particular, rich classes of field theories, the *supersymmetric quiver gauge theories*, can be engineered by considering parallel Dp-branes at the tip of a conical local Calabi-Yau (CY) n-fold $\mathbf{X}_n$, with $p = 9 - 2n$, in type IIB string theory. One obtains the following types of supersymmetric gauge theories in the open-string sector:

- 6d $\mathcal{N} = (0, 1)$ quiver theories on D5-branes at the tip of a CY_2 cone.
- 4d $\mathcal{N} = 1$ quiver theories on D3-branes at the tip of a CY_3 cone.
- 2d $\mathcal{N} = (0, 2)$ quiver theories on D1-branes at the tip of a CY_4 cone.
- 0d $\mathcal{N} = 1$ quiver theories on D-instantons at the tip of a CY_5 cone.

All these quiver gauge theories consist of unitary gauge groups $\prod_I \mathrm{U}(N_I)$ with matter fields in adjoint and bifundamental representations.[1] The 6d case corresponds to D5-branes at the tip of an ADE singularity $\mathbb{C}^2/\Gamma$, and the quiver gauge theories are the corresponding ADE quivers [1]. The 4d case has been thoroughly studied from various points of views — see e.g. [2–15] for a very partial list. It is a special and important case because the D3-branes admit a smooth near-horizon limit of the form $AdS_5 \times X_5$ [3, 4, 16] and the 4d quiver gauge theories flow to non-trivial 4d $\mathcal{N} = 1$ superconformal fixed points.

The 2d and 0d cases had attracted less attention until more recently — see [17, 18] for some early work. A recent breakthrough was the introduction of "brane brick models" [19, 20], which gave an algorithm to determine the $\mathcal{N} = (0, 2)$ quiver gauge theory

[1] That is, fields X_{IJ} in the fundamental of $\mathrm{U}(N_I)$ and in the antifundamental of $\mathrm{U}(N_J)$.

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corresponding to D1-branes probing a *toric* CY_4 singularity, similarly to brane tiling methods for D3-branes at toric CY_3 singularities [11–13]. The brane brick models were derived using mirror symmetry in [21]. There are also hints that a similar structure exists for D(-1)-branes at toric CY_5 singularities.[2] Note that this line of work (and the present paper) is only concerned with the *classical* structure of the $\mathcal{N} = (0,2)$ gauge theory. In a parallel development, there has been some important progress in our understanding of two-dimensional $\mathcal{N} = (0,2)$ gauge theories as full-fledged quantum field theories [23–25]. Incidentally, it was discovered that the simplest SQCD-like $\mathcal{N} = (0,2)$ theories enjoy a beautiful *triality* [23] — an infrared "duality" of order three similar to Seiberg duality. Triality also seems to be a generic property of D1-branes quivers [26]. There has also been some interesting recent work on engineering $\mathcal{N} = (0,2)$ models from F-theory [27–30]. See also [31–42] for related works on quantum aspects of $\mathcal{N} = (0,2)$ theories.

In this paper, we study 2d and 0d quivers from the point of view of *B-branes* on the CY n-fold $\mathbf{X}_n$. A B-brane is simply a (half-BPS) D-brane in the topological B-model on $\mathbf{X}_n$. The B-model is a $g_s = 0$ limit of type II string theory which (somewhat trivially) captures all α' corrections. It can thus be used to accurately describe the local physics of branes at a Calabi-Yau *singularity*. Since the B-model is independent of Kähler deformations, we can use any convenient limit, such as, for instance, the large volume limit of a resolved singularity, to study the quantities of interest. In this way, we loose a lot of important information — for instance, we do not keep track of the central charges of the branes, which determines their stability properties; yet, the B-model is sufficient in order to extract all the information about the *holomorphic* sector of the low-energy open strings. That is, we can read off the matter spectrum and the superpotential interactions of the low-energy quiver gauge theories on Dp-branes from the B-branes alone.[3]

This approach was successfully carried out for D3-branes at CY_3 singularities [6–10, 15]. What we present here is a straightforward extension of some of those earlier works. It provides a string theory derivation of some brane brick models results, without the need to rely on mirror symmetry. Our techniques are also more general, since they are valid beyond the realm of toric geometry.[4]

Mathematically, a B-brane $\mathcal{E}$ on $\mathbf{X}_n$ is an object in the (bounded) derived category of coherent sheaves of $\mathbf{X}_n$:

$$\mathcal{E} \in \mathbf{D}^b(\mathbf{X}_n)\,. \tag{1.1}$$

We can think of B-branes $\mathcal{E}$ as coherent sheaves; more generally they are chain complexes of coherent sheaves (up to certain equivalences). Given two B-branes $\mathcal{E}$, $\mathcal{F}$, we may compute their Ext groups:

$$\mathrm{Ext}^i_{\mathbf{X}_n}(\mathcal{E},\mathcal{F})\,, \tag{1.2}$$

[2] Very recently, these 2d and 0d quivers were also related to cluster algebras [22].

[3] An important caveat is that we need to be given a particular set of B-branes, the "fractional branes", as we will mention. The determination of whether a given set of B-branes is an allowable set of fractional branes may require information beyond the B-model.

[4] This is as a matter of principle. In this paper, all our examples will be toric geometries, somewhat by happenstance, and also so that we can compare our results to the brane brick model literature.

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which are the morphisms in the derived category. Physically, they encode the low-energy modes of the *open strings* stretched between the D-branes $\mathcal{E}$ and $\mathcal{F}$ [43–46]. We refer to [47, 48] for comprehensive reviews of the derived category approach to B-branes.

D-branes quivers from B-branes. Consider a D$(9-2n)$-brane transverse to the Calabi-Yau singularity $\mathbf{X}_n$. Away from the singularity, the brane is locally in flat space. From the point of view of $\mathbf{X}_n$, it is a point-like brane, which is described by a skyscraper sheaf $\mathcal{O}_p$ at a point $p \in \mathbf{X}_n$. When at the singularity, it is expected that $\mathcal{O}_p$ "fractionates" into marginally stable constituents. The resulting "fractional branes" $\{\mathcal{E}_I\}$ realize a gauge group:

$$\prod_I \mathrm{U}(N_I) \tag{1.3}$$

on their worldvolume in the transverse directions. There are also massless open strings connecting the fractional branes among themselves, which realize bifundamental (or adjoint) matter fields X_{IJ}. In this way, the low-energy open string sector at the singularity is described by a *supersymmetric quiver gauge theory*: to each fractional brane $\mathcal{E}_I$, we associate a node in the quiver, denoted by e_I. The matter fields corresponds to various quiver *arrows* connecting the nodes: $e_I \longrightarrow e_J$. There are also interaction terms among the matter fields, which we will discuss below.

In all cases, the fractional branes are such that:

$$\mathrm{Ext}^0_{\mathbf{X}_n}(\mathcal{E}_I, \mathcal{E}_J) = \delta_{IJ}\,. \tag{1.4}$$

These $\mathrm{Ext}^0 \cong \mathrm{Hom}$ groups are identified with *vector multiplets* in $d = 10-2n$ dimensions; a single vector multiplet is assigned to each node e_I, realizing the gauge group $\mathrm{U}(N_I)$. The other Ext^i groups (with $i = 1, \cdots, n-1$) correspond to matter fields charged under the gauge groups.

We should probably emphasize that, in this paper, we will be mostly interested in the supersymmetric quiver as an abstract algebraic object, consisting of nodes, arrows and relations. The assignment of particular gauge groups $\mathrm{U}(N_I)$ is part of the data of a quiver *representation*, and the gauge group ranks can vary depending on the physical setup (that is, which D-branes are we using to probe the singularity). In other words, our concept of supersymmetric quiver can encode many different supersymmetric theories with the same structure but distinct gauge groups.[5]

A crucial property of Ext groups on a Calabi-Yau variety $\mathbf{X}_n$ is the Serre duality relation:

$$\mathrm{Ext}^i_{\mathbf{X}_n}(\mathcal{E}_I, \mathcal{E}_J) \cong \mathrm{Ext}^{n-i}_{\mathbf{X}_n}(\mathcal{E}_J, \mathcal{E}_I)\,, \qquad i = 0, \cdots, n\,. \tag{1.5}$$

This corresponds to the CPT symmetry of the d-dimensional quiver quantum field theory. Generalizing some relatively well-known results for D3-branes, it is natural to propose the following identification of Ext groups with supersymmetry multiplets in various dimensions:

[5]Not all unitary gauge groups are allowed, however. Gauge anomalies provide strong constraints on the allowed quiver ranks.

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D5-brane quivers. For D5-branes on $\mathbf{X}_2$, we have:

$$\mathrm{Ext}^1_{\mathbf{X}_2}(\mathcal{E}_J,\mathcal{E}_I) \qquad \Leftrightarrow \qquad e_I \text{ —— } e_J \qquad \Leftrightarrow \qquad X_{IJ}\,, \tag{1.6}$$

where X_{IJ} are 6d $\mathcal{N}=(0,1)$ hypermultiplets in the bifundamental representation of $\mathrm{U}(n_I)\times\mathrm{U}(n_J)$. Note that the quiver link e_I — e_J is unoriented since the hypermultiplet is non-chiral — this corresponds to the Serre duality $\mathrm{Ext}^1(\mathcal{E}_I,\mathcal{E}_J)\cong\mathrm{Ext}^1(\mathcal{E}_J,\mathcal{E}_I)$ on $\mathbf{X}_2$. In this case, $\mathbf{X}_2$ must be an ADE singularity while the supersymmetric quivers are extended Dynkin diagrams.

D3-brane quivers. For D3-branes on $\mathbf{X}_3$, we have:

$$\mathrm{Ext}^1_{\mathbf{X}_3}(\mathcal{E}_J,\mathcal{E}_I) \qquad \Leftrightarrow \qquad e_I \longrightarrow e_J \qquad \Leftrightarrow \qquad X_{IJ}\,, \tag{1.7}$$

where X_{IJ} are 4d $\mathcal{N}=1$ chiral multiplets in the bifundamental of $\mathrm{U}(n_I)\times\mathrm{U}(n_J)$, or in the adjoint of $\mathrm{U}(n_I)$ if $I=J$. The arrows are oriented. Therefore, such quiver gauge theories are generally *chiral* theories. More precisely, we denote by:

$$d^1_{IJ}\equiv\dim\mathrm{Ext}^1_{\mathbf{X}_3}(\mathcal{E}_J,\mathcal{E}_I) \tag{1.8}$$

the number of arrows from e_I to e_J in the 4d $\mathcal{N}=1$ quiver. D3-brane quivers are "ordinary" quivers (with relations), consisting of nodes and arrows, of the type most studied by both physicists and mathematicians.

D1-brane quivers. D1-branes on $\mathbf{X}_4$ lead to the richer structure of 2d $\mathcal{N}=(0,2)$ quiver gauge theories. Those quivers have *two* distinct types of arrows, corresponding to $(0,2)$ chiral multiplets X_{IJ} and $(0,2)$ fermi multiplets Λ_{IJ}, respectively. We propose the identification:

$$\begin{aligned}
&\mathrm{Ext}^1_{\mathbf{X}_4}(\mathcal{E}_J,\mathcal{E}_I) \qquad \Leftrightarrow \qquad e_I \longrightarrow e_J \qquad \Leftrightarrow \qquad X_{IJ}\,,\\
&\mathrm{Ext}^2_{\mathbf{X}_4}(\mathcal{E}_J,\mathcal{E}_I) \qquad \Leftrightarrow \qquad e_I \text{ - - - } e_J \qquad \Leftrightarrow \qquad \Lambda_{IJ}\,.
\end{aligned} \tag{1.9}$$

Note that the $\mathrm{Ext}^2_{\mathbf{X}_4}(\mathcal{E}_I,\mathcal{E}_J)\cong\mathrm{Ext}^2_{\mathbf{X}_4}(\mathcal{E}_J,\mathcal{E}_I)$ by Serre duality. Thus the second type of arrow is unoriented. This corresponds to the self-duality of the fermi multiplet in such theories. We also define:

$$d^1_{IJ}\equiv\dim\mathrm{Ext}^1_{\mathbf{X}_4}(\mathcal{E}_J,\mathcal{E}_I)\,, \qquad d^2_{IJ}\equiv\dim\mathrm{Ext}^2_{\mathbf{X}_4}(\mathcal{E}_J,\mathcal{E}_I)\,. \tag{1.10}$$

Here d^1_{IJ} is the number of chiral multiplets from e_I to e_J (in bifundamental representations if $I\neq J$ or adjoint representation if $I=J$). Similarly, $d^2_{IJ}=d^2_{JI}$ denotes the number of bifundamental fermi multiplets if $I\neq J$, while $\frac{1}{2}d^2_{IJ}$ is the number of adjoint fermi multiplets if $I=J$.

D(−1)-brane quivers. Finally, we may consider D-instantons on $\mathbf{X}_5$, which results in a quiver with two types of oriented arrows:

$$\begin{aligned}
&\mathrm{Ext}^1_{\mathbf{X}_5}(\mathcal{E}_J,\mathcal{E}_I) \qquad \Leftrightarrow \qquad e_I \longrightarrow e_J \qquad \Leftrightarrow \qquad X_{IJ}\,,\\
&\mathrm{Ext}^2_{\mathbf{X}_5}(\mathcal{E}_J,\mathcal{E}_I) \qquad \Leftrightarrow \qquad e_I \dashrightarrow e_J \qquad \Leftrightarrow \qquad \Lambda_{IJ}\,.
\end{aligned} \tag{1.11}$$

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The corresponding $\mathcal{N}=1$ gauged matrix model contains two types of "matter" multiplets, the chiral and fermi multiplets [49]. In this case, the quantities:

$$d^1_{IJ} \equiv \dim \mathrm{Ext}^1_{\mathbf{X}_5}(\mathcal{E}_J, \mathcal{E}_I)\,, \qquad d^2_{IJ} \equiv \dim \mathrm{Ext}^2_{\mathbf{X}_5}(\mathcal{E}_J, \mathcal{E}_I)\,, \tag{1.12}$$

give the number of arrows of either types from e_I to e_J. We will briefly discuss these gauged matrix models in section 4.

Elusive fractional branes. The above identifications between Ext groups and supersymmetric multiplets in D-brane quivers are *conjectures*, that we may check in many explicit computations. The practical usefulness of these identifications rely on identifying the fractional branes $\mathcal{E}_I$ in the first place, as distinguished objects in the B-brane category on $\mathbf{X}_n$. To the best of our knowledge, this remains an open problem in general. In this note, we will deal with simple examples where we can describe the fractional branes explicitly.

Interactions terms: product structure in the derived category. Importantly, the D-brane quivers have interactions terms, which are encoded in *superpotentials* in various dimensions. On D5-branes, the interactions are fully determined by supersymmetry, while D3-brane quivers have a non-trivial 4d $\mathcal{N}=1$ superpotential $W(X)$. The 2d $\mathcal{N}=(0,2)$ theories have two types of "superpotential" interactions, encoded in holomorphic functions $J(X)$ and $E(X)$ [50]. The 0d $\mathcal{N}=1$ matrix models also have two kinds of holomorphic "superpotentials", distinct from the 2d superpotentials, denoted by $F(X)$ and $H(X)$ [49].[6]

These interactions terms can be recovered from the fractional branes by considering the product structure between Ext groups. Let A denote the graded algebra $\mathrm{Ext}^\bullet$ for a given set of fractional branes, where the grading is by the degree of the Ext groups. (It is also the ghost number of the B-model.) There exists multi-products:

$$m_k : A^{\otimes k} \to A\,, \tag{1.13}$$

of degree $2-k$, satisfying relations amongst themselves, that generate a minimal A_∞ structure.[7] In particular, if $a \in \mathrm{Ext}^\bullet(\mathcal{E}_1, \mathcal{E}_2)$ and $b \in \mathrm{Ext}^\bullet(\mathcal{E}_2, \mathcal{E}_3)$, then $m_2(b,a) \in \mathrm{Ext}^\bullet(\mathcal{E}_1, \mathcal{E}_3)$ is the product obtained by composition. These multi-products correspond to disk correlators in the topological B-model.

It is known that the A_∞ structure encodes the 4d $\mathcal{N}=1$ superpotential of D3-brane quivers [10, 52]. Following the same methods, we will be able to derive the 2d $\mathcal{N}=(0,2)$ and 0d $\mathcal{N}=1$ quiver interactions.

This paper is organized as follows. In section 2, we discuss the construction of 2d $\mathcal{N}=(0,2)$ supersymmetric quiver gauge theories from the knowledge the B-branes on a CY fourfold. In section 3, we discuss triality of 2d $\mathcal{N}=(0,2)$ quivers in this context, and relate triality to mutations of exceptional collections of sheaves. In section 4, we discuss the similar construction of 0d $\mathcal{N}=1$ quiver theories from B-branes on a CY fivefold. A few complementary points are discussed in appendices.

[6] What we call F-term was called J-term in [49]. We choose this notation in order to distinguish between the 2d and 0d interactions.

[7] A minimal A_∞ structure is an A_∞ structure in which $m_1 = 0$ [51].

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2 D1-brane quivers and 2d $\mathcal{N} = (0, 2)$ theories

Two-dimensional gauge theories with $\mathcal{N} = (0, 2)$ supersymmetry are built out of three types of supermultiplets: vector, chiral and fermi multiplets [50]. In Wess-Zumino gauge, the vector multiplet (V, V_z) contains a gauge field A_μ, left-moving gaugini and an auxiliary scalar D, transforming in the adjoint of the Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$, with $\mathbf{G}$ the gauge group.

The charged matter fields consist of chiral multiplets Φ and fermi multiplets Λ — and of their charge-conjugate multiplets, the anti-chiral multiplet $\widetilde{\Phi}$ and the anti-fermi multiplet $\widetilde{\Lambda}$, respectively. They satisfy the half-BPS conditions:

$$\bar{D}_+ \Phi = 0 \,, \qquad \bar{D}_+ \Lambda = E(\Phi) \,. \tag{2.1}$$

The E-potential $E = E(\Phi)$ is itself a chiral superfield, given by a holomorphic function of the chiral multiplets Φ. In components, the chiral superfield reads:

$$\Phi = \phi + \theta^+ \psi_+ - 2i\theta^+ \bar{\theta}^+ \partial_{\bar{z}} \phi \tag{2.2}$$

with ϕ a complex scalar and ψ_+ a right-moving fermion. The fermi superfield is given by:

$$\Lambda_- = \lambda_- - \theta^+ G - 2i\theta^+ \bar{\theta}^+ \partial_{\bar{z}} \lambda_- - \bar{\theta}^+ E \,, \tag{2.3}$$

with λ_- a left-moving fermion an G an auxiliary field. The chiral and fermi multiplets are valued in some representations $\mathfrak{R}_\Phi$ and $\mathfrak{R}_\Lambda$ of $\mathfrak{g}$, respectively. Consequently, the potential $E(\Phi)$ is valued in $\mathfrak{R}_\Lambda$ as well. The canonical kinetic Lagrangian for the matter fields is:

$$\mathscr{L}_{\text{kin}} = \int d\theta^+ d\bar{\theta}^+ \, \left(i\bar{\Phi} \mathcal{D}_z \Phi - \bar{\Lambda}_- \Lambda_- \right) \,, \tag{2.4}$$

with $\mathcal{D}_z$ the gauge covariant derivative, and with the trace over $\mathfrak{g}$ kept implicit. A standard super-Yang-Mills term can also be constructed for the vector multiplet. To every fermi multiplet Λ, we also associate an "$\mathcal{N} = (0, 2)$ superpotential" $J = J(\Phi)$ transforming in the conjugate representation $\bar{\mathfrak{R}}_\Lambda$, such that:

$$\mathrm{Tr}\,(EJ) = 0 \,, \tag{2.5}$$

with the trace over $\mathfrak{g}$, $\mathrm{Tr} : \mathfrak{R}_\Lambda \otimes \bar{\mathfrak{R}}_\Lambda \to \mathbb{C}$. The interaction Lagrangian reads:

$$\mathscr{L}_J = - \int d\theta^+ \, \Lambda_- J(\Phi) - \int d\bar{\theta}^+ \, \bar{\Lambda}_- \bar{J}(\bar{\Phi}) \,, \tag{2.6}$$

with $\bar{J}$ the conjugate potential for the anti-fermi multiplet. This Lagrangian is supersymmetric provided that (2.5) is satisfied. The auxiliary fields G, $\bar{G}$ can be integrated out, which sets $G = \bar{J}$ and $\bar{G} = J$. We then obtain the following Lagrangian for the fermi multiplets:

$$\mathcal{L}_{\text{fermi}} = -2i\bar{\lambda}_- \partial_{\bar{z}} \lambda_- + \bar{E} E + \bar{J} J + \bar{\lambda}_- \frac{\partial E}{\partial \phi} \psi_+ - \lambda_- \frac{\partial \bar{E}}{\partial \bar{\phi}} \bar{\psi}_+ + \lambda_- \frac{\partial J}{\partial \phi} \psi_+ - \bar{\lambda}_- \frac{\partial \bar{J}}{\partial \bar{\phi}} \bar{\psi}_+ \,. \tag{2.7}$$

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Note that there is a symmetry that exchanges fermi and anti-fermi multiplets:

$$\Lambda \leftrightarrow \bar{\Lambda}\,, \qquad E \leftrightarrow J\,, \qquad \bar{E} \leftrightarrow \bar{J}\,. \tag{2.8}$$

In the presence of several fermi multiplets in distinct irreducible representations, each fermi multiplet can be "dualized" independently.[8]

2.1 $\mathcal{N} = (0,2)$ quiver gauge theory from B-branes at a CY_4 singularity

Systems of D1-branes at CY_4 singularities engineer a simple yet rich class of gauge theories with product gauge group:

$$\mathbf{G} = \prod_I \mathrm{U}(N_I)\,. \tag{2.9}$$

To each $\mathrm{U}(N_I)$ gauge group, one associates an $\mathcal{N} = (0,2)$ vector multiplet, denoted by a node e_I in a quiver diagram. The matter fields in chiral multiplets are in bifundamental representations $\mathbf{N}_I \otimes \bar{\mathbf{N}}_J$ between unitary gauge groups. To each chiral multiplet X_{IJ} in the fundamental of $\mathrm{U}(N_I)$ and antifundamental of $\mathrm{U}(N_J)$, we associate a solid arrow $e_I \longrightarrow e_J$ in the quiver diagram. The matter fields in fermi multiplets are also in bifundamental representations. To each bifundamental fermi multiplet Λ_{IJ}, we associate the dashed link e_I - - - e_J in the quiver diagram. While Λ_{IJ} denotes a fermi multiplet in the bifundamental $\mathbf{N}_I \otimes \bar{\mathbf{N}}_J$ of $\mathrm{U}(N_I) \times \mathrm{U}(N_J)$, the associated link in the quiver is unoriented, reflecting the fermi duality (2.8).[9] We may also have chiral and fermi multiplets in the adjoint representation of a single gauge group $\mathrm{U}(N_I)$, corresponding to a special case of the above with $I = J$.

To each Λ_{IJ}, one associates an E-term and a J-term. Given that Λ_{IJ} transform in the bifundamental representation $\mathbf{N}_I \otimes \bar{\mathbf{N}}_J$, by convention, the potential $E_{\Lambda_{IJ}}$ transforms in $\mathbf{N}_I \otimes \bar{\mathbf{N}}_J$ as well, while the potential $J_{\Lambda_{IJ}}$ transforms in the conjugate representation $\bar{\mathbf{N}}_I \otimes \mathbf{N}_J$. In other words, $E_{\Lambda_{IJ}}$ is given by a direct sum of oriented paths p (counted with complex coefficients) from e_I to e_J in the quiver, travelled along chiral multiplet arrows, and $J_{\Lambda_{IJ}}$ is similarly a direct sum of oriented paths $\widetilde{p}$ from e_J to e_I:

$$\begin{aligned} E_{\Lambda_{IJ}}(X) &= \sum_{\text{paths } p} c_p^{IJ} X_{IK_1} X_{K_1K_2} \cdots X_{K_{k-1}J}\,, \\ J_{\Lambda_{IJ}}(X) &= \sum_{\text{paths } \widetilde{p}} \widetilde{c}_{\widetilde{p}}^{IJ} X_{JL_1} X_{L_1L_2} \cdots X_{L_{\widetilde{k}-1}I}\,, \end{aligned} \tag{2.10}$$

where the sum is over all possible paths p and $\widetilde{p}$ of lengths k and $\widetilde{k}$, respectively. The numerical coefficients c_p^{IJ}, $c_{\widetilde{p}}^{IJ}$ are to be given (up to field redefinitions) as part of the definition of the $\mathcal{N} = (0,2)$ supersymmetric quiver. They must be such that the supersymmetry constraint (2.5) holds. This means that, for any *closed loop* P for chiral multiplets

[8] See [53] for a discussion of some subtleties in this symmetry in $(0,2)$ NLSMs.

[9] In practice, we still find it convenient to write oriented dashed arrows for fermi multiplets, reflecting a choice of representation for the fermi multiplets (that is, which is Λ and which is $\bar{\Lambda}$). This is because such a choice is needed to write down the off-shell supersymmetric action.

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in the quiver, we must have:

$$\sum_{\substack{p,\widetilde{p}\\ p+\widetilde{p}=P}} c_p\, \widetilde{c}_{\widetilde{p}} = 0\,, \qquad \forall P\,, \tag{2.11}$$

where the sum is over all pairs of quiver paths $p : e_I \to \cdots \to e_J$ and $\widetilde{p} : e_J \to \cdots \to e_I$ based at fermi multiplets Λ_{IJ} such that the closed path $p + \widetilde{p}$ coincides with P.

2.1.1 From B-branes to quiver

Consider a D1-brane probing a local Calabi-Yau fourfold singularity $\mathbf{X}_4$. Away from the singularity, the D1-brane is described in the B-brane category as a skyscraper sheave $\mathcal{O}_p$ at a point $p \in \mathbf{X}_4$. At the singularity, we expect that the D1-brane fractionates into a finite number n of mutually-stable components:

$$\mathcal{O}_p \cong \mathcal{E}_1 \oplus \cdots \oplus \mathcal{E}_n\,. \tag{2.12}$$

The *fractional branes* $\mathcal{E}_I$, with $I = 1, \cdots, n$, are a distinguished set in the derived category of coherent sheaves on the local CY fourfold. If we normalize the central charge of the D1-brane to $Z(\mathcal{O}_p) = 1$, the fractional branes must be such that their central charge align at a special small-volume point — a "quiver point" — in the quantum Kähler moduli space of $\mathbf{X}_4$, with $Z(\mathcal{E}_I) \in \mathbb{R}_{>0}$ and $\sum_I Z(\mathcal{E}_I) = 1$. In the case of an orbifold of flat space, $\mathbf{X}_4 \cong \mathbb{C}^4/\Gamma$, the "quiver point" is the orbifold point, where perturbative string theory is valid, and the fractional branes are in one-to-one correspondence with the irreducible representations of Γ [1]. We will not study stability issues at all in this work. We will only *assume* that we may identify (or guess) a suitable set of fractional branes. In general, there might be many allowable sets of fractional branes, some of which give the same quiver, and some of which give different quivers. This last possibility should correspond to field theory dualities. We will comment on this point in section 3.

Given the fractional branes:

$$\mathcal{E}_I \in \mathbf{D}^b(\mathbf{X}_4)\,, \tag{2.13}$$

as objects in the B-brane category, we may compute the morphisms between them. For $\mathcal{E}_I$ and $\mathcal{E}_J$ given as coherent sheaves on $\mathbf{X}_4$, the morphisms are elements of the Ext groups:

$$\mathrm{Ext}^i_{\mathbf{X}_4}(\mathcal{E}_I, \mathcal{E}_J)\,, \qquad\qquad i = 0, 1, 2, 3, 4\,. \tag{2.14}$$

These groups encode massless open strings stretched between fractional branes [45]. We should have:

$$\mathrm{Ext}^0_{\mathbf{X}_4}(\mathcal{E}_I, \mathcal{E}_J) = \mathrm{Hom}(\mathcal{E}_I, \mathcal{E}_J) = \delta_{IJ}\, \mathbb{C}\,, \tag{2.15}$$

to obtain a physical quiver. This is because Ext^0 is identified with the massless gauge field in the open string spectrum. In our setup, we identify $\mathrm{Ext}^0(\mathcal{E}_I, \mathcal{E}_J)$ with the $\mathcal{N} = (0, 2)$ vector multiplet at the node e_I of the quiver.

The degree-one Ext groups are identified with the chiral multiplets in the supersymmetric quiver:

$$\mathrm{Ext}^1_{X_4}(\mathcal{E}_J, \mathcal{E}_I) \qquad \Leftrightarrow \qquad e_I \longrightarrow e_J \qquad \Leftrightarrow \qquad X_{IJ}\,, \tag{2.16}$$

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By Serre duality, we have $\mathrm{Ext}^3_{X_4}(\mathcal{E}_J,\mathcal{E}_I)\cong\mathrm{Ext}^1_{X_4}(\mathcal{E}_I,\mathcal{E}_J)$, so that $\mathrm{Ext}^3_{X_4}(\mathcal{E}_I,\mathcal{E}_J)$ is identified with the anti-chiral multiplets $\bar{X}_{IJ}$. This identification of chiral multiplets with Ext^1 is well-known in the case of four-dimensional $\mathcal{N}=1$ quivers associated to D3-branes on a CY threefold [7, 44, 48, 54]. The new ingredient on a CY *fourfold* is that we also have independent *degree-two* Ext groups, with:

$$\mathrm{Ext}^2_{\mathbf{X}_4}(\mathcal{E}_J,\mathcal{E}_I)\cong\mathrm{Ext}^2_{\mathbf{X}_4}(\mathcal{E}_I,\mathcal{E}_J) \tag{2.17}$$

by Serre duality on $\mathbf{X}_4$. It is natural to identify these groups with the fermi multiplets Λ_{IJ} in the $\mathcal{N}=(0,2)$ quiver:

$$\mathrm{Ext}^2_{\mathbf{X}_4}(\mathcal{E}_J,\mathcal{E}_I)\qquad\Leftrightarrow\qquad e_I\text{ - - - }e_J\qquad\Leftrightarrow\qquad\Lambda_{IJ}\,. \tag{2.18}$$

The self-duality relation (2.17) for Ext^2 correspond to the fact that fermi and anti-fermi multiplet are indistinguishable. For each pair of distinct nodes I, J, we may pick the basis of the Ext^2 vector spaces:

$$\{\alpha,\bar{\beta}\}\in\mathrm{Ext}^2_{\mathbf{X}_4}(\mathcal{E}_J,\mathcal{E}_I)\,,\qquad\qquad\{\beta,\bar{\alpha}\}\in\mathrm{Ext}^2_{\mathbf{X}_4}(\mathcal{E}_I,\mathcal{E}_J)\,, \tag{2.19}$$

where α and β correspond to fermi multiplets Λ_{IJ} and Λ_{JI}, while $\bar{\alpha}$ and $\bar{\beta}$ correspond to anti-fermi multiplets $\bar{\Lambda}_{IJ}$ and $\bar{\Lambda}_{JI}$, respectively, and such that Serre duality exchanges α with $\bar{\alpha}$, and β with $\bar{\beta}$. This choice of basis is completely convention-dependent, however. This corresponds exactly to the freedom (2.8) of labelling fermi and anti-fermi multiplets in the supersymmetric field theory.

For $I=J$, $\mathrm{Ext}^2(\mathcal{E}_I,\mathcal{E}_I)$ is self-dual, and each pair of Serre-dual elements correspond to a pair of fermi and anti-fermi multiplets Λ_{II}, $\bar{\Lambda}_{II}$ in the adjoint representation of $\mathrm{U}(N_I)$.

As a simple consistency check of these identifications between Ext groups and $\mathcal{N}=(0,2)$ superfields, it is interesting to look at the product variety $\mathbf{X_4}\cong\mathbf{X}_3\times\mathbb{C}$, with $\mathbf{X}_3$ a CY threefold singularity. This non-isolated singularity preserves $\mathcal{N}=(2,2)$ supersymmetry in two-dimension, and the 2d quiver should simply be the dimensional reduction of the $\mathcal{N}=1$ supersymmetric quiver for D3-branes on $\mathbf{X}_3$. Each 4d $\mathcal{N}=1$ vector multiplet decomposes into one $\mathcal{N}=(0,2)$ vector multiplet and one adjoint fermi multiplet, and each 4d $\mathcal{N}=1$ chiral multiplet decomposes into one $\mathcal{N}=(0,2)$ chiral multiplet and one fermi multiplet. In terms of Ext groups, this means that we should have:

$$\begin{aligned}
&\mathrm{Ext}^0_{\mathbf{X}_4}(\mathcal{E}_I,\mathcal{E}_J)\cong\mathrm{Ext}^0_{\mathbf{X}_3}(\mathcal{E}_I,\mathcal{E}_J)\,,\\
&\mathrm{Ext}^1_{\mathbf{X}_4}(\mathcal{E}_I,\mathcal{E}_J)\cong\mathrm{Ext}^0_{\mathbf{X}_3}(\mathcal{E}_I,\mathcal{E}_J)\oplus\mathrm{Ext}^1_{\mathbf{X}_3}(\mathcal{E}_I,\mathcal{E}_J)\,,\\
&\mathrm{Ext}^2_{\mathbf{X}_4}(\mathcal{E}_I,\mathcal{E}_J)\cong\mathrm{Ext}^1_{\mathbf{X}_3}(\mathcal{E}_I,\mathcal{E}_J)\oplus\mathrm{Ext}^2_{\mathbf{X}_3}(\mathcal{E}_I,\mathcal{E}_J)\,,\\
&\mathrm{Ext}^3_{\mathbf{X}_4}(\mathcal{E}_I,\mathcal{E}_J)\cong\mathrm{Ext}^2_{\mathbf{X}_3}(\mathcal{E}_I,\mathcal{E}_J)\oplus\mathrm{Ext}^3_{\mathbf{X}_3}(\mathcal{E}_I,\mathcal{E}_J)\,,\\
&\mathrm{Ext}^4_{\mathbf{X}_4}(\mathcal{E}_I,\mathcal{E}_J)\cong\mathrm{Ext}^3_{\mathbf{X}_3}(\mathcal{E}_I,\mathcal{E}_J)\,.
\end{aligned} \tag{2.20}$$

This can be shown to be the case in general orbifolds $\mathbb{C}^3/\Gamma\times\mathbb{C}$ — see appendix A. Note that (2.20) is consistent with Serre duality (1.5). One can similarly consider the decomposition $\mathbf{X_4}\cong\mathbf{X}_2\times\mathbb{C}^2$, which preserves 2d $\mathcal{N}=(4,4)$ supersymmetry.

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A comment on conventions. To avoid any possible confusion, let us note that we are using the physicist notation for the chiral multiplets in the $\mathcal{N} = (0,2)$ superpotentials, and the mathematical notation of composition when discussing elements of $\mathrm{Ext}^\bullet$. For instance, we have:

$$x \in \mathrm{Ext}^1(\mathcal{E}_2, \mathcal{E}_1)\,, \qquad y \in \mathrm{Ext}^1(\mathcal{E}_3, \mathcal{E}_2)\,, \qquad x \cdot y \in \mathrm{Ext}^1(\mathcal{E}_3, \mathcal{E}_1)\,, \tag{2.21}$$

where $x \cdot y \equiv x \circ y$. When talking about the fractional branes, we write these maps as:

$$\mathcal{E}_3 \xrightarrow{y} \mathcal{E}_2 \xrightarrow{x} \mathcal{E}_1\,. \tag{2.22}$$

On the other hand, we have chosen the convention that $\mathrm{Ext}(\mathcal{E}_J, \mathcal{E}_I)$ corresponds to the chiral multiplet X_{IJ}, so that the direction of the arrows in the quiver are flipped: a map $\mathcal{E}_J \to \mathcal{E}_I$ corresponds to a quiver arrow $e_I \to e_J$. In our example (2.21), denoting by X and Y the chiral multiplets associated to the Ext group elements, we have:

$$e_1 \xrightarrow{X} e_2 \xrightarrow{Y} e_3 \cong \mathrm{U}(N_1) \xrightarrow{X} \mathrm{U}(N_2) \xrightarrow{Y} \mathrm{U}(N_3)\,, \tag{2.23}$$

where on the right-hand-side we associated a gauge group $\mathrm{U}(N_I)$ to each node e_I. In these conventions, we can write $x \cdot y$ as the matrix product XY for the chiral multiplets.

