TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 187, Issue 1, 1974

B-CONVEXITY AND REFLEXIVITY IN BANACH SPACES

BY

DEAN R. BROWN(1)

ABSTRACT. A proof of James that uniformly nonsquare spaces are reflexive is extended in part to B-convex spaces. A condition sufficient for non-B-convexity and related conditions equivalent to non-B-convexity are given. The following theorem is proved: A Banach space is B-convex if each subspace with basis is B-convex.

0. Introduction. The notion of a B-convex Banach space was introduced by A. Beck [1], [2] as a characterization of those Banach spaces X having the property that a certain strong law of large numbers holds for X valued random variables.

Definition. Let k be a positive integer and ϵ a positive number. X is said to be k, ϵ-convex if for any $\{x_1, \cdots, x_k\}$, $\|x_i\| \leq 1$, $i = 1, \cdots, k$, there is some choice of signs $\xi_1, \cdots, \xi_k$ so that $\|\sum_{i=1}^k \xi_i x_i\| \leq k(1 - \epsilon)$. X is said to be B-convex if it is k, ϵ-convex for some k and ϵ.

Further study of B-convex spaces has been done by R. C. James [6], [7], D. P. Giesy [5] and C. A. Kottman [8]. Giesy showed that B-convex spaces have many of the properties of reflexive spaces. James conjectured that all B-convex spaces are reflexive, and proved the conjecture true for $2, \epsilon$-convex spaces. Both James and Giesy proved the conjecture true for B-convex spaces having an unconditional basis. Kottman extended James' $2, \epsilon$-convex proof to a larger subclass, P-convex spaces. Examples are known of spaces which are reflexive but not B-convex.

§1 of this paper adopts a part of James' $2, \epsilon$-convex theorem to all non-B-convex spaces, presents a condition sufficient for non-B-convexity, and gives related characterizations of non-B-convex spaces, though the conjecture of James remains open. §2 proves a theorem on B-convexity and subspaces with basis analogous to a theorem of Pełczyński on reflexivity and subspaces with basis.

For a Banach space X, $U(X)$ will denote the closed unit ball $\{x: \|x\| \leq 1\}$ of X.

I. Non-B-convexity. In James' proof [6] that $2, \epsilon$-convex spaces are reflexive, he defines for a Banach space X a sequence of numbers K_n, and shows that if X

Received by the editors July 7, 1971.

AMS (MOS) subject classifications (1970). Primary 46B10; Secondary 46B05, 46B15.

Key words and phrases. B-convexity, reflexivity, P-convexity, uniform-convexity, uniformly nonsquare, uniformly non $l_n^{(1)}$, geometry of the unit ball, Schauder basis.

(1) This work represents a part of the author's Ph. D. thesis which was submitted to the Ohio State University under the helpful direction of Professor David W. Dean.

is not reflexive then $K_n \leq 2n$, and in that case X cannot be 2, ϵ-convex. We will extend the second step of this result to show that if K_n is a bounded sequence then X cannot be B-convex. The numbers K'_n, to be defined almost the same as James' K_n, will be used instead of K_n. Another condition, which implies that $\{K'_n\}$ is bounded, is introduced and is shown to be sufficient for non-B-convexity by a much simpler proof.

Let X be a Banach space. For each sequence $\{f_j\}$ of continuous linear functionals with unit norms and each increasing sequence of integers $\{p_1, \cdots, p_{2n}\}$, let $S(p_1, \cdots, p_{2n}, \{f_j\})$ denote the set of all x such that, for all k and i, $3/4 \leq (-1)^{i-1} f_k(x)$ if $p_{2i-1} \leq k \leq p_{2i}$ and $1 \leq i \leq n$. Let

$$K(n, \{f_j\}) = \liminf_{p_1 \to \infty} \liminf_{p_2 \to \infty} \cdots \liminf_{p_{2n} \to \infty} \inf\{\|z\|: z \in S(p_1, \cdots, p_{2n}; \{f_j\})\}$$

and

$$K'_n = \inf\{K(n, \{f_j\}): \|f_j\| = 1 \text{ for all } j\}.$$

James' definition of K_n is similar. It follows from the definitions that $K'_n \leq K_n$ and $K'_n \leq K'_{n+1}$ for all n.

