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Initial Value Problems of One-Dimensional Self-Modulation of Nonlinear Waves in Dispersive Media

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The initial value problems for the nonlinear modulation of dispersive waves are investigated by virtue of the method developed by Zakharov and Shabat. It is studied in general how the modulated waves evolve to decay into solitons moving with their respective speeds or to form the bound state of solitons. The perturbation analysis is applied to investigate the condition for the bisymmetric decay of modulated waves into moving solitons. As a special example, the initial condition of a hyperbolic function type is considered in details. The numerically computed solutions are also shown.

§ 1. Introduction

It is well known that the self-modulation of one-dimensional waves in nonlinear dispersive systems can be described by the so-called nonlinear Schrödinger equation,[1)~4)]

$$iu_t = pu_{xx} + q|u|^2u, \tag{0}$$

where subscripts t and x denote partial differentiation with respect to t and x, respectively. If $pq<0$, a plane wave in this system is stable for the modulation and, otherwise, is unstable. Especially in the latter case there exist special families of solutions, which are called envelope solitons and show various interesting phenomena.[2),5)] In this paper we confine ourselves to the case $pq>0$. Without loss of generality p and q may be put equal to 1/2 and 1, respectively:

$$iu_t = (1/2)u_{xx} + |u|^2u. \tag{1}$$

A solitary wave solution of Eq. (1), $S(x, t)$, is represented by

$$S(x, t) = A\,\mathrm{sech}(Ax)\exp(-iA^2t/2). \tag{2}$$

Since Eq. (1) is invariant under the Galilei transformation,

$$x'=x-Vt, \qquad t'=t,$$
$$u'(x',t')=\exp(iVx-iV^2t/2)u(x,t), \tag{3}$$

we find that the function,

$$u=A\operatorname{sech}[A(x-Vt)]\exp[-iVx+i(V^2-A^2)t/2], \tag{4}$$

is also the solution of Eq. (1). The solution (4) represents the soliton moving with the velocity V, while the solution (2) the soliton being at rest.

Previously, Outi and one of the authors, N.Y., solved Eq. (1) numerically and showed that the envelope soliton is extremely stable.[5] Recently Zakharov and Shabat proposed a method to solve analytically the initial value problem for Eq. (1) with the boundary condition that u tends to zero as $x\to\pm\infty$, together with all its x-derivatives.[6] It is an application of the method which is developed by Gardner, Greene, Kruskal and Miura for the Korteweg-de Vries equation[7] and extended by Lax for a wide class of nonlinear equation.[8] Zakharov and Shabat obtained the important results; the stability of solitons, the existence of infinite numbers of conservation laws and so on. In their work, however, the initial value problem for Eq. (1) is insufficiently considered, that is, it remains still unsolved how the initial disturbance evolves and with what initial conditions it decays into envelope solitons. In this paper, applying the method of Zakharov and Shabat, we study the initial value problem for Eq. (1).

In the rest of this section, the result of Zakharov and Shabat is summarized in the form suitable for our discussion. To begin with, we consider the following eigenvalue equation:

$$iv_x+Uv=\zeta\sigma_3 v, \tag{5}$$

where the eigenfunction v is a two-component column vector,

$$v=\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \tag{6}$$

and σ_3 and U are 2×2 matrices,

$$\sigma_3=\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{7}$$

$$U=\begin{pmatrix} 0 & u \\ u^* & 0 \end{pmatrix}. \tag{8}$$

Here and hereafter, asterisk denotes complex conjugate. The eigenvalue ζ is generally complex. If u in Eq. (8) is the solution of Eq. (1), ζ is independent of time and v develops with increasing time according to

$$iv_t = Av, \tag{9}$$

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\left(\frac{1}{2}\frac{\rho^2-1}{\rho^2+1}\frac{\partial^2}{\partial x^2} - i\zeta\frac{\partial}{\partial x} + C\right)$$
$$+(\rho^2+1)^{-1}\begin{pmatrix} \rho^2|u|^2 & iu_x \\ -i\rho^2 u_x^* & -|u|^2 \end{pmatrix}, \tag{10}$$

where C is a constant independent of x.

Let ζ be real $(=\xi)$ in Eq. (5). Attending to $U\to 0$ as $|x|\to\infty$, we introduce the solutions of Eq. (5), $\phi(x;\xi)$, $\psi(x;\xi)$ and $\bar{\psi}(x;\xi)$, which satisfy the boundary conditions,

$$\phi(x;\xi) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}\exp(-i\xi x) \qquad \text{at} \quad x=-\infty, \tag{11 a}$$

$$\psi(x;\xi) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}\exp(i\xi x) \qquad \text{at} \quad x=\infty, \tag{11 b}$$

$$\bar{\psi}(x;\xi) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}\exp(-i\xi x) \qquad \text{at} \quad x=\infty. \tag{11 c}$$

The function $\bar{\psi}$ is called the adjoint function of ψ (as for the definition of adjoint, see §2). We can write ϕ in terms of ψ and $\bar{\psi}$ as

$$\phi = a(\xi)\bar{\psi} + b(\xi)\psi, \tag{12}$$

where the coefficients a and b satisfy

$$|a(\xi)|^2 + |b(\xi)|^2 = 1. \tag{13}$$

Here, ξ may be analytically continued to the upper half-plane of ζ $(\mathrm{Im}(\zeta)>0)$ in Eq. (12). The zeros of $a(\zeta)$, ζ_k's,

$$a(\zeta_k)=0, \quad (\mathrm{Im}(\zeta_k)>0) \qquad k=1, 2, \cdots, N, \tag{14}$$

determine the set of the discrete eigenvalues of Eq. (5) because the function ϕ tends to zero as $|x|\to\infty$. Now we introduce the quantities

$$\lambda_k = [b(\zeta_k)/a'(\zeta_k)]^{1/2}\exp(i\zeta_k x - i\zeta_k^2 t), \tag{15 a}$$

$$c(x,\xi) = (b(\xi)/a(\xi))\exp(2i\xi x - 2i\xi^2 t), \tag{15 b}$$

where a' is the derivative of a with respect to its argument. The method of the inverse scattering problem tells us[6)] that the solution of Eq. (1) is given by

$$u(x,t) = 2\sum_k \lambda_k^*\psi_{k2}^* + (\pi i)^{-1}\int_{-\infty}^{\infty}\Phi_2^* d\xi, \tag{16 a}$$

$$\int_x^\infty |u(x',t)|^2 dx' = -2i\sum_k \lambda_k\psi_{k1} + \pi^{-1}\int_{-\infty}^{\infty}\Phi_1 d\xi. \tag{16 b}$$