Anomaly-free condition and quiver ranks. Consider an $\mathcal{N} = (0,2)$ quiver with nodes $\{e_I\}$ and gauge group (2.9). For each $\mathrm{U}(N_I)$ factor, the cancellation of the non-abelian anomaly requires:

$$\sum_{J \neq I} \left(d^1_{IJ} + d^3_{IJ} - d^2_{IJ}\right) N_J + 2N_I \left(-1 + d^1_{II} - \frac{1}{2} d^2_{II}\right) = 0\,. \tag{2.24}$$

Here the first sum is over the chiral and fermi multiplets in bifundamental representations, while the second term denote the contribution from the vector multiplet (with $d^0_{II} = 1$) and from adjoint matter. Using Serre duality, this can be written as:

$$\sum_J \sum_{i=0}^{4} (-1)^i N_I \dim \mathrm{Ext}^i_{X_4}(\mathcal{E}_I, \mathcal{E}_J) = 0\,. \tag{2.25}$$

This condition imposes constraint on the allowed ranks N_I in the quiver. If we consider a single D1-brane, the ranks N_I should be fixed from first principle; however, the explicit dictionary between brane-charge basis and quiver-rank basis is not always known. The anomaly-free condition then provides a strong constraint. The solutions to (2.25), as a linear system for the positive integers N_I, correspond to all stable D-brane configurations at the singularity. In particular, the unique solution $\{N_I\}$ such that each N_I is the smallest possible positive integer is expected to correspond to a single D1-brane. In the special case of toric Calabi-Yau singularities, we know from [19, 21, 26] that there exists "toric quiver" with equal ranks, $N_I = N$, corresponding to N D1-branes.

We should also mention that the *abelian* quadratic anomalies, from the $\mathrm{U}(1)_I$ factors in $\mathrm{U}(N_I)$, do not vanish in general. Instead, they should be cancelled by closed string contributions *à la* Green-Schwarz [1, 17, 55].

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2.1.2 A_∞ structure and $\mathcal{N} = (0,2)$ superpotential

To complete the determination of the $\mathcal{N} = (0,2)$ supersymmetric quiver from the fractional branes on $\mathbf{X}_4$, we need to discuss the E- and J-terms (2.10). It is convenient to package them into a gauge-invariant "$(0,2)$ superpotential" $\mathcal{W}$ defined as:[10]

$$\mathcal{W} = \mathrm{Tr}\left(\Lambda_I J^I(X) + \bar{\Lambda}^I E_I(X)\right). \tag{2.26}$$

Here, the index I runs over all the fermi multiplets. This $\mathcal{W}$ can be computed by following the methods of [10], which studied 4d $\mathcal{N} = 1$ quiver theories on D3-branes at CY_3 singularities. On general ground, the superpotential coupling constants are encoded in open string correlation functions. Those can be described in the language of A_∞ algebra — see e.g. [56] and references therein.

An A_∞ algebra is a (graded) algebra A together with a set of multiplications $m_k : A^{\otimes k} \to A$ that satisfy the A_∞ relations:

$$\sum_{r+s+t=n} (-1)^{r+st} m_{n+1-s}(a_1, \cdots, a_r, m_s(a_{r+1}, \cdots, a_{r+s}), a_{r+s+1}, \cdots, a_n) = 0\,, \tag{2.27}$$

for all integer $n > 0$. The first relation states that $(m_1)^2 = 0$, so one can think of $m_1 : A \to A$ as a differential. The Ext group elements between B-branes, on the other hand, generate a *minimal* A_∞ algebra, for which $m_1 = 0$.

To compute the multi-products on the $\mathrm{Ext}^\bullet$ algebra, we proceed as follows. Given an A_∞ algebra $\widetilde{A}$, one defines $H^\bullet(\widetilde{A})$ to be the cohomology of m_1. If $\widetilde{A}$ has no multiplications beyond m_2, then it has been shown [57] that one can define an A_∞ structure on $H^\bullet(\widetilde{A})$ in such a way that there is an A_∞ map:[11]

$$f : H^\bullet(\widetilde{A}) \to \widetilde{A}\,, \tag{2.28}$$

with f_1 equal to a particular representation $H^\bullet(\widetilde{A}) \hookrightarrow \widetilde{A}$ in which cohomology classes map to (noncanonical) representatives in $\widetilde{A}$, and such that $m_1 = 0$ in the A_∞ algebra on $H^\bullet(\widetilde{A})$. One can then use the consistency conditions satisfied by elements of an A_∞ map to solve algebraically for $f_1 \circ m_k$.

In terms of B-branes, the algebra $\widetilde{A}$ is the algebra of complexes of coherent sheaves, with chain maps between complexes. In that construction, m_1 is essentially the BRST charge Q of the B-model. The "physical" open string states then live in the cohomology $H^\bullet(\widetilde{A})$, which gives us the derived category $\mathbf{D}^b(\mathbf{X})$ — we refer to [48] for a pedagogical discussion. We can identify the minimal A_∞ algebra $A \equiv H^\bullet(\widetilde{A})$ with the Ext algebra we are interested in.

Practically, in the examples discussed in this paper, each B-brane will be a single coherent sheaf, which can be represented in the derived category by a locally-free resolution.

[10] This expression is only formal. The $\mathcal{N} = (0,2)$ superpotential that appears in the gauge theory Lagrangian is the usual $\mathrm{Tr}\left(\Lambda_I J^I(X)\right)$, since superspace treats Λ and $\bar{\Lambda}$ asymmetrically. This formal $\mathcal{W}$ first appeared in [22]. It elegantly encodes the algebraic structure of the $\mathcal{N} = (0,2)$ quiver relations. This point is further discussed in appendix C.

[11] That is, a family of maps satisfying certain consistency conditions [57].

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The Ext elements can then be represented by chain maps between resolutions, modulo chain homotopies. The m_2 products in A are given by chain map composition. The higher products can then be computed by the procedure just described.

We elaborate on this procedure in appendix C.3, and we illustrate the computation of the higher products, in a specific example, in appendix D. All of the other examples below will actually have $m_k = 0$ for $k > 2$.

Open string correlators and A_∞ products. Let A denote the Ext algebra associated to a local Calabi-Yau n-fold. There exists a natural "trace map" of degree $-n$, which we denote by $\gamma : A \to \mathbb{C}$. Note that A is a graded algebra, with a of degree q if $a \in \mathrm{Ext}^q$. Serre duality defines a natural pairing of degree $-n$:

$$\langle a, b \rangle \equiv \gamma\left(m_2(a, b)\right) . \tag{2.29}$$

Consider a correlation function of r boundary vertex operators $a_i \in A$ on the open-string worldsheet. In the A_∞ language, this can be written as:

$$\langle a_1 \cdots a_r \rangle \;=\; \langle a_1, m_{r-1}(a_2, \cdots, a_r) \rangle \,, \tag{2.30}$$

in terms of the higher-product m_{r-1} and the pairing (2.29) [10]. Each Ext elements $x \in A$ is *dual* to a "field" X in the supersymmetric quiver — see appendix C for further details. In the case of a 2d $\mathcal{N} = (0, 2)$ quiver describing B-branes on a CY_4 geometry, we have the Ext algebra:

$$A \cong \mathrm{Ext}^0 \oplus \mathrm{Ext}^1 \oplus \mathrm{Ext}^2 \oplus \mathrm{Ext}^3 \oplus \mathrm{Ext}^4 \,, \tag{2.31}$$

where the summands denote all Ext groups between the various fractional branes, of degree $0, \cdots, 4$. Let us denote by $x \in A$ the Ext^1 elements corresponding to the chiral multiplets X, and by $\alpha, \widetilde{\alpha} \in A$ the Ext^2 elements corresponding to the fermi and anti-fermi multiplets $\Lambda, \bar{\Lambda}$, as in (2.19). The coupling constants c_J and c_E appearing as:

$$c_J \,\mathrm{Tr}(\Lambda X_1 \cdots X_r) + c_E \,\mathrm{Tr}(\bar{\Lambda} X'_1 \cdots X'_{r'}) \tag{2.32}$$

in the superpotential (2.26) can be computed as the open-string correlators:

$$c_J = \langle \alpha \, x_1 \cdots x_r \rangle \,, \qquad c_E = \langle \widetilde{\alpha} \, x'_1 \cdots x'_{r'} \rangle \,. \tag{2.33}$$

Explicit formula for the E- and J-terms. We can now spell out the precise formula for the coupling constants appearing in (2.10). Consider a fermi multiplet Λ_{IJ} corresponding to $\alpha \in \mathrm{Ext}^2(\mathcal{E}_J, \mathcal{E}_I)$, and the charge-conjugate anti-fermi multiplet $\bar{\Lambda}_{IJ}$ corresponding to $\bar{\alpha} \in \mathrm{Ext}^2(\mathcal{E}_I, \mathcal{E}_J)$. For each path p as in (2.10), we have the elements $x \in \mathrm{Ext}^1$ corresponding to the chiral multiplets X. We thus have:

$$c_p^{IJ} = \left\langle \bar{\alpha} \; x_{IK_1} \cdots x_{K_{k-1}J} \right\rangle = \gamma\big(m_2(\bar{\alpha}, \, m_k(x_{IK_1}, \, \cdots \, , \, x_{K_{k-1}K}))\big) \,, \tag{2.34}$$

for the E-term coefficients, and

$$c_{\widetilde{p}}^{IJ} = \left\langle \alpha \, x_{JL_1} \, \cdots \, x_{L_{\widetilde{k}-1}I} \right\rangle = \gamma\big(m_2(\alpha, \, m_{\widetilde{k}}(x_{JL_1}, \, \cdots \, , \, x_{L_{\widetilde{k}-1}I}))\big) \,, \tag{2.35}$$

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for the J-term coefficients. We can check this identification for a number of geometries previously studied by independent techniques, and we find perfect agreement.

Last but not least, we should note that, according to the dictionary (2.34)–(2.35), the $\mathrm{Tr}(EJ) = 0$ constraint (2.11) translates into a very non-trivial relation amongst products of open string correlators. In appendix C, we give a general argument for why this constraint will hold for E and J defined by the A_∞ algebra as above. In addition, we will check, in every example below, that the condition $\mathrm{Tr}(EJ) = 0$ indeed holds, thus providing an additional consistency check on our computations. It would be interesting to also understand the first-principle origin of this constraint in the Calabi-Yau fourfold geometry.

2.2 D1-brane on $\mathbb{C}^4$

To illustrate our methods, we start by considering the simplest case, $\mathbf{X}_4 = \mathbb{C}^4$. In flat space, there is a single "fractional brane", the skyscraper sheaf $\mathcal{O}_p$, which corresponds to a single transverse D1-brane. Consider $\mathcal{O}_p$ at the origin of $\mathbb{C}^4$, without loss of generality. One can show that:

$$\mathrm{Ext}^0(\mathcal{O}_p, \mathcal{O}_p) \cong \mathrm{Ext}^4(\mathcal{O}_p, \mathcal{O}_p) \cong \mathbb{C}\,, \tag{2.36}$$

and

$$\mathrm{Ext}^1(\mathcal{O}_p, \mathcal{O}_p) \cong \mathrm{Ext}^3(\mathcal{O}_p, \mathcal{O}_p) \cong \mathbb{C}^4\,, \qquad \mathrm{Ext}^2(\mathcal{O}_p, \mathcal{O}_p) \cong \mathbb{C}^6\,. \tag{2.37}$$

From this result, we directly read off the $\mathcal{N} = (0,2)$ supermultiplet content according to the general rules. We have a single vector multiplet, 4 chiral multiplets and 3 fermi multiplets. If there are N fractional branes at a point, all these fields are in the adjoint of a $\mathrm{U}(N)$ gauge group. This reproduces the field content of maximally supersymmetric $\mathcal{N} = (8,8)$ Yang-Mills theory in 2d, as expected. To compute the interaction terms, we will need to describe the Ext algebra more explicitly.

2.2.1 An explicit basis for $\mathrm{Ext}^\bullet(\mathcal{O}_p, \mathcal{O}_p)$

The Ext algebra can be computed from the Koszul resolution of $\mathcal{O}_p$, which reads:

$$0 \to \mathcal{O} \xrightarrow{D} \mathcal{O}^4 \xrightarrow{C} \mathcal{O}^6 \xrightarrow{B} \mathcal{O}^4 \xrightarrow{A} \mathcal{O} \to \mathcal{O}_p \to 0\,, \tag{2.38}$$

where:[12]

$$A = \begin{pmatrix} x & y & z & w \end{pmatrix}\,,$$

$$B = \begin{pmatrix} y & z & w & 0 & 0 & 0 \\ -x & 0 & 0 & z & w & 0 \\ 0 & -x & 0 & -y & 0 & w \\ 0 & 0 & -x & 0 & -y & -z \end{pmatrix}\,, \qquad C = \begin{pmatrix} z & w & 0 & 0 \\ -y & 0 & w & 0 \\ 0 & -y & -z & 0 \\ x & 0 & 0 & w \\ 0 & x & 0 & -z \\ 0 & 0 & x & y \end{pmatrix}\,, \qquad D = \begin{pmatrix} w \\ -z \\ y \\ -x \end{pmatrix}\,.$$

[12]Here and in the following, we denote a map $M : \mathbb{C}^n \to \mathbb{C}^m$ by an $m \times n$ matrix, so that composition of maps corresponds to matrix multiplication (for instance, $A \circ B = AB$).

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Let us present explicit expressions for the generators of $\mathrm{Ext}^\bullet$. We will use the notation:

$$X_j^i \in \mathrm{Ext}^i(\mathcal{O}_p, \mathcal{O}_p)\,, \qquad j = 1, \cdots, \dim \mathrm{Ext}^i(\mathcal{O}_p, \mathcal{O}_p)\,. \tag{2.39}$$

Every Ext element can be represented by a chain map between two copies of the Koszul resolution; the actual Ext element is given by the corresponding element in its cohomology, by the definition of Ext as a derived functor. First of all, $\mathrm{Ext}^0(\mathcal{O}_p, \mathcal{O}_p)$ is spanned by the single element:

$$\begin{array}{ccccccccc}
\mathcal{O} & \xrightarrow{D} & \mathcal{O}^4 & \xrightarrow{C} & \mathcal{O}^6 & \xrightarrow{B} & \mathcal{O}^4 & \xrightarrow{A} & \mathcal{O} \\
{\scriptstyle 1}\downarrow & & {\scriptstyle 1}\downarrow & & {\scriptstyle 1}\downarrow & & {\scriptstyle 1}\downarrow & & {\scriptstyle 1}\downarrow \\
\mathcal{O} & \xrightarrow{D} & \mathcal{O}^4 & \xrightarrow{C} & \mathcal{O}^6 & \xrightarrow{B} & \mathcal{O}^4 & \xrightarrow{A} & \mathcal{O}
\end{array}$$

Secondly, $\mathrm{Ext}^1(\mathcal{O}_p, \mathcal{O}_p)$ is spanned by maps of the form:

$$\begin{array}{ccccccccc}
 & & \mathcal{O} & \xrightarrow{D} & \mathcal{O}^4 & \xrightarrow{C} & \mathcal{O}^6 & \xrightarrow{B} & \mathcal{O}^4 & \xrightarrow{A} & \mathcal{O} \\
 & & {\scriptstyle \alpha}\downarrow & & {\scriptstyle \beta}\downarrow & & {\scriptstyle \gamma}\downarrow & & {\scriptstyle \sigma}\downarrow & & \\
\mathcal{O} & \xrightarrow{D} & \mathcal{O}^4 & \xrightarrow{C} & \mathcal{O}^6 & \xrightarrow{B} & \mathcal{O}^4 & \xrightarrow{A} & \mathcal{O} & &
\end{array}$$

A basis can be obtained by taking

$$\alpha = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

and demanding the diagram be anti-commutative. For example, when $\alpha = (1,0,0,0)^t$, we can take

$$\beta = \begin{pmatrix} 0&1&0&0 \\ 0&0&1&0 \\ 0&0&0&0 \\ 0&0&0&1 \\ 0&0&0&0 \\ 0&0&0&0 \end{pmatrix}, \quad \gamma = \begin{pmatrix} 0&0&1&0&0&0 \\ 0&0&0&0&1&0 \\ 0&0&0&0&0&1 \\ 0&0&0&0&0&0 \end{pmatrix}, \quad \sigma = \begin{pmatrix} 0&0&0&1 \end{pmatrix}.$$

Similarly, $\mathrm{Ext}^2(\mathcal{O}_p, \mathcal{O}_p)$ is spanned by maps of the form:

$$\begin{array}{ccccccccc}
 & & & & \mathcal{O} & \xrightarrow{D} & \mathcal{O}^4 & \xrightarrow{C} & \mathcal{O}^6 & \xrightarrow{B} & \mathcal{O}^4 & \xrightarrow{A} & \mathcal{O} \\
 & & & & {\scriptstyle \varphi}\downarrow & & {\scriptstyle \psi}\downarrow & & {\scriptstyle \omega}\downarrow & & & & \\
\mathcal{O} & \xrightarrow{D} & \mathcal{O}^4 & \xrightarrow{C} & \mathcal{O}^6 & \xrightarrow{B} & \mathcal{O}^4 & \xrightarrow{A} & \mathcal{O} & & & &
\end{array}$$

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As before, we can choose φ to be one of the unit column vectors with six entries, and then make the diagram commutative. For example,

$$\varphi = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \qquad \psi = \begin{pmatrix} 1\,0\,0\,0 \\ 0\,0\,0\,0 \\ 0\,0\,0\,0 \\ 0\,0\,0\,1 \end{pmatrix}, \qquad \omega = \begin{pmatrix} 0\,0\,0\,1\,0\,0 \end{pmatrix}.$$

$\mathrm{Ext}^3(\mathcal{O}_p, \mathcal{O}_p)$ is spanned by maps of the form:

$$\begin{array}{ccccccccccc} & & & & & & \mathcal{O} & \xrightarrow{D} & \mathcal{O}^4 & \xrightarrow{C} & \mathcal{O}^6 & \xrightarrow{B} & \mathcal{O}^4 & \xrightarrow{A} & \mathcal{O} \\ & & & & & & {\scriptstyle\rho}\downarrow & & {\scriptstyle\tau}\downarrow & & & & & & \\ \mathcal{O} & \xrightarrow{D} & \mathcal{O}^4 & \xrightarrow{C} & \mathcal{O}^6 & \xrightarrow{B} & \mathcal{O}^4 & \xrightarrow{A} & \mathcal{O} & & & & & & \end{array}$$

Here ρ is one of the unit vectors with four entries and τ is such that $A \cdot \rho = -\tau \cdot D$. For example, when $\rho = (0,0,0,1)^t$, $\tau = (-1,0,0,0)$. Finally, $\mathrm{Ext}^4(\mathcal{O}_p, \mathcal{O}_p)$ is spanned by:

$$\begin{array}{ccccccccccc} & & & & & & \mathcal{O} & \xrightarrow{D} & \mathcal{O}^4 & \xrightarrow{C} & \mathcal{O}^6 & \xrightarrow{B} & \mathcal{O}^4 & \xrightarrow{A} & \mathcal{O} \\ & & & & & & {\scriptstyle 1}\downarrow & & & & & & & & \\ \mathcal{O} & \xrightarrow{D} & \mathcal{O}^4 & \xrightarrow{C} & \mathcal{O}^6 & \xrightarrow{B} & \mathcal{O}^4 & \xrightarrow{A} & \mathcal{O} & & & & & & \end{array}$$

2.2.2 Multiplication of maps

The multiplication rule can be determined by composing these maps. For example, $X_3^1 \cdot X_2^1$ is computed by:

$$\begin{array}{ccccccccccc} & & & & & & \mathcal{O} & \xrightarrow{D} & \mathcal{O}^4 & \xrightarrow{C} & \mathcal{O}^6 & \xrightarrow{B} & \mathcal{O}^4 & \xrightarrow{A} & \mathcal{O} \\ & & & & & & {\scriptstyle\alpha_2^1}\downarrow & & {\scriptstyle\beta_2^1}\downarrow & & {\scriptstyle\gamma_2^1}\downarrow & & {\scriptstyle\sigma_2^1}\downarrow & & \\ & & & & \mathcal{O} & \xrightarrow{D} & \mathcal{O}^4 & \xrightarrow{C} & \mathcal{O}^6 & \xrightarrow{B} & \mathcal{O}^4 & \xrightarrow{A} & \mathcal{O} & & \\ & & & & {\scriptstyle\alpha_3^1}\downarrow & & {\scriptstyle\beta_3^1}\downarrow & & {\scriptstyle\gamma_3^1}\downarrow & & {\scriptstyle\sigma_3^1}\downarrow & & & & \\ \mathcal{O} & \xrightarrow{D} & \mathcal{O}^4 & \xrightarrow{C} & \mathcal{O}^6 & \xrightarrow{B} & \mathcal{O}^4 & \xrightarrow{A} & \mathcal{O} & & & & & & \end{array}$$

with:

$$\alpha_2^1 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \ \beta_2^1 = \begin{pmatrix} -1\,0\,0\,0 \\ 0\;\;0\,0\,0 \\ 0\;\;0\,1\,0 \\ 0\;\;0\,0\,0 \\ 0\;\;0\,0\,1 \\ 0\;\;0\,0\,0 \end{pmatrix}, \ \gamma_2^1 = \begin{pmatrix} 0\,-1\,0\;\;0\;\;0\,0 \\ 0\;\;0\;\;0\,-1\,0\,0 \\ 0\;\;0\;\;0\;\;0\;\;0\,0 \\ 0\;\;0\;\;0\;\;0\;\;0\,1 \end{pmatrix}, \ \sigma_2^1 = \begin{pmatrix} 0\,0\,-1\,0 \end{pmatrix},$$

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$$\alpha_3^1 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}\,,\ \beta_3^1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\,,\ \gamma_3^1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}\,,\ \sigma_3^1 = \begin{pmatrix} 0 & 1 & 0 & 0 \end{pmatrix}\,,$$

from which we see that $\beta_3^1 \cdot \alpha_2^1 = -\varphi_3$, and so on and so forth, so that $X_3^1 \cdot X_2^1 = -X_3^2$. Proceeding in this way, we find the multiplication rules:

$$m_2(X_i^1, X_j^1) = X_i^1 \cdot X_j^1 = \begin{pmatrix} 0 & X_1^2 & X_2^2 & X_4^2 \\ -X_1^2 & 0 & X_3^2 & X_5^2 \\ -X_2^2 & -X_3^2 & 0 & X_6^2 \\ -X_4^2 & -X_5^2 & -X_6^2 & 0 \end{pmatrix}\,. \tag{2.40}$$

The product $X_i^1 \cdot X_j^1$ is given by the matrix element ij in (2.40). One can also compute the products:

$$X_6^2 \cdot X_1^2 = -X_5^2 \cdot X_2^2 = X_4^2 \cdot X_3^2 = X_1^4\,,$$

which commute. All other products between degree-two maps vanish. This shows that the Serre dual of X_1^2, X_2^2, X_3^3 are $X_6^2, -X_5^2, X_4^2$ respectively.

One can also show that the higher products vanish in this case — that is, $m_k = 0$ if $k > 2$. Therefore, any nonzero correlation function can be reduced to one of the following:

$$\begin{aligned} &\langle X_1^2\ X_3^1\ X_4^1\rangle = 1\,, && \langle X_2^2\ X_2^1\ X_4^1\rangle = -1\,, \\ &\langle X_3^2\ X_1^1\ X_4^1\rangle = 1\,, && \langle X_4^2\ X_2^1\ X_3^1\rangle = 1\,, \\ &\langle X_5^2\ X_1^1\ X_3^1\rangle = -1\,, && \langle X_6^2\ X_1^1\ X_2^1\rangle = 1\,. \end{aligned} \tag{2.41}$$

2.2.3 The $\mathbb{C}^4$ quiver: $\mathcal{N} = (8,8)$ SYM

From (2.36)–(2.37), we see that the $\mathcal{N} = (0,2)$ gauge theory corresponding to D1-branes on $\mathbb{C}^4$ has the field content of $\mathcal{N} = (8,8)$ SYM. We can also verify that the product structure encoded in (2.41) reproduces the correct supersymmetric interactions. In $\mathcal{N} = (0,2)$ notation, this theory consists of four chiral multiplets, denoted Σ and Φ_a $(a = 1,2,3)$, and three fermi multiplets Λ_a $(a = 1,2,3)$, with the E and J terms:

$$E_a = [\Sigma, \Phi_a]\,, \qquad J^a = \epsilon^{abc}\Phi_b\Phi_c\,. \tag{2.42}$$

This is reproduced by our computation, with the identifications:

$$\Sigma = -X_4^1\,, \qquad \Phi_a = (X_3^1\,,\ -X_2^1\,,\ X_1^1)\,, \tag{2.43}$$

for the chiral multiplets, and

$$\Lambda_a = (X_6^2\,,\ -X_5^2\,,\ X_4^2)\,, \qquad \bar{\Lambda}_a = (X_1^2\,,\ X_2^2\,,\ X_3^2) \tag{2.44}$$

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for the fermi multiplets, as one can easily check using (2.34)–(2.35), and

$$E_a = \sum_{ij} \langle \bar{\Lambda}_a, m_2(\phi_i, \phi_j) \rangle \ \phi_i \phi_j ,$$
$$J^a = \sum_{ij} \langle \Lambda_a, m_2(\phi_i, \phi_j) \rangle \ \phi_i \phi_j ,$$

for $\{\phi_i\}$ the set of all chiral superfields — here, by abuse of notation, we identified the quiver fields with the corresponding Ext elements in the open-string correlators. Note that the condition (2.5) is satisfied, $\mathrm{Tr}(E_a J^a) = 0$. The interaction terms (2.42) display an SU(3) flavory symmetry. On-shell, there is a larger SU(4) flavor symmetry, with $(\Lambda_a, \bar{\Lambda}_a)$ sitting in the **6** of SU(4). It will often be the case that the flavor symmetry displayed by the $\mathcal{N} = (0,2)$ quiver is smaller than the symmetry expected from the CY_4 geometry. Those larger geometric symmetries can be thought to arise in the infrared of the gauge theory, as accidental symmetries [20].

2.3 Orbifolds $\mathbb{C}^4/\Gamma$

The next simplest class of examples are supersymmetric orbifolds of flat space. Consider the CY_4 singularity $\mathbb{C}^4/\Gamma$, with Γ a discrete subgroup of SU(4). There exists one fractional brane $\mathcal{E}_I$ for each irreducible representation ρ_I of Γ [46]. We also denote by ρ_I the trivial line bundle $\mathcal{O}$ with the corresponding Γ-equivariant structure. The fractional branes are given by:

$$\mathcal{E}_I = \rho_I \otimes \mathcal{O}_p \,, \tag{2.45}$$

with $\mathcal{O}_p$ the skyscraper sheaf supported at the origin. In the following, we consider a few examples with Γ abelian, for simplicity.

2.3.1 $\mathbb{C}^4/\mathbb{Z}_2(1,1,1,1)$

Consider $\mathbb{C}^4/\mathbb{Z}_2$, where the generator of $\mathbb{Z}_2$ acts on the $\mathbb{C}^4$ coordinates (x,y,z,w) as:

$$(x,y,z,w) \ \mapsto \ (-x,-y,-z,-w) \,. \tag{2.46}$$

We have two fractional branes:

$$\mathcal{E}_0 = \rho_0 \otimes \mathcal{O}_p \,, \qquad \mathcal{E}_1 = \rho_1 \otimes \mathcal{O}_p \,. \tag{2.47}$$

for the trivial and non-trivial representation of $\mathbb{Z}_2$, respectively. The dimensions of the Ext groups can be computed following the methods of [46]. We have:

$$\begin{array}{lll}
\mathrm{Ext}^0(\mathcal{E}_0,\mathcal{E}_0) \cong \mathbb{C} \,, & \mathrm{Ext}^1(\mathcal{E}_0,\mathcal{E}_0) \cong 0 \,, & \mathrm{Ext}^2(\mathcal{E}_0,\mathcal{E}_0) \cong \mathbb{C}^6 \,, \\
\mathrm{Ext}^0(\mathcal{E}_1,\mathcal{E}_1) \cong \mathbb{C} \,, & \mathrm{Ext}^1(\mathcal{E}_1,\mathcal{E}_1) \cong 0 \,, & \mathrm{Ext}^2(\mathcal{E}_1,\mathcal{E}_1) \cong \mathbb{C}^6 \,, \\
\mathrm{Ext}^0(\mathcal{E}_0,\mathcal{E}_1) \cong 0 \,, & \mathrm{Ext}^1(\mathcal{E}_0,\mathcal{E}_1) \cong \mathbb{C}^4 \,, & \mathrm{Ext}^2(\mathcal{E}_0,\mathcal{E}_1) \cong 0 \,, \\
\mathrm{Ext}^0(\mathcal{E}_1,\mathcal{E}_0) \cong 0 \,, & \mathrm{Ext}^1(\mathcal{E}_1,\mathcal{E}_0) \cong \mathbb{C}^4 \,, & \mathrm{Ext}^2(\mathcal{E}_1,\mathcal{E}_0) \cong 0 \,,
\end{array} \tag{2.48}$$

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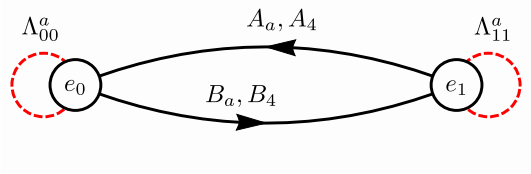
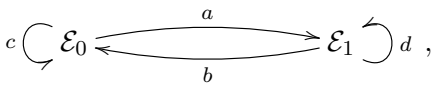


Figure 1. The $\mathbb{C}^2/\mathbb{Z}_2(1,1,1,1)$ quiver. The index a takes the values $a = 1,2,3$. There are three adjoint fermi multiplets Λ^a at each nodes, and two sets of four chiral multiplets, $A_i = (A_a, A_4)$ and $B_i = (B_a, B_4)$, in bifundamental representations. The quiver arrows have multiplicities equal to the number of distinct chiral or fermi multiplets.

with the higher Ext groups determined by Serre duality. We can also recover this spectrum from the results of section 2.2. Let us replace X in (2.39) by a, b, c, d according to the following diagram:



which encodes all possible Ext groups. From the Koszul resolution (2.38) and the fact that the maps A, B, C, D are all odd under $\mathbb{Z}_2$, we see that the superscript of a and b can only take values $1, 3$, while the superscript of c and d can only take values $0, 2, 4$, in agreement with (2.48). This gives us the $\mathcal{N} = (0,2)$ quiver indicated in figure 1.

The B-model correlation functions can be read off from (2.41). The $\mathcal{N} = (0,2)$ superpotential immediately follows. Let $\Lambda^1_{00}, \Lambda^2_{00}, \Lambda^3_{00}, \Lambda^1_{11}, \Lambda^2_{11}, \Lambda^3_{11}$ denote the fermi superfields corresponding to $c_4^2, c_5^2, c_6^2, d_4^2, d_5^2, d_6^2$, respectively. Note that they are Serre dual to $c_3^2, -c_2^2, c_1^2, d_3^2, -d_2^2, d_1^2$. Let us also denote the chiral superfields corresponding to a_j^1, b_j^1 by A_j, B_j. We then have, for instance:

$$\begin{aligned} J_{\Lambda^1_{00}} &= \sum_{i,j} \left\langle c_4^2 b_i^1 a_j^1 \right\rangle \, B_i A_j = B_2 A_3 - B_3 A_2 \,, \\ E_{\Lambda^1_{00}} &= \sum_{i,j} \left\langle c_3^2 b_i^1 a_j^1 \right\rangle \, B_i A_j = B_1 A_4 - B_4 A_1 \,, \end{aligned} \tag{2.49}$$

and so on and so forth. It is convenient to introduce the notation:

$$\Lambda^a_{00}\,, \qquad \Lambda^a_{11}\,, \qquad A_i = (A_a, A_4)\,, \qquad B_i = (B_a, B_4)\,, \tag{2.50}$$

with the index $a = 1,2,3$, to emphasize an SU(3) flavor symmetry. The interaction terms are given by:

$$\begin{aligned} J_{\Lambda^a_{00}} &= \epsilon^{abc} B_b A_c \,, & J_{\Lambda^a_{11}} &= A_a B_4 - A_4 B_a \,, \\ E_{\Lambda^a_{00}} &= B_a A_4 - B_4 A_a \,, & E_{\Lambda^a_{11}} &= \epsilon^{abc} A_b B_c \,. \end{aligned} \tag{2.51}$$

This satisfies $\mathrm{Tr}(EJ) = 0$, and it is in perfect agreement with the results of [19]. Note that, while the Lagrangian of the theory only has an SU(3) × U(1) global symmetry, the E and J terms of either node, taken together, fit into the **6** of SU(4), while the fields A_i and B_i each sit in the **4** of SU(4). This is the sign of an enhanced global symmetry in the infrared of the gauge theory, which can also be seen in the geometry.