Theorem 1.1. *If the sequence $\{K'_n\}$ for a Banach space X is bounded, then X is not B-convex.*

Proof. For any positive integer k and any $0 < \delta < 2$ we will show X is not k, δ-convex by showing there are $x_1, \cdots, x_k \in U(X)$ such that for any choice of signs $\xi_1, \cdots, \xi_k$ we have $\|\sum_{i=1}^k \xi_i x_i\| > k(1-\delta)$. Since the sequence $\{K'_n\}$ is bounded, and monotone, we can choose m such that $K'_{2m}/K'_{3m2^k} > 1 - \delta/3$. Let $3m2^k = M$. Choose μ, $\{f_j\}$ where $\|f_j\| = 1$, and ϵ such that $0 < \mu < (K'_M)^2\delta/3K'_{2m}$, $K'_M + \mu > K(M, \{f_j\})$ and $0 < \epsilon < (K'_M)^2\delta/3(K'_{2m} + K'_M)$. From these inequalities, it follows that

$$(K(2m, \{f_j\}) - \epsilon)/(K(M, \{f_j\}) + \epsilon) > 1 - \delta.$$

As will be described below, it is possible to choose an increasing set of integers $P = \{p_{i,j}: i = 1, \cdots, k; j = 1, \cdots, 2M\}$ having the following properties:

(1) For each $i = 1, \cdots, k$ there is $u_i \in S(p_{i,1}, \cdots, p_{i,2M}; \{f_j\})$ such that $\|u_i\| \leq K(M, \{f_j\}) + \epsilon$.

(2) For each choice of signs $\xi_1, \cdots, \xi_k$ there is an increasing set of integers $\{\sigma_1, \cdots, \sigma_{4m}\} \subset P$ such that

(2a) $(1/k)\sum_{i=1}^k \xi_i u_i \in S(\sigma_1, \cdots, \sigma_{4m}; \{f_j\})$, and

(2b) any element of $S(\sigma_1, \cdots, \sigma_{4m}; \{f_j\})$ has norm greater than or equal to $K(2m, \{f_j\}) - \epsilon$.

Let $x_i = u_i/K(M, \{f_j\}) + \epsilon$ for $i = 1, \cdots, k$. From property (1), $\|x_i\| \le 1$. From property (2), for any choice of signs $\xi_1, \cdots, \xi_k$ we have

$$\left\| \frac{1}{k} \sum_{i=1}^{k} \xi_i x_i \right\| \ge \frac{K(2m, \{f_j\}) - \epsilon}{K(M, \{f_j\}) + \epsilon} > 1 - \delta$$

which completes the proof except for the choice of P.

The choice of P is rather tedious. Integers are chosen successively in m blocks of increasing integers:

$$P_1 = \{p_{i,j} : i = j, \cdots, k;\ j = 1, \cdots, 2M/m\}$$

$$P_2 = \{p_{i,j} : i = 1, \cdots, k;\ j = (2M/m) + 1, \cdots, 4M/m\}$$

$$\vdots$$

$$P_m.$$

Let the k-tuples of signs $(\xi_1, \cdots, \xi_k)$ be denoted $\Xi_1, \cdots, \Xi_{2^k}$. The integers of P_1 are chosen successively in 2^k sets of increasing integers $P_1(\Xi_1), \cdots, P_1(\Xi_{2^k})$. The number of integers in $P_1(\Xi_n)$ depends on Ξ_n; as will be shown, four are chosen for each plus sign in Ξ_n and eight for each minus sign, so that P_1 has $6k2^k = 2kM/m$ integers.

Property (2a) is provided by the order of choice of the integers. This order may be illustrated by supposing $\Xi_n = (\xi_1, \cdots, \xi_k)$ where $\xi_\eta = -1$ ($\eta \neq 1$ or k) and $\xi_i = +1$ for $i \neq \eta$. Suppose for each $i = 1, \cdots, k$, the last integers of $P_1(\Xi_{n-1})$ are $p_{i,j(i)}$. The order of choice of integers of $P_1(\Xi_n)$ is shown in Figure 1. The integers $\sigma_1, \cdots, \sigma_4$ of property (2) for this choice of signs are $p_{k,j(k)+1}, p_{1,j(1)+1}, p_{k,j(k)+3}, p_{1,j(1)+4}$. The sets $P_2, P_3, \cdots, P_m$ are chosen in succession by the same method. The integers $\sigma_1, \cdots, \sigma_{4m}$ for a choice of signs Ξ_n are in the set $\{P_1(\Xi_n), \cdots, P_m(\Xi_n)\}$. Properties (1) and (2b) are provided by requiring that the integers chosen satisfy appropriate inequalities at each step.