In the above expressions $\psi_{k1}, \psi^*_{k2}, \Phi_1, \Phi^*_2$ are the solutions of the simultaneous equations,

$$\Phi_1 - c(x,\xi)[(1+T)/2]\Phi^*_2 = -c(x,\xi)\sum_k \frac{\lambda^*_k}{\xi-\zeta^*_k}\psi^*_{k2}, \tag{17 a}$$

$$c^*(x,\xi)[(1-T)/2]\Phi_1 + \Phi^*_2 = c^*(x,\xi) + c^*(x,\xi)\sum_k \frac{\lambda_k}{\xi-\zeta_k}\psi_{k1}, \tag{17 b}$$

$$\lambda_j^{-1}\psi_{j1} + \sum_k(\zeta_j-\zeta^*_k)^{-1}\lambda^*_k\psi^*_{k2} = \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\Phi^*_2(\xi)}{\xi-\zeta_j}d\xi, \tag{18 a}$$

$$-\sum_k(\zeta^*_j-\zeta_k)^{-1}\lambda_k\psi_{k1} + \lambda_j^{*-1}\psi^*_{j2} = 1 + \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\Phi_1(\xi)}{\xi-\zeta^*_j}d\xi, \tag{18 b}$$

where T is the Hilbert transformation operator,

$$T\Phi = (\pi i)^{-1}P\int_{-\infty}^{\infty}\frac{\Phi(\xi')}{\xi'-\xi}d\xi'$$

(P denotes principal value). Once a and b are obtained for an initial value, $u(x, t=0)$, then we can get $u(x,t)$ at an arbitrary instant through Eqs. (15), (16), (17) and (18).

According to Zakharov and Shabat, the soliton solution corresponds to the discrete eigenvalue $\zeta_k = \xi_k + i\eta_k$ (ξ_k and η_k are both real);

$$\begin{aligned} S_k(x,t) &= 2\eta_k \operatorname{sech}[2\eta_k(x-2\xi_k t - x_k)] \\ &\qquad \times \exp[-2i\xi_k x + 2i(\xi_k^2-\eta_k^2)t], \\ x_k &= (2\eta_k)^{-1}\ln[b(\zeta_k)/a'(\zeta_k)]/(2\eta_k). \end{aligned} \tag{19}$$

Comparing Eq. (19) with Eq. (4) ,we find that the real and imaginary part of ζ_k are connected with the velocity and the amplitude of the soliton, respectively, such as $V=2\xi_k$ and $A=2\eta_k$.

In §2, the symmetry properties of Eq. (5) are investigated. If the initial value of Eq. (1) is a real and not antisymmetric function of x, it is found from such symmetry properties that $u(x,t)$ does not decay into the series of solitons moving with their respective velocities but indicates the formation of the bound state of solitons. In §3, the perturbation method is developed to make clear the condition for the decay of the initial modulated wave into the series of moving solitons. It is found that the bound state is, in general, not stable but decays into a series of solitons under an appropriate perturbation. In §4, the long-time asymptotic behaviors of modulated waves are considered. The non-soliton part is shown to decay as $t^{-1/2}$ and the soliton part to play the main role in the temporal development of solutions at $t\to\infty$. In §5, as a solvable example the special type of initial values, $u(x, t=0) = A\operatorname{sech}(x)$, is studied. The results of the numerical computation are also given.

§ 2. Eigenvalue problem

2-1 *The Galilei and Gauge transformations*

In the preceding section we note that Eq. (1) is invariant under the Galilei transformation. We now examine the invariance property of Eq. (5). Substituting Eq. (3) into Eq. (5), U is transformed to

$$U'(x')=\begin{pmatrix} e^{i\phi/2} & 0 \\ 0 & e^{-i\phi/2} \end{pmatrix} U(x) \begin{pmatrix} e^{-i\phi/2} & 0 \\ 0 & e^{i\phi/2} \end{pmatrix}, \tag{20 a}$$

$$\phi = Vx - V^2 t/2 + \alpha, \tag{20 b}$$

where α is an arbitrary constant. If the eigenfunction $v(x)$ is transformed as

$$v'(x')=\begin{pmatrix} e^{i\phi/2} & 0 \\ 0 & e^{-i\phi/2} \end{pmatrix} v(x), \tag{21}$$

then Eq. (5) becomes

$$iv'_{x'} + U'v' = (\zeta - V/2)\sigma_3 v'. \tag{22}$$

It is seen that in the frame moving with the velocity V the eigenvalue is reduced by $V/2$ compared with that in the rest frame. This reflects the fact that, as is shown in Eq. (19), the velocity of the soliton is given by $2\mathrm{Re}(\zeta_k)$.

When ϕ is constant, i.e., $\phi=\alpha$ in Eqs. (20a) and (21), the eigenvalue is unchanged because $V=0$. This implies that Eq. (1) is invariant under the gauge transformation, $u'=\exp(i\alpha)u(x)$.