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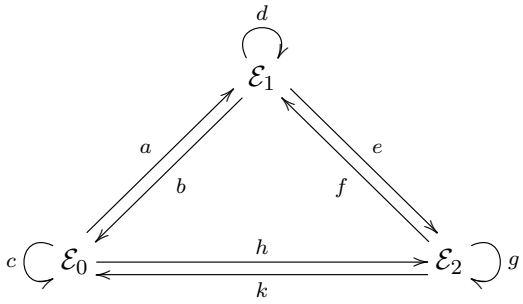
2.3.2 $\mathbb{C}^4/\mathbb{Z}_3(1,1,2,2)$

Consider $\mathbb{C}^4/\mathbb{Z}_3$ with the orbifold action:

$$(x,y,z,w) \mapsto (\omega x, \omega y, \omega^2 z, \omega^2 w)\,, \qquad \omega = e^{\frac{2\pi i}{3}}\,. \tag{2.52}$$

As before, we denote by ρ_i $(i=1,2,3)$ the trivial line bundle with equivariant structure i, in conventions in which ρ_0 has the trivial equivariant structure, and $\rho_1^* = \rho_2$. The possible Ext groups can be organized in the following diagram:



They are of the form:

$$\begin{aligned}
&\mathrm{Ext}^0(\mathcal{E}_i,\mathcal{E}_j) \cong \mathbb{C}\delta_{ij}\,,\\
&\mathrm{Ext}^1(\mathcal{E}_i,\mathcal{E}_j) \cong \mathbb{C}^2\delta_{i,j+1} \oplus \mathbb{C}^2\delta_{i,j+2}\\
&\mathrm{Ext}^2(\mathcal{E}_i,\mathcal{E}_j) \cong \mathbb{C}^4\delta_{ij} \oplus \mathbb{C}\delta_{i,j+1} \oplus \mathbb{C}\delta_{i,j+2}
\end{aligned} \tag{2.53}$$

with i defined mod 3, and the higher Ext groups determined by Serre duality. From the orbifold weights (2.52) on the coordinates, we can determine the weights for the sheaves in the Koszul resolution of $\mathcal{E}_i$. The result is:

$$\begin{pmatrix} i \end{pmatrix} \xrightarrow{D} \begin{pmatrix} i+2\\ i+2\\ i+1\\ i+1 \end{pmatrix} \xrightarrow{C} \begin{pmatrix} i+1\\ i\\ i\\ i\\ i\\ i+2 \end{pmatrix} \xrightarrow{B} \begin{pmatrix} i+2\\ i+2\\ i+1\\ i+1 \end{pmatrix} \xrightarrow{A} \begin{pmatrix} i \end{pmatrix}.$$

Thus the spectrum is given explicitly by:

$$\begin{aligned}
&X_1^1, X_2^1, X_6^2, X_1^3, X_2^3\,, && \text{for} \quad X = a, e, k\,,\\
&X_3^1, X_4^1, X_1^2, X_3^3, X_4^3\,, && \text{for} \quad X = b, h, f\,,\\
&X_1^0, X_2^2, X_3^2, X_4^2, X_5^2, X_1^4\,, && \text{for} \quad X = c, d, g\,.
\end{aligned}$$

Let us denote by $\Lambda_{01}, \Lambda_{12}, \Lambda_{20}$ the fermi multiplets corresponding to b_1^2, f_1^2, h_1^2, respectively, with the charge-conjugate fermi multiplets $\widetilde{\Lambda}_{01}, \widetilde{\Lambda}_{12}, \widetilde{\Lambda}_{20}$ corresponding to the Serre dual elements a_6^2, e_6^2, k_6^2; let us denote by $\Lambda_{00}^1, \Lambda_{00}^2, \Lambda_{11}^1, \Lambda_{11}^2, \Lambda_{22}^1, \Lambda_{22}^2$ the fermi multiplets corresponding to $c_4^2, c_5^2, d_4^2, d_5^2, g_4^2, g_5^2$ (dual to $c_3^2, -c_2^2, d_3^2, -d_2^2, g_3^2, -g_2^2$), respectively. We also

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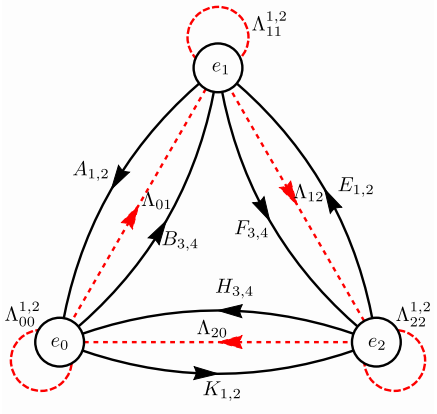


Figure 2. The $\mathbb{C}^3/\mathbb{Z}_3(1,1,2,2)$ supersymmetric quiver, with all the chiral and fermi multiplets indicated explicitly.

denote by $A_i, B_j, \cdots$ the chiral multiplets associated to $a_i^1, b_j^1, \cdots$, so that we have the 12 chiral multiplets:

$$A_1, A_2, B_3, B_4, \qquad E_1, E_2, F_3, F_4, \qquad K_1, K_2, H_3, H_4\,, \tag{2.54}$$

in the spectrum. The corresponding $\mathcal{N} = (0,2)$ supersymmetric quiver is shown in figure 2. We can directly compute the superpotential terms from (2.41). One finds:

$$\begin{aligned}
&J_{\Lambda_{00}^1} = K_2H_3 - B_3A_2\,, \quad J_{\Lambda_{11}^1} = A_2B_3 - F_3E_2\,, \quad J_{\Lambda_{22}^1} = E_2F_3 - H_3K_2\,,\\
&J_{\Lambda_{00}^2} = B_3A_1 - K_1H_3\,, \quad J_{\Lambda_{11}^2} = F_3E_1 - A_1B_3\,, \quad J_{\Lambda_{22}^2} = H_3K_1 - E_1F_3\,,\\
&J_{\Lambda_{01}} = F_3H_4 - F_4H_3\,, \quad J_{\Lambda_{12}} = H_3B_4 - H_4B_3\,, \quad J_{\Lambda_{20}} = B_3F_4 - B_4F_3\,.
\end{aligned} \tag{2.55}$$

and

$$\begin{aligned}
&E_{\Lambda_{00}^1} = K_1H_4 - B_4A_1\,, \quad E_{\Lambda_{11}^1} = A_1B_4 - F_4E_1\,, \quad E_{\Lambda_{22}^1} = E_1F_4 - H_4K_1\,,\\
&E_{\Lambda_{00}^2} = K_2H_4 - B_4A_2\,, \quad E_{\Lambda_{11}^2} = A_2B_4 - F_4E_2\,, \quad E_{\Lambda_{22}^2} = E_2F_4 - H_4K_2\,,\\
&E_{\Lambda_{01}} = K_1E_2 - K_2E_1\,, \quad E_{\Lambda_{12}} = A_1K_2 - A_2K_1\,, \quad E_{\Lambda_{20}} = E_1A_2 - E_2A_1\,.
\end{aligned} \tag{2.56}$$

One can check that $\mathrm{Tr}(EJ) = 0$. This again agrees with the results of [19]. Note that this quiver theory has only a $\mathrm{U}(1)^3$ (toric) flavor symmetry, though there is an expected enhancement to $\mathrm{SU}(2)^2 \times \mathrm{U}(1)$ in the infrared.

2.3.3 $\mathbb{C}^4/\mathbb{Z}_4(1,1,1,1)$

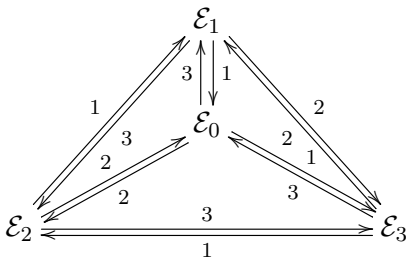
Consider $\mathbb{C}^/\mathbb{Z}_4$, where $\mathbb{Z}_4$ acts on the coordinates $\mathbb{C}^4$ coordinates as

$$(x,y,z,w) \;\mapsto\; (\omega x, \omega y, \omega z, \omega w), \qquad \omega = e^{\frac{2\pi i}{4}}\,. \tag{2.57}$$

We have four fractional branes, $\mathcal{E}_i$, $i = 0, 1, 2, 3$. In this case, the weights for the sheaves in the Koszul resolution of $\mathcal{E}_i$ are given as follows:

$$\begin{pmatrix} i \end{pmatrix} \xrightarrow{D} \begin{pmatrix} i+1 \\ i+1 \\ i+1 \\ i+1 \end{pmatrix} \xrightarrow{C} \begin{pmatrix} i+2 \\ i+2 \\ i+2 \\ i+2 \\ i+2 \\ i+2 \end{pmatrix} \xrightarrow{B} \begin{pmatrix} i+3 \\ i+3 \\ i+3 \\ i+3 \end{pmatrix} \xrightarrow{A} \begin{pmatrix} i \end{pmatrix}.$$

The Ext groups can be summarized by the diagram:



Here, the number attached to each arrow is the degree, and we omitted the degree-0 and degree-4 operators from one sheaf to itself, which also survive the orbifold projection and correspond to $\mathcal{N} = (0, 2)$ vector multiplets. We can similarly compute the interaction terms. They will be presented in section 2.4 below, after we reconsider the same quiver in a different guise. Any other supersymmetry-preserving orbifold of $\mathbb{C}^4$ can be worked out similarly.

2.4 Fractional branes on a local $\mathbb{P}^3$

Another interesting class of examples are given by Calabi-Yau fourfold singularities $\mathbf{X}_4$ that admit a crepant resolution:

$$\pi : \widetilde{\mathbf{X}}_4 \to \mathbf{X}_4 \,. \tag{2.58}$$

One of the simplest such singularity is the $\mathbb{C}^4/\mathbb{Z}_4$ orbifold (2.57), which admits a crepant resolution as the total space of the canonical line bundle over $\mathbb{P}^3$:

$$\widetilde{\mathbf{X}}_4 = \mathrm{Tot}\left(\mathcal{O}(-4) \to \mathbb{P}^3\right) . \tag{2.59}$$

For a Calabi-Yau threefold the total space of the canonical line bundle over a Fano surface, nice bases of fractional branes can be found in terms of strongly exceptional collections [7, 9, 15, 58, 59]. We can similarly construct a well-behaved set of sheaves on $\widetilde{\mathbf{X}}_4$ starting from what is known as a *strongly exceptional collection of sheaves* on $\mathbb{P}^3$. We will discuss this procedure in section 3.3. In the rest of this section, we will just postulate the sets of fractional branes, without further explanation.

We will discuss two distinct sets of fractional branes on (2.59), which give rise to two distinct supersymmetric quivers. In section 3, we will show that those two quivers are related by a field theory infrared duality, and by a mutation of the corresponding

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exceptional collections. We should emphasize that these two quiver gauge theories are only two relatively simple examples among an infinite number of dual theories for D1-branes probing the same CY_4 geometry. We refer to appendix B for a review of the simpler case of a CY threefold.

2.4.1 Fractional branes and Ext algebra (I)

Consider the following strongly exceptional collection on $\mathbb{P}^3$:

$$\{\Omega^3(3)[3], \Omega^2(2)[2], \Omega(1)[1], \mathcal{O}\} \cong \{\wedge^3 Q^*[3], \wedge^2 Q^*[2], Q^*[1], \mathcal{O}\}\,. \tag{2.60}$$

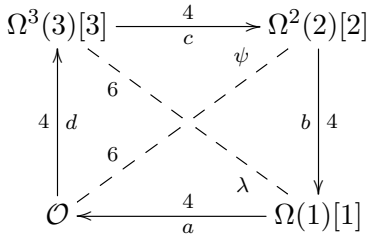
Let i denote the embedding $i : \mathbb{P}^3 \hookrightarrow \widetilde{\mathbf{X}}_4$. The four fractional branes are identified with $\mathcal{E}_j = i_*\Omega^j(j)[j]$ $(j = 0, 1, 2, 3)$, namely:

$$\mathcal{E}_0 = i_*\mathcal{O}\,, \quad \mathcal{E}_1 = i_*\Omega(1)[1]\,, \quad \mathcal{E}_2 = i_*\Omega^2(2)[2]\,, \quad \mathcal{E}_3 = i_*\Omega^3(3)[3]\,. \tag{2.61}$$

One can compute the Ext groups explicitly. One finds:[13]

$$\dim \mathrm{Ext}^1(\mathcal{E}_I, \mathcal{E}_J) = \begin{pmatrix} 0 & 0 & 0 & 4 \\ 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 4 & 0 \end{pmatrix}, \qquad \dim \mathrm{Ext}^2(\mathcal{E}_I, \mathcal{E}_J) = \begin{pmatrix} 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 6 \\ 6 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 \end{pmatrix}, \tag{2.62}$$

where $I, J = 0, 1, 2, 3$. The corresponding quiver diagram for the Ext groups reads:



Here the arrows stand for elements of Ext^1 and the dashed lines stand for elements of Ext^2, with the multiplicities indicated. This coincides with the $[\mathbb{C}^4/\mathbb{Z}_4]$ orbifold quiver in section 2.3.3.

Interestingly, the Ext groups fill out irreducible representations of GL(4), which are induced from the underlying GL(4) symmetry of $\mathbb{P}^3$. The precise representations can be worked out from the Bott-Borel-Weil theorem [60, 61]. The Ext^1 elements a, b, c, d naturally span the $\mathbf{4}$ or $\mathbf{4}'$ (fundamental or fully anti-symmetric representations), while the Ext^2 elements ψ and λ fall into $\mathbf{6}$'s (anti-symmetric representations) of GL(4).[14]

[13] One needs to use the fact that $\mathrm{Ext}^i(\mathcal{F}[m], \mathcal{G}[n]) = \mathrm{Ext}^{i-m+n}(\mathcal{F}, \mathcal{G})$.

[14] We refer to appendix A of [62] for a short introduction to the Bott-Borell-Weyl theorem in a related context. For our purposes in this paper, we only wish to point out the geometric origin of the SU(4) global symmetry in the $\mathcal{N} = (0, 2)$ supersymmetric quivers to be discussed below.

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An explicit basis for Ext$^\bullet$. Let us compute the Ext generators explicitly.[15] We take (x_i, y_i, z_i) to be the coordinates on the patch U_i such that the i-th homogeneous coordinate of $\mathbb{P}^3$ is nonzero, $i = 0, 1, 2, 3$. We also take w_i to be the coordinate of the fiber of $\mathcal{O}(-4)$ over U_i. A sheaf of the form $i_*\mathcal{E}$ has a Koszul resolution:

$$0 \to \mathcal{E}(4) \xrightarrow{w_0} \mathcal{E} \to i_*\mathcal{E} \to 0\,.$$

Every state in the Ext quiver can be represented by a chain map between the corresponding locally-free resolutions of sheaves, as follows:

$\mathrm{Ext}^1(i_*\Omega(1)[1], i_*\mathcal{O})$ is generated by $a_i \in \check{C}^0(X, \mathrm{Hom}^1(i_*\Omega(1)[1], i_*\mathcal{O}))$:

$$a_1: \begin{array}{ccc} \Omega(5) & \longrightarrow & \Omega(1) \\ {\scriptstyle (-1,0,0)}\downarrow & & \downarrow{\scriptstyle (1,0,0)} \\ \mathcal{O}(4) & \longrightarrow & \mathcal{O} \end{array} \qquad a_2: \begin{array}{ccc} \Omega(5) & \longrightarrow & \Omega(1) \\ {\scriptstyle (0,-1,0)}\downarrow & & \downarrow{\scriptstyle (0,1,0)} \\ \mathcal{O}(4) & \longrightarrow & \mathcal{O} \end{array}$$

$$a_3: \begin{array}{ccc} \Omega(5) & \longrightarrow & \Omega(1) \\ {\scriptstyle (0,0,-1)}\downarrow & & \downarrow{\scriptstyle (0,0,1)} \\ \mathcal{O}(4) & \longrightarrow & \mathcal{O} \end{array} \qquad a_4: \begin{array}{ccc} \Omega(5) & \longrightarrow & \Omega(1) \\ \downarrow{\scriptstyle (-x_0,-y_0,-z_0)} & & \downarrow{\scriptstyle (x_0,y_0,z_0)} \\ \mathcal{O}(4) & \longrightarrow & \mathcal{O} \end{array}$$

$\mathrm{Ext}^1(i_*\Omega^2(2)[2], i_*\Omega(1)[1])$ is generated by $b_i \in \check{C}^0(X, \mathrm{Hom}^1(i_*\Omega^2(2)[2], i_*\Omega(1)[1]))$:

$$b_1: \begin{array}{ccc} \Omega^2(6) & \longrightarrow & \Omega^2(2) \\ \left(\begin{smallmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right)\downarrow & & \downarrow\left(\begin{smallmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right) \\ \Omega(5) & \longrightarrow & \Omega(1) \end{array} \qquad b_2: \begin{array}{ccc} \Omega^2(6) & \longrightarrow & \Omega^2(2) \\ \left(\begin{smallmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{smallmatrix}\right)\downarrow & & \downarrow\left(\begin{smallmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{smallmatrix}\right) \\ \Omega(5) & \longrightarrow & \Omega(1) \end{array}$$

$$b_3: \begin{array}{ccc} \Omega^2(6) & \longrightarrow & \Omega^2(2) \\ \left(\begin{smallmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{smallmatrix}\right)\downarrow & & \downarrow\left(\begin{smallmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{smallmatrix}\right) \\ \Omega(5) & \longrightarrow & \Omega(1) \end{array} \qquad b_4: \begin{array}{ccc} \Omega^2(6) & \longrightarrow & \Omega^2(2) \\ \left(\begin{smallmatrix} 0 & -z_0 & y_0 \\ z_0 & 0 & -x_0 \\ -y_0 & x_0 & 0 \end{smallmatrix}\right)\downarrow & & \downarrow\left(\begin{smallmatrix} 0 & z_0 & -y_0 \\ -z_0 & 0 & x_0 \\ y_0 & -x_0 & 0 \end{smallmatrix}\right) \\ \Omega(5) & \longrightarrow & \Omega(1) \end{array}$$

$\mathrm{Ext}^1(i_*\Omega^3(3)[3], i_*\Omega^2(2)[2])$ is generated by $c_i \in \check{C}^0(X, \mathrm{Hom}^1(i_*\Omega^3(3)[3], i_*\Omega^2(2)[2]))$:

$$c_1: \begin{array}{ccc} \mathcal{O}(3) & \longrightarrow & \mathcal{O}(-1) \\ \left(\begin{smallmatrix} -1 \\ 0 \\ 0 \end{smallmatrix}\right)\downarrow & & \downarrow\left(\begin{smallmatrix} 1 \\ 0 \\ 0 \end{smallmatrix}\right) \\ Q(3) & \longrightarrow & Q(-1) \end{array} \qquad c_2: \begin{array}{ccc} \mathcal{O}(3) & \longrightarrow & \mathcal{O}(-1) \\ \left(\begin{smallmatrix} 0 \\ -1 \\ 0 \end{smallmatrix}\right)\downarrow & & \downarrow\left(\begin{smallmatrix} 0 \\ 1 \\ 0 \end{smallmatrix}\right) \\ Q(3) & \longrightarrow & Q(-1) \end{array}$$

$$c_3: \begin{array}{ccc} \mathcal{O}(3) & \longrightarrow & \mathcal{O}(-1) \\ \left(\begin{smallmatrix} 0 \\ 0 \\ -1 \end{smallmatrix}\right)\downarrow & & \downarrow\left(\begin{smallmatrix} 0 \\ 0 \\ 1 \end{smallmatrix}\right) \\ Q(3) & \longrightarrow & Q(-1) \end{array} \qquad c_4: \begin{array}{ccc} \mathcal{O}(3) & \longrightarrow & \mathcal{O}(-1) \\ \left(\begin{smallmatrix} -x_0 \\ -y_0 \\ -z_0 \end{smallmatrix}\right)\downarrow & & \downarrow\left(\begin{smallmatrix} x_0 \\ y_0 \\ z_0 \end{smallmatrix}\right) \\ Q(3) & \longrightarrow & Q(-1) \end{array}$$

[15] In the following, we use the notation $X = \widetilde{\mathbf{X}}_4$ to avoid clutter.

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$\mathrm{Ext}^1(i_*\mathcal{O}, i_*\Omega^3(3)[3])$ is generated by $d_i \in \check{C}^3(X, \mathrm{Hom}^{-2}(i_*\mathcal{O}, i_*\Omega^3(3)[3]))$:

$$d_1: \quad \begin{array}{ccc} & & \mathcal{O}(4) \\ & & \downarrow{\scriptstyle x_0^{-1}y_0^{-1}z_0^{-1}} \\ \mathcal{O}(3) & \longrightarrow & \mathcal{O}(-1) \end{array} \qquad d_2: \quad \begin{array}{ccc} & & \mathcal{O}(4) \\ & & \downarrow{\scriptstyle x_0^{-2}y_0^{-1}z_0^{-1}} \\ \mathcal{O}(3) & \longrightarrow & \mathcal{O}(-1) \end{array}$$

$$d_3: \quad \begin{array}{ccc} & & \mathcal{O}(4) \\ & & \downarrow{\scriptstyle x_0^{-1}y_0^{-2}z_0^{-1}} \\ \mathcal{O}(3) & \longrightarrow & \mathcal{O}(-1) \end{array} \qquad d_4: \quad \begin{array}{ccc} & & \mathcal{O}(4) \\ & & \downarrow{\scriptstyle x_0^{-1}y_0^{-1}z_0^{-2}} \\ \mathcal{O}(3) & \longrightarrow & \mathcal{O}(-1) \end{array}$$

$\mathrm{Ext}^2(i_*\Omega^2(2)[2], i_*\mathcal{O})$ is generated by $\psi_n \in \check{C}^0(X, \mathrm{Hom}^2(i_*\Omega^2(2)[2], i_*\mathcal{O}))$:

$$\psi_1: \quad \begin{array}{ccc} & & Q(3) \\ & & \downarrow{\scriptstyle (1,0,0)} \\ \mathcal{O}(4) & \longrightarrow & Q(-1) \\ \downarrow{\scriptstyle (1,0,0)} & & \\ \mathcal{O} & & \end{array} \qquad \psi_2: \quad \begin{array}{ccc} & & Q(3) \\ & & \downarrow{\scriptstyle (0,1,0)} \\ \mathcal{O}(4) & \longrightarrow & Q(-1) \\ \downarrow{\scriptstyle (0,1,0)} & & \\ \mathcal{O} & & \end{array}$$

$$\psi_3: \quad \begin{array}{ccc} & & Q(3) \\ & & \downarrow{\scriptstyle (0,0,1)} \\ \mathcal{O}(4) & \longrightarrow & Q(-1) \\ \downarrow{\scriptstyle (0,0,1)} & & \\ \mathcal{O} & & \end{array} \qquad \psi_4: \quad \begin{array}{ccc} & & Q(3) \\ & & \downarrow{\scriptstyle (y_0,-x_0,0)} \\ \mathcal{O}(4) & \longrightarrow & Q(-1) \\ \downarrow{\scriptstyle (y_0,-x_0,0)} & & \\ \mathcal{O} & & \end{array}$$

$$\psi_5: \quad \begin{array}{ccc} & & Q(3) \\ & & \downarrow{\scriptstyle (z_0,0,-x_0)} \\ \mathcal{O}(4) & \longrightarrow & Q(-1) \\ \downarrow{\scriptstyle (z_0,0,-x_0)} & & \\ \mathcal{O} & & \end{array} \qquad \psi_6: \quad \begin{array}{ccc} & & Q(3) \\ & & \downarrow{\scriptstyle (0,z_0,-y_0)} \\ \mathcal{O}(4) & \longrightarrow & Q(-1) \\ \downarrow{\scriptstyle (0,z_0,-y_0)} & & \\ \mathcal{O} & & \end{array}$$

We denote the Serre dual of ψ by ψ', with $\psi'_n \in \check{C}^3(X, \mathrm{Hom}^{-1}(i_*\mathcal{O}, i_*\Omega^2(2)[2]))$:

$$\psi'_1: \quad \begin{array}{ccc} & & \mathcal{O}(4) \\ & & \downarrow{\scriptstyle \begin{pmatrix} x_0^{-1}y_0^{-1}z_0^{-1} \\ 0 \\ 0 \end{pmatrix}} \\ Q(3) & \longrightarrow & Q(-1) \end{array} \qquad \psi'_2: \quad \begin{array}{ccc} & & \mathcal{O}(4) \\ & & \downarrow{\scriptstyle \begin{pmatrix} 0 \\ x_0^{-1}y_0^{-1}z_0^{-1} \\ 0 \end{pmatrix}} \\ Q(3) & \longrightarrow & Q(-1) \end{array}$$

$$\psi'_3: \quad \begin{array}{ccc} & & \mathcal{O}(4) \\ & & \downarrow{\scriptstyle \begin{pmatrix} 0 \\ 0 \\ x_0^{-1}y_0^{-1}z_0^{-1} \end{pmatrix}} \\ Q(3) & \longrightarrow & Q(-1) \end{array} \qquad \psi'_4: \quad \begin{array}{ccc} & & \mathcal{O}(4) \\ & & \downarrow{\scriptstyle \begin{pmatrix} x_0^{-1}y_0^{-2}z_0^{-1} \\ 0 \\ 0 \end{pmatrix}} \\ Q(3) & \longrightarrow & Q(-1) \end{array}$$

$$\psi'_5: \quad \begin{array}{ccc} & & \mathcal{O}(4) \\ & & \downarrow{\scriptstyle \begin{pmatrix} x_0^{-1}y_0^{-1}z_0^{-2} \\ 0 \\ 0 \end{pmatrix}} \\ Q(3) & \longrightarrow & Q(-1) \end{array} \qquad \psi'_6: \quad \begin{array}{ccc} & & \mathcal{O}(4) \\ & & \downarrow{\scriptstyle \begin{pmatrix} 0 \\ x_0^{-1}y_0^{-1}z_0^{-2} \\ 0 \end{pmatrix}} \\ Q(3) & \longrightarrow & Q(-1) \end{array}$$

Note that, since the maps given by:

$$\begin{pmatrix} x_0^{-1}y_0^{-2}z_0^{-1} \\ x_0^{-2}y_0^{-1}z_0^{-1} \\ 0 \end{pmatrix}, \begin{pmatrix} x_0^{-1}y_0^{-1}z_0^{-2} \\ 0 \\ x_0^{-2}y_0^{-1}z_0^{-1} \end{pmatrix}, \begin{pmatrix} 0 \\ x_0^{-1}y_0^{-1}z_0^{-2} \\ x_0^{-1}y_0^{-2}z_0^{-1} \end{pmatrix}$$

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are exact, $\psi_4', \psi_5', \psi_6'$ can be equivalently represented by:

$$\begin{pmatrix} 0 \\ -x_0^{-2}y_0^{-1}z_0^{-1} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -x_0^{-2}y_0^{-1}z_0^{-1} \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -x_0^{-1}y_0^{-2}z_0^{-1} \end{pmatrix},$$

respectively.

$\mathrm{Ext}^2(i_*\Omega^3(3)[3], i_*\Omega(1)[1])$ is generated by $\lambda_n \in \check{C}^0(X, \mathrm{Hom}^2(i_*\Omega^3(3)[3], i_*\Omega(1)[1]))$:

$$\lambda_1: \begin{array}{ccc} \mathcal{O}(3) & \longrightarrow & \mathcal{O}(-1) \\ \left(\begin{smallmatrix}1\\0\\0\end{smallmatrix}\right)\downarrow & & \downarrow\left(\begin{smallmatrix}1\\0\\0\end{smallmatrix}\right) \\ Q^*(4) & \longrightarrow & Q^* \end{array} \qquad \lambda_2: \begin{array}{ccc} \mathcal{O}(3) & \longrightarrow & \mathcal{O}(-1) \\ \left(\begin{smallmatrix}0\\1\\0\end{smallmatrix}\right)\downarrow & & \downarrow\left(\begin{smallmatrix}0\\1\\0\end{smallmatrix}\right) \\ Q^*(4) & \longrightarrow & Q^* \end{array}$$

$$\lambda_3: \begin{array}{ccc} \mathcal{O}(3) & \longrightarrow & \mathcal{O}(-1) \\ \left(\begin{smallmatrix}0\\0\\1\end{smallmatrix}\right)\downarrow & & \downarrow\left(\begin{smallmatrix}0\\0\\1\end{smallmatrix}\right) \\ Q^*(4) & \longrightarrow & Q^* \end{array} \qquad \lambda_4: \begin{array}{ccc} \mathcal{O}(3) & \longrightarrow & \mathcal{O}(-1) \\ \left(\begin{smallmatrix}y_0\\-x_0\\0\end{smallmatrix}\right)\downarrow & & \downarrow\left(\begin{smallmatrix}y_0\\-x_0\\0\end{smallmatrix}\right) \\ Q^*(4) & \longrightarrow & Q^* \end{array}$$

$$\lambda_5: \begin{array}{ccc} \mathcal{O}(3) & \longrightarrow & \mathcal{O}(-1) \\ \left(\begin{smallmatrix}z_0\\0\\-x_0\end{smallmatrix}\right)\downarrow & & \downarrow\left(\begin{smallmatrix}z_0\\0\\-x_0\end{smallmatrix}\right) \\ Q^*(4) & \longrightarrow & Q^* \end{array} \qquad \lambda_6: \begin{array}{ccc} \mathcal{O}(3) & \longrightarrow & \mathcal{O}(-1) \\ \left(\begin{smallmatrix}0\\z_0\\-y_0\end{smallmatrix}\right)\downarrow & & \downarrow\left(\begin{smallmatrix}0\\z_0\\-y_0\end{smallmatrix}\right) \\ Q^*(4) & \longrightarrow & Q^* \end{array}$$

We denote the Serre dual of λ by λ', with $\lambda_n' \in \check{C}^3(X, \mathrm{Hom}^{-1}(i_*\Omega(1)[1], i_*\Omega^3(3)[3]))$:

$$\lambda_1': \begin{array}{ccc} & & Q^*(4) \\ & & \downarrow{\scriptstyle (x_0^{-1}y_0^{-1}z_0^{-1},0,0)} \\ \mathcal{O}(3) & \longrightarrow & \mathcal{O}(-1) \\ & & \downarrow \\ & & Q^*(4) \end{array} \qquad \lambda_2': \begin{array}{ccc} & & Q^*(4) \\ & & \downarrow{\scriptstyle (0,x_0^{-1}y_0^{-1}z_0^{-1},0)} \\ \mathcal{O}(3) & \longrightarrow & \mathcal{O}(-1) \\ & & \downarrow \\ & & Q^*(4) \end{array}$$

$$\lambda_3': \begin{array}{ccc} & & Q^*(4) \\ & & \downarrow{\scriptstyle (0,0,x_0^{-1}y_0^{-1}z_0^{-1})} \\ \mathcal{O}(3) & \longrightarrow & \mathcal{O}(-1) \\ & & \downarrow \\ & & Q^*(4) \end{array} \qquad \lambda_4': \begin{array}{ccc} & & Q^*(4) \\ & & \downarrow{\scriptstyle (x_0^{-1}y_0^{-2}z_0^{-1},0,0)} \\ \mathcal{O}(3) & \longrightarrow & \mathcal{O}(-1) \\ & & \downarrow \\ & & Q^*(4) \end{array}$$

$$\lambda_5': \begin{array}{ccc} & & Q^*(4) \\ & & \downarrow{\scriptstyle (x_0^{-1}y_0^{-1}z_0^{-2},0,0)} \\ \mathcal{O}(3) & \longrightarrow & \mathcal{O}(-1) \end{array} \qquad \lambda_6': \begin{array}{ccc} & & Q^*(4) \\ & & \downarrow{\scriptstyle (0,x_0^{-1}y_0^{-1}z_0^{-2},0)} \\ \mathcal{O}(3) & \longrightarrow & \mathcal{O}(-1) \end{array}$$

Note that, since the maps given by $(x_0^{-1}y_0^{-2}z_0^{-1}, x_0^{-2}y_0^{-1}z_0^{-1}, 0)$, $(x_0^{-1}y_0^{-1}z_0^{-2}, 0, x_0^{-2}y_0^{-1}z_0^{-1})$, and $(0, x_0^{-1}y_0^{-1}z_0^{-2}, x_0^{-1}y_0^{-2}z_0^{-1})$, are exact, $\lambda_4', \lambda_5', \lambda_6'$ can be equivalently represented by $(0, -x_0^{-2}y_0^{-1}z_0^{-1}, 0)$, $(0, 0, -x_0^{-2}y_0^{-1}z_0^{-1})$, $(0, 0, -x_0^{-1}y_0^{-2}z_0^{-1})$, respectively.

Finally, $\mathrm{Ext}^4(i_*\mathcal{O}, i_*\mathcal{O})$ is generated by $t \in \check{C}^3(X, \mathrm{Hom}^1(i_*\mathcal{O}, i_*\mathcal{O}))$:

$$t: \begin{array}{ccc} & & \mathcal{O}(4) \\ & & \downarrow{\scriptstyle x_0^{-1}y_0^{-1}z_0^{-1}} \\ \mathcal{O}(4) & \longrightarrow & \mathcal{O} \end{array}$$

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$\mathrm{Ext}^4(i_*\Omega^3(3)[3], i_*\Omega^3(3)[3])$ is generated by $t' \in \check{C}^3(X, \mathrm{Hom}^1(i_*\Omega^3(3)[3], i_*\Omega^3(3)[3]))$:

$$t' : \quad \begin{array}{ccc} \mathcal{O}(3) & \longrightarrow & \mathcal{O}(-1) \\ {\scriptstyle x_0^{-1}y_0^{-1}z_0^{-1}}\big\downarrow & & \\ \mathcal{O}(3) & \longrightarrow & \mathcal{O}(-1) \end{array}$$

And similarly for the other two Ext^4 generators.

From this data, we determine the multiplication rules $m_2(x, y)$ by composition. One finds:

$$a_i \cdot b_j = \begin{pmatrix} \psi_2 & \psi_3 & 0 & \psi_6 \\ -\psi_1 & 0 & \psi_3 & -\psi_5 \\ 0 & -\psi_1 & -\psi_2 & \psi_4 \\ -\psi_4 & -\psi_5 & -\psi_6 & 0 \end{pmatrix}, \qquad b_i \cdot c_j = \begin{pmatrix} -\lambda_2 & \lambda_1 & 0 & \lambda_4 \\ -\lambda_3 & 0 & \lambda_1 & \lambda_5 \\ 0 & -\lambda_3 & \lambda_2 & \lambda_6 \\ -\lambda_6 & \lambda_5 & -\lambda_4 & 0 \end{pmatrix}, \tag{2.63}$$

and

$$c_i \cdot d_j = \begin{pmatrix} \psi'_1 & 0 & \psi'_4 & \psi'_5 \\ \psi'_2 & -\psi'_4 & 0 & \psi'_6 \\ \psi'_3 & -\psi_5 & -\psi'_6 & 0 \\ 0 & \psi'_1 & \psi'_2 & \psi'_3 \end{pmatrix}, \qquad d_i \cdot a_j = \begin{pmatrix} -\lambda'_1 & -\lambda'_2 & -\lambda'_3 & 0 \\ 0 & \lambda'_4 & \lambda'_5 & -\lambda'_1 \\ -\lambda'_4 & 0 & \lambda'_6 & -\lambda'_2 \\ -\lambda'_5 & -\lambda'_6 & 0 & -\lambda'_3 \end{pmatrix}, \tag{2.64}$$

with all other products $\mathrm{Ext}^1 \cdot \mathrm{Ext}^1$ vanishing. (All higher products also vanish.) It is convenient to define the basis:

$$\begin{aligned} (\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4) &\equiv (a_3, -a_2, a_1, a_4)\,, \qquad & (\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3, \mathbf{b}_4) &\equiv (b_1, b_2, b_3, b_4)\,, \\ (\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3, \mathbf{c}_4) &\equiv (c_3, -c_2, c_1, c_4)\,, \qquad & (\mathbf{d}_1, \mathbf{d}_2, \mathbf{d}_3, \mathbf{d}_4) &\equiv (d_4, -d_3, d_2, -d_1)\,. \end{aligned} \tag{2.65}$$

such that the matrices $(\mathbf{a}_i \cdot \mathbf{b}_j)$, $(\mathbf{b}_i \cdot \mathbf{c}_j)$, $(\mathbf{c}_i \cdot \mathbf{d}_j)$ and $(\mathbf{d}_i \cdot \mathbf{a}_j)$ are all antisymmetric. This is simply a manifestation of the GL(4) symmetry mentioned above.

2.4.2 Supersymmetric quiver (I)

From the above results, we have a complete description of the 2d $\mathcal{N} = (0, 2)$ supersymmetric quiver for D1-branes on the $\mathbb{C}^4/\mathbb{Z}_4$ singularity. The chiral multiplets are identified with the Ext^1 group elements according to:

$$A_i = \mathbf{a}_i\,, \qquad B_j = \mathbf{b}_j\,, \qquad C_k = \mathbf{c}_k\,, \qquad D_l = \mathbf{d}_l\,, \tag{2.66}$$

with $i, j, k, l \in 1, \cdots, 4$, and the elements $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$ defined in (2.65). As expected, the quiver theory has an SU(4) global symmetry, with the fields (2.66) in the **4** of SU(4). The fermi multiplets $\Lambda^n_{02} \sim \psi_n$ and $\Lambda^m_{13} \sim \lambda_m$ naturally fit in the **6** of SU(4), which we denote by $\Lambda^n = \Lambda^{ij} = -\Lambda^{ji}$. We define the fermi multiplets in terms of the elements of Ext^2 according to:

$$\begin{aligned} (\Lambda^{12}_{02}, \Lambda^{13}_{02}, \Lambda^{14}_{02}, \Lambda^{23}_{02}, \Lambda^{24}_{02}, \Lambda^{34}_{02}) &= (-\psi_1, -\psi_2, \psi_4, -\psi_3, \psi_5, \psi_6)\,, \\ (\Lambda^{13}_{2}, \Lambda^{13}_{13}, \Lambda^{14}_{13}, \Lambda^{23}_{13}, \Lambda^{24}_{13}, \Lambda^{34}_{13}) &= (-\lambda_1, -\lambda_2, \lambda_4, -\lambda_3, \lambda_5, \lambda_6)\,, \end{aligned} \tag{2.67}$$

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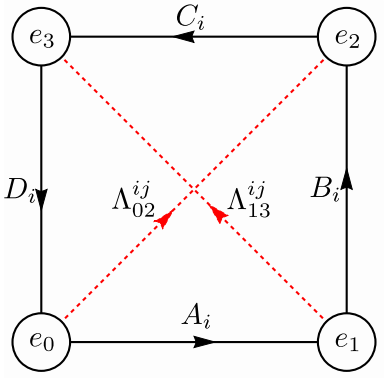


Figure 3. The $\mathbb{C}^4/\mathbb{Z}_4(1,1,1,1)$ supersymmetric quiver — "theory (I)".

and similarly for the (Serre dual) anti-fermi multiplets. The supersymmetric quiver is displayed in figure 3. The SU(4)-preserving interactions terms encoded in (2.63)–(2.64) take the simple form:

$$\begin{aligned} J_{\Lambda^{ij}_{02}} &= \epsilon^{ijkl} C_k D_l \,, & J_{\Lambda^{ij}_{13}} &= \epsilon^{ijkl} D_k A_l \,, \\ E_{\Lambda^{ij}_{02}} &= A_i B_j - A_j B_i \,, & E_{\Lambda^{ij}_{13}} &= B_i C_j - B_j C_i \,. \end{aligned} \tag{2.68}$$

This satisfies $\mathrm{Tr}(EJ) = 0$. It again agrees with the results of [19] for the $\mathbb{C}^4/\mathbb{Z}_4$ orbifold.