The proof of Theorem 1.1 is rather tedious. We now present a strengthening of the condition "$\{K_n'\}$ bounded" which will be sufficient for non-B-convexity in a much simpler way.

Condition 1. For some $0 < \epsilon < 1$, $U(X^*)$ contains a sequence $\{f_j\}$ such that for any $m \ge 1$ and any choice of signs $\xi_1, \cdots, \xi_m$ there is $x \in U(X)$ satisfying $f_j(\xi_j x) > \epsilon$, $j = 1, \cdots, m$.

Theorem 1.2. *If a Banach space satisfies Condition* 1, *then the sequence* $\{K_n'\}$ *for that space is bounded.*

Proof. We will show $K_n' \le 3/4\epsilon$ by showing that for $\{f_j\}$ as asserted in Con-

Figure 1. Order of choice of $P_1(\Xi_n)$

For order of choice, read down first column, then down second column, etc.

$P_{1,j(1)+1}$	$P_{1,j(1)+2}$	$P_{1,j(1)+3}$	$P_{1,j(1)+4}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$P_{\eta-1,j(\eta-1)+1}$	$P_{\eta-1,j(\eta-1)+2}$	$P_{\eta-1,j(\eta-1)+3}$	$P_{\eta-1,j(\eta-1)+4}$
$P_{\eta,j(\eta)+1}$			
$P_{\eta,j(\eta)+2}$			
$P_{\eta,j(\eta)+3}$	$P_{\eta,j(\eta)+4}$	$P_{\eta,j(\eta)+5}$	$P_{\eta,j(\eta)+6}$
			$P_{\eta,j(\eta)+7}$
			$P_{\eta,j(\eta)+8}$
$P_{\eta+1,j(\eta+1)+1}$	$P_{\eta+1,j(\eta+1)+2}$	$P_{\eta+1,j(\eta+1)+3}$	$P_{\eta+1,j(\eta+1)+4}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$P_{k,j(k)+1}$	$P_{k,j(k)+2}$	$P_{k,j(k)+3}$	$P_{k,j(k)+4}$

dition 1, and any $p_1, \cdots, p_{2n}$ there is $y \in S(p_1, \cdots, p_{2n}; \{f_j/\|f_j\|\})$ and $\|y\| \leq 3/4\epsilon$. Choose ξ_k, $k = 1, \cdots, p_{2n}$, so that if $p_{2i-1} \leq k \leq p_{2i}$, $1 \leq i \leq n$, then $\xi_k = (-1)^{i-1}$. Let $y = 3x/4\epsilon$, where x is that element asserted by Condition 1.

Theorem 1.3. *The following conditions are equivalent to non-B-convexity*:

Condition 2. *For some* $0 < \epsilon < 1$ *and any* $k \geq 2$, $U(X^*)$ *contains* $f_1, \cdots, f_k$ *such that for any choice of signs* $\xi_1, \cdots, \xi_k$ *there is* $x \in U(X)$ *satisfying* $f_j(\xi_j x) > \epsilon$, $j = 1, \cdots, k$.

Condition 3. *For some* $0 < \epsilon < 1$ *and any* $k \geq 2$, $U(X)$ *contains* $x_1, \cdots, x_k$ *such that for any choice of signs* $\xi_1, \cdots, \xi_k$ *there is* $f \in U(X^*)$ *satisfying* $f(\xi_j x_j) > \epsilon$, $j = 1, \cdots, k$.