2-2 *Discrete eigenvalue*

We consider the case that the eigenfunction of Eq. (5) satisfies the boundary condition, $v=0$ as $|x|\to\infty$ so that the eigenvalues constitute a discrete set. Let the eigenvalues and the corresponding eigenfunctions be $\zeta_1, \zeta_2, \cdots, \zeta_N$ and $v_1, v_2, \cdots, v_N$;

$$i\frac{dv_n(x)}{dx} + U(x)v_n(x) = \zeta_n \sigma_3 v_n(x), \qquad n=1, 2, \cdots, N. \tag{23}$$

For the following discussions we may introduce the Hermitian matrices defined by

$$\sigma_1=\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \sigma_2=\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \tag{24}$$

which are called Pauli's spin matrices together with σ_3 given by Eq. (7). Between them hold the relations,

$$\begin{aligned}&\sigma_i\sigma_j+\sigma_j\sigma_i=2\delta_{ij},\\&[\sigma_i,\sigma_j]=\sigma_i\sigma_j-\sigma_j\sigma_i=2i\sigma_k.\end{aligned}\tag{25}$$

In the second relation i, j, k denote 1, 2, 3 and its cyclic permutations. We can rewrite $U(x)$ by using σ_1 and σ_2

$$U(x)=\mathrm{Re}(u(x))\sigma_1-\mathrm{Im}(u(x))\sigma_2. \tag{8'}$$

We now investigate the orthogonality of the set of eigenfunctions. Multiplying Eq. (23) by σ_2 from the left and transposing the resulting equation, we find

$$-i\frac{dv_m^T}{dx}\sigma_2-v_m^TU^*\sigma_2=i\zeta_mv_m^T\sigma_1,$$

where the superscript T denotes transpose. Multiplying it by v_n from the right and Eq. (22) by $v_m^T\sigma_2$ from the left and subtracting one from the other, we obtain

$$(\zeta_n-\zeta_m)\int_{-\infty}^{\infty}v_m^T\sigma_1v_ndx=0.$$

In deriving the above equation the boundary conditions, v_n, $v_m\to0$ as $|x|\to\infty$, are taken into account. This immediately yields the ortho-normal condition,

$$\int_{-\infty}^{\infty}v_m^T\sigma_1v_ndx=\delta_{nm}. \tag{26}$$

Next we proceed to study the symmetry properties of Eq. (5).

(I) *If $u(x)$ satisfies $u(-x)=u^*(x)$, $v_n(x)$ has the symmetry,*

$$v_n(-x)=\beta\sigma_2v_n(x),\qquad \beta=\pm1.$$

Proof: Replacing x with $-x$ in Eq. (23) and multiplying by σ_2 from the left, we find

$$i\frac{d}{dx}[\sigma_2v_n(-x)]+U(x)[\sigma_2v_n(-x)]=\zeta_n\sigma_3[\sigma_2v_n(-x)].$$

Since $[\sigma_2v_n(-x)]$ is also the eigenfunction associated with ζ_n and behaves just like $v_n(x)$ in the asymptotic region, i.e.,

$$[\sigma_2v_n(-x)]\to0\qquad\text{for}\quad |x|\to\infty,$$

it can be concluded that

$$\sigma_2v_n(-x)=\beta v_n(x).$$

Using the relation twice, $\beta=\pm1$ can be shown.

(I′) *If* $u(-x)=-u^*(x)$, $v_n(x)$ *satisfies the symmetry property* $v_n(-x)=\beta\sigma_1 v_n(x)$, *where* $\beta=\pm 1$.

The proof goes parallel to that in (I) by putting σ_2 in place of σ_1 in the above verification.

(II) *If* $u(x)$ *is a symmetric (or antisymmetric) function of* x, *i.e.,* $u(-x)=\pm u(x)$, $-\zeta_n^*$ *is also the eigenvalue as well as* ζ_n *and the corresponding eigenfunction* w_n *is* $w_n^{(s)}(x)=\sigma_1 v_n^*(-x)$ $(w_n^{(a)}(x)=\sigma_2 v_n^*(-x))$. *The suffix s (or a) to the eigenfunction* w_n *specifies that u is symmetric (or antisymmetric).*

Proof: Let us consider the case $u(-x)=u(x)$. Replacing x with $-x$ in Eq. (23) and taking complex conjugate, we obtain

$$i\frac{d}{dx}[\sigma_1 v_n^*(-x)]+U(x)[\sigma_1 v_n^*(-x)]=-\zeta_n^*\sigma_3[\sigma_1 v_n^*(-x)].$$

Compared with Eq. (23), the above equation implies that $-\zeta_n^*$ is also the eigenvalue and the associated eigenfunction $w_n^{(s)}(x)$ equals to $\sigma_1 v_n^*(-x)$ with the arbitrariness of the proportional coefficient. For the case $u(-x)=-u(x)$, the proof is performed by replacing σ_1 with σ_2 in the above verification.

These symmetry properties are useful to take a general view of the solution of Eq. (1). As is noted in §1, the real part of the eigenvalue, ξ_n, corresponds to the velocity of a soliton and the imaginary part, η_n, the amplitude. Then, it can be seen that if $u(x, t)$, whose initial value has the symmetry $u(x, t=0)=\pm u(-x, t=0)$, breaks to the series of solitons, the decay is bisymmetric corresponding to the eigenvalues ζ_n and $-\zeta_n^*$.

If $u(x)$ is real, the symmetry property (I) yields

$$w_n^{(s)}(-x)=\sigma_1[-\beta\sigma_2 v_n^*(-x)]=\beta\sigma_2 w_n^{(s)}(x),$$
$$w_n^{(a)}(-x)=\sigma_2[\beta\sigma_1 v_n^*(-x)]=-\beta\sigma_1 w_n^{(a)}(x),$$

that is to say, $w_n^{(s)}(x)$ has the same parity as $v_n(x)$, while $w_n^{(a)}(x)$ has the opposite one. When $u(-x)=-u(x)$, therefore, provided that ζ_n is pure imaginary $(\zeta_n=-\zeta_n^*)$ the eigenvalues degenerate corresponding to the positive and negative parity eigenfunctions.

(III) *If* $u(x)$ *is real and not antisymmetric, then the eigenvalue* ζ_n *is pure imaginary, i.e.,* $\mathrm{Re}(\zeta_n)=0$.