2.4.3 Fractional branes and Ext algebra (III)

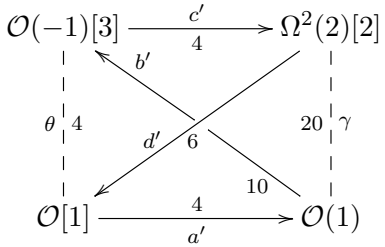
Another interesting set of fractional branes on $\widetilde{\mathbf{X}}_4 = \mathrm{Tot}(\mathcal{O}(-4) \to \mathbb{P}^3)$ is given by:

$$\mathcal{E}_0 = i_*\mathcal{O}[1] \,, \qquad \mathcal{E}_1 = i_*\mathcal{O}(1) \,, \qquad \mathcal{E}_2 = \Omega^2(2)[2] \,, \qquad \mathcal{E}_3 = i_*\mathcal{O}(-1)[3] \,. \tag{2.69}$$

Here we simply postulate this set of fractional branes, which we will further discuss in section 3. The Ext groups between the sheaves (2.69) have a slightly more complicated structure than in the previous example. One finds:

$$\dim \mathrm{Ext}^1(\mathcal{E}_I, \mathcal{E}_J) = \begin{pmatrix} 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 10 \\ 6 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 \end{pmatrix} , \qquad \dim \mathrm{Ext}^2(\mathcal{E}_I, \mathcal{E}_J) = \begin{pmatrix} 0 & 0 & 0 & 4 \\ 0 & 0 & 20 & 0 \\ 0 & 20 & 0 & 0 \\ 4 & 0 & 0 & 0 \end{pmatrix} , \tag{2.70}$$

This is conveniently summarized in the following Ext quiver:



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Comparing to the Ext quiver of section 2.4.1, we see that the upper row is the same (since $\Omega^3(3) \cong \mathcal{O}(-1)$ on $\mathbb{P}^3$). We can thus identity c' with the elements c there. All Ext groups again form irreducible representations of GL(4). In particular, the elements b' span the **10** (symmetric representation) and the elements γ span the **20′** (the mixed-symmetry three-tensor).

An explicit basis for Ext$^\bullet$. We now turn to the explicit description of the Ext algebra. $\mathrm{Ext}^1(i_*\Omega^2(2)[2], i_*\mathcal{O}[1])$ is generated by $d'_n \in \check{C}^0(X, \mathrm{Hom}^1(i_*\Omega^2(2)[2], i_*\mathcal{O}[1])), n = 1, \ldots, 6$:

$$d'_1: \begin{array}{ccc} Q(3) & \longrightarrow & Q(-1) \\ {\scriptstyle (-1,0,0)}\downarrow & & \downarrow{\scriptstyle (1,0,0)} \\ \mathcal{O}(4) & \longrightarrow & \mathcal{O} \end{array} \qquad d'_2: \begin{array}{ccc} Q(3) & \longrightarrow & Q(-1) \\ {\scriptstyle (0,-1,0)}\downarrow & & \downarrow{\scriptstyle (0,1,0)} \\ \mathcal{O}(4) & \longrightarrow & \mathcal{O} \end{array}$$

$$d'_3: \begin{array}{ccc} Q(3) & \longrightarrow & Q(-1) \\ {\scriptstyle (0,0,-1)}\downarrow & & \downarrow{\scriptstyle (0,0,1)} \\ \mathcal{O}(4) & \longrightarrow & \mathcal{O} \end{array} \qquad d'_4: \begin{array}{ccc} Q(3) & \longrightarrow & Q(-1) \\ {\scriptstyle (-y_0,x_0,0)}\downarrow & & \downarrow{\scriptstyle (y_0,-x_0,0)} \\ \mathcal{O}(4) & \longrightarrow & \mathcal{O} \end{array}$$

$$d'_5: \begin{array}{ccc} Q(3) & \longrightarrow & Q(-1) \\ {\scriptstyle (-z_0,0,x_0)}\downarrow & & \downarrow{\scriptstyle (z_0,0,-x_0)} \\ \mathcal{O}(4) & \longrightarrow & \mathcal{O} \end{array} \qquad d'_6: \begin{array}{ccc} Q(3) & \longrightarrow & Q(-1) \\ {\scriptstyle (0,-z_0,y_0)}\downarrow & & \downarrow{\scriptstyle (0,z_0,-y_0)} \\ \mathcal{O}(4) & \longrightarrow & \mathcal{O} \end{array}$$

$\mathrm{Ext}^1(i_*\mathcal{O}[1], i_*(1))$ is generated by $a'_i \in \check{C}^0(X, \mathrm{Hom}^1(i_*\mathcal{O}[1], i_*(1))), i = 1, 2, 3, 4$:

$$a'_1: \begin{array}{ccc} \mathcal{O}(4) & \longrightarrow & \mathcal{O} \\ {\scriptstyle 1}\downarrow & & \downarrow{\scriptstyle -1} \\ \mathcal{O}(5) & \longrightarrow & \mathcal{O}(1) \end{array} \qquad a'_2: \begin{array}{ccc} \mathcal{O}(4) & \longrightarrow & \mathcal{O} \\ {\scriptstyle x_0}\downarrow & & \downarrow{\scriptstyle -x_0} \\ \mathcal{O}(5) & \longrightarrow & \mathcal{O}(1) \end{array}$$

$$a'_3: \begin{array}{ccc} \mathcal{O}(4) & \longrightarrow & \mathcal{O} \\ {\scriptstyle y_0}\downarrow & & \downarrow{\scriptstyle -y_0} \\ \mathcal{O}(5) & \longrightarrow & \mathcal{O}(1) \end{array} \qquad a'_4: \begin{array}{ccc} \mathcal{O}(4) & \longrightarrow & \mathcal{O} \\ {\scriptstyle z_0}\downarrow & & \downarrow{\scriptstyle -z_0} \\ \mathcal{O}(5) & \longrightarrow & \mathcal{O}(1) \end{array}$$

$\mathrm{Ext}^1(i_*\mathcal{O}(-1)[3], i_*\Omega^2(2)[2])$ is generated by $c'_i \in \check{C}^0(X, \mathrm{Hom}^1(i_*\mathcal{O}(-1)[3], i_*\Omega^2(2)[2]))$, $i = 1, 2, 3, 4$:

$$c'_1: \begin{array}{ccc} \mathcal{O}(3) & \longrightarrow & \mathcal{O}(-1) \\ {\scriptstyle \begin{pmatrix}-1\\0\\0\end{pmatrix}}\downarrow & & \downarrow{\scriptstyle \begin{pmatrix}1\\0\\0\end{pmatrix}} \\ Q(3) & \longrightarrow & Q(-1) \end{array} \qquad c'_2: \begin{array}{ccc} \mathcal{O}(3) & \longrightarrow & \mathcal{O}(-1) \\ {\scriptstyle \begin{pmatrix}0\\-1\\0\end{pmatrix}}\downarrow & & \downarrow{\scriptstyle \begin{pmatrix}0\\1\\0\end{pmatrix}} \\ Q(3) & \longrightarrow & Q(-1) \end{array}$$

$$c'_3: \begin{array}{ccc} \mathcal{O}(3) & \longrightarrow & \mathcal{O}(-1) \\ {\scriptstyle \begin{pmatrix}0\\0\\-1\end{pmatrix}}\downarrow & & \downarrow{\scriptstyle \begin{pmatrix}0\\0\\1\end{pmatrix}} \\ Q(3) & \longrightarrow & Q(-1) \end{array} \qquad c'_4: \begin{array}{ccc} \mathcal{O}(3) & \longrightarrow & \mathcal{O}(-1) \\ {\scriptstyle \begin{pmatrix}-x_0\\-y_0\\-z_0\end{pmatrix}}\downarrow & & \downarrow{\scriptstyle \begin{pmatrix}x_0\\y_0\\z_0\end{pmatrix}} \\ Q(3) & \longrightarrow & Q(-1) \end{array}$$

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$\mathrm{Ext}^1(i_*\mathcal{O}(1), i_*\mathcal{O}(-1)[3])$ is generated by $b'_l \in \check{C}^3(X, \mathrm{Hom}^{-2}(i_*\mathcal{O}(1), i_*\mathcal{O}(-1)[3])), l = 1, \ldots, 10$:

$$
\begin{array}{lcc}
b'_1: & \begin{array}{ccc} \mathcal{O}(5) & \longrightarrow & \mathcal{O}(1) \\ {\scriptstyle x_0^{-1}y_0^{-1}z_0^{-1}}\downarrow & & \\ \mathcal{O}(3) & \longrightarrow & \mathcal{O}(-1) \end{array} & \begin{array}{ccc} \mathcal{O}(5) & \longrightarrow & \mathcal{O}(1) \\ {\scriptstyle x_0^{-2}y_0^{-1}z_0^{-1}}\downarrow & & \\ \mathcal{O}(3) & \longrightarrow & \mathcal{O}(-1) \end{array} \\
b'_3: & \begin{array}{ccc} \mathcal{O}(5) & \longrightarrow & \mathcal{O}(1) \\ {\scriptstyle x_0^{-1}y_0^{-2}z_0^{-1}}\downarrow & & \\ \mathcal{O}(3) & \longrightarrow & \mathcal{O}(-1) \end{array} & \begin{array}{ccc} \mathcal{O}(5) & \longrightarrow & \mathcal{O}(1) \\ {\scriptstyle x_0^{-1}y_0^{-1}z_0^{-2}}\downarrow & & \\ \mathcal{O}(3) & \longrightarrow & \mathcal{O}(-1) \end{array} \\
b'_5: & \begin{array}{ccc} \mathcal{O}(5) & \longrightarrow & \mathcal{O}(1) \\ {\scriptstyle x_0^{-3}y_0^{-1}z_0^{-1}}\downarrow & & \\ \mathcal{O}(3) & \longrightarrow & \mathcal{O}(-1) \end{array} & \begin{array}{ccc} \mathcal{O}(5) & \longrightarrow & \mathcal{O}(1) \\ {\scriptstyle x_0^{-1}y_0^{-3}z_0^{-1}}\downarrow & & \\ \mathcal{O}(3) & \longrightarrow & \mathcal{O}(-1) \end{array} \\
b'_7: & \begin{array}{ccc} \mathcal{O}(5) & \longrightarrow & \mathcal{O}(1) \\ {\scriptstyle x_0^{-1}y_0^{-1}z_0^{-3}}\downarrow & & \\ \mathcal{O}(3) & \longrightarrow & \mathcal{O}(-1) \end{array} & \begin{array}{ccc} \mathcal{O}(5) & \longrightarrow & \mathcal{O}(1) \\ {\scriptstyle x_0^{-2}y_0^{-2}z_0^{-1}}\downarrow & & \\ \mathcal{O}(3) & \longrightarrow & \mathcal{O}(-1) \end{array} \\
b'_9: & \begin{array}{ccc} \mathcal{O}(5) & \longrightarrow & \mathcal{O}(1) \\ {\scriptstyle x_0^{-2}y_0^{-1}z_0^{-2}}\downarrow & & \\ \mathcal{O}(3) & \longrightarrow & \mathcal{O}(-1) \end{array} & \begin{array}{ccc} \mathcal{O}(5) & \longrightarrow & \mathcal{O}(1) \\ {\scriptstyle x_0^{-1}y_0^{-2}z_0^{-2}}\downarrow & & \\ \mathcal{O}(3) & \longrightarrow & \mathcal{O}(-1) \end{array}
\end{array}
$$

$\mathrm{Ext}^2(i_*\Omega^2(2)[2], i_*\mathcal{O}(1))$ is generated by $\gamma_s \in \check{C}^0(X, \mathrm{Hom}^2(i_*\Omega^2(2)[2], i_*\mathcal{O}(1))), s = 1, \ldots, 20$:

$$
\begin{array}{lcc}
\gamma_1: & \begin{array}{ccc} Q(3) & \longrightarrow & Q(-1) \\ {\scriptstyle (1,0,0)}\downarrow & & {\scriptstyle (1,0,0)}\downarrow \\ \mathcal{O}(5) & \longrightarrow & \mathcal{O}(1) \end{array} & \gamma_2: \begin{array}{ccc} Q(3) & \longrightarrow & Q(-1) \\ {\scriptstyle (0,1,0)}\downarrow & & {\scriptstyle (0,1,0)}\downarrow \\ \mathcal{O}(5) & \longrightarrow & \mathcal{O}(1) \end{array} \\
\gamma_3: & \begin{array}{ccc} Q(3) & \longrightarrow & Q(-1) \\ {\scriptstyle (0,0,1)}\downarrow & & {\scriptstyle (0,0,1)}\downarrow \\ \mathcal{O}(5) & \longrightarrow & \mathcal{O}(1) \end{array} & \gamma_4: \begin{array}{ccc} Q(3) & \longrightarrow & Q(-1) \\ {\scriptstyle (x_0,0,0)}\downarrow & & {\scriptstyle (x_0,0,0)}\downarrow \\ \mathcal{O}(5) & \longrightarrow & \mathcal{O}(1) \end{array} \\
\gamma_5: & \begin{array}{ccc} Q(3) & \longrightarrow & Q(-1) \\ {\scriptstyle (0,x_0,0)}\downarrow & & {\scriptstyle (0,x_0,0)}\downarrow \\ \mathcal{O}(5) & \longrightarrow & \mathcal{O}(1) \end{array} & \gamma_6: \begin{array}{ccc} Q(3) & \longrightarrow & Q(-1) \\ {\scriptstyle (0,0,x_0)}\downarrow & & {\scriptstyle (0,0,x_0)}\downarrow \\ \mathcal{O}(5) & \longrightarrow & \mathcal{O}(1) \end{array} \\
\gamma_7: & \begin{array}{ccc} Q(3) & \longrightarrow & Q(-1) \\ {\scriptstyle (y_0,0,0)}\downarrow & & {\scriptstyle (y_0,0,0)}\downarrow \\ \mathcal{O}(5) & \longrightarrow & \mathcal{O}(1) \end{array} & \gamma_8: \begin{array}{ccc} Q(3) & \longrightarrow & Q(-1) \\ {\scriptstyle (0,y_0,0)}\downarrow & & {\scriptstyle (0,y_0,0)}\downarrow \\ \mathcal{O}(5) & \longrightarrow & \mathcal{O}(1) \end{array} \\
\gamma_9: & \begin{array}{ccc} Q(3) & \longrightarrow & Q(-1) \\ {\scriptstyle (0,0,y_0)}\downarrow & & {\scriptstyle (0,0,y_0)}\downarrow \\ \mathcal{O}(5) & \longrightarrow & \mathcal{O}(1) \end{array} & \gamma_{10}: \begin{array}{ccc} Q(3) & \longrightarrow & Q(-1) \\ {\scriptstyle (z_0,0,0)}\downarrow & & {\scriptstyle (z_0,0,0)}\downarrow \\ \mathcal{O}(5) & \longrightarrow & \mathcal{O}(1) \end{array}
\end{array}
$$

$$\gamma_{11}:\ \begin{array}{ccc} Q(3) & \longrightarrow & Q(-1) \\ {\scriptstyle (0,z_0,0)}\downarrow & & {\scriptstyle (0,z_0,0)}\downarrow \\ \mathcal{O}(5) & \longrightarrow & \mathcal{O}(1) \end{array} \qquad \gamma_{12}:\ \begin{array}{ccc} Q(3) & \longrightarrow & Q(-1) \\ {\scriptstyle (0,0,z_0)}\downarrow & & {\scriptstyle (0,0,z_0)}\downarrow \\ \mathcal{O}(5) & \longrightarrow & \mathcal{O}(1) \end{array}$$

$$\gamma_{13}:\ \begin{array}{ccc} Q(3) & \longrightarrow & Q(-1) \\ \downarrow{\scriptstyle (0,-z_0^2,y_0z_0)} & & \downarrow{\scriptstyle (0,-z_0^2,y_0z_0)} \\ \mathcal{O}(5) & \longrightarrow & \mathcal{O}(1) \end{array} \qquad \gamma_{14}:\ \begin{array}{ccc} Q(3) & \longrightarrow & Q(-1) \\ \downarrow{\scriptstyle (0,y_0z_0,-y_0^2)} & & \downarrow{\scriptstyle (0,y_0z_0,-y_0^2)} \\ \mathcal{O}(5) & \longrightarrow & \mathcal{O}(1) \end{array}$$

$$\gamma_{15}:\ \begin{array}{ccc} Q(3) & \longrightarrow & Q(-1) \\ \downarrow{\scriptstyle (0,x_0z_0,-x_0y_0)} & & \downarrow{\scriptstyle (0,x_0z_0,-x_0y_0)} \\ \mathcal{O}(5) & \longrightarrow & \mathcal{O}(1) \end{array} \qquad \gamma_{16}:\ \begin{array}{ccc} Q(3) & \longrightarrow & Q(-1) \\ \downarrow{\scriptstyle (-z_0^2,0,x_0z_0)} & & \downarrow{\scriptstyle (-z_0^2,0,x_0z_0)} \\ \mathcal{O}(5) & \longrightarrow & \mathcal{O}(1) \end{array}$$

$$\gamma_{17}:\ \begin{array}{ccc} Q(3) & \longrightarrow & Q(-1) \\ \downarrow{\scriptstyle (x_0z_0,0,-x_0^2)} & & \downarrow{\scriptstyle (x_0z_0,0,-x_0^2)} \\ \mathcal{O}(5) & \longrightarrow & \mathcal{O}(1) \end{array} \qquad \gamma_{18}:\ \begin{array}{ccc} Q(3) & \longrightarrow & Q(-1) \\ \downarrow{\scriptstyle (y_0z_0,0,-x_0y_0)} & & \downarrow{\scriptstyle (y_0z_0,0,-x_0y_0)} \\ \mathcal{O}(5) & \longrightarrow & \mathcal{O}(1) \end{array}$$

$$\gamma_{19}:\ \begin{array}{ccc} Q(3) & \longrightarrow & Q(-1) \\ \downarrow{\scriptstyle (x_0y_0,-x_0^2,0)} & & \downarrow{\scriptstyle (x_0y_0,-x_0^2,0)} \\ \mathcal{O}(5) & \longrightarrow & \mathcal{O}(1) \end{array} \qquad \gamma_{20}:\ \begin{array}{ccc} Q(3) & \longrightarrow & Q(-1) \\ \downarrow{\scriptstyle (-y_0^2,x_0y_0,0)} & & \downarrow{\scriptstyle (-y_0^2,x_0y_0,0)} \\ \mathcal{O}(5) & \longrightarrow & \mathcal{O}(1) \end{array}$$

We denote the Serre dual of γ by γ', with $\gamma'_s \in \check{C}^3(X, \mathrm{Hom}^{-1}(i_*\mathcal{O}(1), i_*\Omega^2(2)[2]))$:

$$\gamma'_1:\ \begin{array}{ccc} \mathcal{O}(5) & \longrightarrow & \mathcal{O}(1) \\ \downarrow{\scriptstyle \begin{pmatrix} x_0^{-1}y_0^{-1}z_0^{-1} \\ 0 \\ 0 \end{pmatrix}} & & \\ Q(3) & \longrightarrow & Q(-1) \end{array} \qquad \gamma'_2:\ \begin{array}{ccc} \mathcal{O}(5) & \longrightarrow & \mathcal{O}(1) \\ \downarrow{\scriptstyle \begin{pmatrix} 0 \\ x_0^{-1}y_0^{-1}z_0^{-1} \\ 0 \end{pmatrix}} & & \\ Q(3) & \longrightarrow & Q(-1) \end{array}$$

$$\gamma'_3:\ \begin{array}{ccc} \mathcal{O}(5) & \longrightarrow & \mathcal{O}(1) \\ \downarrow{\scriptstyle \begin{pmatrix} 0 \\ 0 \\ x_0^{-1}y_0^{-1}z_0^{-1} \end{pmatrix}} & & \\ Q(3) & \longrightarrow & Q(-1) \end{array} \qquad \gamma'_4:\ \begin{array}{ccc} \mathcal{O}(5) & \longrightarrow & \mathcal{O}(1) \\ \downarrow{\scriptstyle \begin{pmatrix} x_0^{-2}y_0^{-1}z_0^{-1} \\ 0 \\ 0 \end{pmatrix}} & & \\ Q(3) & \longrightarrow & Q(-1) \end{array}$$

$$\gamma'_5:\ \begin{array}{ccc} \mathcal{O}(5) & \longrightarrow & \mathcal{O}(1) \\ \downarrow{\scriptstyle \begin{pmatrix} 0 \\ x_0^{-2}y_0^{-1}z_0^{-1} \\ 0 \end{pmatrix}} & & \\ Q(3) & \longrightarrow & Q(-1) \end{array} \qquad \gamma'_6:\ \begin{array}{ccc} \mathcal{O}(5) & \longrightarrow & \mathcal{O}(1) \\ \downarrow{\scriptstyle \begin{pmatrix} 0 \\ 0 \\ x_0^{-2}y_0^{-1}z_0^{-1} \end{pmatrix}} & & \\ Q(3) & \longrightarrow & Q(-1) \end{array}$$

$$\gamma'_7:\ \begin{array}{ccc} \mathcal{O}(5) & \longrightarrow & \mathcal{O}(1) \\ \downarrow{\scriptstyle \begin{pmatrix} x_0^{-1}y_0^{-2}z_0^{-1} \\ 0 \\ 0 \end{pmatrix}} & & \\ Q(3) & \longrightarrow & Q(-1) \end{array} \qquad \gamma'_8:\ \begin{array}{ccc} \mathcal{O}(5) & \longrightarrow & \mathcal{O}(1) \\ \downarrow{\scriptstyle \begin{pmatrix} 0 \\ x_0^{-1}y_0^{-2}z_0^{-1} \\ 0 \end{pmatrix}} & & \\ Q(3) & \longrightarrow & Q(-1) \end{array}$$

$$\gamma'_9:\ \begin{array}{ccc} \mathcal{O}(5) & \longrightarrow & \mathcal{O}(1) \\ \downarrow{\scriptstyle \begin{pmatrix} 0 \\ 0 \\ x_0^{-1}y_0^{-2}z_0^{-1} \end{pmatrix}} & & \\ Q(3) & \longrightarrow & Q(-1) \end{array} \qquad \gamma'_{10}:\ \begin{array}{ccc} \mathcal{O}(5) & \longrightarrow & \mathcal{O}(1) \\ \downarrow{\scriptstyle \begin{pmatrix} x_0^{-1}y_0^{-1}z_0^{-2} \\ 0 \\ 0 \end{pmatrix}} & & \\ Q(3) & \longrightarrow & Q(-1) \end{array}$$

$$\gamma'_{11}:\ \begin{array}{ccc} \mathcal{O}(5) & \longrightarrow & \mathcal{O}(1) \\ \downarrow{\scriptstyle \begin{pmatrix} 0 \\ x_0^{-1}y_0^{-1}z_0^{-2} \\ 0 \end{pmatrix}} & & \\ Q(3) & \longrightarrow & Q(-1) \end{array} \qquad \gamma'_{12}:\ \begin{array}{ccc} \mathcal{O}(5) & \longrightarrow & \mathcal{O}(1) \\ \downarrow{\scriptstyle \begin{pmatrix} 0 \\ 0 \\ x_0^{-1}y_0^{-1}z_0^{-2} \end{pmatrix}} & & \\ Q(3) & \longrightarrow & Q(-1) \end{array}$$

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$$\gamma'_{13}: \quad \begin{array}{ccc} \mathcal{O}(5) & \longrightarrow & \mathcal{O}(1) \\ \left(\begin{smallmatrix} 0 \\ 0 \\ x_0^{-1}y_0^{-2}z_0^{-2} \end{smallmatrix}\right)\Big\downarrow & & \\ Q(3) & \longrightarrow & Q(-1) \end{array} \qquad \gamma'_{14}: \quad \begin{array}{ccc} \mathcal{O}(5) & \longrightarrow & \mathcal{O}(1) \\ \left(\begin{smallmatrix} 0 \\ x_0^{-1}y_0^{-2}z_0^{-2} \\ 0 \end{smallmatrix}\right)\Big\downarrow & & \\ Q(3) & \longrightarrow & Q(-1) \end{array}$$

$$\gamma'_{15}: \quad \begin{array}{ccc} \mathcal{O}(5) & \longrightarrow & \mathcal{O}(1) \\ \left(\begin{smallmatrix} 0 \\ x_0^{-2}y_0^{-1}z_0^{-2} \\ 0 \end{smallmatrix}\right)\Big\downarrow & & \\ Q(3) & \longrightarrow & Q(-1) \end{array} \qquad \gamma'_{16}: \quad \begin{array}{ccc} \mathcal{O}(5) & \longrightarrow & \mathcal{O}(1) \\ \left(\begin{smallmatrix} 0 \\ 0 \\ x_0^{-2}y_0^{-1}z_0^{-2} \end{smallmatrix}\right)\Big\downarrow & & \\ Q(3) & \longrightarrow & Q(-1) \end{array}$$

$$\gamma'_{17}: \quad \begin{array}{ccc} \mathcal{O}(5) & \longrightarrow & \mathcal{O}(1) \\ \left(\begin{smallmatrix} x_0^{-2}y_0^{-1}z_0^{-2} \\ 0 \\ 0 \end{smallmatrix}\right)\Big\downarrow & & \\ Q(3) & \longrightarrow & Q(-1) \end{array} \qquad \gamma'_{18}: \quad \begin{array}{ccc} \mathcal{O}(5) & \longrightarrow & \mathcal{O}(1) \\ \left(\begin{smallmatrix} x_0^{-1}y_0^{-2}z_0^{-2} \\ 0 \\ 0 \end{smallmatrix}\right)\Big\downarrow & & \\ Q(3) & \longrightarrow & Q(-1) \end{array}$$

$$\gamma'_{19}: \quad \begin{array}{ccc} \mathcal{O}(5) & \longrightarrow & \mathcal{O}(1) \\ \left(\begin{smallmatrix} x_0^{-2}y_0^{-2}z_0^{-1} \\ 0 \\ 0 \end{smallmatrix}\right)\Big\downarrow & & \\ Q(3) & \longrightarrow & Q(-1) \end{array} \qquad \gamma'_{20}: \quad \begin{array}{ccc} \mathcal{O}(5) & \longrightarrow & \mathcal{O}(1) \\ \left(\begin{smallmatrix} 0 \\ x_0^{-2}y_0^{-2}z_0^{-1} \\ 0 \end{smallmatrix}\right)\Big\downarrow & & \\ Q(3) & \longrightarrow & Q(-1) \end{array}$$

$\mathrm{Ext}^2(i_*\mathcal{O}(-1)[3], i_*\mathcal{O}[1])$ is generated by $\theta_i \in \check{C}^0(X, \mathrm{Hom}^2(i_*\mathcal{O}(-1)[3], i_*\mathcal{O}[1])), i = 1, \ldots, 4$:

$$\theta_1: \quad \begin{array}{ccc} \mathcal{O}(3) & \longrightarrow & \mathcal{O}(-1) \\ {\scriptstyle 1}\big\downarrow & & \big\downarrow{\scriptstyle 1} \\ \mathcal{O}(4) & \longrightarrow & \mathcal{O} \end{array} \qquad \theta_2: \quad \begin{array}{ccc} \mathcal{O}(3) & \longrightarrow & \mathcal{O}(-1) \\ {\scriptstyle x_0}\big\downarrow & & \big\downarrow{\scriptstyle x_0} \\ \mathcal{O}(4) & \longrightarrow & \mathcal{O} \end{array}$$

$$\theta_3: \quad \begin{array}{ccc} \mathcal{O}(3) & \longrightarrow & \mathcal{O}(-1) \\ {\scriptstyle y_0}\big\downarrow & & \big\downarrow{\scriptstyle y_0} \\ \mathcal{O}(4) & \longrightarrow & \mathcal{O} \end{array} \qquad \theta_4: \quad \begin{array}{ccc} \mathcal{O}(3) & \longrightarrow & \mathcal{O}(-1) \\ {\scriptstyle z_0}\big\downarrow & & \big\downarrow{\scriptstyle z_0} \\ \mathcal{O}(4) & \longrightarrow & \mathcal{O} \end{array}$$

We denote the Serre dual of θ by θ', with $\theta'_i \in \check{C}^3(X, \mathrm{Hom}^{-1}(i_*\mathcal{O}[1], i_*\mathcal{O}(-1)[3]))$:

$$\theta'_1: \quad \begin{array}{ccc} \mathcal{O}(4) & \longrightarrow & \mathcal{O} \\ {\scriptstyle x_0^{-1}y_0^{-1}z_0^{-1}}\big\downarrow & & \\ \mathcal{O}(3) & \longrightarrow & \mathcal{O}(-1) \end{array} \qquad \theta'_2: \quad \begin{array}{ccc} \mathcal{O}(4) & \longrightarrow & \mathcal{O} \\ {\scriptstyle x_0^{-2}y_0^{-1}z_0^{-1}}\big\downarrow & & \\ \mathcal{O}(3) & \longrightarrow & \mathcal{O}(-1) \end{array}$$

$$\theta'_3: \quad \begin{array}{ccc} \mathcal{O}(4) & \longrightarrow & \mathcal{O} \\ {\scriptstyle x_0^{-1}y_0^{-2}z_0^{-1}}\big\downarrow & & \\ \mathcal{O}(3) & \longrightarrow & \mathcal{O}(-1) \end{array} \qquad \theta'_4: \quad \begin{array}{ccc} \mathcal{O}(4) & \longrightarrow & \mathcal{O} \\ {\scriptstyle x_0^{-1}y_0^{-1}z_0^{-2}}\big\downarrow & & \\ \mathcal{O}(3) & \longrightarrow & \mathcal{O}(-1) \end{array}$$

$\mathrm{Ext}^4(i_*\mathcal{O}[1], i_*\mathcal{O}[1])$ is generated by $t \in \check{C}^3(X, \mathrm{Hom}^1(i_*\mathcal{O}[1], i_*\mathcal{O}[1]))$:

$$t: \quad \begin{array}{ccc} \mathcal{O}(4) & \longrightarrow & \mathcal{O} \\ {\scriptstyle x_0^{-1}y_0^{-1}z_0^{-1}}\big\downarrow & & \\ \mathcal{O}(4) & \longrightarrow & \mathcal{O} \end{array}$$

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$\mathrm{Ext}^4(i_*\mathcal{O}(1), i_*\mathcal{O}(1))$ is generated by $t' \in \check{C}^3(X, \mathrm{Hom}^1(i_*\mathcal{O}(1), i_*\mathcal{O}(1)))$:

$$t' : \quad \begin{array}{ccc} & & \mathcal{O}(5) \longrightarrow \mathcal{O}(1) \\ & & x_0^{-1}y_0^{-1}z_0^{-1}\big\downarrow \\ \mathcal{O}(5) & \longrightarrow & \mathcal{O}(1) \end{array}$$

From these data, we can again compute the product rules, which are all of the form $m_2(x, y) = x \cdot y$ and can be obtained by composition. One finds:

$$b'_i \cdot a'_j = \begin{pmatrix} \theta'_1 & 0 & 0 & 0 \\ \theta'_2 & \theta'_1 & 0 & 0 \\ \theta'_3 & 0 & \theta'_1 & 0 \\ \theta'_4 & 0 & 0 & \theta'_1 \\ 0 & \theta'_2 & 0 & 0 \\ 0 & 0 & \theta'_3 & 0 \\ 0 & 0 & 0 & \theta'_4 \\ 0 & \theta'_3 & \theta'_2 & 0 \\ 0 & \theta'_4 & 0 & \theta'_2 \\ 0 & 0 & \theta'_4 & \theta'_3 \end{pmatrix}, \qquad d'_i \cdot c'_j = \begin{pmatrix} \theta_1 & 0 & 0 & \theta_2 \\ 0 & \theta_1 & 0 & \theta_3 \\ 0 & 0 & \theta_1 & \theta_4 \\ 0 & \theta_4 & -\theta_3 & 0 \\ \theta_4 & 0 & -\theta_2 & 0 \\ \theta_3 & -\theta_2 & 0 & 0 \end{pmatrix},$$

$$a'_i \cdot d'_j = \begin{pmatrix} -\gamma_1 & -\gamma_2 & -\gamma_3 & \gamma_9 - \gamma_{11} & \gamma_6 - \gamma_{10} & \gamma_5 - \gamma_7 \\ -\gamma_4 & -\gamma_5 & -\gamma_6 & -\gamma_{15} & -\gamma_{17} & -\gamma_{19} \\ -\gamma_7 & -\gamma_8 & -\gamma_9 & -\gamma_{14} & -\gamma_{18} & \gamma_{20} \\ -\gamma_{10} & -\gamma_{11} & -\gamma_{12} & \gamma_{13} & \gamma_{16} & \gamma_{15} - \gamma_{18} \end{pmatrix},$$

$$c'_i \cdot b'_j = \begin{pmatrix} -\gamma'_1 & -\gamma'_4 & -\gamma'_7 & -\gamma'_{10} & 0 & \gamma'_{20} & \gamma'_{16} & -\gamma'_{19} & -\gamma'_{17} & -\gamma'_{18} \\ -\gamma'_2 & -\gamma'_5 & -\gamma'_8 & -\gamma'_{11} & \gamma'_{19} & 0 & \gamma'_{13} & -\gamma'_{20} & -\gamma'_{15} & -\gamma'_{14} \\ -\gamma'_3 & -\gamma'_6 & -\gamma'_9 & -\gamma'_{12} & \gamma'_{17} & \gamma'_{14} & 0 & \gamma'_{15} + \gamma'_{18} & -\gamma'_{16} & -\gamma'_{13} \\ 0 & -\gamma'_1 & -\gamma'_2 & -\gamma'_3 & -\gamma'_4 & -\gamma'_8 & -\gamma'_{12} & -\gamma'_5 - \gamma'_7 & -\gamma'_6 - \gamma'_{10} & -\gamma'_9 - \gamma'_{11} \end{pmatrix}.$$

2.4.4 Supersymmetric quiver (III)

To present the final result for the $\mathcal{N} = (0, 2)$ quiver theory associated to the fractional branes (2.69), it is convenient to take advantage of the SU(4) global symmetry. Let us introduce the chiral multiplets:

$$\begin{aligned} A''^i &= (A''^1, A''^2, A''^3, A''^4) = (a'_a, -a'_3, a'_2, -a'_1)\,, \\ C_i &= (C_1, C_2, C_3, C_4) = (c'_3, -c'_2, c'_1, c'_4)\,, \end{aligned} \tag{2.71}$$

which are identified with the Ext^1 elements a', c' as indicated. We also introduce the fields $M_{ij} = M_{ji} \sim b'$ and $D''_{ij} = -D''_{ji} \sim d'$ in the **10** and **6** of SU(4), respectively:

$$M_{ij} = \begin{pmatrix} b'_7 & -b'_{10} & b'_9 & -b'_4 \\ -b'_{10} & b'_6 & -b'_8 & b'_3 \\ b'_9 & -b'_8 & b'_5 & -b'_2 \\ -b'_4 & b'_3 & -b'_2 & b'_1 \end{pmatrix}, \qquad D''_{ij} = \begin{pmatrix} 0 & -d'_1 & -d'_2 & d'_4 \\ d'_1 & 0 & -d'_3 & d'_5 \\ d'_2 & d'_3 & 0 & d_6 \\ -d'_4 & -d'_5 & -d'_6 & 0 \end{pmatrix}. \tag{2.72}$$