Proof. We first show non-B-convexity implies Condition 2. Choose any $0 < \epsilon < 1$. For a given $k \geq 2$ choose δ such that $k - 1 + \epsilon < k(1 - 2\delta)$. Since X is not B-convex, by [5, Theorem II-3], X^* is not B-convex. Hence there is $f_1, \cdots, f_k \in U(X^*)$ such that for all choices of signs $\xi_1, \cdots, \xi_k$, $\|\Sigma_{i=1}^k \xi_i f_i\| > k(1 - \delta)$. Thus, for each choice of signs there is $x \in U(X)$ such that $\Sigma_{i=1}^k \xi_i f_i(x) > k(1 - 2\delta)$. If there were some j such that $f_j(\xi_j x) \leq \epsilon$, then $\Sigma_{i=1}^k \xi_i f_i(x) \leq k - 1 + \epsilon < k(1 - 2\delta)$.

To prove the converse, suppose X is B-convex and that Condition 2 is satisfied. Since X^* is also B-convex, by [5, Lemma I-4], $\lim_{k\to\infty} a_k(X^*) = 0$, where

$$a_k(X^*) = \sup\left\{\inf\left\{\frac{1}{k}\sum_{i=1}^{k} \xi_i x_i : \xi_1, \cdots, \xi_k = \pm 1\right\} : f_1, \cdots, f_k \in U(X^*)\right\}$$

and we may choose k so that $a_k(X^*) < \epsilon$. Therefore, for $f_1, \cdots, f_k$ asserted by Condition 2 there is some choice of signs such that $\|\Sigma_{j=1}^k \xi_j f_j\| < k\epsilon$. But by Condition 2, $f_j(\xi_j x) > \epsilon$, so that $\Sigma_{j=1}^k \xi_j f_j(x) > k\epsilon$ and $\|\Sigma_{j=1}^k \xi_j f_j\| > k\epsilon$ which is a contradiction.

Condition 3 may be shown equivalent to non-B-convexity by similar proofs.

Corollary 1.4. *If a Banach space satisfies Condition* 1 *it is not B-convex.*

We do not know whether nonreflexivity implies $\{K'_n\}$ bounded or Condition 1. There is however a large class of nonreflexive, non-B-convex spaces satisfying Condition 1, listed in the following easily proved proposition.

Proposition 1.5. *The Banach spaces* c_0, 1_1, *and all spaces containing* c_0 *or* 1_1 *satisfy Condition* 1.

2. *B*-convexity and basis. It is known that a Banach space is reflexive if each subspace with basis is reflexive [9]. In this section we show that a Banach space is B-convex if each subspace with basis is B-convex.

As usual we say that $\{x_i\} \subset X$ is a basis for X if for each $x \in X$ there is a unique sequence of numbers $\{a_i\}$ so that $\lim_{n\to\infty} \|\Sigma_{i=1}^n a_i x_i - x\| = 0$. It is well known that $\{x_i\}$ is a basis for X if there is some number k so that for any integers n and q and any sequence of numbers $\{a_i\}$ we have

$$\left\|\sum_{i=1}^{n} a_i x_i\right\| \le k \left\|\sum_{i=1}^{n+q} a_i x_i\right\|.$$

We will prove the main theorem of this section by constructing a basic sequence in an arbitrary non-B-convex space in such a way that the span of this sequence is not B-convex. The technique for construction of this basic sequence is an adaption of the method of Day [3] and Gelbaum [4]. It relies on the following lemmas:

Lemma 2.1. *If* X *is finite dimensional, for any* $\epsilon > 0$ *there is* $\{f_i\}_{i=1}^n \subset U(X^*)$ *such that, for any* x,

$$\|x\| \le (1+\epsilon)\max\{f_i(x) : i = 1, \cdots, n\}.$$

Lemma 2.2. *If* $\{f_i\}_{i=1}^n \subset X^*$ *and* $Y = \bigcap_{i=1}^n f_i^{-1}(0)$, *then* Y *is a space of finite codimension in* X; *that is, there is a finite dimensional subspace* Z *so that* $X = Y \oplus Z$.

Theorem 2.3. *If* X *is not B-convex it contains a subspace with basis not B-convex.*

Proof. Let $\{\delta_i\}$ be a sequence of positive numbers less than one tending to zero. Let $\{k_i\}$ be a sequence of integers tending to infinity. Let $p(0) = 0$, $p(m) = \sum_{i=1}^m k_i$, $m = 1, 2, \cdots$.