Proof: From Eq. (23) and its Hermitian conjugate, we find

$$\mathrm{Re}(\zeta_n)(n|\sigma_2|n)=(n|\mathrm{Im}(u(x))\sigma_3|n), \tag{27}$$

where Re and Im denote the real and imaginary part, respectively. The matrix element $(m|Q(x)|n)$ is defined by

$$(m|Q(x)|n)=\int_{-\infty}^{\infty} v_m^\dagger Q(x) v_n dx, \tag{28}$$

where dagger represents Hermitian conjugate. In deriving Eq. (27) the identity, $[U, \sigma_1]=2i\,\mathrm{Im}(u)\sigma_3$, is used. It can be seen from Eq. (27) that $\mathrm{Re}(\zeta_n)$ vanishes if u is real and $(n|\sigma_2|n)\neq 0$. When u is a real and antisymmetric function of x, the symmetry property (I′) gives the relation

$$(n|\sigma_2|n)=\beta^2\int_{-\infty}^{\infty} v_n^\dagger(-x)\sigma_1\sigma_2\sigma_1 v_n(-x)dx$$
$$=-(n|\sigma_2|n),$$

and then $(n|\sigma_2|n)=0$. With the exception of this case, $(n|\sigma_2|n)$ is expected not to vanish. Indeed, the example in §5 assures $(n|\sigma_2|n)\neq 0$.

(IV) *If the initial value takes the form as $u=\exp(iVx)\times r(x)$, where $r(x)$ is a real and not antisymmetric function of x, all the eigenvalues have the common real part, $-V/2$.*

This can be easily shown by applying the discussion of the Galilei transformation in §2-1 to the theorem (III).

The conclusion is as follows: When $u(x, t=0)=\exp(iVx)\times r(x)$, where $r(x)$ is real and not antisymmetric for x, the solution does not decay to the series of solitons moving with the different velocities, but indicates the formation of the bound state of solitons (as for the definition of the bound state of solitons, see Ref. 6)). In this case, the real parts are common to all the eigenvalues, that is, the relative velocities of the solitons vanish.

(V) *If u is a real and not antisymmetric function of x, it holds that*

$$v_n^*(x)=i\gamma\sigma_3 v_n(x), \qquad \gamma=\pm 1. \tag{29}$$

Proof: Remembering that $\mathrm{Re}(\zeta_n)=0$ and taking the complex conjugate of Eq. (23), we see that $v_n^*(x)\propto\sigma_3 v_n(x)$. Substitution of Eq. (29) into the normalization condition (26) proves that $\gamma=\pm 1$.

2-3 *Continuous eigenvalue states*

Consider the case where the eigenvalue of Eq. (5) is real, i.e., $\zeta=\xi(=\text{real})$:

$$i\frac{dv}{dx}+Uv=\xi\sigma_3 v. \tag{30}$$

It can be readily shown that the adjoint function of v, which is defined by

$$\bar{v}=i\sigma_2 v^*, \tag{31}$$

is also the solution of Eq. (30);

$$i\frac{d\bar{v}}{dx}+U\bar{v}=\xi\sigma_3\bar{v}. \tag{30'}$$

It follows from Eqs. (30) and (30′) that

$$\frac{d}{dx}(v^\dagger v)=\frac{d}{dx}(\bar{v}^\dagger v)=\frac{d}{dx}(v^\dagger \bar{v})=\frac{d}{dx}(\bar{v}^\dagger \bar{v})=0. \tag{32}$$

Using the boundary conditions (11a∼c) in Eq. (32), we find

$$\phi^\dagger\phi=\psi^\dagger\psi=\bar{\psi}^\dagger\bar{\psi}=1, \tag{33 a}$$

$$\bar{\psi}^\dagger\psi=\psi^\dagger\bar{\psi}=0. \tag{33 b}$$

These are brought into Eq. (12) to yield

$$a=\bar{\psi}^\dagger\phi, \qquad b=\psi^\dagger\phi. \tag{34}$$

§ 3. Perturbation analysis

As is pointed out in §2, as long as the real (not antisymmetric) initial value is considered the solution does not decay into moving solitons but indicates the formation of the bound state of solitons pulsating with the proper frequency. We here develop the perturbation analysis in order to make clear under what initial values the solutions evolve to decay into moving solitons.

Assume that u undergoes a small change in Eq. (5), i.e.,

$$u \to u'=u+\Delta u, \tag{35}$$

and then the corresponding U varies by the quantity ΔU,

$$\Delta U=\begin{pmatrix} 0 & \Delta u \\ \Delta u^* & 0 \end{pmatrix}. \tag{36}$$

We here investigate the variation of the eigenvalue which is caused by the small change in u.

3-1 *Shifts of eigenvalues* I

Suppose that the change in u makes ζ_n and v_n into $\zeta_n+\Delta\zeta_n$ and $v_n+\Delta v_n$, respectively. Equation (23) becomes in the first order of the variations

$$\left[i\frac{d}{dx}+(U-\zeta_n\sigma_3)\right]\Delta v_n+(\Delta U-\Delta\zeta_n\sigma_3)v_n=0.$$

Multiplying by $v_n^T\sigma_2$ from the left and integrating with respect to x over $(-\infty, \infty)$, we see that

$$\begin{aligned}\Delta\zeta_n&=-i\int_{-\infty}^{\infty} v_n^T\sigma_2\Delta U v_n dx\\ &=-\int_{-\infty}^{\infty} v_n^T \operatorname{Re}(\Delta u)\sigma_3 v_n dx+i\int_{-\infty}^{\infty} v_n^T \operatorname{Im}(\Delta u) v_n dx.\end{aligned}$$

If u is a real and not antisymmetric function of x, Eq. (29) holds and the above expression is then written in the form

$$\Delta\zeta_n=\gamma(n|\mathrm{Im}(\Delta u)\sigma_3|n)+i\gamma(n|\mathrm{Re}(\Delta u)|n). \tag{37}$$

Note that the matrix elements $(n|\mathrm{Im}(\Delta u)\sigma_3|n)$ and $(n|\mathrm{Re}(\Delta u)|n)$ are real in view of their definitions. Equation (37) indicates that if $(n|\mathrm{Im}(\Delta u)\sigma_3|n)\neq 0$ the perturbation Δu makes the real part of the eigenvalue non-vanishing, i.e., the solution of Eq. (1) for the initial value, $u(x)+\Delta u(x)$, breaks up into moving solitons with the respective velocities $2\mathrm{Re}(\Delta\zeta_n)$.