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We define the fermi multiplets $\Lambda^i_{03} \sim \theta'$ and $\Gamma_{ijk} \sim \gamma$ in the **4** and **20′** of SU(4), respectively. We have:

$$\Lambda^i_{03} = (\Lambda^1_{03}, \Lambda^2_{03}, \Lambda^3_{03}, \Lambda^4_{03}) = (\theta'_4, -\theta'_3, \theta'_2, -\theta'_1)\,. \tag{2.73}$$

in terms of the Ext^2 elements θ'. The fields Γ_{ijk} are such that:

$$\Gamma_{ijk} = -\Gamma_{jik}\,, \qquad \epsilon^{ijkl}\Gamma_{jkl} = 0\,. \tag{2.74}$$

We choose the explicit set of 20 fields:

$$\begin{aligned}
\Gamma_{ij1} &= \begin{pmatrix} 0 & -\gamma'_{13} & \gamma'_{16} & -\gamma'_{12} \\ & 0 & \Gamma_{231} & \gamma'_{11} \\ & & 0 & -\gamma'_{10} \\ & & & 0 \end{pmatrix}, \qquad
\Gamma_{ij2} = \begin{pmatrix} 0 & -\gamma'_{14} & \gamma'_{18} & \gamma'_{9} \\ & 0 & \gamma'_{20} & -\gamma'_{8} \\ & & 0 & \gamma'_{7} \\ & & & 0 \end{pmatrix}, \\
\Gamma_{ij3} &= \begin{pmatrix} 0 & \gamma'_{15} & -\gamma'_{17} & -\gamma'_{6} \\ & 0 & \gamma'_{19} & \gamma'_{5} \\ & & 0 & -\gamma'_{4} \\ & & & 0 \end{pmatrix}, \qquad
\Gamma_{ij4} = \begin{pmatrix} 0 & \Gamma_{124} & \Gamma_{134} & \gamma'_{3} \\ & 0 & \Gamma_{234} & -\gamma'_{2} \\ & & 0 & \gamma'_{1} \\ & & & 0 \end{pmatrix},
\end{aligned} \tag{2.75}$$

which are identified with the Ext^2 elements as indicated (the fields $\Gamma_{231}, \Gamma_{124}, \Gamma_{134}, \Gamma_{234}$ are redundant). The supersymmetric quiver is shown in figure 4. It is convenient to introduce the notation:

$$\widetilde{D}^{ij} = \frac{1}{2}\epsilon^{ijkl} D''_{kl}\,. \tag{2.76}$$

The interaction terms for Λ_{03} are given by:

$$E_{\Lambda^i_{03}} = -\widetilde{D}^{il} C_l\,, \qquad J_{\Lambda^i_{03}} = M_{ij} A''^{j}\,. \tag{2.77}$$

The E-terms for Γ read:

$$E_{\Gamma_{ijk}} = C_i M_{jk} - C_j M_{ik}\,. \tag{2.78}$$

To write down the J-terms, it is more convenient to use the explicit choice of 20 components as in (2.75). We find:

$$J_{\Gamma_{ijk}} = -A''^{k}\widetilde{D}^{ij} \qquad \text{if } \; k = i \;\text{ or }\; k = j\,. \tag{2.79}$$

The J-terms of the remaining 8 fields are given explicitly by:

$$\begin{aligned}
J_{\Gamma_{123}} &= -A''^{3}\widetilde{D}^{12} + A''^{1}\widetilde{D}^{23}\,, & J_{\Gamma_{132}} &= -A''^{2}\widetilde{D}^{13} - A''^{1}\widetilde{D}^{23}\,, \\
J_{\Gamma_{142}} &= -A''^{2}\widetilde{D}^{14} - A''^{4}\widetilde{D}^{12}\,, & J_{\Gamma_{143}} &= -A''^{3}\widetilde{D}^{14} - A''^{4}\widetilde{D}^{13}\,, \\
J_{\Gamma_{241}} &= -A''^{1}\widetilde{D}^{24} + A''^{4}\widetilde{D}^{21}\,, & J_{\Gamma_{243}} &= -A''^{3}\widetilde{D}^{24} - A''^{4}\widetilde{D}^{21}\,, \\
J_{\Gamma_{341}} &= -A''^{1}\widetilde{D}^{34} + A''^{4}\widetilde{D}^{13}\,, & J_{\Gamma_{342}} &= -A''^{2}\widetilde{D}^{34} + A''^{4}\widetilde{D}^{23}\,.
\end{aligned} \tag{2.80}$$

One can check that:

$$\sum_{i=1}^{4} \mathrm{Tr}(E_{\Lambda^i_{03}} J_{\Lambda^i_{03}}) = -\sum_{s=1}^{20} \mathrm{Tr}(E_{\Gamma_{(s)}} J_{\Gamma_{(s)}}) = \mathrm{Tr}\left(C_i M_{jk} A''^{k} \widetilde{D}^{ij}\right)\,, \tag{2.81}$$

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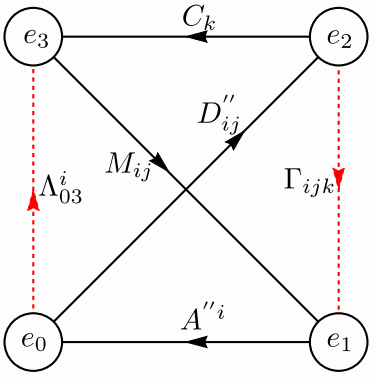

Figure 4. Another $\mathbb{C}^4/\mathbb{Z}_4(1,1,1,1)$ supersymmetric quiver, which we dub "Theory (III)".

and therefore $\mathrm{Tr}(EJ) = 0$, as required. Interestingly, this supersymmetric quiver cannot be realized as a brane brick model [19]. This is an example of a "non-toric quiver" (even though the CY_4 geometry is itself toric, in this case).

Since both quiver theories (I) and (III) appear to describe the low-energy dynamics of D1-branes at the $[\mathbb{C}^4/\mathbb{Z}_4]$ singularity, we expect that these two gauge theories are related by an infrared duality. It is indeed the case, as we will discuss in section 3.

2.5 Fractional branes on a local $\mathbb{P}^1 \times \mathbb{P}^1$

As our last example, we consider a toric singularity which is not an orbifold. Let $\mathbf{X}_4$ be the real cone over the seven-manifold known as $Q^{1,1,1}$:

$$\mathbf{X}_4 = C(Q^{1,1,1}), \qquad Q^{1,1,1} \cong \frac{\mathrm{SU}(2)\times\mathrm{SU}(2)\times\mathrm{SU}(2)}{\mathrm{U}(1)\times\mathrm{U}(1)}. \tag{2.82}$$

This singularity was also discussed in [19]. In order to describe fractional D1-branes on $\mathbf{X}_4$, we will consider the following crepant resolution, the local $\mathbb{P}^1 \times \mathbb{P}^1$ geometry:

$$\widetilde{\mathbf{X}}_4 = \mathrm{Tot}(\mathcal{O}(-1,-1)\oplus\mathcal{O}(-1,-1)\to\mathbb{P}^1\times\mathbb{P}^1). \tag{2.83}$$

We can construct a set of fractional branes on (2.83) in terms of a strongly exceptional collection on $\mathbb{P}^1\times\mathbb{P}^1$, similarly to the local $\mathbb{P}^3$ example. We choose the collection $\{\mathcal{O}(-1,-1), \mathcal{O}(0,-1), \mathcal{O}(-1,0), \mathcal{O}\}$ on $\mathbb{P}^1\times\mathbb{P}^1$. The corresponding fractional branes on the resolved singularity (2.83) are then given by:

$$\mathcal{E}_0 = i_*\mathcal{O}, \quad \mathcal{E}_1 = i_*\mathcal{O}(-1,0)[1], \quad \mathcal{E}_2 = i_*\mathcal{O}(0,-1)[1], \quad \mathcal{E}_3 = i_*\mathcal{O}(-1,-1)[2]. \tag{2.84}$$

As before, i denotes the embedding $i : \mathbb{P}^1\times\mathbb{P}^1 \hookrightarrow \widetilde{\mathbf{X}}_4$. The normal bundle of $\mathbb{P}^1\times\mathbb{P}^1$ in $\widetilde{\mathbf{X}}_4$ is $\mathcal{N} = \mathcal{O}(-1,-1)\oplus\mathcal{O}(-1,-1)$, thus $\wedge^2\mathcal{N} = \mathcal{O}(-2,-2)$, from which we can compute:

$$\begin{aligned}
\mathrm{Ext}^i(i_*\mathcal{O}(-1,-1)[2], i_*\mathcal{O}(0,-1)[1]) &= \mathbb{C}^2\delta^{i1},\\
\mathrm{Ext}^i(i_*\mathcal{O}(-1,-1)[2], i_*\mathcal{O}(-1,0)[1]) &= \mathbb{C}^2\delta^{i1},\\
\mathrm{Ext}^i(i_*\mathcal{O}(-1,-1)[2], i_*\mathcal{O}) &= \mathbb{C}^2\delta^{i3}\oplus\mathbb{C}^4\delta^{i2},
\end{aligned}$$

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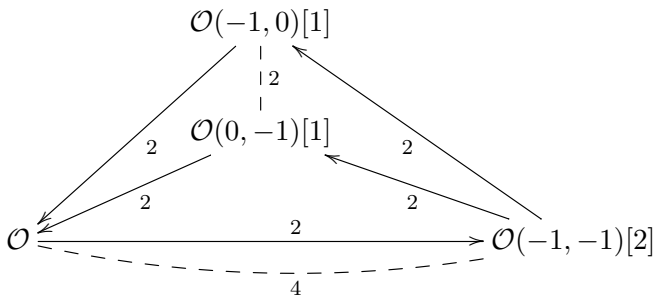
$$
\begin{aligned}
\mathrm{Ext}^i(i_*\mathcal{O}(0,-1)[1], i_*\mathcal{O}(-1,0)[1]) &= \mathbb{C}^2\delta^{i2}\,,\\
\mathrm{Ext}^i(i_*\mathcal{O}(0,-1)[1], i_*\mathcal{O}) &= \mathbb{C}^2\delta^{i1}\,,\\
\mathrm{Ext}^i(i_*\mathcal{O}(-1,0)[1], i_*\mathcal{O}) &= \mathbb{C}^2\delta^{i1}\,,\\
\mathrm{Ext}^i(i_*\mathcal{O}(-1,-1)[2], i_*\mathcal{O}(-1,-1)[2]) &= \mathrm{Ext}^i_X(i_*\mathcal{O}(0,-1)[1], i_*\mathcal{O}(0,-1)[1])\\
&= \mathrm{Ext}^i(i_*\mathcal{O}(-1,0)[1], i_*\mathcal{O}(-1,0)[1])\\
&= \mathrm{Ext}^i_X(i_*\mathcal{O}, i_*\mathcal{O}) = \mathbb{C}\delta^{i0}\oplus\mathbb{C}\delta^{i4}\,.
\end{aligned}
\tag{2.85}
$$

The corresponding $\mathrm{Ext}^{1,2}$ quiver diagram reads:



where the solid lines represent Ext^1 elements, the dashed lines represent Ext^2 elements, and the number labeling each line is the corresponding degeneracy.

2.5.1 The Ext algebra on a local $\mathbb{P}^1\times\mathbb{P}^1$

Let us compute the A_∞ structure satisfied by the Ext group elements. If we denote by x_0, x_1 the homogeneous coordinates on the first $\mathbb{P}^1$ and y_0, y_1 the homogeneous coordinates on the second $\mathbb{P}^1$, then $\widetilde{\mathbf{X}}_4$ can be covered by four open sets $U_{ij}, i, j = 0, 1$, defined by

$$
U_{ij} = \{x_i \neq 0, y_j \neq 0\}\,.
$$

We also define local coordinates $x = x_1/x_0, w = x_0/x_1, u = y_1/y_0, v = y_0/y_1$ in the corresponding open sets, and define y_{ij}, z_{ij} to be the coordinates of the fibers in U_{ij}. Thus, we have the transition functions $y_{01} = uy_{00}, z_{11} = xuz_{00}$, and so forth. We have the following Koszul resolutions of the fractional branes:

$$
\begin{aligned}
&0\to\mathcal{O}(2,2)\xrightarrow{\binom{-z_{00}}{y_{00}}}\mathcal{O}(1,1)\oplus\mathcal{O}(1,1)\xrightarrow{(\,y_{00}\;\;z_{00}\,)}\mathcal{O}\to i_*\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}\to 0\\
&0\to\mathcal{O}(1,2)\xrightarrow{\binom{-z_{00}}{y_{00}}}\mathcal{O}(0,1)\oplus\mathcal{O}(0,1)\xrightarrow{(\,y_{00}\;\;z_{00}\,)}\mathcal{O}(-1,0)\to i_*\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(-1,0)\to 0\\
&0\to\mathcal{O}(2,1)\xrightarrow{\binom{-z_{00}}{y_{00}}}\mathcal{O}(1,0)\oplus\mathcal{O}(1,0)\xrightarrow{(\,y_{00}\;\;z_{00}\,)}\mathcal{O}(0,-1)\to i_*\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(0,-1)\to 0\\
&0\to\mathcal{O}(1,1)\xrightarrow{\binom{-z_{00}}{y_{00}}}\mathcal{O}\oplus\mathcal{O}\xrightarrow{(\,y_{00}\;\;z_{00}\,)}\mathcal{O}(-1,-1)\to i_*\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(-1,-1)\to 0
\end{aligned}
$$

where all the bundle maps are written on coordinate patch U_{00}. Every state in the Ext quiver diagram can be represented by a chain map between two of the above complexes.

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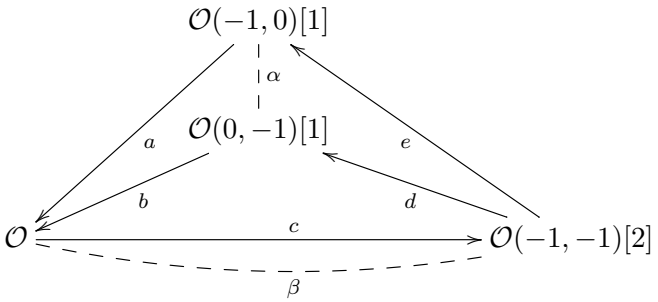
Let us introduce the notation:



$\mathrm{Ext}^1(i_*\mathcal{O}(-1,0)[1], i_*\mathcal{O})$ is generated by $a_1, a_2 \in \check{C}^0(U_{ij}, \mathrm{Hom}^1(i_*\mathcal{O}(-1,0)[1], i_*\mathcal{O}))$:

$$
\begin{array}{rccccc}
 & \mathcal{O}(1,2) & \longrightarrow & \mathcal{O}(0,1)\oplus\mathcal{O}(0,1) & \longrightarrow & \mathcal{O}(-1,0) \\
a_1 \ : & {\scriptstyle 1}\downarrow & & {\scriptstyle \left(\begin{smallmatrix}-1 & 0\\ 0 & -1\end{smallmatrix}\right)}\downarrow & & {\scriptstyle 1}\downarrow \\
 & \mathcal{O}(2,2) & \longrightarrow & \mathcal{O}(1,1)\oplus\mathcal{O}(1,1) & \longrightarrow & \mathcal{O} \\
 & \mathcal{O}(1,2) & \longrightarrow & \mathcal{O}(0,1)\oplus\mathcal{O}(0,1) & \longrightarrow & \mathcal{O}(-1,0) \\
a_2 \ : & {\scriptstyle x}\downarrow & & {\scriptstyle \left(\begin{smallmatrix}-x & 0\\ 0 & -x\end{smallmatrix}\right)}\downarrow & & {\scriptstyle x}\downarrow \\
 & \mathcal{O}(2,2) & \longrightarrow & \mathcal{O}(1,1)\oplus\mathcal{O}(1,1) & \longrightarrow & \mathcal{O}
\end{array}
$$

$\mathrm{Ext}^1(i_*\mathcal{O}(0,-1)[1], i_*\mathcal{O})$ is generated by $b_1, b_2 \in \check{C}^0(U_{ij}, \mathrm{Hom}^1(i_*\mathcal{O}(0,-1)[1], i_*\mathcal{O}))$:

$$
\begin{array}{rccccc}
 & \mathcal{O}(2,1) & \longrightarrow & \mathcal{O}(1,0)\oplus\mathcal{O}(1,0) & \longrightarrow & \mathcal{O}(0,-1) \\
b_1 \ : & {\scriptstyle 1}\downarrow & & {\scriptstyle \left(\begin{smallmatrix}-1 & 0\\ 0 & -1\end{smallmatrix}\right)}\downarrow & & {\scriptstyle 1}\downarrow \\
 & \mathcal{O}(2,2) & \longrightarrow & \mathcal{O}(1,1)\oplus\mathcal{O}(1,1) & \longrightarrow & \mathcal{O}
\end{array}
$$

and

$$
\begin{array}{rccccc}
 & \mathcal{O}(2,1) & \longrightarrow & \mathcal{O}(1,0)\oplus\mathcal{O}(1,0) & \longrightarrow & \mathcal{O}(0,-1) \\
b_2 \ : & {\scriptstyle u}\downarrow & & {\scriptstyle \left(\begin{smallmatrix}-u & 0\\ 0 & -u\end{smallmatrix}\right)}\downarrow & & {\scriptstyle u}\downarrow \\
 & \mathcal{O}(2,2) & \longrightarrow & \mathcal{O}(1,1)\oplus\mathcal{O}(1,1) & \longrightarrow & \mathcal{O}
\end{array}
$$

The representatives for the Ext^1 elements $d_{1,2}$ and $e_{1,2}$ are given by same maps as in $a_{1,2}$ and $b_{1,2}$, respectively.

$\mathrm{Ext}^2(i_*\mathcal{O}(-1,0)[1], i_*\mathcal{O}(0,-1)[1])$ is generated by $\alpha_i \in \check{C}^1(U_{ij}, \mathrm{Hom}^1(i_*\mathcal{O}(-1,0), i_*\mathcal{O}(0,-1)))$, $i = 1, 2$:

$$
\begin{array}{rccccc}
 & & \mathcal{O}(1,2) & \longrightarrow & \mathcal{O}(0,1)\oplus\mathcal{O}(0,1) & \longrightarrow & \mathcal{O}(-1,0) \\
\alpha_1 \ : & & {\scriptstyle \left(\begin{smallmatrix}\alpha\\ 0\end{smallmatrix}\right)}\downarrow & & {\scriptstyle (\,0\ -\alpha\,)}\downarrow & & \\
 & \mathcal{O}(2,1) & \longrightarrow & \mathcal{O}(1,0)\oplus\mathcal{O}(1,0) & \longrightarrow & \mathcal{O}(0,-1) & \\
 & & \mathcal{O}(1,2) & \longrightarrow & \mathcal{O}(0,1)\oplus\mathcal{O}(0,1) & \longrightarrow & \mathcal{O}(-1,0) \\
\alpha_2 \ : & & {\scriptstyle \left(\begin{smallmatrix}0\\ \alpha\end{smallmatrix}\right)}\downarrow & & {\scriptstyle (\,\alpha\ 0\,)}\downarrow & & \\
 & \mathcal{O}(2,1) & \longrightarrow & \mathcal{O}(1,0)\oplus\mathcal{O}(1,0) & \longrightarrow & \mathcal{O}(0,-1) &
\end{array}
$$

with:

$$(\alpha)_{01} = (\alpha)_{02} = u^{-1}\,, \qquad (\alpha)_{13} = (\alpha)_{23} = -u^{-1}\,, \qquad (\alpha)_{03} = (\alpha)_{12} = 0\,.$$

The Serre dual elements to $\alpha_{1,2}$, denoted by $\alpha'_{1,2}$, can be defined in the following way:

$$\begin{array}{rccccc}
 & & \mathcal{O}(2,1) & \longrightarrow & \mathcal{O}(1,0)\oplus\mathcal{O}(1,0) & \longrightarrow \mathcal{O}(0,-1) \\
\alpha'_1\ : & & \left(\begin{smallmatrix}\alpha'\\0\end{smallmatrix}\right)\big\downarrow & & \left(\begin{smallmatrix}0 & -\alpha'\end{smallmatrix}\right)\big\downarrow & \\
 & \mathcal{O}(1,2) & \longrightarrow \mathcal{O}(0,1)\oplus\mathcal{O}(0,1) & \longrightarrow & \mathcal{O}(-1,0) & \\
 & & \mathcal{O}(2,1) & \longrightarrow & \mathcal{O}(1,0)\oplus\mathcal{O}(1,0) & \longrightarrow \mathcal{O}(0,-1) \\
\alpha'_2\ : & & \left(\begin{smallmatrix}0\\\alpha'\end{smallmatrix}\right)\big\downarrow & & \left(\begin{smallmatrix}\alpha' & 0\end{smallmatrix}\right)\big\downarrow & \\
 & \mathcal{O}(1,2) & \longrightarrow \mathcal{O}(0,1)\oplus\mathcal{O}(0,1) & \longrightarrow & \mathcal{O}(-1,0) &
\end{array}$$

with:

$$(\alpha')_{02} = (\alpha)_{03} = (\alpha')_{12} = (\alpha')_{13} = x^{-1}\,, \quad (\alpha')_{01} = (\alpha')_{23} = 0\,.$$

$\mathrm{Ext}^1(i_*\mathcal{O}, i_*\mathcal{O}(-1,-1)[2])$ is generated by:

$$\begin{array}{rccccc}
 & & \mathcal{O}(2,2) & \longrightarrow & \mathcal{O}(1,1)\oplus\mathcal{O}(1,1) & \longrightarrow \mathcal{O} \\
c_1\ : & & \left(\begin{smallmatrix}c\\0\end{smallmatrix}\right)\big\downarrow & & \left(\begin{smallmatrix}0 & -c\end{smallmatrix}\right)\big\downarrow & \\
 & \mathcal{O}(1,1) & \longrightarrow \mathcal{O}\oplus\mathcal{O} & \longrightarrow & \mathcal{O}(-1,-1) & \\
 & & \mathcal{O}(2,2) & \longrightarrow & \mathcal{O}(1,1)\oplus\mathcal{O}(1,1) & \longrightarrow \mathcal{O} \\
c_2\ : & & \left(\begin{smallmatrix}0\\c\end{smallmatrix}\right)\big\downarrow & & \left(\begin{smallmatrix}c & 0\end{smallmatrix}\right)\big\downarrow & \\
 & \mathcal{O}(1,1) & \longrightarrow \mathcal{O}\oplus\mathcal{O} & \longrightarrow & \mathcal{O}(-1,-1) &
\end{array}$$

Closedness requires:

$$(c)_{123} + (c)_{013} = (c)_{023} + (c)_{012}.$$

If the two sides of the above identity were both zero, c would be exact. We deduce that one of $(c)_{013}$ and $(c)_{123}$ is $\pm x^{-1}u^{-1}$ and the other is zero, and similarly for $(c)_{023}$ and $(c)_{012}$. Different choices only differ by exact terms and sign convention. In the following, we will fix:

$$(c)_{012} = (c)_{013} = 0\,, \qquad (c)_{023} = (c)_{123} = x^{-1}u^{-1}.$$

$\mathrm{Ext}^2(i_*\mathcal{O}, i_*\mathcal{O}(-1,-1)[2])$ is generated by the elements $\beta_1, \cdots, \beta_4$:

$$\begin{array}{ccccc}
 & \mathcal{O}(2,2) & \longrightarrow \mathcal{O}(1,1)\oplus\mathcal{O}(1,1) & \longrightarrow \mathcal{O} \\
 & \beta\big\downarrow & & \\
\mathcal{O}(1,1) & \longrightarrow \mathcal{O}\oplus\mathcal{O} & \longrightarrow \mathcal{O}(-1,-1) &
\end{array}$$

β_1 is defined by

$$(\beta_1)_{123} + (\beta_1)_{013} = x^{-1}u^{-1}\,, \qquad (\beta_1)_{123}(\beta_1)_{013} = 0\,,$$
$$(\beta_1)_{012} + (\beta_1)_{023} = x^{-1}u^{-1}\,, \qquad (\beta_1)_{012}(\beta_1)_{023} = 0\,.$$

Again, different choices do not affect the cohomology class they represent. β_2, β_3 and β_4 are defined similarly with $x^{-1}u^{-1}$ replaced by $x^{-2}u^{-1}$ for β_2, $x^{-1}u^{-2}$ for β_3 and $x^{-2}u^{-2}$ for β_4. The Serre dual elements are given by $\beta_1', \cdots, \beta_4'$ defined by

$$\begin{array}{rccccc}
 & \mathcal{O}(1,1) & \longrightarrow & \mathcal{O}\oplus\mathcal{O} & \longrightarrow & \mathcal{O}(-1,-1) \\
\beta_1' : & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle \left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)} & & \downarrow{\scriptstyle 1} \\
 & \mathcal{O}(2,2) & \longrightarrow & \mathcal{O}1,1\oplus\mathcal{O}(1,1) & \longrightarrow & \mathcal{O} \\
 & \mathcal{O}(1,1) & \longrightarrow & \mathcal{O}\oplus\mathcal{O} & \longrightarrow & \mathcal{O}(-1,-1) \\
\beta_2' : & \downarrow{\scriptstyle x} & & \downarrow{\scriptstyle \left(\begin{smallmatrix}x&0\\0&x\end{smallmatrix}\right)} & & \downarrow{\scriptstyle x} \\
 & \mathcal{O}(2,2) & \longrightarrow & \mathcal{O}1,1\oplus\mathcal{O}(1,1) & \longrightarrow & \mathcal{O} \\
 & \mathcal{O}(1,1) & \longrightarrow & \mathcal{O}\oplus\mathcal{O} & \longrightarrow & \mathcal{O}(-1,-1) \\
\beta_3' : & \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle \left(\begin{smallmatrix}u&0\\0&u\end{smallmatrix}\right)} & & \downarrow{\scriptstyle u} \\
 & \mathcal{O}(2,2) & \longrightarrow & \mathcal{O}1,1\oplus\mathcal{O}(1,1) & \longrightarrow & \mathcal{O} \\
 & \mathcal{O}(1,1) & \longrightarrow & \mathcal{O}\oplus\mathcal{O} & \longrightarrow & \mathcal{O}(-1,-1) \\
\beta_4' : & \downarrow{\scriptstyle xu} & & \downarrow{\scriptstyle \left(\begin{smallmatrix}xu&0\\0&xu\end{smallmatrix}\right)} & & \downarrow{\scriptstyle xu} \\
 & \mathcal{O}(2,2) & \longrightarrow & \mathcal{O}1,1\oplus\mathcal{O}(1,1) & \longrightarrow & \mathcal{O}
\end{array}$$

The generator of Ext^4 at each node has the following form:

$$\begin{array}{ccccc}
\mathcal{O}(m+2,n+2) & \longrightarrow & \mathcal{O}(m+1,n+1)\oplus\mathcal{O}(m+1,n+1) & \longrightarrow & \mathcal{O}(m,n) \\
 & \searrow^{t} & & & \\
\mathcal{O}(m+2,n+2) & \longrightarrow & \mathcal{O}(m+1,n+1)\oplus\mathcal{O}(m+1,n+1) & \longrightarrow & \mathcal{O}(m,n)
\end{array}$$

with

$$(t)_{123} + (t)_{013} = x^{-1}u^{-1}\,, \qquad (t)_{123}(t)_{013} = 0\,,$$
$$(t)_{012} + (t)_{023} = x^{-1}u^{-1}\,, \qquad (t)_{012}(t)_{023} = 0\,.$$

It can be shown that α_1' is Serre dual to α_2, α_2' is Serre dual to $-\alpha_1$ and β_i' is Serre dual to β_i:

$$m_2(\alpha_1', \alpha_2) = t\,, \quad m_2(\alpha_2', \alpha_1) = -t\,, \quad m_2(\beta_i, \beta_j) = \delta_{ij}\, t\,,$$

From the composition of the chain maps, one can compute the products:

$$\begin{aligned}
&m_2(b_1, d_1) = \beta_1'\,, \quad m_2(b_1, d_2) = \beta_2'\,, \quad m_2(b_2, d_1) = \beta_3'\,, \quad m_2(b_2, d_2) = \beta_4'\,, \\
&m_2(a_1, e_1) = \beta_1'\,, \quad m_2(a_2, e_1) = \beta_2'\,, \quad m_2(a_1, e_2) = \beta_3'\,, \quad m_2(a_2, e_2) = \beta_4'\,,
\end{aligned} \tag{2.86}$$

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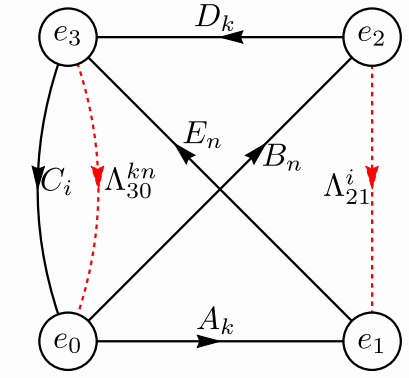

Figure 5. The $C(Q^{1,1,1})$ supersymmetric quiver.

amongst the Ext^1 elements. In addition, this model also has non-zero higher products, whose computation is rather more technical [10, 57]. We discuss it in appendix D. One finds the non-zero products:

$$\begin{aligned} m_3(d_1,c_1,a_2) &= -m_3(d_2,c_1,a_1) = \alpha_1\,, & m_3(d_1,c_2,a_2) &= -m_3(d_2,c_2,a_1) = \alpha_2\,, \\ m_3(e_1,c_2,b_2) &= -m_3(e_2,c_2,b_1) = -\alpha'_2\,, & m_3(e_2,c_1,b_1) &= -m_3(e_1,c_1,b_2) = \alpha'_1\,, \end{aligned}$$

and:

$$\begin{aligned} m_4(c_1,b_2,d_2,c_2) &= m_4(c_2,a_2,e_2,c_1) = -\beta_1\,, \\ m_4(c_2,a_1,e_2,c_1) &= m_4(c_1,b_2,d_1,c_2) = \beta_2\,, \\ m_4(c_1,a_2,e_1,c_2) &= m_4(c_2,b_1,d_2,c_1) = -\beta_3\,, \\ m_4(c_2,b_1,d_1,c_1) &= m_4(c_1,a_1,e_1,c_2) = \beta_4\,, \end{aligned} \tag{2.87}$$

with all other products amongst the Ext^1 elements vanishing.

2.5.2 The local $\mathbb{P}^1 \times \mathbb{P}^1$ quiver

Given the above result, it is straightforward to write down the corresponding quiver gauge theory, shown in figure 5. From the geometric structure (2.82), one would expect that the corresponding supersymmetric quiver theory has an $\mathrm{SU}(2)^3$ global symmetry. However, the A_∞ structure only preserves the minimal "toric" flavor symmetry $\mathrm{U}(1)^3$, which is the apparent symmetry of the quiver gauge theory.

The $\mathcal{N} = (0,2)$ quiver has four pairs of chiral multiplet, which are identified with the above Ext^1 elements according to:

$$A_k = a_k\,, \qquad B_n = -b_n\,, \qquad C_i = c_i\,, \qquad D_k = d_k\,, \qquad E_n = e_n\,, \tag{2.88}$$

with $k,n,i \in 1,2$. The k and n index are related to the $\mathrm{SU}(2) \times \mathrm{SU}(2)$ induced from the $\mathbb{P}^1 \times \mathbb{P}^1$ geometry; however, the interaction terms break this symmetry to its maximal torus. The quiver has the fermi multiplets:

$$\Lambda^i_{21} = \left(\Lambda^1_{21}, \Lambda^2_{21}\right) = (\alpha_1, \alpha_2)\,, \qquad \Lambda^{kn}_{03} = \left(\Lambda^{11}_{03}, \Lambda^{12}_{03}, \Lambda^{21}_{03}, \Lambda^{22}_{03}\right) = (\beta_1, \beta_3, \beta_2, \beta_4)\,, \tag{2.89}$$

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which are identified with the Ext^2 elements as indicated. From the A_∞ product structure discussed above, we find the interaction terms:

$$\begin{aligned} J_{\Lambda^1_{21}} &= E_2C_2B_1 - E_1C_2B_2\,, & E_{\Lambda^1_{21}} &= D_1C_1A_2 - D_2C_1A_1\,, \\ J_{\Lambda^2_{21}} &= E_1C_1B_2 - E_2C_1B_1\,, & E_{\Lambda^2_{21}} &= D_1C_2A_2 - D_2C_2A_1\,, \end{aligned} \tag{2.90}$$

and

$$\begin{aligned} J_{\Lambda^{11}_{30}} &= A_1E_1 - B_1D_1\,, & E_{\Lambda^{11}_{30}} &= C_1B_2D_2C_2 - C_2A_2E_2C_1\,, \\ J_{\Lambda^{12}_{30}} &= A_1E_2 - B_2D_1\,, & E_{\Lambda^{12}_{30}} &= C_2B_1D_2C_1 - C_1A_2E_1C_2\,, \\ J_{\Lambda^{21}_{30}} &= A_2E_1 - B_1D_2\,, & E_{\Lambda^{21}_{30}} &= C_2A_1E_2C_1 - C_1B_2D_1C_2\,, \\ J_{\Lambda^{22}_{30}} &= A_2E_2 - B_2D_2\,, & E_{\Lambda^{22}_{30}} &= C_1A_1E_1C_2 - C_2B_1D_1C_1\,. \end{aligned} \tag{2.91}$$

This satisfies $\mathrm{Tr}(EJ) = 0$ and agrees with [19].

3 Triality and mutations of exceptional collections

For some D3-brane quiver theories, it was proposed long ago that Seiberg duality in the gauge theory can be understood in terms of *mutations* of the underlying branes [58]. More precisely, for a singularity $\mathbf{X}_3$ whose crepant resolution $\widetilde{\mathbf{X}}_3$ is the total space of the canonical line bundle over a del Pezzo surface B_2, we can construct the fractional branes on $\widetilde{\mathbf{X}}_3$ in terms of an exceptional collection of B-branes on B_2 [6, 7, 58], and Seiberg dualities can be realized as mutations of the exceptional collection [8]. (See appendix B for an explicit example.)

We may consider the Calabi-Yau fourfold analogue of this setup, which involves the singularity $\mathbf{X}_4$ whose crepant resolution is $\widetilde{\mathbf{X}}_4 = \mathrm{Tot}(\mathcal{K} \to B_3)$, with B_3 a Fano threefold and $\mathcal{K}$ its canonical line bundle. The fractional branes on $\mathbf{X}_4$ can be similarly constructed from the data of an exceptional collection $\{\mathcal{E}\}$ of sheaves on B_3, in principle. In the previous section, we considered the simplest possible example, $B_3 = \mathbb{P}^3$. A mutation of the exceptional collection gives another exceptional collection $\{\mathcal{E}'\}$, and we can again consider the corresponding $\mathcal{N} = (0,2)$ quiver gauge theory. It is natural to suspect that the geometric operation amounts to a field theory *duality* between the different $\mathcal{N} = (0,2)$ quiver gauge theories. A well-studied example[16] of an $\mathcal{N} = (0,2)$ gauge theory duality is the *triality* of Gadde, Gukov and Putrov (GGP) [23]. We will show, in the simplest example of local $\mathbb{P}^3$, that indeed *mutation is triality*. This obviously deserves further study, which we leave for future work.

3.1 Triality acting on $\mathcal{N} = (0,2)$ supersymmetric quivers

Let us first review GGP triality and its action on quiver gauge theories [23]. The *triality transformation* can be formulated as a local operation at a single node e_0 of an $\mathcal{N} = (0,2)$ supersymmetric quiver without adjoint matter fields, as depicted in figure 6. The central node e_0 is a $\mathrm{U}(N_0)$ gauge group, while the nodes e_1, e_2, e_3 realize a "flavor" group

[16] Other two-dimensional dualities are also known amongst $\mathcal{N} = (2,2)$ and $\mathcal{N} = (0,2)$ gauge theories, see for example [63, 64].