The subspace to be constructed will be the closed span of a sequence $\{x_i\}$ having the following properties:

(1) there is a sequence $\{\epsilon_i\}$ tending to zero so that for each $m = 1, 2, \cdots$, the space $[x_i]_{i=p(m-1)+1}^{p(m)}$ is an ϵ_m isometric image of $1_1^{k_m}$, in particular,

$$(1-\epsilon_m)\sum_{i=p(m-1)+1}^{p(m)} |a_i| \le \left\| \sum_{i=p(m-1)+1}^{p(m)} a_i x_i \right\| \le \sum_{i=p(m-1)+1}^{p(m)} |a_i|.$$

(2) For any $\{a_i\}$, n, q,

$$\left\|\sum_{i=1}^{n} a_i x_i\right\| \le (3+\delta_n)\left\|\sum_{i=1}^{n+q} a_i x_i\right\|.$$

Property (1) shows that the subspace is not B-convex, and property (2) shows that $\{x_i\}$ is a basis.

For $\{\delta_i\}$, $\{k_i\}$ as above there are $\{\epsilon_i\}$, $\{\eta_i\}$ such that

(3) $\epsilon_i \longrightarrow 0$.

(4) If $1 \le n \le p(1)$ then $(1+\eta_1)/(1-\epsilon_1) \le 1 + \delta_n$.

(5) If $p(m) < n \le p(m+1)$ then

$$1 + \eta_m + (2 + \eta_m + \eta_{m+1})/(1-\epsilon_{m+1}) \le 3 + \delta_n.$$

$[x_i]_{i=1}^{p(m)}$ will be denoted by $L_{p(m)}$. $\{x_i\}$ will be constructed in blocks by induction on m. In the induction step, from a previously constructed subspace Λ_{m-1}, $\{x_i\}_{i=p(m-1)+1}^{p(m)}$ will be chosen satisfying (1). Then a subspace Λ_m of Λ_{m-1}, of finite codimension in X, will be constructed so that $\Lambda_m \cap L_{p(m)} = 0$ and the projection $P_m\colon \Lambda_m \oplus L_{p(m)} \longrightarrow L_{p(m)}$ satisfies $\|P_m\| < 1 + \eta_m$. Finally (2) will be proved.

Let $m = 1$. Since X is not B-convex, we can choose $\{x_i\}_{i=1}^{k_1=p(1)} \subset U(X)$ satisfying (1). To construct Λ_1 choose $\{f_i\}_{i=1}^{q(1)} \subset U(L_{p(1)}^*)$ by Lemma 2.1 and extend them to X without increase of norm so that if $x \in L_{p(1)}$,

$$\|x\| \le (1+\eta_1)\max\{f_i(x)\colon i = 1, \cdots, q(1)\}.$$

Let $\Lambda_1 = \bigcap_{i=1}^{q(1)} f_i^{-1}(0)$. By Lemma 2.2, Λ_1 is of finite codimension in X, and $L_{p(1)} \cap \Lambda_1 = 0$, so there is a projection $P_1\colon L_{p(1)} \oplus \Lambda_1 \longrightarrow L_{p(1)}$. To see that $\|P_1\| \le 1 + \eta_1$ we have for any $x \in L_{p(1)}$, $\lambda \in \Lambda_1$, and some $i = 1, \cdots, q(1)$, $\|P(x+\lambda)\| = \|x\| \le (1+\eta_1)f_i(x) = (1+\eta_1)f_i(x+\lambda) \le (1+\eta_1)\|x+\lambda\|$.

Now suppose we have $\{x_i\}_{i=1}^{p(1)}$ satisfying (1), $\{f_i\}_{i=1}^{q(m-1)} \subset U(X^*)$, and for

$n = 1, 2, \cdots, m-1$, $\Lambda_n = \bigcap_{i=1}^{q(n)} f_i^{-1}(0)$, such that $\|P_n\| \leq 1 + \eta_n$ where $P_n\colon \Lambda_n \oplus L_{p(n)} \to L_{p(n)}$. Since Λ_{m-1} is of finite codimension in X, by [5, Theorem 2.12], it is not B-convex so that there are $\{x_i\}_{i=p(m-1)+1}^{p(m)} \subset U(\Lambda_{m-1})$ satisfying (1). By Lemma 2.1 choose $\{f_i\}_{i=q(m-1)+1}^{q(m)} \subset U(X^*)$ so that if $x \in L_{p(m)}$,

$$\|x\| \leq (1 + \eta_m) \max\{f_i(x)\colon i = q(m-1)+1, \cdots, q(m)\}.$$

Let $\Lambda_m = \bigcap_{i=1}^{q(m)} f_i^{-1}(0)$. Then exactly as in the $m = 1$ case, $L_{p(m)} \cap \Lambda_m = 0$ and $\|P_m\| \leq 1 + \eta_m$.