If u is real and either symmetric or antisymmetric for x, the symmetry properties (I) and (I′) provide that

$$(n|\mathrm{Im}(\Delta u)\sigma_3|n)=-(n|\mathrm{Im}(\Delta u(-x))\sigma_3|n).$$

Hence, in this case we can see that if $\mathrm{Im}(\Delta u)$ is a symmetric function, $(n|\mathrm{Im}(\Delta u)\sigma_3|n)$ vanishes, i.e., $\mathrm{Re}(\Delta\zeta_n)=0$ and the soliton bound state does not resolve into moving solitons even in the presence of perturbation Δu.

3-2 *Shifts of eigenvalues* II

We deal with the double-humped initial values given by

$$u(x)=u_0(x-x_0)+\exp(i\alpha)u_0(x+x_0), \tag{38}$$

where $u_0(x)$ is a real and symmetric function of x and x_0, α are considered to be real. Corresponding to this $u(x)$, the eigenvalue equation is

$$i\frac{d}{dx}w_n(x)+U(x)w_n(x)=\zeta_n\sigma_3 w_n(x), \tag{39}$$

$$U(x)=U_1(x)+U_2(x), \tag{40}$$

$$U_1(x)=\sigma_1 u_0(x-x_0), \tag{41 a}$$

$$U_2(x)=[\cos(\alpha)\sigma_1-\sin(\alpha)\sigma_2]u_0(x+x_0). \tag{41 b}$$

We now define the eigenfunction $v_n(x)$ by

$$i\frac{d}{dx}v_n(x)+U_0(x)v_n(x)=\zeta_n^{(0)}\sigma_3 v_n(x), \tag{42}$$

where $U_0(x)=\sigma_1 u_0(x)$. Since $U_0(x)$ is the real and symmetric function of x, it is obvious from (I), (III) and (V) in §2-2 that $\zeta_n^{(0)}$ is pure imaginary and

$$v_n(-x)=\beta\sigma_2 v_n(x), \qquad \beta=\pm 1, \tag{43 a}$$

$$v_n^*(x)=i\gamma\sigma_3 v_n(x), \qquad \gamma=\pm 1. \tag{43 b}$$

To apply the perturbation analysis, we assume that x_0 is so large that the overlapping of two humps in $u(x)$ is small and that the eigenfunction $w_n(x)$ is approximated as

$$\left.\begin{aligned} w_n(x) &= w_{\pm}^{(n)}(x) + \Delta v(x), \\ w_{\pm}^{(n)}(x) &= [v_1^{(n)}(x) \pm v_2^{(n)}(x)]/\sqrt{2}\,, \end{aligned}\right\} \tag{44}$$

where Δv is the first order quantity of the overlapping, and $v_1^{(n)}$, $v_2^{(n)}$ satisfy the equations

$$i\frac{d}{dx}v_{1,2}^{(n)} + U_{1,2}v_{1,2}^{(n)} = \zeta_n^{(0)}\sigma_3 v_{1,2}^{(n)}. \tag{45}$$

The eigenfunction $v_{1,2}^{(n)}$ can be connected with $v_n(x)$ in Eq. (42);

$$v_1^{(n)}(x) = v_n(x - x_0), \tag{46 a}$$

$$v_2^{(n)}(x) = [\cos(\alpha/2) + i\sin(\alpha/2)\sigma_3]v_n(x + x_0) \tag{46 b}$$

(as for $v_2^{(n)}(x)$, remember the discussions in §2-1). Substituting Eqs. (46a) and (46b) into Eq. (44), transforming x into $-x$ and using Eq. (43a), we obtain the symmetry property

$$w_{\pm}^{(n)}(-x) = \pm\beta[\cos(\alpha/2)\sigma_2 + \sin(\alpha/2)\sigma_1]w_{\pm}^{(n)}(x). \tag{47}$$

Equation (47) yields the orthogonality between $w_+^{(n)}$ and $w_-^{(n)}$;

$$\int_{-\infty}^{\infty} w_+^{(n)T}(x)\sigma_1 w_-^{(n)}(x)dx = 0. \tag{48}$$

The norm of $w_{\pm}^{(n)}$ is calculated from Eqs. (44), (46a) and (46b);

$$\begin{aligned} \| w_{\pm}^{(n)}(x) \| &= \int_{-\infty}^{\infty} w_{\pm}^{(n)T}(x)\sigma_1 w_{\pm}^{(n)}(x)dx \\ &= 1 \pm \Big[\cos(\alpha/2)\int_{-\infty}^{\infty} v_n^T(x - x_0)\sigma_1 v_n(x + x_0)dx \\ &\quad + \sin(\alpha/2)\int_{-\infty}^{\infty} v_n^T(x - x_0)\sigma_2 v_n(x + x_0)dx\Big], \end{aligned} \tag{49}$$

where $v_n(x)$ is assumed to satisfy the ortho-normal condition (26). The deviation of $\| w_{\pm}^{(n)}(x) \|$ from unity is proportional to the overlapping integral, which can be taken to be small.

Substituting Eq. (44) into Eq. (39), letting $\zeta_n^{(\pm)} = \zeta_n^{(0)} + \Delta\zeta_n^{(\pm)}$ and using Eq. (45), we get, up to the first order for the overlapping integral,

$$\begin{aligned} \Delta\zeta_n^{(\pm)} = &-(i/2)\Big[\int_{-\infty}^{\infty} v_1^{(n)T}\sigma_2 U_2 v_1^{(n)}dx + \int_{-\infty}^{\infty} v_2^{(n)T}\sigma_2 U_1 v_2^{(n)}dx\Big] \\ &\mp (i/2)\Big[\int_{-\infty}^{\infty} v_2^{(n)T}\sigma_2 U_2 v_1^{(n)}dx + \int_{-\infty}^{\infty} v_1^{(n)T}\sigma_2 U_1 v_2^{(n)}dx\Big], \end{aligned} \tag{50}$$

where the double sign corresponds to that in Eq. (44). Using Eqs. (41), (43) and (46), we can show

$$\int_{-\infty}^{\infty} v_1^{(n)T}\sigma_2 U_2 v_1^{(n)} dx = \int_{-\infty}^{\infty} v_2^{(n)T}\sigma_2 U_1 v_2^{(n)} dx$$
$$= -\gamma \cos\alpha (n|u_0(x+2x_0)|n)$$
$$+ i\gamma \sin\alpha (n|\sigma_3 u_0(x+2x_0)|n),$$

$$\int_{-\infty}^{\infty} v_2^{(n)T}\sigma_2 U_2 v_1^{(n)} dx = \int_{-\infty}^{\infty} v_1^{(n)T}\sigma_2 U_1 v_2^{(n)} dx$$
$$= -\gamma \cos(\alpha/2)(n|u_0(x)\exp[2x_0(d/dx)]|n)$$
$$- i\gamma \sin(\alpha/2)(n|\sigma_3 u_0(x)\exp[2x_0(d/dx)]|n).$$