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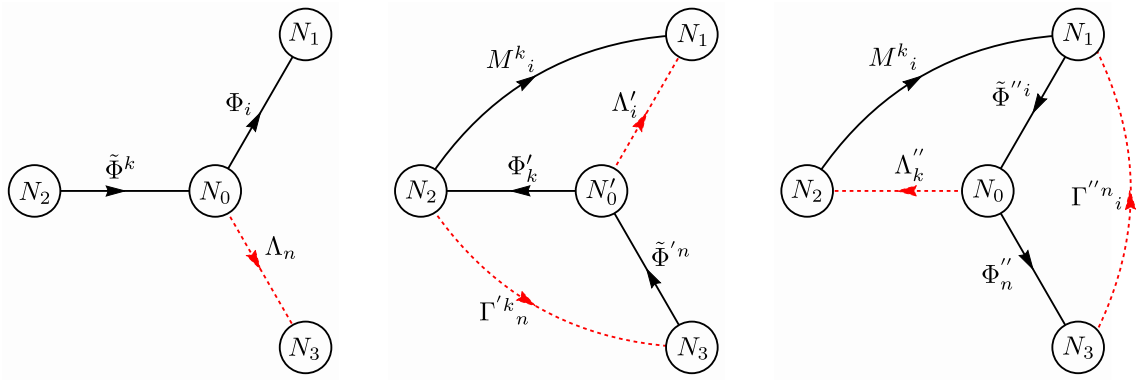

(a) Theory (i). **(b)** Theory (ii). **(c)** Theory (iii).

Figure 6. Triality as a local operation at a quiver node. The transformation of the interaction terms under triality is discussed in the main text.

$\mathrm{U}(N_1) \times \mathrm{U}(N_2) \times \mathrm{U}(N_3)$ from the point of view of $\mathrm{U}(N_0)$.[17] In the "original" theory, shown in figure 6a, we have chiral multiplets Φ^i in the fundamental representation of $\mathrm{U}(N_0)$, chiral multiplets $\widetilde{\Phi}_k$ in the antifundamental representation of $\mathrm{U}(N_0)$, and fermi multiplets Λ_n in the fundamental representation of $\mathrm{U}(N_0)$. (The flavor indices i, k, n run over $i = 1, \cdots, N_1$; $k = 1, \cdots, N_2$; $n = 1, \cdots, N_3$.) We must have:

$$2N_0 = N_1 + N_2 - N_3\,, \tag{3.1}$$

to cancel the non-abelian gauge anomaly. The theory can also have non-trivial interaction terms. Let Ξ and X denote any additional fermi and chiral multiplets, respectively, distinct from Λ and $\Phi, \widetilde{\Phi}$, in any larger $\mathcal{N} = (0,2)$ quiver in which figure 6a might be embedded. We have:

$$E_\Xi(X, \widetilde{\Phi}\Phi)\,, \qquad J_\Xi(X, \widetilde{\Phi}\Phi)\,, \qquad E_\Lambda(X, \Phi, \widetilde{\Phi})\,, \qquad J_\Lambda(X, \Phi, \widetilde{\Phi})\,, \tag{3.2}$$

which must be such that $\mathrm{Tr}(EJ) = \mathrm{Tr}(\sum_\Xi E_\Xi J_\Xi + \sum_\Lambda E_\Lambda J_\Lambda) = 0$.

The "triality move" can be described as follows: given the above Theory (i) with gauge group $\mathrm{U}(N_0)$, we obtain Theory (ii) as shown in figure 6b. The dual gauge group is $\mathrm{U}(N'_0)$ with dual rank given by the number of antifundamental chiral multiplets minus N_0:

$$N'_0 \equiv N_2 - N_0\,. \tag{3.3}$$

The dual charged matter fields in chiral and fermi multiplets, denoted by Φ'_k, $\widetilde{\Phi}'^n$ and Λ'_i, transform under the "flavor" group as indicated on the figure. In addition, the new theory also contains some "mesonic fields" $M^k{}_i$ and $\Gamma'^k{}_n$. Those fields are identified with the following $\mathrm{U}(N_0)$-invariant combinations of matter fields in Theory (i):

$$M^k{}_i = \widetilde{\Phi}^k \Phi_i\,, \qquad \Gamma'^k{}_n = \widetilde{\Phi}^k \Lambda_n\,. \tag{3.4}$$

[17] For simplicity, we write down a single arrow $e_0 \to e_i$ $(i = 1, 2, 3)$ for the matter fields of the $\mathrm{U}(N_0)$ gauge group at node e_0. In general, the "effective flavor group" $\mathrm{U}(N_i)$ at the node e_0 corresponds to a combination of both quiver gauge groups and actual flavor symmetries, which may be broken explicitly by interaction terms. We choose the slightly schematic depiction of figure 6 to avoid clutter.

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To fully specify the new theory, we need to determine the new interaction terms. Given that the original theory has interaction terms (3.2), the interaction terms for any "spectator" fermi multiplet Ξ are obtained by substituting $\widetilde{\Phi}\Phi = M$ inside E_Ξ and J_Ξ:

$$E_\Xi(X, M)\,, \qquad J_\Xi(X, M)\,. \tag{3.5}$$

In addition, the interaction terms of the new fermi multiplets Λ' and Γ' are given by:[18]

$$\begin{aligned} E_{\Lambda'_i} &= \Phi'_k M^k{}_i\,, & E_{\Gamma'^k{}_n} &= -M^k{}_i \left(\frac{\partial E_{\Lambda_n}}{\partial \Phi_i}\right)\,, \\ J_{\Lambda'_i} &= \left(\frac{\partial E_{\Lambda_n}}{\partial \Phi_i}\right) \widetilde{\Phi}'^n\,, & J_{\Gamma'^k{}_n} &= \widetilde{\Phi}'^n \Phi'_k - \frac{\partial J_{\Lambda_n}}{\partial \widetilde{\Phi}_k}\,, \end{aligned} \tag{3.6}$$

as holomorphic functions of X, Φ' and $\widetilde{\Phi}'$. One easily sees that the constraint $\mathrm{Tr}(EJ) = 0$ is again satisfied (given that it is satisfied in the original theory).

Theory (iii), shown in figure 6c, is similarly obtained from Theory (ii) by the *same* triality operation. The new gauge group is $\mathrm{U}(N''_0)$ with:

$$N''_0 = N_3 - N'_0\,. \tag{3.7}$$

The new matter fields Λ''_k, Φ''^i and $\widetilde{\Phi}''_n$ are as indicated. We also have the new mesons M' and Γ'', which are identified with the $\mathrm{U}(N'_0)$-invariant combinations:

$$M'^n{}_k = \widetilde{\Phi}'^n \Phi'_k\,, \qquad \Gamma''^n{}_i = \widetilde{\Phi}'^n \Lambda'_i\,, \tag{3.8}$$

in Theory (ii). Applying the rules above for the interaction terms, one finds that:

$$J_{\Gamma'^k_n} = M'^n{}_k - \frac{\partial J_{\Lambda_n}}{\partial \widetilde{\Phi}_k}\,, \tag{3.9}$$

in particular. This implies that Γ' and M' are both massive, and can be integrated out by imposing the linear relation $J_{\Gamma'} = 0$. We are left with the mesons M and Γ'' only, as shown in figure 6c.

Finally, one can check that another triality move, starting from Theory (iii), gives a theory which is identical to Theory (i) after integrating out all the massive fields. Thus, we confirm that the triality operation is indeed a "duality" of order three. More precisely, this is the case if we act repeatedly on a *single* node of a given $\mathcal{N} = (0, 2)$ quiver. If we act subsequently on different nodes, one uncovers very rich, infinite-dimensional "triality trees".

3.2 Triality and the $\mathbb{C}^4/\mathbb{Z}_4$ quiver

Let us now discuss an example of the triality operation on a full-fledged D1-brane quiver. Consider the $\mathbb{C}^4/\mathbb{Z}_4$ singularity with crepant resolution the local $\mathbb{P}^3$ geometry. Two distinct quiver gauge theories were derived in section 2.4, which we dubbed "Theory (I)" and "Theory (III)". They are reproduced in figure 7a and 7c, respectively.

[18] These transformation rules were left implicit in most of the literature. They where recently studied explicitly in [22].

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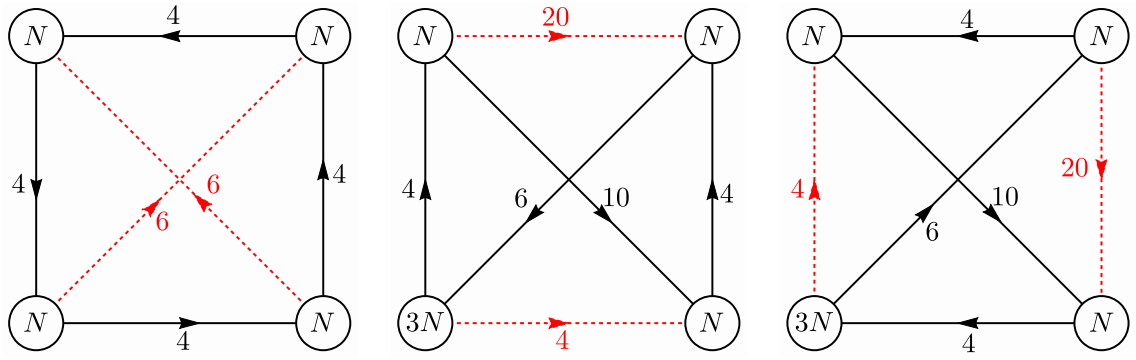

(a) Theory (I). **(b)** Theory (II). **(c)** Theory (III).

Figure 7. Three $\mathcal{N} = (0,2)$ quiver theories describing N D1-branes at the resolved orbifold $\widetilde{\mathbb{C}^4/\mathbb{Z}_4}(1,1,1,1)$. The numbers denote the multiplicities of the arrows.

From Theory (I) to (II). It is straightforward to apply the "triality operation" of subsection 3.1 to these quiver gauge theories. For definiteness, conside "Theory (I)", whose field content is shown in figure 8a. The interaction terms read:

$$\begin{aligned} J_{\Lambda_{02}^{ij}} &= \epsilon^{ijkl} C_k D_l \,, & J_{\Lambda_{13}^{ij}} &= \epsilon^{ijkl} D_k A_l \,, \\ E_{\Lambda_{02}^{ij}} &= A_i B_j - A_j B_i \,, & E_{\Lambda_{13}^{ij}} &= B_i C_j - B_j C_i \,. \end{aligned} \tag{3.10}$$

A triality operation on the node e_0 (lower left) leads to the quiver shown in figure 8b, with the chiral and fermi multiplets as indicated. In particular, we have the mesonic fields $\widetilde{M}$ and $\widetilde{\Gamma}'$, which are given in terms of the elementary fields of Theory (I) by:

$$\widetilde{M}_{ij} = D_i A_j \,, \qquad \widetilde{\Gamma}'_{ijk} = D_i \Lambda_{02}^{jk} \,. \tag{3.11}$$

By contruction, the mesons $\widetilde{M}_{ij}$ sit in the $\mathbf{4} \otimes \mathbf{4}$ of SU(4), which decomposes into $\mathbf{10} \oplus \mathbf{6}$. Similarly, the fermionic fields $\widetilde{\Gamma}'$ sit in the $\mathbf{4} \otimes \mathbf{6} \cong \mathbf{20} \oplus \mathbf{4}'$. From the matter content shown in figure 8b, we see that the $\mathbf{6}$ component of $\widetilde{M}_{ij}$ and the $\mathbf{4}'$ components of $\widetilde{\Gamma}'_{ijk}$ can become massive by pairing with Λ_{13}^{ij} and C_i, respectively. To see that this indeed happens, we simply look at the interaction terms, which are obtained by applying the triality rules (3.5)–(3.6). In particular, from (3.10) we find:

$$J_{\Lambda_{13}^{ij}} = \epsilon^{ijkl} \widetilde{M}_{kl} \,, \tag{3.12}$$

which states that the antisymmetric part of $\widetilde{M}_{ij}$ is massive, and can be set to zero in the low-energy theory. Let us denote by $M_{ij} = \frac{1}{2}(\widetilde{M}_{ij} + \widetilde{M}_{ji})$ the remaining light mesons, which span the $\mathbf{10}$ of SU(4). Similarly, it follows from (3.6) and (3.10) that the fields C_i are massive. The corresponding constraint reads:

$$J_{\widetilde{\Gamma}'_{ijk}} = A'^{jk} D'^{i} + \epsilon^{ijkl} C_l = 0 \,. \tag{3.13}$$

This sets the $\mathbf{4}'$ (fully antisymmetric) component of $\widetilde{\Gamma}'_{ijk}$ to zero. Let us denote by:

$$\Gamma'_{ijk} = \{\widetilde{\Gamma}'_{ijk}\} \,, \tag{3.14}$$

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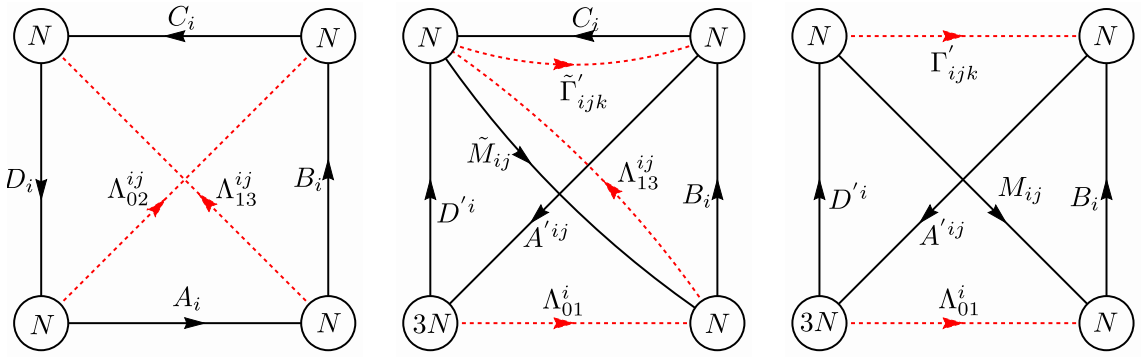

(a) Theory (I). **(b)** Triality-transformed. **(c)** Theory (II).

Figure 8. A triality move on node e_0 (lower left) of Theory (I) gives the quiver shown in the middle. Integrating out the massive fields, which corresponds to cancelling fermi-chiral loops between pairs of nodes, we obtain Theory (II).

the remaining fields, spanning the **20** of SU(4). Here and in the following, the notation $\{X_{ijk}\}$ denotes the projection of the three-tensor X_{ijk} with two antisymmetrized indices onto the **20** of SU(4). We are thus left with the quiver shown in figure 8c. The interaction terms are given explicitly by:

$$\begin{aligned} J_{\Lambda^i_{01}} &= B_k A'^{ik}\,, & J_{\Gamma'_{ijk}} &= \{A'^{jk} D'^{i}\}\,, \\ E_{\Lambda^i_{01}} &= D'^{j} M_{ji}\,, & E_{\Gamma'_{ijk}} &= -M_{ij} B_k + M_{ik} B_j\,. \end{aligned} \tag{3.15}$$

One can again verify that $\mathrm{Tr}(EJ) = 0$.

From Theory (II) to (III). Starting from Theory (II) with the interaction terms (3.15), we can again perform a triality operation on node e_0. The process of integrating out massive fields is similar, as depicted in figure 9. At the intermediate step (figure 9a), we have the new mesons $\widetilde{N}$ and $\widetilde{\Gamma}$, which are identified with the fundamental fields of Theory (II) according to:

$$\widetilde{N}^{ijk} = A'^{ij} D'^{k}\,, \qquad \widetilde{\Gamma}^{ij}{}_{k} = A'^{ij} \Lambda^k_{01}\,. \tag{3.16}$$

We see from (3.15) that the **20** part of $\widetilde{N}$ couple with Γ' to form a mass term $J_{\Gamma'_{ijk}} = \{\widetilde{N}^{ijk}\}$. Setting $\{\widetilde{N}^{ijk}\}$ to zero, we are left with chiral fields in the **4** of SU(4), which we denote by C_i, defined such that:

$$\widetilde{N}^{ijk} = -\epsilon^{ijkl} C_l\,. \tag{3.17}$$

Similarly, we have the following mass term for B_i:

$$J_{\widetilde{\Gamma}^{ij}{}_{k}} = A''^{k} D''_{ij} - \delta^k_i B_j\,. \tag{3.18}$$

Integrating out B_i, we are left with the **20** component of $\widetilde{\Gamma}^{ij}{}_{k}$. It is convenient to define the new fields:

$$\Gamma_{ijk} = -\frac{1}{2}\epsilon_{ijln}\{\widetilde{\Gamma}^{ln}{}_{k}\}\,, \qquad \widetilde{D}^{ij} = \frac{1}{2}\epsilon^{ijkl} D''_{kl}\,. \tag{3.19}$$

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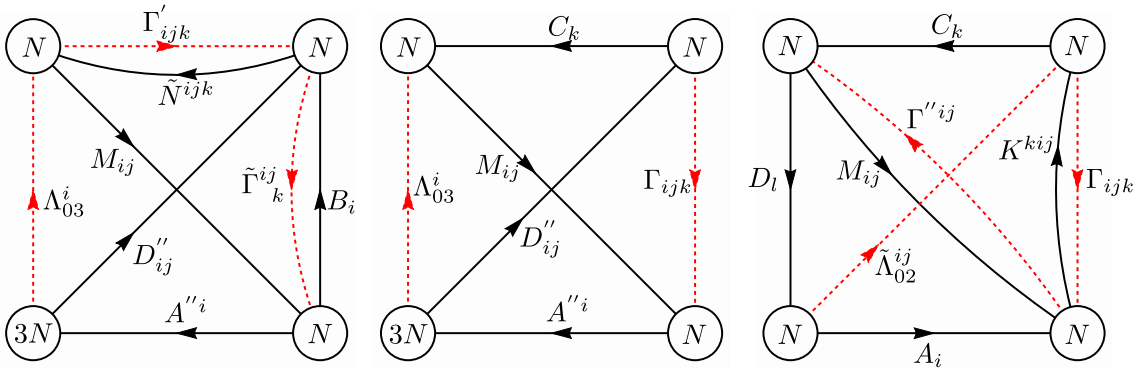

(a) Triality transformation on Theory (II).

(b) Theory (III).

(c) Triality transformation on Theory (III).

Figure 9. A triality move on node e_0 of Theory (II) gives the quiver shown on the left. Integrating out the massive fields, we obtain Theory (III), shown in the middle. Another triality move on Theory (III) gives the theory on the right, which is equivalent to Theory (I).

We then obtain Theory (III) shown in figure 9b, with the interaction terms:

$$\begin{aligned} J_{\Lambda^i_{03}} &= M_{ij}A''^j\,, & J_{\Gamma_{ijk}} &= -\{A'^k\widetilde{D}^{ij}\}\,, \\ E_{\Lambda^i_{03}} &= -\widetilde{D}^{ij}C_j\,, & E_{\Gamma_{ijk}} &= C_iM_{jk} - C_iM_{jk}\,. \end{aligned} \tag{3.20}$$

This is precisely the $\mathbb{C}^4/\mathbb{Z}_4$ quiver derived in section 2.4.4.

From Theory (III) to (I). Finally, we can close this triality cycle by performing a triality operation on node e_0 of Theory (III). The intermediate step is shown in figure 9c. We have the new mesons K and Γ'', which are identified with the fundamental fields of Theory (III) as:

$$K^{kij} = A''^k\widetilde{D}^{ij}\,, \qquad \Gamma''^{ij} = A''^i\Lambda^j_{03}\,. \tag{3.21}$$

It follows from $J_{\Gamma_{ijk}} = -\{K^{kij}\}$ that the **20** component of K^{kij} is massive. The remaining light fields, denoted by B_i, are defined by:

$$K^{kij} = \epsilon^{ijkl}B_l\,. \tag{3.22}$$

Similarly, we have the term:

$$J_{\Gamma''_{ij}} = D_jA_i - M_{ij}\,, \tag{3.23}$$

which gives a mass to M_{ij} and the symmetric part of Γ''^{ij}. If we define the new fermi multiplets:

$$\Lambda^{ij}_{02} = \frac{1}{2}\epsilon^{ijkl}(\widetilde{\Lambda}_{02})_{kl}\,, \qquad \Lambda^{ij}_{13} = -\epsilon^{ijkl}(\Gamma'')_{kl}\,, \tag{3.24}$$

we precisely reproduce Theory (I) in figure 8a, with the interaction terms (3.10).

These three $\mathcal{N} = (0,2)$ quiver gauge theories are thus related by a triality cycle. Note that the quiver ranks of Theory (I) are (N,N,N,N), while the quiver ranks of both Theory (II) and (III) are $(3N,N,N,N)$. In each case, this is the only rank assignment compatible with the non-abelian anomaly-free condition. (Abelian anomalies are not cancelled; they

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are expected to be cancelled by the contribution of bulk modes in string theory.) Theories (II) and (III) are examples of "non-toric" quivers.

3.3 Triality from mutation — a conjecture

We expect that the triality relations of $\mathcal{N} = (0,2)$ quiver gauge theories are realized in string theory in the same way that all known Seiberg-like dualities are realized: by a change of "brane basis". This intuition was realized in the type IIB mirror picture in [21], where triality was related to certain permutations of Lagrangian 4-cycles. We would like to understand the analogous notion in the B-model.

Fractional branes from strongly exceptional collections. Following previous work [7–9, 15], we consider the local Fano setup. We focus on $B_3 = \mathbb{P}^3$, although we expect that most of the following is valid more generally.[19] Let us denote by E_k the sheaves on B_3. A sheaf E is called *exceptional* if $\mathrm{Ext}^i_{B_3}(E,E) = \delta_{i,0}\mathbb{C}$. A strongly exceptional collection:

$$\mathbf{E} = \{E_1, \cdots, E_n\} \tag{3.25}$$

on B_3 is a collection of exceptional objects such that[20]

$$\mathrm{Ext}^i_{B_3}(E_k, E_l) = 0 \quad \forall\, i \neq 0\,,\ \forall k,l\,, \qquad \mathrm{Ext}^i_{B_3}(E_k, E_l) = 0 \quad \forall\, i,\ k > l\,. \tag{3.26}$$

In particular, each sheaf in $\mathbf{E}$ is exceptional. To describe fractional branes, we also need our collection to be "maximal" in some appropriate sense. Let $b_n = \dim H^n(B_3, \mathbb{R})$ denote the Betti numbers of B_3. We call the strongly exceptional collection $\mathbf{E}$ *complete* if it contains $n = 2+b_2+b_4$ sheaves — physically, this corresponds to the most general D-brane wrapping the 0-, 2-, 4- and 6-cycles [8]. We have $n = 4$ on $\mathbb{P}^3$.

Given a complete strongly exceptional collection (3.25) on B_3, we propose that there exists a good set of fractional branes on $\widetilde{\mathbf{X}}_4 = \mathrm{Tot}(\mathcal{K} \to B_3)$ given by:

$$\mathcal{E}_I = i_* E_{n-I}[I]\,, \qquad I = 0, \cdots, n-1\,. \tag{3.27}$$

From the strong exceptionality condition on $\mathbf{E}$, it follows that the sheaves $\widetilde{E}_I \equiv E_{n-I}[I]$ are ordered such that $\mathrm{Ext}^1(\widetilde{E}_I, \widetilde{E}_J)$ is non-vanishing only if $I = J+1$. Thus, we have:

$$E_1[n-1] \longrightarrow E_2[n-2] \longrightarrow \cdots \longrightarrow E_{n-1}[1] \longrightarrow E_n\,, \tag{3.28}$$

where the arrows denote the $\mathrm{Ext}^1_{B_3}(\widetilde{E}_I, \widetilde{E}_J)$ elements. The pushforward to $\widetilde{\mathbf{X}}_4$ will "close the quiver", by adding additional Ext groups due to the contribution of the embedding.

As an example, consider the following strongly exceptional collections on $\mathbb{P}^3$: $\mathbf{E}_I \equiv \{\Omega^3(3)\,,\ \Omega^2(2)\,,\ \Omega(1)\,,\ \mathcal{O}\}$. The intermediate quiver (3.28) reads:

$$\Omega^3(3)[3] \xrightarrow{4} \Omega^2(2)[2] \xrightarrow{4} \Omega(1)[1] \xrightarrow{4} \mathcal{O}\,, \tag{3.29}$$

with the dimension of the Ext^1 groups indicated over the arrows. The corresponding fractional branes were discussed in section 2.4.1.

[19] Including more general local geometries, such as the local $\mathbb{P}^1 \times \mathbb{P}^1$ of section 2.5.

[20] An *exceptional collection* $\mathbf{E}$ is such that $\mathrm{Ext}^i_{B_3}(E_k, E_l) = 0$ for $k > l$, $\forall i$. In this section, we consider the stronger condition of strong exceptionality, following [8].

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Triality and mutations. A natural geometric operation on these fractional branes is provided by mutations of exceptional collections [65]. Consider the strongly exceptional collection (3.25). A *mutation at position* k, with $k < n$, is a braiding operation on the exceptional collection:

$$(E_1, \cdots, E_k, E_{k+1}, \cdots E_n) \quad \rightsquigarrow \quad (E_1, \cdots, L_{E_k} E_{k+1}, E_k, \cdots\cdots E_n)\,. \tag{3.30}$$

Here, the new sheaf $L_{E_k} E_{k+1}$ at position k is given by a *left mutation*. Note that a left mutation of an exceptional pair of sheaves (E, F) produces another exceptional pair $(L_E F, E)$. The precise definition of $L_E F$ can be found in [65]. For our purposes here, we just note the properties $L_{E[1]} F = L_E F$ and $L_E(F[1]) = (L_E F)[1]$ under the translation functor.

The effect of (3.30) on the fractional branes may also be called a *mutation at node* e_I, with $I = n - k > 0$. Given the ordered fractional branes (3.27), a mutation at e_I corresponds to:

$$(\mathcal{E}_1, \cdots, \mathcal{E}_{I-1}, \mathcal{E}_I, \cdots, \mathcal{E}_{n-1}) \quad \rightsquigarrow \quad (\mathcal{E}_1, \cdots, \mathcal{E}_I[-1], L_{\mathcal{E}_I}(\mathcal{E}_{I-1})[1], \cdots, \mathcal{E}_{n-1})\,. \tag{3.31}$$

Here we defined the new fractional brane:

$$L_{\mathcal{E}_I}(\mathcal{E}_{I-1}) = i_* \left(L_{E_k} E_{k+1}[n-k-1]\right)\,, \qquad I = n - k\,, \tag{3.32}$$

by abuse of notation. We conjecture that mutations of a strongly exceptional collection which preserve the strongly exceptional condition realize the field-theory triality operation of section 3.1.[21] This proposal passes some obvious sanity checks. First of all, note that the pair $(\mathcal{E}_{I-1}, \mathcal{E}_I)$ involved in the mutation has:

$$\mathrm{Ext}^{i+1}(\mathcal{E}_I, \mathcal{E}_{I-1}) = \delta_{i,0} \mathbb{C}^{n_a}\,. \tag{3.33}$$

According to our general rules, n_a is the number of incoming arrows at node e_I — in the language of section 3.1, the number of antifundamental chiral multiplets $\widetilde{\Phi}$ under $\mathrm{U}(N_0)$ is $n_a N_A$, and we have:

$$\mathcal{E}_I \xrightarrow{n_a} \mathcal{E}_{I-1} \qquad \cong \qquad e_I \xleftarrow{n_a} e_{I-1}\,. \tag{3.34}$$

The condition that the new collection is strongly exceptional leads to:

$$\mathrm{Ext}^{i+1}(L_{\mathcal{E}_I}(\mathcal{E}_{I-1})[1], \mathcal{E}_I[-1]) = \delta_{i,0} \mathbb{C}^{n_a}\,. \tag{3.35}$$

in the new quiver.[22] This means that we now have *outgoing* arrow from e_I to e_{I-1} in the supersymmetric quiver: $e_I \xrightarrow{n_A} e_{I-1}$. This matches the fact that the antifundamental multiplets are dualized to fundamental multiplets under triality ($\widetilde{\Phi} \rightsquigarrow \Phi'$). We also see that:

$$\mathrm{Ext}^3(\mathcal{E}_I, \mathcal{E}_J) = \mathrm{Ext}^2(\mathcal{E}_I[-1], \mathcal{E}_J)\,, \qquad \mathrm{Ext}^2(\mathcal{E}_I, \mathcal{E}_J) = \mathrm{Ext}^1(\mathcal{E}_I[-1], \mathcal{E}_J)\,. \tag{3.36}$$

[21] While a mutation of an exceptional collection gives another exceptional collection, it is not clear that is also preserves the strongly exceptional condition. This will be the case in our examples. See [8] for further discussions.

[22] More precisely, the dimension n_a in (3.33) and (3.35) might still differ. They must actually match for our conjecture to hold.

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These relations imply that the fundamental chiral multiplets of the original theory are dualized to fermi multiplets ($\Phi \rightsquigarrow \Lambda'$), and the fermi multiplets are dualized to antifundamental chiral multiplets ($\Lambda \rightsquigarrow \widetilde{\Phi}'$). In this way, we elegantly reproduce the simplest aspects of the triality, as summarized in figures 6a and 6b.

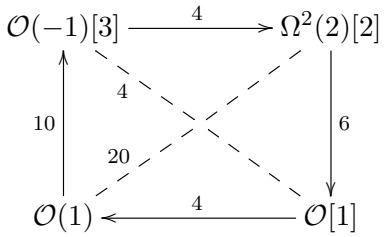
Examples. Consider $B_3 = \mathbb{P}^3$, as discussed above. For definiteness, we start from the strongly exceptional collection:

$$\mathbf{E}_{III} = \left\{\mathcal{O}(-1)\,,\,\Omega^2(2)\,,\,\mathcal{O}\,,\,\mathcal{O}(1)\right\}\,. \tag{3.37}$$

The corresponding fractional branes were discussed in section 2.4.3 — they were dubbed $\{\mathcal{E}_1, \mathcal{E}_0, \mathcal{E}_2, \mathcal{E}_3\}$, where the ordering matters. The intermediate Ext^1 quiver (3.28) reads:

$$\mathcal{O}(-1)[3] \xrightarrow{4} \Omega^2(2)[2] \xrightarrow{6} \mathcal{O}(1)[1] \xrightarrow{4} \mathcal{O}(1)\,, \tag{3.38}$$

and the full $\mathrm{Ext}^{1,2}$ quiver on the local CY_4 reads:



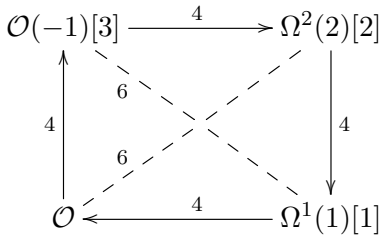
This gives the supersymmetric quiver that we called "Theory (III)" above. Now, consider a mutation at $\mathcal{E}_0$, which is a mutation at the third position in (3.37), at $E_3 = \mathcal{O}$. It is a well-known result that:

$$L_{\mathcal{O}}\mathcal{O}(1) = \Omega^1(1) \tag{3.39}$$

on $\mathbb{P}^3$. Therefore, the new strongly exceptional collection is given:

$$\mathbf{E}_I = \left\{\mathcal{O}(-1)\,,\,\Omega^2(2)\,,\,\Omega^1(1)\,,\,\mathcal{O}\right\}\,. \tag{3.40}$$

The corresponding $\mathrm{Ext}^{1,2}$ quiver on the CY fourfold reads:



corresponding to "Theory (I)" above. (Recall that $\mathcal{O}(-1) \cong \Omega^3(3)$ on $\mathbb{P}^3$.) This matches the field theory expectation: a mutation at $\mathcal{E}_0 = i_*\mathcal{O}[1]$ should be a triality operation on node e_0 of Theory (III). This indeed gives Theory (I), as explained in section 3.2.

A triality operation at node e_0 of Theory (I) gives theory (II). Unfortunately, we cannot directly realize it by mutation, because $\mathcal{E}_0$ corresponds to the last sheaf in the exceptional

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collection (3.40). However, remark that Theory (I) has a $\mathbb{Z}_4$ symmetry that rotates the four nodes of the quiver. Therefore, a triality at any node of Theory (I) gives Theory (II), up to a rotation of the nodes. We can then consider any other mutation of adjacent sheaves in (3.40) to obtain Theory (II). Consider a mutation at position 1. One can show that:

$$L_{\mathcal{O}(-1)}\Omega^2(2) = \mathcal{O}(-2)\,. \tag{3.41}$$

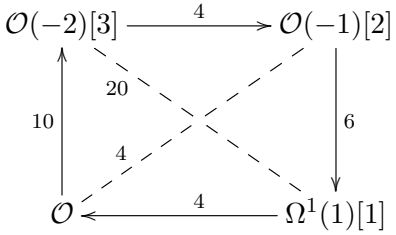
We thus obtain the new strongly exceptional collection on $\mathbb{P}^3$:

$$\mathbf{E}_{II} = \left\{\mathcal{O}(-2)\,,\,\mathcal{O}(-1)\,,\,\Omega^1(1)\,,\,\mathcal{O}\right\}\,. \tag{3.42}$$

The corresponding fractional branes are:

$$\mathcal{E}'_0 = i_*\mathcal{O}(-2)[3]\,, \qquad \mathcal{E}'_1 = i_*\mathcal{O}(-1)[2]\,, \qquad \mathcal{E}'_2 = i_*\Omega^1(1)[1]\,, \qquad \mathcal{E}'_3 = i_*\mathcal{O}\,. \tag{3.43}$$

One can again compute the $\mathrm{Ext}^{1,2}$ quiver. It reads:



Relabelling the fractional branes $(\mathcal{E}'_0, \mathcal{E}'_1, \mathcal{E}'_2, \mathcal{E}'_3) = (\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_0, \mathcal{E}_3)$, we precisely reproduce the "Theory (II)" quiver shown in figure 7b.

4 D-instanton quivers and gauged matrix models

Zero-dimensional "gauge theories" — gauged matrix models (GMM) — naturally arise as the low-energy description of D-instantons in type IIB string theory [66]. In particular, gauged matrix models with $\mathcal{N} = 1$ supersymmetry can describe D-instantons at Calabi-Yau fivefold singularities [49].

Since 0d $\mathcal{N} = 1$ superspace is spanned by a single Grassmanian coordinate θ, any superfield is of the form $X = x + \theta y$, with x, y some variables of opposite Grassmann parity.[23] The single supersymmetry is generated by $\mathcal{Q} = \partial_\theta$. A generic $\mathcal{N} = 1$ GMM can be conveniently described using three elementary supermultiplets. The $\mathcal{N} = 1$ *chiral multiplet* $(\Phi, \bar{\Phi})$ consists of a complex boson $\phi, \bar{\phi}$ and a fermion $\bar{\psi}$. In superspace, we have:

$$\Phi = \phi\,, \qquad \bar{\Phi} = \bar{\phi} + \theta\bar{\psi}\,. \tag{4.1}$$

The chiral multiplet Φ has a single component, with $\mathcal{Q}\phi = 0$, while the anti-chiral multiplet $\bar{\Phi}$ has two components, with $\mathcal{Q}\bar{\phi} = \bar{\psi}$ and $\mathcal{Q}\bar{\psi} = 0$. The bosons ϕ and $\bar{\phi}$ should be considered

[23] In this section, by an abuse of language, we call the Grassmann-even and Grassmann-odd integration variables in the matrix integral the "bosons" and "fermions", respectively.