To show (2) holds we first observe, for any a_i, $i = 1, 2, \cdots$,

A. $\|\Sigma_{i=1}^{p(m)} a_i x_i\| \leq (1 + \eta_m)\|\Sigma_{i=1}^{p(m)+q} a_i x_i\|$ for any $q = 1, 2, \cdots$, since $\|P_m\| \leq 1 + \eta_m$.

Further, we observe

B. If $p(m-1) < n \leq p(m)$ then

$$\left\| \sum_{i=p(m-1)+1}^{n} a_i x_i \right\| \leq \frac{1}{1-\epsilon_m} \left\| \sum_{i=p(m-1)+1}^{p(m)} a_i x_i \right\|,$$

since

$$\left\| \sum_{i=p(m-1)+1}^{n} a_i x_i \right\| \leq \sum_{i=p(m-1)+1}^{n} |a_i| \leq \sum_{i=p(m-1)+1}^{p(m)} |a_i| \leq \frac{1}{1-\epsilon_m} \left\| \sum_{p(m-1)+1}^{p(m)} a_i x_i \right\|,$$

using the inequalities of (1).

(2) can be proved in four cases as follows:

Case 1. $1 \leq n < n + q < p(1)$. Using B, with $m = 1$, and (4) we obtain $\|\Sigma_{i=1}^{n} a_i x_i\| \leq (1 + \delta_n)\|\Sigma_{i=1}^{n+q} a_i x_i\|$.

Case 2. $1 \leq n \leq p(1) < n + q$. The above inequality can be proved using B and A, where $m = 1$, and (4).

Case 3. There is m so that $p(m) < n < n + q \leq p(m+1)$. Inequality (2) can be proved by using A, B with m replaced by $m + 1$, and (5).

Case 4. There is m so that $p(m) < n \leq p(m+1) < n + q$. Inequality (2) can be proved by using A, B with m replaced by $m + 1$, A with m replaced with $m + 1$, and (5).

REFERENCES

1. A. Beck, *A convexity condition in Banach spaces and the strong law of large numbers*, Proc. Amer. Math. Soc. 13 (1962), 329–334. MR 24 #A3681.

2. ———, *On the strong law of large numbers*, Ergodic Theory (Proc. Internat. Sympos., Tulane Univ., New Orleans, La., 1961), Academic Press, New York, 1963, pp. 21–53. MR 28 #3470.

3. M. M. Day, *On the basis problem in normed spaces*, Proc. Amer. Math. Soc. 13 (1962), 655–658. MR 25 #1435.

4. B. Gelbaum, *Notes on Banach spaces and bases*, An. Acad. Brasil Ci. 30 (1958), 29–36. MR 20 #5419.

5. D. P. Giesy, *On a convexity condition in normed linear spaces*, Trans. Amer. Math. Soc. 125 (1966), 114–146. MR 34 #4866.

6. R. C. James, *Uniformly non-square Banach spaces*, Ann. of Math. (2) 80 (1964), 542–550. MR 30 #4139.

7. ———, *Convexity and smoothness*, Proc. Colloquium on Convexity (Copenhagen, 1965), Københavns Univ. Mat. Inst., Copenhagen, 1967, pp. 165–167. MR 36 #657.

8. C. A. Kottman, *Packing and reflexivity in Banach spaces*, Trans. Amer. Math. Soc. 150 (1970), 565–576. MR 42 #827.

9. A. Pełczyński, *A note on the paper of I. Singer, 'Basic sequences and reflexivity of Banach spaces,'* Studia Math. 21 (1961/62), 371–374. MR 26 #4156.

DEPARTMENT OF MATHEMATICS, YOUNGSTOWN STATE UNIVERSITY, YOUNGSTOWN, OHIO 44503