We finally obtain

$$\begin{aligned}\Delta\zeta_n^{(\pm)} = \gamma[&\sin\alpha(n|\sigma_3 u_0(x+2x_0)|n)\\ &\mp \sin(\alpha/2)(n|\sigma_3 u_0(x)\exp[2x_0(d/dx)]|n)]\\ &+ i\gamma[\cos\alpha(n|u_0(x+2x_0)|n)\\ &\pm \cos(\alpha/2)(n|u_0(x)\exp[2x_0(d/dx)]|n)].\end{aligned} \tag{51}$$

Note that the matrix elements in Eq. (51) are all real.

When $\alpha=0$, i.e., $u(x)$ is real and symmetric, $\Delta\zeta_n^{(\pm)}$ becomes pure imaginary. This is also expected from the general discussion in §2-2 (III).

When $\alpha=\pi$, i.e., $u(x)$ is real and antisymmetric, $\Delta\zeta_n^{(\pm)}$ possesses the real part;

$$\mathrm{Re}[\Delta\zeta_n^{(\pm)}(\alpha=\pi)] = \mp\gamma(n|\sigma_3 u_0(x)\exp[2x_0(d/dx)]|n), \tag{52 a}$$

$$\mathrm{Im}[\Delta\zeta_n^{(\pm)}(\alpha=\pi)] = -\gamma(n|u_0(x+2x_0)|n). \tag{52 b}$$

Equations (52a) and (52b) imply that the solutions develop to decay into paired solitons and each pair consists of solitons with the equal amplitude moving in the opposite directions at the equal speed. This case corresponds to the exceptional case shown in §2-2 (III).

For arbitrary α's, it is shown from Eq. (51) that the solutions break up into even number of moving solitons which are different both in the speed and in the amplitude.

Of course, in the above discussions the coefficients $b(\xi)$ does not vanish in general, so that the non-soliton part affects the behavior of solutions. When the long time asymptotic solutions are considered, however, the non-soliton part becomes unimportant (see §4).

3-3 *Variations of $a(\zeta)$ and $b(\zeta)$*

Assume that ϕ and ψ vary as

$$\phi' = \phi + \Delta\phi, \qquad \psi' = \psi + \Delta\psi, \tag{53}$$

corresponding to the small change Δu, $\Delta\phi$ and $\Delta\psi$ satisfying the boundary conditions,

$$\begin{aligned} &\Delta\phi=0 \quad \text{at} \quad x=-\infty, \\ &\Delta\psi=0 \quad \text{at} \quad x=+\infty. \end{aligned} \tag{54}$$

Then a and b also undergo small changes, i.e.,

$$a'=a+\Delta a, \qquad b'=b+\Delta b. \tag{55}$$

Substituting Eq. (53) into Eq. (29) and making use of the first order perturbation theory, we find

$$i\frac{d}{dx}\Delta\phi+(U-\xi\sigma_3)\Delta\phi=-\Delta U\phi. \tag{56}$$

The corresponding solution is easily obtained as

$$\Delta\phi=i\psi(x,\xi)\int_{-\infty}^{x}\psi^\dagger\Delta U\phi dx'+i\bar{\psi}(x,\xi)\int_{-\infty}^{x}\bar{\psi}^\dagger\Delta U\phi dx'. \tag{57 a}$$

Similarly, we have

$$\begin{pmatrix}\Delta\psi\\ \Delta\bar{\psi}\end{pmatrix}=-i\psi(x,\xi)\int_{x}^{\infty}\psi^\dagger\Delta U\begin{pmatrix}\psi\\ \bar{\psi}\end{pmatrix}dx'-i\bar{\psi}(x,\xi)\int_{x}^{\infty}\psi^\dagger\Delta U\begin{pmatrix}\psi\\ \bar{\psi}\end{pmatrix}dx'. \tag{57 b}$$

We here notice Eq. (57b) directly assures that $\Delta\bar{\psi}$ satisfies the relation (30), i.e., $\Delta\bar{\psi}=i\sigma_2\Delta\psi^*$. Substituting Eqs. (53) and (55) into Eq. (11) and taking Eqs. (57a) and (57b) into account, we get

$$\Delta a=ib\int_{-\infty}^{\infty}\bar{\psi}^\dagger\Delta U\psi dx'-ia\int_{-\infty}^{\infty}\psi^\dagger\Delta U\psi dx', \tag{58 a}$$

$$\Delta b=ib\int_{-\infty}^{\infty}\psi^\dagger\Delta U\psi dx'+ia\left\{\int_{-\infty}^{\infty}\bar{\psi}^\dagger\Delta U\psi dx'\right\}^*, \tag{58 b}$$

where we use the relations,

$$\int_{-\infty}^{\infty}\bar{\psi}^\dagger\Delta U\bar{\psi}dx=-\int_{-\infty}^{\infty}\psi^\dagger\Delta U\psi dx,$$

$$\int_{-\infty}^{\infty}\psi^\dagger\Delta U\bar{\psi}dx=\left\{\int_{-\infty}^{\infty}\bar{\psi}^\dagger\Delta U\psi dx\right\}^*.$$

Remembering Eq. (13), we can show that the shifts of the eigenvalues studied in §3-1 are connected with Δa through

$$\Delta\zeta_n=-\Delta a(\zeta_n)\Big/\frac{d}{d\zeta_n}a(\zeta_n). \tag{59}$$

From Eqs. (13) and (58a), this reduces to

$$\Delta\zeta_n=-i\left\{b(\zeta_n)\Big/\frac{d}{d\zeta_n}a(\zeta_n)\right\}\int_{-\infty}^{\infty}\bar{\psi}^\dagger\Delta U\psi dx. \tag{59'}$$

We note that the above result does not contradict Eq. (36) if the relation

$$v_n(x)=\left\{-ib(\zeta_n)\Big/\frac{d}{d\zeta_n}a(\zeta_n)\right\}^{1/2}\psi(x,\zeta_n)$$

is satisfied.