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as complex conjugate in the matrix integral, while there is a single fermion $\bar{\psi}$. The second type of multiplet is the fermi multiplet Λ, with a single fermionic component λ, such that:

$$\Lambda = \lambda + \theta F_\lambda \,, \qquad\qquad F_\lambda = F_\lambda(\phi) \,. \tag{4.2}$$

Here, the $\mathcal{N} = 1$ superpotential F_λ is an holomorphic function of the bosons ϕ in chiral multiplets. Given the chiral multiplets ϕ_i and fermi multiplets λ_a, one can write the supersymmetric action:

$$S_F = \int d\theta \, \bar{F}^a(\bar{\Phi})\Lambda_a = \bar{F}^a(\bar{\phi})F_a(\phi) + \bar{\psi}^i \frac{\partial \bar{F}^a}{\partial \bar{\phi}^i}\lambda_a \,. \tag{4.3}$$

Another quadratic action in the fermions can be written in terms of an holomorphic potential $H^{ab}(\phi) = -H^{ba}(\phi)$:

$$S_H = H^{ab}(\phi)\lambda_a\lambda_b \,. \tag{4.4}$$

This is supersymmetric provided that $H^{ab}F_b = 0$. The third type of sypersymmetry multiplet is the gaugino multiplet, which implements a gauge constraint on field space. The gaugino multplet $\mathcal{V}$ consists of two components, the fermion χ and the real boson D, with:

$$\mathcal{V} = \chi + \theta D \,. \tag{4.5}$$

Given a theory of chiral and fermi multiplets with some non-trivial Lie group symmetry, we can gauge a subgroup $\mathbf{G}$ (with Lie algebra $\mathfrak{g}$) of that symmetry by introducing an $\mathfrak{g}$-valued gaugino multiplet, with the action:

$$S_{\text{gauge}} = \int d\theta \, \left(\frac{1}{2} D\chi - i\xi\chi + i\bar{\phi}\chi\phi \right) \,, \tag{4.6}$$

with χ acting on ϕ in the appropriate representation, and an overall trace over the gauge group is implicit. Here ξ is a 0d Fayet-Iliopoulos parameter. Integrating out D, we obtain:

$$S_{\text{gauge}} \cong \mu^2 - i\psi\chi\phi \,. \tag{4.7}$$

where $\mu \equiv \bar{\phi}\phi - \xi$ (schematically), which is the moment map (minus the "level" ξ) of the $\mathbf{G}$ action on the bosonic field space.

4.1 $\mathcal{N} = 1$ gauged matrix model from B-branes at a CY_5 singularity

D-instantons at CY_5 singularities engineer precisely such gauged matrix models with gauge group $\prod_I \mathrm{U}(N_I)$. For each node e_I in the 0d $\mathcal{N} = 1$ quiver, we have a $\mathrm{U}(N_I)$ gaugino multiplet. The matter fields are either chiral or fermi multiplets, in adjoint or bifundamental representations. We have thus a quiver with two type of oriented arrows: $e_I \to e_J$ for chiral multiplets X_{IJ}, and $e_I \dashrightarrow e_J$ for fermi multiplets Λ_{IJ}. Finally, we also have the F- and H-type interaction terms. To each fermi multiplet Λ_{IJ}, we associate the element F_{IJ}, a direct sum over oriented paths p from e_I to e_J, of length k:

$$F_{IJ}(X) = \sum_{\text{paths } p} c_p^{IJ} X_{IK_1} X_{K_1K_2} \cdots X_{K_{k-1}J} \,, \tag{4.8}$$

similarly to (2.10), with given coefficients c_p^{IJ}. In addition, to every pair of fermi multiplets Λ_{IJ} and Λ_{KL}, we associate the H-term action $S_{H^{IJ,KL}}$, which is a sum over closed loops $\widetilde{p}$ from e_I back to itself, which includes both Λ_{IJ} and Λ_{KL}, in addition to chiral multiplets X:

$$S_{H^{IJ,KL}} = \sum_{\text{paths } \widetilde{p}} c_{\widetilde{p}}^{IJ,KL} \operatorname{Tr}(\Lambda_{IJ} X_{JM_1} \cdots X_{M_{k-1}K} \Lambda_{KL} X_{LN_1} \cdots X_{N_{k'-1}I}) \,. \tag{4.9}$$

Note that the closed path $\widetilde{p}$ has length $k + k' + 2$, including the two fermions.

This quiver structure naturally arises from open strings between fractional $D(-1)$-branes at a CY_5 singularity, where each node e_I corresponds to a fractional brane $\mathcal{E}_I$. As before, we must have:

$$\operatorname{Ext}^0_{\mathbf{X}_5}(\mathcal{E}_I, \mathcal{E}_J) = \mathbb{C}\delta_{IJ} \,. \tag{4.10}$$

The non-vanishing Ext^0 elements are identified with the gaugino multiplets. The degree-one Ext groups are identified with chiral multiplets:

$$\operatorname{Ext}^1_{\mathbf{X}_5}(\mathcal{E}_J, \mathcal{E}_I) \qquad \Leftrightarrow \qquad e_I \longrightarrow e_J \qquad \Leftrightarrow \qquad X_{IJ} \,, \tag{4.11}$$

in bifundamental (if $I \neq J$) or adjoint (if $I = J$) representations. Similarly, the degree-two Ext groups are identified with the fermi multiplets:

$$\operatorname{Ext}^2_{\mathbf{X}_5}(\mathcal{E}_J, \mathcal{E}_I) \qquad \Leftrightarrow \qquad e_I \dashrightarrow e_J \qquad \Leftrightarrow \qquad \Lambda_{IJ} \,. \tag{4.12}$$

By Serre duality, we also have $\operatorname{Ext}^4_{\mathbf{X}_5}(\mathcal{E}_J, \mathcal{E}_I) \cong \operatorname{Ext}^1_{\mathbf{X}_5}(\mathcal{E}_I, \mathcal{E}_J)$ and $\operatorname{Ext}^3_{\mathbf{X}_5}(\mathcal{E}_J, \mathcal{E}_I) \cong \operatorname{Ext}^2_{\mathbf{X}_5}(\mathcal{E}_I, \mathcal{E}_J)$.

Interaction terms. The F-terms (4.3) and H-terms (4.4) also arise naturally in the B-model. As discussed in section 2.1.2, the Ext-group generators satisfy an A_∞ algebra with multi-products m_k. Consider a fermi multiplet Λ_{IJ} corresponding to $\alpha \in \operatorname{Ext}^2(\mathcal{E}_J, \mathcal{E}_I)$, and let us denote by $\bar{\alpha} \in \operatorname{Ext}^3(\mathcal{E}_I, \mathcal{E}_J)$ the Serre dual generator. For each path p as in (4.8), we have the elements $x \in \operatorname{Ext}^1$ corresponding to the chiral multiplets X. We propose that:

$$c_p^{IJ} = \langle \bar{\alpha}\, x_{IK_1} \cdots x_{K_{k-1}J} \rangle = \gamma\big(m_2(\bar{\alpha},\, m_k(x_{IK_1} \cdots x_{K_{k-1}J}))\big) \,, \tag{4.13}$$

for the F-term coefficients in (4.8). Similarly, consider the fermi multiplets Λ_{IJ} and Λ_{KL} corresponding to $\alpha \in \operatorname{Ext}^2(\mathcal{E}_J, \mathcal{E}_I)$ and $\beta \in \operatorname{Ext}^2(\mathcal{E}_L, \mathcal{E}_K)$, respectively. We propose that the H-term coefficients in (4.9) are given by:

$$\begin{aligned} c_{\widetilde{p}}^{IJ,KL} &= \Big\langle \alpha\, x_{JM_1} \cdots x_{M_{k-1}K}\, \beta\, x_{LN_1} \cdots x_{N_{k'-1}I} \Big\rangle \\ &= \gamma\big(m_2(\alpha,\, m_{\widetilde{k}}(x_{JM_1}, \cdots, x_{M_{k-1}K}, \beta, x_{LN_1}, \cdots, x_{N_{k'-1}I}))\big) \,, \end{aligned} \tag{4.14}$$

with $\widetilde{k} = k + k' + 1$. We will check this prescription in some examples below. Note that this corresponds exactly to computing the formal 0d $\mathcal{N} = 1$ superpotential:

$$\mathcal{W} = \operatorname{Tr}\Big(\bar{\Lambda}^a F_a(X) + \Lambda_a \Lambda_b H^{ab}(X)\Big) \,, \tag{4.15}$$

which was recently introduced in [22].

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4.2 $D(-1)$-brane on $\mathbb{C}^5$

We can work out the very simplest case, a $D(-1)$ brane on $\mathbf{X}_5 = \mathbb{C}^5$, exactly like in section 2.2. Consider the skyscraper sheaf $\mathcal{O}_p$ at the origin of $\mathbb{C}^5$. We have:

$$\begin{aligned} &\mathrm{Ext}^0(\mathcal{O}_p,\mathcal{O}_p) \cong \mathrm{Ext}^5(\mathcal{O}_p,\mathcal{O}_p) \cong \mathbb{C}\,, \\ &\mathrm{Ext}^1(\mathcal{O}_p,\mathcal{O}_p) \cong \mathrm{Ext}^4(\mathcal{O}_p,\mathcal{O}_p) \cong \mathbb{C}^5\,, \\ &\mathrm{Ext}^2(\mathcal{O}_p,\mathcal{O}_p) \cong \mathrm{Ext}^3(\mathcal{O}_p,\mathcal{O}_p) \cong \mathbb{C}^{10}\,. \end{aligned} \tag{4.16}$$

Using the above dictionary to $\mathcal{N}=1$ superfields, this reproduces the expected field content of the maximally-supersymmetric $\mathcal{N}=16$ matrix model, as we will review below.

4.2.1 The Ext algebra of $\mathbb{C}^5$

Proceeding as before, the Koszul resolution of $\mathcal{O}_p$ on $\mathbb{C}^5$ reads:

$$0 \to \mathcal{O} \xrightarrow{E} \mathcal{O}^5 \xrightarrow{D} \mathcal{O}^{10} \xrightarrow{C} \mathcal{O}^{10} \xrightarrow{B} \mathcal{O}^5 \xrightarrow{A} \mathcal{O} \to \mathcal{O}_p \to 0, \tag{4.17}$$

where:

$$A = \begin{pmatrix} x & y & z & w & u \end{pmatrix}, \qquad B = \begin{pmatrix} y & z & w & u & 0 & 0 & 0 & 0 & 0 & 0 \\ -x & 0 & 0 & 0 & z & w & u & 0 & 0 & 0 \\ 0 & -x & 0 & 0 & -y & 0 & 0 & w & u & 0 \\ 0 & 0 & -x & 0 & 0 & -y & 0 & -z & 0 & u \\ 0 & 0 & 0 & -x & 0 & 0 & -y & 0 & -z & -w \end{pmatrix},$$

$$C = \begin{pmatrix} z & w & u & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -y & 0 & 0 & w & u & 0 & 0 & 0 & 0 & 0 \\ 0 & -y & 0 & -z & 0 & u & 0 & 0 & 0 & 0 \\ 0 & 0 & -y & 0 & -z & -w & 0 & 0 & 0 & 0 \\ x & 0 & 0 & 0 & 0 & 0 & w & u & 0 & 0 \\ 0 & x & 0 & 0 & 0 & 0 & -z & 0 & u & 0 \\ 0 & 0 & x & 0 & 0 & 0 & 0 & -z & -w & 0 \\ 0 & 0 & 0 & x & 0 & 0 & y & 0 & 0 & u \\ 0 & 0 & 0 & 0 & x & 0 & 0 & y & 0 & -w \\ 0 & 0 & 0 & 0 & 0 & x & 0 & 0 & y & z \end{pmatrix}, \; D = \begin{pmatrix} w & u & 0 & 0 & 0 \\ -z & 0 & u & 0 & 0 \\ 0 & -z & -w & 0 & 0 \\ y & 0 & 0 & u & 0 \\ 0 & y & 0 & -w & 0 \\ 0 & 0 & y & z & 0 \\ -x & 0 & 0 & 0 & u \\ 0 & -x & 0 & 0 & -w \\ 0 & 0 & -x & 0 & z \\ 0 & 0 & 0 & -x & -y \end{pmatrix}, \; E = \begin{pmatrix} u \\ -w \\ z \\ -y \\ x \end{pmatrix}.$$

Similarly to section 2.2, we choose as bases of the Ext groups the commutative diagrams whose leftmost nonzero vertical map has 1 at an entry and 0 elsewhere. We denote them by X_j^i, following the same conventions. The multiplication rule is again determined by composition. The products $m_2(X_i^1, X_j^1) = X_i^1 \cdot X_j^1$ are given by:

	X_1^1	X_2^1	X_3^1	X_4^1	X_5^1
X_1^1	0	$-X_1^2$	$-X_2^2$	$-X_4^2$	$-X_7^2$
X_2^1	X_1^2	0	$-X_3^2$	$-X_5^2$	$-X_8^2$
X_3^1	X_2^2	X_3^2	0	$-X_6^2$	$-X_9^2$
X_4^1	X_4^2	X_5^2	X_6^2	0	$-X_{10}^2$
X_5^1	X_7^2	X_8^2	X_9^2	X_{10}^2	0

(4.18)

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The elements in this table are the products of the elements in the first column multiplied by elements in the first row. (For example, $X_1^1 \cdot X_2^1 = -X_1^2$.) Similarly, we have non-zero products $m_2(X_i^1, X_j^2)$ mapping $\mathrm{Ext}^1 \otimes \mathrm{Ext}^2$ to Ext^3, according to:

	X_1^2	X_2^2	X_3^2	X_4^2	X_5^2	X_6^2	X_7^2	X_8^2	X_9^2	X_{10}^2
X_1^1	0	0	X_1^3	0	X_2^3	X_3^3	0	X_5^3	X_6^3	X_8^3
X_2^1	0	$-X_1^3$	0	$-X_2^3$	0	X_4^3	$-X_5^3$	0	X_7^3	X_9^3
X_3^1	X_1^3	0	0	$-X_3^3$	$-X_4^3$	0	$-X_6^3$	$-X_7^3$	0	X_{10}^3
X_4^1	X_2^3	X_3^3	X_4^3	0	0	0	$-X_8^3$	$-X_9^3$	$-X_{10}^3$	0
X_5^1	X_5^3	X_6^3	X_7^3	X_8^3	X_9^3	X_{10}^3	0	0	0	0

(4.19)

We also find the following Serre dual elements to X_i^2:

	X_1^2	X_2^2	X_3^2	X_4^2	X_5^2	X_6^2	X_7^2	X_8^2	X_9^2	X_{10}^2
dual:	X_{10}^3	$-X_9^3$	X_8^3	X_7^3	$-X_6^3$	X_5^3	$-X_4^3$	X_3^3	$-X_2^3$	X_1^3

(4.20)

Using the multiplication rule (and the cyclic property of the open-string correlators), we see that any nonzero correlation function can be computed in terms the following $\langle X^3 X^1 X^1 \rangle$-type correlators:

$$\begin{aligned}
&\langle X_1^3 X_5^1 X_4^1 \rangle = 1\,, && \langle X_2^3 X_5^1 X_3^1 \rangle = -1\,, \quad , && \langle X_3^3 X_5^1 X_2^1 \rangle = 1\,,\\
&\langle X_4^3 X_5^1 X_1^1 \rangle = -1\,, && \langle X_5^3 X_4^1 X_3^1 \rangle = 1\,, && \langle X_6^3 X_4^1 X_2^1 \rangle = -1\,,\\
&\langle X_7^3 X_4^1 X_1^1 \rangle = 1\,, && \langle X_8^3 X_3^1 X_2^1 \rangle = 1\,, && \langle X_9^3 X_3^1 X_1^1 \rangle = -1\,,\\
&\langle X_{10}^3 X_2^1 X_1^1 \rangle = 1\,, &&&&
\end{aligned} \tag{4.21}$$

and the following $\langle X^2 X^2 X^1 \rangle$-type correlators:

$$\begin{aligned}
&\langle X_3^2 X_{10}^2 X_1^1 \rangle = 1\,, && \langle X_5^2 X_9^2 X_1^1 \rangle = -1\,, && \langle X_6^2 X_8^2 X_1^1 \rangle = 1\,,\\
&\langle X_2^2 X_{10}^2 X_2^1 \rangle = -1\,, && \langle X_4^2 X_9^2 X_2^1 \rangle = 1\,, && \langle X_7^2 X_6^2 X_2^1 \rangle = -1\,,\\
&\langle X_1^2 X_{10}^2 X_3^1 \rangle = 1\,, && \langle X_4^2 X_8^2 X_3^1 \rangle = -1\,, && \langle X_5^2 X_7^2 X_3^1 \rangle = 1\,,\\
&\langle X_1^2 X_9^2 X_4^1 \rangle = -1\,, && \langle X_2^2 X_8^2 X_4^1 \rangle = 1\,, && \langle X_3^2 X_7^2 X_4^1 \rangle = -1\,,\\
&\langle X_1^2 X_6^2 X_5^1 \rangle = 1\,, && \langle X_2^2 X_5^2 X_5^1 \rangle = -1\,, && \langle X_3^2 X_4^2 X_5^1 \rangle = 1.
\end{aligned} \tag{4.22}$$

4.2.2 The $\mathbb{C}^5$ quiver: $\mathcal{N} = 16$ SYM

Consider the $\mathcal{N} = 16$ supersymmetric GMM with gauge group $\mathrm{U}(N)$, corresponding to N $D(-1)$-branes in flat space. Its field content can be deduced from dimensional reduction of 2d $\mathcal{N} = (8,8)$ SYM in section 2.2.3. In $\mathcal{N} = 1$ language, we have a single $\mathrm{U}(N)$ gaugino multiplet, 5 chiral multiplets in the adjoint representation, and 10 fermi multiplets in the adjoint representation. It is convenient to denote the chiral and fermi multiplets by Φ_n and $\Lambda_{mn} = -\Lambda_{nm}$, with $n = 1, \cdots 5$, since Φ_n and Λ_{nm} transform in the **5** and **10** of an SU(5)

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flavor symmetry. This spectrum is reproduced by the Ext groups above. We identify the fields with the Ext elements according to $X_n^1 = \phi_n$, $n = 1, \cdots, 5$, and:

$$\begin{array}{lllll} X_1^2 = \lambda_{21}\,, & X_2^2 = \lambda_{31}\,, & X_3^2 = \lambda_{32}\,, & X_4^2 = \lambda_{41}\,, & X_5^2 = \lambda_{42}\,, \\ X_6^2 = \lambda_{43}\,, & X_7^2 = \lambda_{51}\,, & X_8^2 = \lambda_{52}\,, & X_9^2 = \lambda_{53}\,, & X_{10}^2 = \lambda_{54}\,. \end{array} \tag{4.23}$$

The interaction terms are determined by the F- and H-terms [49]:

$$F_{mn} = \phi_m\phi_n - \phi_n\phi_m\,, \qquad H^{mn,pq} = \epsilon^{mnpqr}\phi_r\,. \tag{4.24}$$

One can check that the open-string correlators (4.21)–(4.22) precisely reproduce these interactions. Note that, to check that the H-term:

$$S_H = \frac{1}{4}\epsilon^{mnpqr}\,\mathrm{Tr}(\lambda_{mn}\lambda_{pq}\phi_r) \tag{4.25}$$

is supersymmetric, we need to use the Jacobi identity for U(N). This is equivalent to the non-trivial condition $H^{ab}F_b = 0$ mentioned above, which must always be realized by the B-brane correlators.

4.3 Orbifolds $\mathbb{C}^5/\Gamma$

Given the above results for $\mathbb{C}^5$, we can easily study various $\mathcal{N} = 1$-preserving orbifolds $\mathbb{C}^5/\Gamma$, where Γ is any discrete subgroup of SU(5).

4.3.1 $\mathbb{C}^5/\mathbb{Z}_5(1,1,1,1,1)$

Consider for instance $\mathbb{C}^5/\mathbb{Z}_5$, where $\mathbb{Z}_5$ acts as:

$$(x, y, z, w, t) \mapsto (\omega x, \omega y, \omega z, \omega w, \omega t)\,, \qquad \omega = e^{\frac{2\pi i}{5}} \tag{4.26}$$

on the $\mathbb{C}^5$ coordinates. We have five fractional branes denoted by $\mathcal{E}_i$, $i = 0, \cdots, 4$. The weights for the sheaves in the Koszul resolution of $\mathcal{E}_i$ are given by:

$$\begin{pmatrix} i \end{pmatrix} \xrightarrow{E} \begin{pmatrix} i+1 \\ i+1 \\ i+1 \\ i+1 \\ i+1 \end{pmatrix} \xrightarrow{D} \begin{pmatrix} i+2 \\ i+2 \\ i+2 \\ i+2 \\ i+2 \\ i+2 \\ i+2 \\ i+2 \\ i+2 \\ i+2 \end{pmatrix} \xrightarrow{C} \begin{pmatrix} i+3 \\ i+3 \\ i+3 \\ i+3 \\ i+3 \\ i+3 \\ i+3 \\ i+3 \\ i+3 \\ i+3 \end{pmatrix} \xrightarrow{B} \begin{pmatrix} i+4 \\ i+4 \\ i+4 \\ i+4 \\ i+4 \end{pmatrix} \xrightarrow{A} \begin{pmatrix} i \end{pmatrix}.$$

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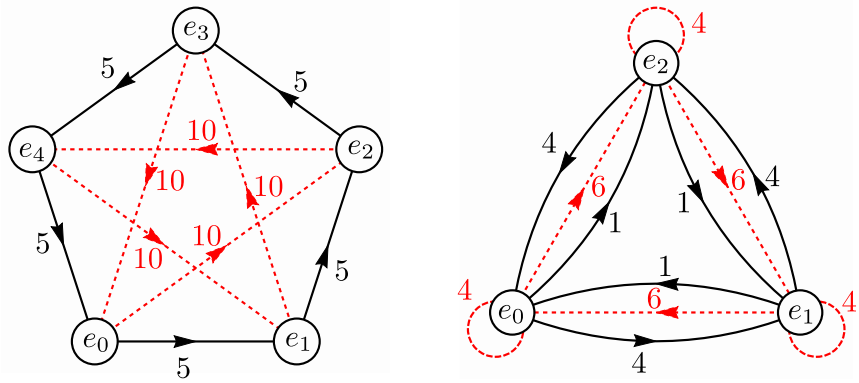

(a) $\mathbb{C}^5/\mathbb{Z}_5(1,1,1,1,1)$ quiver. **(b)** $\mathbb{C}^5/\mathbb{Z}_3(1,1,1,1,2)$ quiver.

Figure 10. Examples of 0d $\mathcal{N}=1$ quivers for orbifold singularities $\mathbb{C}^5/\Gamma$. The numbers on the chiral and fermi multiplet arrows indicate their multiplicities.

We then find the spectrum:

$$\mathrm{Ext}^0_{[\mathbb{C}^5/\mathbb{Z}_5]}(\mathcal{E}_i,\mathcal{E}_j)=\begin{cases}\mathrm{Span}_{\mathbb{C}}\{X_1^0\} & \text{if } j\equiv i \bmod 5\,,\\ 0 & \text{otherwise,}\end{cases}$$

$$\mathrm{Ext}^1_{[\mathbb{C}^5/\mathbb{Z}_5]}(\mathcal{E}_i,\mathcal{E}_j)=\begin{cases}\mathrm{Span}_{\mathbb{C}}\{X_1^1,X_2^1,X_3^1,X_4^1,X_5^1\} & \text{if } j+1\equiv i \bmod 5\,,\\ 0 & \text{otherwise,}\end{cases}$$

$$\mathrm{Ext}^2_{[\mathbb{C}^5/\mathbb{Z}_5]}(\mathcal{E}_i,\mathcal{E}_j)=\begin{cases}\mathrm{Span}_{\mathbb{C}}\{X_1^2,X_2^2,X_3^2,X_4^2,X_5^2,X_6^2,X_7^2,X_8^2,X_9^2,X_{10}^2\} & \text{if } j+2\equiv i \bmod 5\,,\\ 0 & \text{otherwise.}\end{cases}$$

The higher Ext groups are obtained by Serre duality. The correlation functions can be read off from (4.21)–(4.22). Let us introduce the chiral multiplets:

$$\Phi_I^n : e_I \longrightarrow e_{I+1}\,, \qquad \Lambda_I^{mn} : e_I \dashrightarrow e_{I+2}\,, \tag{4.27}$$

with I an integer mod 5, $m,n=1,\cdots,5$, and $\Lambda_I^{mn}=-\Lambda_I^{nm}$. The gauged matrix model quiver is shown in figure 10a. The interaction terms are:

$$F_{\Lambda_I^{mn}}=\Phi_I^m\Phi_{I+1}^n-\Phi_I^n\Phi_{I+1}^m\,, \qquad H^{\Lambda_I^{mn},\Lambda_{I+2}^{pq}}=\epsilon_{mnpqr}\Phi_{I-1}^r\,. \tag{4.28}$$

Note the obvious SU(5) flavor symmetry. This quiver was discussed in [49, 67, 68].

4.3.2 $\mathbb{C}^5/\mathbb{Z}_3(11112)$

As a last example, consider the $\mathbb{C}^5/\mathbb{Z}_3$ orbifold:

$$(x,y,z,w,t)\mapsto(\omega x,\omega y,\omega z,\omega w,\omega^2 t)\,, \qquad \omega=e^{\frac{2\pi i}{3}}\,. \tag{4.29}$$

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We have three fractional branes $\mathcal{E}_i$, $i = 0, 1, 2$. The weights for the sheaves in the Koszul resolution of $\mathcal{E}_i$ are:

$$\left(\begin{array}{c} i \end{array}\right) \xrightarrow{E} \begin{pmatrix} i+2 \\ i+1 \\ i+1 \\ i+1 \\ i+1 \end{pmatrix} \xrightarrow{D} \begin{pmatrix} i \\ i \\ i+2 \\ i \\ i+2 \\ i+2 \\ i \\ i+2 \\ i+2 \\ i+2 \end{pmatrix} \xrightarrow{C} \begin{pmatrix} i+1 \\ i+1 \\ i+1 \\ i \\ i+1 \\ i+1 \\ i \\ i+1 \\ i \\ i \end{pmatrix} \xrightarrow{B} \begin{pmatrix} i+2 \\ i+2 \\ i+2 \\ i+2 \\ i+1 \end{pmatrix} \xrightarrow{A} \left(\begin{array}{c} i \end{array}\right).$$

The spectrum consists of:

$$\begin{aligned}
\mathrm{Ext}^0_{[\mathbb{C}^5/\mathbb{Z}_3]}(\mathcal{E}_i, \mathcal{E}_j) &= \begin{cases} \mathrm{Span}_{\mathbb{C}}\{X_1^0\} & \text{if } j \equiv i \bmod 3\,, \\ 0 & \text{otherwise,} \end{cases} \\
\mathrm{Ext}^1_{[\mathbb{C}^5/\mathbb{Z}_3]}(\mathcal{E}_i, \mathcal{E}_j) &= \begin{cases} 0 & \text{if } j \equiv i \bmod 3\,, \\ \mathrm{Span}_{\mathbb{C}}\{X_2^1, X_3^1, X_4^1, X_5^1\} & \text{if } j+1 \equiv i \bmod 3\,, \\ \mathrm{Span}_{\mathbb{C}}\{X_1^1\} & \text{if } j+2 \equiv i \bmod 3\,, \end{cases} \\
\mathrm{Ext}^2_{[\mathbb{C}^5/\mathbb{Z}_3]}(\mathcal{E}_i, \mathcal{E}_j) &= \begin{cases} \mathrm{Span}_{\mathbb{C}}\{X_1^2, X_2^2, X_4^2, X_7^2\} & \text{if } j \equiv i \bmod 3\,, \\ & \text{if } j+1 \equiv i \bmod 3\,, \\ \mathrm{Span}_{\mathbb{C}}\{X_3^2, X_5^2, X_6^2, X_8^2, X_9^2, X_{10}^2\} & \text{if } j+2 \equiv i \bmod 3\,. \end{cases}
\end{aligned} \tag{4.30}$$

The corresponding 0d $\mathcal{N} = 1$ quiver is shown in figure 10b. The correlation functions can be read off from (4.21)–(4.22). Taking advantage of the residual SU(4) flavor symmetry, let us introduce the chiral multiplets:

$$A_I : e_I \xrightarrow{X_1^1} e_{I-1}\,, \qquad B_I^a : e_I \xrightarrow{X_1^{a+1}} e_{I+1}\,, \tag{4.31}$$

with $a = 1, \cdots 4$, and I an integer mod 3. Similarly, we define the fermi multiplets:

$$\Lambda_I^{ab} : e_I \xrightarrow{\lambda_{a+1,b+1}} e_{I-1}\,, \qquad \Xi_I^a : e_I \xrightarrow{\lambda_{a+1,1}} e_I\,, \tag{4.32}$$

where the Ext^1 elements $\lambda_{mn} = -\lambda_{nm}$ are defined as in (4.23). In this notation, the interaction terms read:

$$\begin{aligned}
F_{\Lambda_I^{ab}} &= B_I^a B_{I+1}^b - B_I^b B_{I+1}^a\,, & F_{\Xi_I^a} &= -A_I B_{I-1}^a\,, \\
H_{\Xi_I^a, \Lambda_I^{bc}} &= -\epsilon_{abcd} B_{I-1}^d\,, & H_{\Lambda_I^{ab}, \Xi_{I-1}^c} &= \epsilon_{abcd} B_{I-1}^d \\
H_{\Lambda_I^{ab}, \Lambda_{I-1}^{cd}} &= \epsilon_{abcd} A_{I+1}\,.
\end{aligned} \tag{4.33}$$

Many more $\mathcal{N} = 1$ matrix models can be worked out in this way. It would also be instructive to study fractional branes on local Fano fourfold varieties, such as the resolution of the $\mathbb{C}^5/\mathbb{Z}_5(1,1,1,1,1)$ to $\mathrm{Tot}(\mathcal{O}(-5) \to \mathbb{P}^4)$. We leave this and many other related questions for future work.

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A Dimensional reductions

Fourfolds versus threefolds. Let X be a Calabi-Yau orbifold $[X_c/G]$ of complex dimension 3, with a set of fractional branes $\{\mathcal{E}_i\}$ supported at a point $p \in X_c$, a fixed point of the G-action. Let N_3 denote the normal bundle $N_{p/X}$.

Let us build another Calabi-Yau orbifold $Y = \mathbb{C} \times X$, which again has an isomorphic set of fractional branes $\{\mathcal{E}_i\}$, supported at $x \equiv \{0\} \times p \in \mathbb{C} \times X$, of codimension four. Let N denote the normal bundle to x in Y, and ρ_0 the structure sheaf with trivial G-equivariant structure. Then,

$$\begin{aligned} N &= \rho_0 \oplus N_3\,, \\ \wedge^2 N &= \rho_0 \otimes N_3 \oplus \wedge^2 N_3 \;=\; N_3 \oplus \wedge^2 N_3\,, \\ \wedge^3 N &= \rho_0 \otimes \wedge^2 N_3 \oplus \wedge^3 N_3 \;=\; \wedge^2 N_3 \oplus \wedge^3 N_3\,, \\ \wedge^4 N &= \rho_0 \otimes \wedge^3 N_3 \;=\; \wedge^3 N_3\,. \end{aligned}$$

We then have:

$$\begin{aligned} \mathrm{Ext}^0_Y(\mathcal{E}_i,\mathcal{E}_j) &= H^0(x,\rho_i^*\otimes\rho_j)^G \\ &= \mathrm{Ext}^0_X(\mathcal{E}_i,\mathcal{E}_j)\,, \\ \mathrm{Ext}^1_Y(\mathcal{E}_i,\mathcal{E}_j) &= H^0(x,\rho_i^*\otimes\rho_j\otimes N)^G \;=\; H^0(x,\rho_i^*\otimes\rho_j\otimes(\rho_0\oplus N_3))^G \\ &= \mathrm{Ext}^0_X(\mathcal{E}_i,\mathcal{E}_j) \;\oplus\; \mathrm{Ext}^1_X(\mathcal{E}_i,\mathcal{E}_j)\,, \\ \mathrm{Ext}^2_Y(\mathcal{E}_i,\mathcal{E}_j) &= H^0(x,\rho_i^*\otimes\rho_j\otimes\wedge^2 N)^G \;=\; H^0(x,\rho_i^*\otimes\rho_j\otimes(N_3\oplus\wedge^2 N_3))^G \\ &= \mathrm{Ext}^1_X(\mathcal{E}_i,\mathcal{E}_j) \;\oplus\; \mathrm{Ext}^2_X(\mathcal{E}_i,\mathcal{E}_j)\,, \\ \mathrm{Ext}^3_Y(\mathcal{E}_i,\mathcal{E}_j) &= H^0(x,\rho_i^*\otimes\rho_j\otimes\wedge^3 N)^G \;=\; H^0(x,\rho_i^*\otimes\rho_j\otimes(\wedge^2 N_3\oplus\wedge^3 N_3))^G \\ &= \mathrm{Ext}^2_X(\mathcal{E}_i,\mathcal{E}_j) \;\oplus\; \mathrm{Ext}^3_X(\mathcal{E}_i,\mathcal{E}_j)\,, \\ \mathrm{Ext}^4_Y(\mathcal{E}_i,\mathcal{E}_j) &= H^0(x,\rho_i^*\otimes\rho_j\otimes\wedge^4 N)^G \;=\; H^0(x,\rho_i^*\otimes\rho_j\otimes\wedge^3 N_3)^G \\ &= \mathrm{Ext}^3_X(\mathcal{E}_i,\mathcal{E}_j)\,. \end{aligned}$$

This directly confirms (2.20) in the case of an orbifold singularity. We conjecture that it holds more generally.

Fourfolds versus twofolds. Similarly, we may consider X a Calabi-Yau orbifold $[X_c/G]$ of complex dimension 2, with a set of fractional branes $\{\mathcal{E}_i\}$ supported at a point $p \in X_c$, a fixed point of the G-action. Let N_X denote the normal bundle $N_{p/X}$. Let us build another Calabi-Yau orbifold $Y = \mathbb{C}^2 \times X$, which again has an isomorphic set of fractional branes

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$\{\mathcal{E}_i\}$, supported at $x \equiv \{(0,0)\} \times p \in \mathbb{C}^2 \times X$, of codimension four. Let N denote the normal bundle to x in Y, and ρ_0 the structure sheaf with trivial G-equivariant structure. Then,

$$\begin{aligned} N &= \rho_0^2 \oplus N_X \,, \\ \wedge^2 N &= \rho_0 \otimes \rho_0 \oplus (\rho_0 \otimes N_X)^{\oplus 2} \oplus \wedge^2 N_X \;=\; \rho_0 \oplus N_X^{\oplus 2} \oplus \wedge^2 N_X, \\ \wedge^3 N &= \rho_0 \otimes \rho_0 \otimes N_X \oplus (\rho_0 \otimes \wedge^2 N_X)^{\oplus 2} \;=\; N_X \oplus (\wedge^2 N_X)^{\oplus 2} \,, \\ \wedge^4 N &= \rho_0 \otimes \rho_0 \otimes \wedge^2 N_X \;=\; \wedge^2 N_X \,. \end{aligned}$$

We then find:

$$\begin{aligned} \mathrm{Ext}^0_Y(\mathcal{E}_i, \mathcal{E}_j) &= H^0(x, \rho_i^* \otimes \rho_j)^G \\ &= \mathrm{Ext}^0_X(\mathcal{E}_i, \mathcal{E}_j) \,, \\ \mathrm{Ext}^1_Y(\mathcal{E}_i, \mathcal{E}_j) &= H^0(x, \rho_i^* \otimes \rho_j \otimes N)^G \;=\; H^0(x, \rho_i^* \otimes \rho_j \otimes (\rho_0^2 \oplus N_X))^G \\ &= \mathrm{Ext}^0_X(\mathcal{E}_i, \mathcal{E}_j) \oplus \mathrm{Ext}^0_X(\mathcal{E}_i, \mathcal{E}_j) \oplus \mathrm{Ext}^1_X(\mathcal{E}_i, \mathcal{E}_j) \,, \\ \mathrm{Ext}^2_Y(\mathcal{E}_i, \mathcal{E}_j) &= H^0(x, \rho_i^* \otimes \rho_j \otimes \wedge^2 N)^G \;=\; H^0(x, \rho_i^* \otimes \rho_j \otimes (\rho_0 \oplus (N_X)^{\oplus 2} \oplus \wedge^2 N_X))^G \\ &= \mathrm{Ext}^0_X(\mathcal{E}_i, \mathcal{E}_j) \oplus \mathrm{Ext}^1_X(\mathcal{E}_i, \mathcal{E}_j) \oplus \mathrm{Ext}^1_X(\mathcal{E}_i, \mathcal{E}_j) \oplus \mathrm{Ext}^2_X(\mathcal{E}_i, \mathcal{E}_j) \,, \\ \mathrm{Ext}^3_Y(\mathcal{E}_i, \mathcal{E}_j) &= H^0(x, \rho_i^* \otimes \rho_j \otimes \wedge^3 N)^G \;=\; H^0(x, \rho_i^* \otimes \rho_j \otimes (N_X \oplus (\wedge^2 N_X)^{\oplus 2}))^G \\ &= \mathrm{Ext}^1_X(\mathcal{E}_i, \mathcal{E}_j) \oplus \mathrm{Ext}^2_X(\mathcal{E}_i, \mathcal{E}_j) \oplus \mathrm{Ext}^2_X(\mathcal{E}_i, \mathcal{E}_j) \,, \\ \mathrm{Ext}^4_Y(\mathcal{E}_i, \mathcal{E}_j) &= H^0(x, \rho_i^* \otimes \rho_j \otimes \wedge^4 N)^G \;=\; H^0(x, \rho_i^* \otimes \rho_j \otimes \wedge^2 N_X)^G \\ &= \mathrm{Ext}^2_X(\mathcal{E}_i, \mathcal{E}_j) \,. \end{aligned}$$

This decomposition corresponds to the dimensional reduction of a 6d $\mathcal{N} = 1$ quiver theory (or, equivalently, of a 4d $\mathcal{N} = 2$ theory) to 2d, giving rise to an $\mathcal{N} = (4,4)$ quiver theory. Each $\mathcal{N} = (4,4)$ vector multiplet splits into one $\mathcal{N} = (2,2)$ vector multiplet, two chiral multiplets and one fermi multiplet. Each $\mathcal{N} = (4,4)$ hypermultiplet splits into two chiral and two fermi multiplets. This is precisely the decomposition seen here.