§ 4. Long-time asymptotic behavior of solutions

It is rather difficult to solve the singular integral equations (17) and (18) unless $b(\xi)$ vanishes. The long-time asymptotic solutions, however, can be easily obtained, showing the non-soliton part decays as $t^{-1/2}$ in the asymptotic region, $t\to\infty$.

The coefficient $c(x,\xi)$ (or $c^*(x,\xi)$) in Eq. (17) includes the rapid oscillating exponential factors with respect to ξ as $t\to\infty$. Thus, one can show that

$$\int_{-\infty}^{\infty}\Phi_1 d\xi,\qquad \int_{-\infty}^{\infty}\Phi_2^* d\xi=O(t^{-1/2}),\tag{60 a}$$

$$\int_{-\infty}^{\infty}\Phi_1/(\xi-\zeta_j^*)d\xi,\qquad \int_{-\infty}^{\infty}\Phi_2^*/(\xi-\zeta_j)d\xi=O(t^{-1/2})\tag{60 b}$$

as $t\to\infty$ (see Appendix). Inserting Eqs. (60a) and (60b) into Eqs. (16) and (18), we see that the non-soliton part of solutions, which includes the factor $b(\xi)$, does not contribute as $t\to\infty$ and, hence, only the soliton part is important.

In the asymptotic region $t\to\infty$, we can estimate the norm of the soliton part of solutions: Equation (16b) yields

$$\begin{aligned}\|u\|_\infty&=\int_{-\infty}^{\infty}|u(x,t=\infty)|^2dx\\&=-2i\Big[\sum_{k=1}^{N}\lambda_k\psi_{k1}\Big]_{x=-\infty}.\end{aligned}\tag{61}$$

With the aid of Eq. (60b) and the fact that $\lambda_j(x=-\infty)=\infty$ Eq. (18b) reduces

$$\sum_{k=1}^{N}[\lambda_k\psi_{k1}]_{x=-\infty}/(\zeta_k-\zeta_j^*)=1.\tag{62}$$

Following Ref. 6), we can solve Eq. (62);

$$[\lambda_k\psi_{k1}]_{x=-\infty}=\prod_{j=1}^{N}(\zeta_k-\zeta_j^*)\Big/\prod_{j\neq k}(\zeta_k-\zeta_j),\qquad k=1,2,\cdots,N.$$

We can then obtain

$$\sum_{k=1}^{N}[\lambda_k\psi_{k1}]_{x=-\infty}=(2i\pi)^{-1}\int d\zeta\prod_{j=1}^{N}(\zeta-\zeta_j^*)/(\zeta-\zeta_j),$$

where the integration is performed along the closed path involving all poles,

$\zeta_1, \zeta_2, \cdots, \zeta_N$, to yield

$$\|u\|_\infty = -2i\sum_{j=1}^{N}(\zeta_j-\zeta_j^*) = 4\sum_{j=1}^{N}\eta_j. \tag{63}$$

On the other hand, the norm of the soliton corresponding to the discrete eigenvalue with the imaginary part η_j is calculated from Eq. (19), to yield

$$\|u\|_\infty = \sum_{j=1}^{N}\|S_j(x, t)\|. \tag{64}$$

It must be noted here that the norm of the long-time asymptotic solutions is equal to the sum of that of each soliton.

§ 5. Example ; $u(x, t=0)=A\,\mathrm{sech}\,x$

We consider the case $u(x, t=0)=A\,\mathrm{sech}\,x$ as an example solved easily by means of the inverse-problem method. Eliminating v_2 in Eq. (5) and using $u=A\,\mathrm{sech}\,x$, we find

$$\begin{aligned} s(1-s)\frac{d^2}{ds^2}v_1+(1/2-s)\frac{d}{ds}v_1& \\ +\left[A^2+\frac{\zeta^2+i\zeta(1-2s)}{4s(1-s)}\right]v_1&=0, \end{aligned} \tag{65}$$

where $s=(1-\tanh x)/2$. We here note that the boundary $x=\infty$ corresponds to $s=0$ and $x=-\infty$ to $s=1$. Further transformation of the dependent variable v_1 into $s^\alpha(1-s)^\beta w_1$ reduces Eq. (65) to the hypergeometric equation, presenting the two linearly independent solutions

$$\begin{aligned} v_1^{(1)}(s)&=s^{i\zeta/2}(1-s)^{-i\zeta/2}F(-A, A, i\zeta+1/2; s), \\ v_1^{(2)}(s)&=s^{1/2-i\zeta/2}(1-s)^{-i\zeta/2} \\ &\quad\times F(1/2-i\zeta+A, 1/2-i\zeta-A, 3/2-i\zeta; s), \end{aligned} \tag{66}$$

where $F(\alpha, \beta, \gamma; s)$ is the hypergeometric function. The two linearly independent solutions $v_2^{(1)}$ and $v_2^{(2)}$ can be obtained by replacing ζ with $-\zeta$ in Eq. (66), i.e.,

$$\begin{aligned} v_2^{(1)}(s)&=s^{-i\zeta/2}(1-s)^{i\zeta/2}F(-A, A, -i\zeta+1/2; s), \\ v_2^{(2)}(s)&=s^{1/2+i\zeta/2}(1-s)^{i\zeta/2} \\ &\quad\times F(1/2+i\zeta+A, 1/2+i\zeta-A, 3/2+i\zeta; s). \end{aligned} \tag{67}$$