B Fractional D3-branes on a local $\mathbb{P}^2$

Consider the well-known case of fractional D3-branes on the Calabi-Yau threefold:

$$\widetilde{\mathbf{X}}_3 = \mathrm{Tot}(\mathcal{O}(-3) \to \mathbb{P}^2) \,, \tag{B.1}$$

which is a crepant resolution of the orbifold singularity $\mathbf{X}_3 = \mathbb{C}^3/\mathbb{Z}_3$. The corresponding 4d $\mathcal{N} = 1$ quiver gauge theory is very well studied — see e.g. [2, 3, 58, 69]. In this appendix, we review this 4d $\mathcal{N} = 1$ quiver using the B-brane language. This will help to illustrate, in a more familiar context, the tools that we similarly use to study D1-brane quivers.

B.1 Fractional branes and supersymmetric quivers

Let us discuss two particular sets of fractional branes. Below, we will see how they are related by mutation of exceptional collections, providing a geometric realization of Seiberg duality [58].

B.1.1 A first set of fractional branes: theory (I)

Fractional branes on the resolution (B.1) can be constructed from the data of a strongly exceptional collection on $\mathbb{P}^2$, as in section 3.3. Let us first consider the exceptional collection:

$$\mathbf{E}_I = \{\Omega^2(2)\,,\ \Omega^1(1)\,,\ \mathcal{O}\}\,. \tag{B.2}$$

The corresponding three fractional branes on $\widetilde{\mathbf{X}}_3$ are:

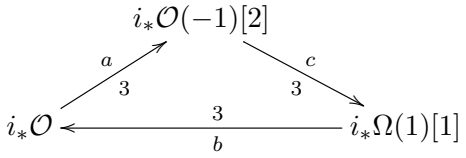
$$\mathcal{E}_0 = i_*\mathcal{O}\,, \qquad \mathcal{E}_1 = i_*\Omega^1(1)[1]\,, \qquad \mathcal{E}_2 = i_*\Omega^2(2)[2]\,, \tag{B.3}$$

where i is the inclusion from $\mathbb{P}^2$ into $\widetilde{\mathbf{X}}_3$.

Let z_0, z_1, z_2 be the homogeneous coordinates of $\mathbb{P}^2$ and U_i be the open set in which $z_i \neq 0$. Denote the local coordinates in U_i by (x_i, y_i) and the coordinate of the fiber of $\mathcal{O}(-3)$ in U_i by w_i. We have $w_1 = x_0^3 w_0, w_2 = y_0^3 w_0 = y_1^3 w_1$. In the following we will take Koszul resolutions:

$$\begin{aligned} 0 \to \mathcal{O}(k+3) &\xrightarrow{w_0} \mathcal{O}(k) \to i_*\mathcal{O}_{\mathbb{P}^2}(k) \to 0 \\ 0 \to \Omega(k+3) &\xrightarrow{w_0} \Omega(k) \to i_*\Omega_{\mathbb{P}^2}(k) \to 0\,. \end{aligned}$$

It is straightforward to compute the Ext groups themselves. The Ext^1 quiver reads:



A basis of the Ext groups can be chosen as follows:

$\mathrm{Ext}^1(i_*\mathcal{O}, i_*\mathcal{O}(-1)[2])$ is generated by $a_i \in \check{C}^2(\mathrm{Hom}^{-1}(i_*\mathcal{O}, i_*\mathcal{O}(-1)[2]))$:

$$\begin{array}{ccc} \mathcal{O}(3) & \longrightarrow & \mathcal{O} \\ {\scriptstyle a_i}\big\downarrow & & \\ \mathcal{O}(2) & \longrightarrow & \mathcal{O}(-1) \end{array}$$

$$a_1 = \frac{1}{x_0 y_0}\,, \qquad a_2 = \frac{1}{x_0^2 y_0}\,, \qquad a_3 = \frac{1}{x_0 y_0^2}\,.$$

$\mathrm{Ext}^1(i_*\Omega(1)[1], i_*\mathcal{O})$ is generated by $b_i \in \check{C}^0(\mathrm{Hom}^1(i_*\Omega(1)[1], i_*\mathcal{O}))$:

$$b_1: \begin{array}{ccc} \Omega(4) & \longrightarrow & \Omega(1) \\ {\scriptstyle (x_0,y_0)}\big\downarrow & & {\scriptstyle (-x_0,-y_0)}\big\downarrow \\ \mathcal{O}(3) & \longrightarrow & \mathcal{O} \end{array} \qquad b_2: \begin{array}{ccc} \Omega(4) & \longrightarrow & \Omega(1) \\ {\scriptstyle (-1,0)}\big\downarrow & & {\scriptstyle (1,0)}\big\downarrow \\ \mathcal{O}(3) & \longrightarrow & \mathcal{O} \end{array} \qquad b_3: \begin{array}{ccc} \Omega(4) & \longrightarrow & \Omega(1) \\ {\scriptstyle (0,-1)}\big\downarrow & & {\scriptstyle (0,1)}\big\downarrow \\ \mathcal{O}(3) & \longrightarrow & \mathcal{O} \end{array}$$

$\mathrm{Ext}^1(i_*\mathcal{O}(-1)[2], i_*\Omega(1)[1])$ is generated by $c_i \in \check{C}^0(\mathrm{Hom}^1(i_*\mathcal{O}(-1)[2], i_*\Omega(1)[1]))$:

$$c_1: \begin{array}{ccc} \mathcal{O}(2) & \longrightarrow & \mathcal{O}(-1) \\ {\scriptstyle \binom{y_0}{-x_0}}\big\downarrow & & {\scriptstyle \binom{-y_0}{x_0}}\big\downarrow \\ \Omega(4) & \longrightarrow & \Omega(1) \end{array} \qquad c_2: \begin{array}{ccc} \mathcal{O}(2) & \longrightarrow & \mathcal{O}(-1) \\ {\scriptstyle \binom{0}{1}}\big\downarrow & & {\scriptstyle \binom{0}{-1}}\big\downarrow \\ \Omega(4) & \longrightarrow & \Omega(1) \end{array} \qquad c_3: \begin{array}{ccc} \mathcal{O}(2) & \longrightarrow & \mathcal{O}(-1) \\ {\scriptstyle \binom{-1}{0}}\big\downarrow & & {\scriptstyle \binom{1}{0}}\big\downarrow \\ \Omega(4) & \longrightarrow & \Omega(1) \end{array}$$

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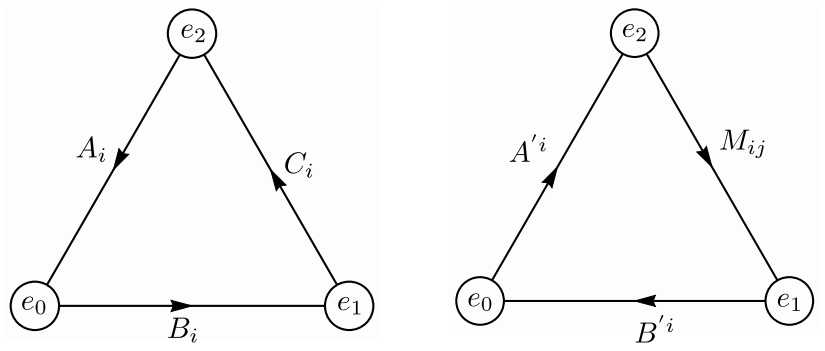


(a) $\mathbb{C}^3/\mathbb{Z}_3$ quiver. **(b)** A Seiberg dual quiver.

Figure 11. The $\mathbb{C}^3/\mathbb{Z}_3$ quiver and a Seiberg dual. The arrows A, B, C and A', B' each have multiplicity 3, while the arrow M has multiplicity 6.

The generator of $\mathrm{Ext}^3(i_*\mathcal{O}, i_*\mathcal{O})$ can be chosen to be $t \in \check{C}^2(X, \mathrm{Hom}^1(i_*\mathcal{O}, i_*\mathcal{O}))$ with

$$t|_{U_0} = \frac{1}{x_0 y_0}\,.$$

One can then compute:

$$\gamma(m_2(a_i, m_2(b_j, c_k))) = \epsilon_{ijk}\,. \tag{B.4}$$

Note that there is a GL(3) symmetry inherited from $\mathbb{P}^2$, and a corresponding SU(3) flavor symmetry in the $\mathcal{N} = 1$ gauge theory.

The $\mathcal{N} = 1$ quiver gauge theory is the one shown in figure 11a, with a gauge group $\mathrm{U}(N) \times \mathrm{U}(N) \times \mathrm{U}(N)$. The bifundamental chiral multiplets A_i, B_i, C_i correspond to the Ext^1 elements a_i, b_i, c_i, and the product structure (B.4) leads to the $\mathcal{N} = 1$ superpotential:

$$W = \mathrm{Tr}(\epsilon^{ijk} A_i B_j C_k)\,. \tag{B.5}$$

This quiver can also be obtained by orbifold projection from 4d $\mathcal{N} = 4$ theory [2, 3].

B.1.2 A second set of fractional branes: theory (II)

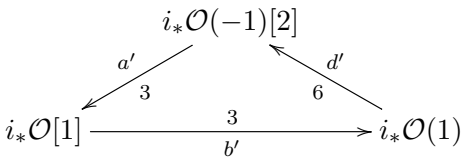
Consider another strongly exceptional collection on $\mathbb{P}^2$:

$$\mathbf{E}_{II} = \{\mathcal{O}(-1)\,,\ \mathcal{O}\,, \mathcal{O}(1)\}\,. \tag{B.6}$$

The corresponding fractional branes are:

$$\mathcal{E}_0 = i_*\mathcal{O}[1]\,, \qquad \mathcal{E}_1 = i_*\mathcal{O}(1)\,, \qquad \mathcal{E}_2 = i_*\mathcal{O}(-1)[2]\,. \tag{B.7}$$

We repeat the same analysis as before. The Ext^1 quiver reads:



The corresponding $\mathcal{N} = 1$ quiver is shown in figure 11b.

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$\mathrm{Ext}^1(i_*\mathcal{O}(-1)[2], i_*\mathcal{O}[1])$ is generated by $a'_i \in \check{C}^0(\mathrm{Hom}^1(i_*\mathcal{O}(-1)[2], i_*\mathcal{O}[1]))$:

$$a'_1: \begin{array}{ccc} \mathcal{O}(2) & \longrightarrow & \mathcal{O}(-1) \\ {\scriptstyle -1}\downarrow & & \downarrow{\scriptstyle 1} \\ \mathcal{O}(3) & \longrightarrow & \mathcal{O} \end{array} \qquad a'_2: \begin{array}{ccc} \mathcal{O}(2) & \longrightarrow & \mathcal{O}(-1) \\ {\scriptstyle -x_0}\downarrow & & \downarrow{\scriptstyle x_0} \\ \mathcal{O}(3) & \longrightarrow & \mathcal{O} \end{array} \qquad a'_3: \begin{array}{ccc} \mathcal{O}(2) & \longrightarrow & \mathcal{O}(-1) \\ {\scriptstyle -y_0}\downarrow & & \downarrow{\scriptstyle y_0} \\ \mathcal{O}(3) & \longrightarrow & \mathcal{O} \end{array}$$

$\mathrm{Ext}^1(i_*\mathcal{O}[1], i_*\mathcal{O}(1))$ is generated by $b'_i \in \check{C}^0(\mathrm{Hom}^1(i_*\mathcal{O}[1], i_*\mathcal{O}(1)))$:

$$b'_1: \begin{array}{ccc} \mathcal{O}(3) & \longrightarrow & \mathcal{O} \\ {\scriptstyle -1}\downarrow & & \downarrow{\scriptstyle 1} \\ \mathcal{O}(4) & \longrightarrow & \mathcal{O}(1) \end{array} \qquad b'_2: \begin{array}{ccc} \mathcal{O}(3) & \longrightarrow & \mathcal{O} \\ {\scriptstyle -x_0}\downarrow & & \downarrow{\scriptstyle x_0} \\ \mathcal{O}(4) & \longrightarrow & \mathcal{O}(1) \end{array} \qquad b'_3: \begin{array}{ccc} \mathcal{O}(3) & \longrightarrow & \mathcal{O} \\ {\scriptstyle -y_0}\downarrow & & \downarrow{\scriptstyle y_0} \\ \mathcal{O}(4) & \longrightarrow & \mathcal{O}(1) \end{array}$$

$\mathrm{Ext}^1(i_*\mathcal{O}(1), i_*\mathcal{O}(-1)[2])$ is generated by $d'_n \in \check{C}^2(\mathrm{Hom}^{-1}(i_*\mathcal{O}(1), i_*\mathcal{O}(-1)[2]))$:

$$\begin{array}{ccc} \mathcal{O}(4) & \longrightarrow & \mathcal{O}(1) \\ {\scriptstyle d'_i}\downarrow & & \\ \mathcal{O}(2) & \longrightarrow & \mathcal{O}(-1) \end{array}$$

$$d'_1 = \frac{1}{x_0 y_0}\,, \qquad d'_2 = \frac{1}{x_0 y_0^2}\,, \qquad d'_3 = \frac{1}{x_0^2 y_0}\,,$$
$$d'_4 = \frac{1}{x_0^2 y_0^2}\,, \qquad d'_5 = \frac{1}{x_0^3 y_0}\,, \qquad d'_6 = \frac{1}{x_0 y_0^3}\,.$$

The field theory is shown in figure 11b. The fields A'^i, B'^i are both in the **3** of the SU(3) flavor symmetry, while the fields $M_{ij} = M_{ji}$ span the **6** of SU(3). They are identified with the Ext^1 elements according to:

$$A'^i = a_i\,, \qquad B'^i = b_i\,, \qquad M_{ij} = \begin{pmatrix} d'_1 & d'_3 & d'_2 \\ d'_3 & d'_5 & d'_4 \\ d'_2 & d'_4 & d'_6 \end{pmatrix}\,. \tag{B.8}$$

One can then derive the superpotential:

$$W = A'^i M_{ij} A'^j\,. \tag{B.9}$$

Moreover, due to the non-abelian anomaly-cancellation condition, the gauge group must be $\mathrm{U}(2N) \times \mathrm{U}(N) \times \mathrm{U}(N)$. This is also what is obtained from the usual rules of Seiberg duality.

B.2 Seiberg duality as mutation

The two $\mathcal{N} = 1$ quiver theories of figure 11 are related by a Seiberg duality on node e_0. Consider for instance the "Theory (I)". A Seiberg duality at node e_0 reverses the arrows A_i and B_j while generating the new mesons $\widetilde{M}_{ij}$, with the identification $\widetilde{M}_{ij} = A_i B_j$ across the duality. The superpotential dual to (B.5) reads:

$$W = \epsilon^{ijk}\widetilde{M}_{ij}C_k + A'^i\widetilde{M}_{ij}B'^j\,. \tag{B.10}$$

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This contains a mass term for C_i and the antisymmetric part of $\widetilde{M}_{ij}$. Integrating those fields out, we are left with "Theory (II)", including the superpotential (B.9).

Similarly, if we start from Theory (II) and perform a Seiberg duality at node e_0, we flip the arrows A'^i, B'^j, and generate the dual mesons $N^{ij} = A'^i B'^j$, with the superpotential:

$$W = M_{ij} N^{ji} + B_i N^{ij} A_j \,. \tag{B.11}$$

Integrating out the massive fields — M_{ij} and the symmetric part of N^{ij} — we recover Theory (I) and (B.5), with the identification $N^{ij} = -\epsilon^{ijk} C_k$.

Mutation of exceptional collection. It was proposed in [58] that Seiberg duality could be realized as mutation on exceptional collections of sheaves. Start with Theory (II) and the corresponding exceptional collection $\mathbf{E}_{II}$ (B.6). Using the left mutation:

$$L_{\mathcal{O}} \mathcal{O}(1) = \Omega^1(1) \tag{B.12}$$

on $\mathbb{P}^2$, we see that a left mutation of the collection $\mathbf{E}_{II}$ at the second sheaf precisely gives the collection $\mathbf{E}_I$ in (B.2):

$$\{\mathcal{O}(-1)\,,\ \mathcal{O}\,, \mathcal{O}(1)\} \quad \rightsquigarrow \quad \{\mathcal{O}(-1)\,,\ \Omega^1(1)\,,\ \mathcal{O}\}\,. \tag{B.13}$$

Therefore, the Seiberg duality at node e_0 of Theory (I) is indeed realized by a mutation of the underlying sheaves. This observation has been generalized to a number of other cases [8].

C A_∞ structure and $\mathcal{N} = (0,2)$ quiver

In this appendix, we discuss the A_∞ structure of the $\mathrm{Ext}^\bullet$ algebra, and how it is related to the structure of the $\mathcal{N} = (0,2)$ quiver. This discussion is a straightforward generalization of a similar discussion for 4d $\mathcal{N} = 1$ quivers by Aspinwall and Katz [10]. See also [51, 68, 70].

C.1 An algebraic preliminary

Let V be a graded vector space, and let $T(V)$ be the associated graded tensor algebra:

$$T(V) = \bigoplus_{n=1}^{\infty} V^{\otimes n}\,. \tag{C.1}$$

Let $\mathbf{d}$ be an derivative operator of degree 1 acting on $T(V)$, satisfying the graded Leibniz rule:

$$\mathbf{d}(A \otimes B) = \mathbf{d}A \otimes B + (-1)^{|A|} A \otimes \mathbf{d}B\,, \tag{C.2}$$

with $A, B \in T(V)$, and $|A|$ denoting the degree of A. We also require that:

$$\mathbf{d}^2 = 0\,. \tag{C.3}$$

Using the Leibniz rule, the action of $\mathbf{d}$ on $T(V)$ is determined by its action on V itself. Let us decompose $\mathbf{d}$ as:

$$\mathbf{d}\big|_V = d_1 + d_2 + \cdots\,, \qquad \text{with} \qquad d_k : V \to V^{\otimes k}\,. \tag{C.4}$$

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Let $V[1]$ denote the vector space V with all degrees decreased by one, and let $s : V \to V[1]$ denote the corresponding map of degree -1. Given this data, we can define an A_∞ algebra A as being the *dual* of $V[1]$:

$$A = (V[1])^* \,, \tag{C.5}$$

together with the multi-products:

$$m_k \;:\; A^{\otimes k} \to A \tag{C.6}$$

given by the dual of the map $s^{\otimes k} \cdot d_k \cdot s^{-1} : V[1] \to V[1]^{\otimes k}$. The nilpotency condition (C.3) is equivalent to the following A_∞ relation on the multi-products:

$$\sum_{r+s+t=n} (-1)^{r+st} m_{n+1-s}(\mathbf{1}^{\otimes r} \otimes m_s \otimes \mathbf{1}^{\otimes t}) = 0 \,, \qquad \forall n > 0 \,, \tag{C.7}$$

where the sum is over all $r, t \geq 0$, $s > 0$, such that $r + s + t = n$ [51].

C.2 Ext algebra and $\mathcal{N} = (0, 2)$ quiver

In our physical setup, the vector space A is spanned by the various Ext^i groups ($i = 0, \cdots, 4$) among the fractional branes on a CY_4 singularity. Schematically:

$$A \cong \mathrm{Ext}^0 \oplus \mathrm{Ext}^1 \oplus \mathrm{Ext}^2 \oplus \mathrm{Ext}^3 \oplus \mathrm{Ext}^4 \,. \tag{C.8}$$

The grading of A is given by the degree i of Ext^i. Any $z \in A$ of degree q is associated to a local vortex operator in the B-model, with the degree identified with the ghost number. Given $z \in A$, let $z^{(1)}$ denote the corresponding one-form descendant. The one-form operators can be used to deform the B-model according to [10, 70]:

$$S \to S + \sum_i Z_i \, z_i^{(1)} \,. \tag{C.9}$$

The coupling Z_i is identified with a "quiver field" in the space-time (D1-brane) theory. Note that Z_i has degree $1 - q_i$ if z_i has degree q_i. The quiver fields are elements of the vector space V, in the notation of subsection C.1.

Let us denote by $\bar{z} \in A$ the Serre dual of $z \in A$, with the Ext algebra A given by (C.8). Let us then choose a basis of A according to:

$$\{z_i\} = \{e_0 \,,\, x_\alpha \,,\, \alpha_I \,,\, \bar{\alpha}_I \,,\, \bar{x}_\alpha \,,\, \bar{e}_0\} \,, \tag{C.10}$$

with:

$$e_0 \in \mathrm{Ext}^0 \,, \quad x_\alpha \in \mathrm{Ext}^1 \,, \quad \alpha_I, \bar{\alpha}_I \in \mathrm{Ext}^2 \,, \quad \bar{x}_\alpha \in \mathrm{Ext}^3 \,, \quad \bar{e}_0 \in \mathrm{Ext}^4 \,. \tag{C.11}$$

As discussed in the main text, the choice of basis for Ext^2 is arbitrary. Any given choice introduces a distinction between the elements α and the Serre dual elements $\bar{\alpha}$, which is a matter of convention.

The dual vector space V spans the "quiver fields". We choose a basis of V:

$$\{Z_i\} = \{e \,,\, X_\alpha \,,\, \Lambda_I \,,\, \bar{\Lambda}^I \,,\, \bar{X}^\alpha \,,\, \bar{e}\} \,, \tag{C.12}$$

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dual to (C.10). The element e correspond to the vector multiplets, while X_α and Λ_I correspond to the chiral and fermi multiplets, respectively. Note the degrees:

	e	X_α	Λ_I	$\bar{\Lambda}^I$	$\bar{X}^\alpha$	$\bar{e}$
degree:	1	0	-1	-1	-2	-3

In particular, the chiral multiplets have degree 0. Given this explicit basis of V, we define a derivative $\mathbf{d}$ as follows. First, let us introduce the $\mathcal{N} = (0,2)$ "superpotential":

$$\mathcal{W} = \mathrm{Tr}\left(\Lambda_I \otimes J^I(X) + \bar{\Lambda}^I \otimes E_I(X)\right), \tag{C.13}$$

with $J_I(X)$ and $E_I(X)$ some arbitrary functions of the chiral multiplets X_α. This $\mathcal{W}$ is an arbitrary gauge-invariant function of degree -1 that is independent of e, except that we need to impose the constraint:

$$\mathrm{Tr}(E_I \otimes J^I) = 0\,. \tag{C.14}$$

Let us also define the derivatives:

$$\partial_\alpha \mathcal{W} \equiv \frac{\partial \mathcal{W}}{\partial X_\alpha} \tag{C.15}$$

by left derivation on $\mathcal{W}$ — that is, we use the cyclic property of the trace to write (C.13) with X_α on the left, and the derivative with respect to X_α is defined as the sum of all possible forms of $\mathcal{W}$ with X_α in front, with X_α removed. Given the superpotential, we define the degree-one derivative $\mathbf{d}$ on V as:[24]

$$\begin{aligned}
\mathbf{d}e &= -e \otimes e\,,\\
\mathbf{d}X_\alpha &= X_\alpha \otimes e - e \otimes X_\alpha\,,\\
\mathbf{d}\Lambda_I &= E_I(X) - \Lambda_I \otimes e - e \otimes \Lambda_I\,,\\
\mathbf{d}\bar{\Lambda}^I &= J^I(X) - \bar{\Lambda}^I \otimes e - e \otimes \bar{\Lambda}^I\,,\\
\mathbf{d}\bar{X}^\alpha &= \partial_\alpha \mathcal{W} - e \otimes \bar{X}^\alpha + \bar{X}^\alpha \otimes e\,,\\
\mathbf{d}\bar{e} &= \bar{X}^\alpha \otimes X_\alpha - X_\alpha \otimes \bar{X}^\alpha + \bar{\Lambda}^I \otimes \Lambda_I + \Lambda_I \otimes \bar{\Lambda}^I - \bar{e} \otimes e - e \otimes \bar{e}\,.
\end{aligned} \tag{C.16}$$

By direct computation, one can check that $\mathbf{d}^2 = 0$. The relations:

$$\mathbf{d}^2 e = 0\,, \qquad \mathbf{d}^2 X_\alpha = 0\,, \qquad \mathbf{d}^2 \Lambda_I = 0\,, \qquad \mathbf{d}^2 \bar{\Lambda}^I = 0 \tag{C.17}$$

are obvious.[25] The key relation is:

$$\mathbf{d}^2 \bar{X}^\alpha = 0\,, \tag{C.18}$$

which holds true if and only if the non-trivial constraint (C.14) is satisfied. This is nothing but the requirement that the $\mathcal{N} = (0,2)$ superpotential be properly supersymmetric. Since we explicitly displayed a nilpotent derivative $\mathbf{d}$ on the vector space V spanned by the quiver

[24] This is the analogue of equations (30) and (39) of [10].

[25] To check the last two relations, one uses that:

$$\mathbf{d}F(X) = F(X) \otimes e - e \otimes F(X)\,,$$

for any degree-zero holomorphic function $F(X)$, which follows from the second line in (C.16).

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fields, it follows from the general discussion above that the multi-products m_k acting on the Ext vector space A satisfy the A_∞ relations (C.7). In this way, we see clearly that the A_∞ relations on a CY_4 are intimately related to the supersymmetry constraint (C.14).

We should also note that the differential $\mathbf{d}$ defined in (C.16) has:

$$d_1 = 0\,, \tag{C.19}$$

where d_k is defined as in (C.4), if and only if the potentials E_I and J^I do not contain any linear terms in X_α. In such a case, we have $m_1 = 0$ in the dual Ext algebra, which gives us a minimal A_∞ structure. Linear terms in E_I or J^I are mass terms, and the corresponding fields can always be integrated out, as discussed in examples in section 3. Therefore, (C.19) always holds for the low-energy quiver.

Similarly, we see from (C.4) that there exists non-zero higher products m_k for $k = 2, \cdots, k_{\max}$, with $k_{\max}$ the highest order in the fields X_α that appear in the potentials E_I, J^I. In the simplest case when E_I, J^I are all quadratic in the chiral multiplets, we have $m_k = 0$ for $k \geq 3$, and the A_∞ algebra reduces to an associative algebra with a product given by m_2.

C.3 General procedure to compute the higher products

Let us discuss in more detail the procedure to compute the higher products of the $\mathrm{Ext}^\bullet$ A_∞ algebra [10], which we outlined in section 2.1.2. Consider an A_∞ algebra $\widetilde{A}$ and the A_∞ map:

$$f : H^\bullet(\widetilde{A}) \to \widetilde{A}\,. \tag{C.20}$$

Let the first map:

$$f_1 = i : H^\bullet(\widetilde{A}) \to \widetilde{A}\,, \tag{C.21}$$

be the inclusion map defined by picking representatives of cohomology classes, and let $d = \widetilde{m}_1 : \widetilde{A} \to \widetilde{A}$ denote the differential on $\widetilde{A}$. The first A_∞ constraint on the maps f_k reads:

$$i \circ m_2(\alpha, \beta) = i(\alpha) \circ i(\beta) + d f_2(\alpha, \beta)\,.$$

We can compute $i(\alpha) \circ i(\beta)$, and use the result to define $m_2(\alpha, \beta)$ and $f_2(\alpha, \beta)$. The next A_∞ constraint is of the form:

$$\begin{aligned} &i \circ m_3(\alpha, \beta, \gamma) = \\ &\quad f_2(\alpha, m_2(\beta, \gamma)) - f_2(m_2(\alpha, \beta), \gamma) + i(\alpha) \circ f_2(\beta, \gamma) - f_2(\alpha, \beta) \circ i(\gamma) + d f_3(\alpha, \beta, \gamma). \end{aligned}$$

Using the previously-computed m_2 and f_2, this expression allows us to compute m_3 and f_3. Proceeding inductively in this fashion, one can construct m_k and f_k to any order k.

D Higher products on a local $\mathbb{P}^1 \times \mathbb{P}^1$

In this appendix, we spell out the computation of the higher products on the local $\mathbb{P}^1 \times \mathbb{P}^1$ geometry of section 2.5, using the procedure summarized in appendix C.3.

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Consider the Ext element representations discussed in section 2.5.1. From the composition of the chain maps, one finds the products:

$$\begin{aligned} &m_2(b_1,d_1)=\beta_1'\,, \quad m_2(b_1,d_2)=\beta_2'\,, \quad m_2(b_2,d_1)=\beta_3'\,, \quad m_2(b_2,d_2)=\beta_4'\,,\\ &m_2(a_1,e_1)=\beta_1'\,, \quad m_2(a_2,e_1)=\beta_2'\,, \quad m_2(a_1,e_2)=\beta_3'\,, \quad m_2(a_2,e_2)=\beta_4'\,. \end{aligned} \tag{D.1}$$

It follows that $f_2(b,d)=f_2(a,e)=0$. Define the 1-cochains λ and δ as follows:

$$\begin{aligned} &(\lambda)_{01}=(\lambda)_{02}=(\lambda)_{03}=(\lambda)_{12}=(\lambda)_{13}=0\,, \quad (\lambda)_{23}=x^{-1}u^{-1}\,,\\ &(\delta)_{03}=(\delta)_{12}=(\delta)_{23}=0\,, \quad (\delta)_{01}=(\delta)_{02}=-(\delta)_{03}=u^{-1}. \end{aligned}$$

One can compute

$$\begin{aligned} &d_1\cdot c_1=d\lambda_1\,, \qquad d_1\cdot c_2=d\lambda_2\,,\\ &d_2\cdot c_2=d\delta_1\,, \qquad d_2\cdot c_2=d\delta_2\,, \end{aligned}$$

where the chain maps are defined by

$$\begin{aligned} &\lambda_1=\begin{pmatrix}-\lambda\\0\end{pmatrix}(0,-\lambda)\,, \qquad \lambda_2=\begin{pmatrix}0\\-\lambda\end{pmatrix}(\lambda,0)\,,\\ &\delta_1=\begin{pmatrix}-\delta\\0\end{pmatrix}(0,-\delta)\,, \qquad \delta_2=\begin{pmatrix}0\\-\delta\end{pmatrix}(\delta,0)\,, \end{aligned}$$

between the corresponding complexes. This implies $m_2(d,c)=0$ and:

$$\begin{aligned} &f_2(d_1,c_1)=-\lambda_1\,, \quad f_2(d_1,c_2)=-\lambda_2\,,\\ &f_2(d_2,c_1)=-\delta_1\,, \quad f_2(d_2,c_2)=-\delta_2\,. \end{aligned} \tag{D.2}$$

Similarly, one can show:

$$\begin{aligned} &c_1\cdot a_1=d\lambda_1\,, \quad c_2\cdot a_1=d\lambda_2\,,\\ &c_1\cdot a_2=d\delta_1\,, \quad c_2\cdot a_2=d\delta_2\,. \end{aligned}$$

Thus, $m_2(c,a)=0$ and:

$$\begin{aligned} &f_2(c_1,a_1)=-\lambda_1\,, \quad f_2(c_2,a_1)=-\lambda_2\,,\\ &f_2(c_1,a_2)=-\delta_1\,, \quad f_2(c_2,a_2)=-\delta_2\,. \end{aligned}$$

Plugging these results into the A_∞ map constraint:

$$im_3(d,c,a)=f_2(d,m_2(c,a))-f_2(m_2(d,c),a)+d\cdot f_2(c,a)-f_2(d,c)\cdot a+df_3(d,c,a)\,,$$

we get:

$$\begin{aligned} &m_3(d_1,c_1,a_1)=0\,, && m_3(d_1,c_1,a_2)=\alpha_1\,,\\ &m_3(d_1,c_2,a_1)=0\,, && m_3(d_1,c_2,a_2)=\alpha_2\,,\\ &m_3(d_2,c_1,a_1)=-\alpha_1\,, && m_3(d_2,c_1,a_2)=0\,,\\ &m_3(d_2,c_2,a_1)=-\alpha_2\,, && m_3(d_2,c_2,a_2)=0\,. \end{aligned} \tag{D.3}$$

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Similarly, if we define τ and θ by:

$$(\tau)_{01} = (\tau)_{03} = (\tau)_{13} = (\tau)_{23} = 0\,, \quad (\tau)_{02} = (\tau)_{12} = x^{-1}u^{-1}\,,$$
$$(\theta)_{01} = (\theta)_{02} = (\theta)_{12} = (\theta)_{23} = 0\,, \quad (\theta)_{03} = (\theta)_{13} = -x^{-1}\,,$$

we get $m_2(e,c) = 0, m_2(c,b) = 0$ and

$$\begin{aligned} &f_2(e_1,c_1) = \tau_1\,, \quad f_2(e_1,c_2) = \tau_2\,, \\ &f_2(e_2,c_1) = \theta_1\,, \quad f_2(e_2,c_2) = \theta_2\,, \\ &f_2(c_1,b_1) = -\tau_1\,, \quad f_2(c_2,b_1) = -\tau_2\,, \\ &f_2(c_1,b_2) = -\theta_1\,, \quad f_2(c_2,b_2) = -\theta_2\,, \end{aligned} \tag{D.4}$$

where:

$$\tau_1 = \begin{pmatrix} \tau \\ 0 \end{pmatrix} (0,\tau)\,, \qquad \tau_2 = \begin{pmatrix} 0 \\ \tau \end{pmatrix} (-\tau,0)\,,$$
$$\theta_1 = \begin{pmatrix} \theta \\ 0 \end{pmatrix} (0,\theta)\,, \qquad \theta_2 = \begin{pmatrix} 0 \\ \theta \end{pmatrix} (-\theta,0)\,.$$

Plugging these results into:

$$im_3(e,c,b) = f_2(e,m_2(c,b)) - f_2(m_2(e,c),b) + e\cdot f_2(c,b) - f_2(e,c)\cdot b + df_3(e,c,b)\,,$$

we get:

$$\begin{aligned} &m_3(e_1,c_1,b_1) = 0\,, \qquad &&m_3(e_1,c_1,b_2) = -\alpha_1'\,, \\ &m_3(e_1,c_2,b_1) = 0\,, \qquad &&m_3(e_1,c_2,b_2) = -\alpha_2'\,, \\ &m_3(e_2,c_1,b_1) = \alpha_1'\,, \qquad &&m_3(e_2,c_1,b_2) = 0\,, \\ &m_3(e_2,c_2,b_1) = \alpha_2'\,, \qquad &&m_3(e_2,c_2,b_2) = 0\,. \end{aligned} \tag{D.5}$$

This completes the computation of the three-product m_3. As a consistency check, one can verify that the m_2 and m_3 just computed satisfy the relevant A_∞ relations.

By the same procedure, we could also find the m_4 product, while the higher products vanish. We can use various short-cuts to the correct answer, however. For instance, we can impose $\mathrm{Tr}(EJ) = 0$ in the gauge theory quiver, which is equivalent to imposing the A_∞ relations. This leads to the result (2.87) for the m_4 products amongst the Ext^1 elements.

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