Let us construct the functions ϕ, ψ and $\bar{\psi}$ which satisfy the boundary conditions (11a)~(11c) for a continuous eigenvalue ξ, and seek for the coefficients a, b defined by Eq. (12). After the tedious but straightforward calculations, we obtain

$$\psi=\begin{pmatrix} A(\xi+i/2)^{-1}v_1^{(2)} \\ v_2^{(1)} \end{pmatrix}, \qquad \bar{\psi}=\begin{pmatrix} v_2^{(1)*} \\ -A(\xi-i/2)^{-1}v_1^{(2)*} \end{pmatrix}, \tag{68}$$

$$a(\xi)=[\Gamma(-i\xi+1/2)]^2/[\Gamma(-i\xi+A+1/2)\Gamma(-i\xi-A+1/2)], \tag{69 a}$$

$$\begin{aligned} b(\xi)&=i|\Gamma(i\xi+1/2)|^2/[\Gamma(A)\Gamma(1-A)] \\ &=i\sin(\pi A)/\cosh(\pi\xi). \end{aligned} \tag{69 b}$$

As is noted in §1, the discrete eigenvalues of Eq. (5) are given as the zeros of $a(\zeta)$, which is the analytic continuation of Eq. (69a) into the upper half-plane of ζ. We then obtain

$$\zeta_r=i(A-r+1/2), \tag{70}$$

where r must be positive integers satisfying $A-r+1/2>0$, i.e., $\mathrm{Im}(\zeta_r)>0$.

From Eq. (15)∼(17) and (68)∼(70), $u(x, t)$ is found at an arbitrary instant.

5-1 $A=N$

At first we consider the case $A=N$ (positive integer). It follows from Eq. (70) that there exist N discrete eigenvalues, all of which are purely imaginary, i.e., the solitons are in the bound state. It is easily seen from Eq. (69b) that $b(\xi)$ vanishes, so that the solution in this case consists only of soliton part. By means of Eq. (70), $a(\zeta)$ reduces to

$$a(\zeta)=\prod_{r=1}^{N}(\zeta-\zeta_r)/(\zeta-\zeta_r^*), \tag{71}$$

which coincides with the expression for $a(\zeta)$ obtained from the general discussion under the assumption of $b(\xi)=0$ by Zakharov and Shabat.[6] From Eqs. (69b) and (70), we obtain

$$b(\zeta_k)=(-1)^{k-1}i. \tag{72}$$

The coefficient $c(x, \xi)$ defined by Eq. (15b) is equal to zero and Eqs. (17a) and (17b) yield

$$\Phi_1=\Phi_2^*=0.$$

Substituting the above results into Eqs. (18a) and (18b) and using Eqs. (15a), (71) and (72), we can solve the equations for ψ_{k1} and ψ_{k2}^*. The eigenfunctions for $A=1$ and $A=2$ are given in Table I.

We can write down explicitly the solutions in the cases $N=1$ and 2;

$$u(x, t)=\exp(-it/2)\,\mathrm{sech}\,x \qquad \text{for} \quad N=1,$$

Table I. The eigenvalues and the eigenfunctions for $A=1, 2$. The coefficients β and γ in §§2-2 (I) and (V) are also given.

$A=1$ (One-eigenvalue state):

$$r=1;\quad \zeta=i/2,\quad v=(\operatorname{sech} x/2)\begin{pmatrix} i^{-1/2}e^{-x/2} \\ i^{1/2}e^{x/2} \end{pmatrix},$$
$$\beta=1,\quad \gamma=1.$$

$A=2$ (Two-eigenvalue state):

$$r=1;\quad \zeta=3i/2,\quad v=(3/8)^{1/2}\operatorname{sech}^2 x\begin{pmatrix} i^{-1/2}e^{-x/2} \\ i^{1/2}e^{x/2} \end{pmatrix},$$
$$\beta=1,\quad \gamma=1.$$
$$r=2;\quad \zeta=i/2,\quad v=8^{-1/2}\operatorname{sech}^2 x\begin{pmatrix} i^{-1/2}e^{-x/2}(2e^{x}-e^{-x}) \\ i^{1/2}e^{x/2}(e^{x}-2e^{-x}) \end{pmatrix},$$
$$\beta=-1,\quad \gamma=1.$$

$$u(x,t)=4\exp(-it/2)[\operatorname{ch}(4x)+4\operatorname{ch}(2x)+3\cos(4t)]^{-1}$$
$$\times[\operatorname{ch}(3x)+3\exp(-4it)\operatorname{ch}(x)]\qquad \text{for}\quad N=2.$$

When $N=1$, the solution corresponds to the one-soliton state, keeping its initial shape in the course of time. When $N=2$, on the other hand, the solution represents the bound state of solitons and its envelope pulsates with the frequency $\pi/2$. Beyond this point, the calculations rapidly become lengthier. Therefore we merely illustrate the results for $N=1$, 2, 3 in Fig. 1.

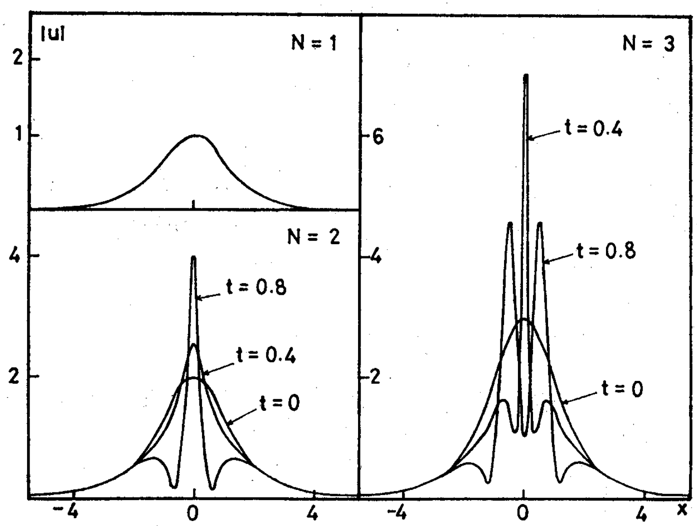

Fig. 1. Exact solutions for $u(x, t=0)=N\operatorname{sech} x$.

5-2 $A=N+\alpha$, $|\alpha|<1/2$

In this case, $a(\zeta)$ has N zeros, in other words, there exist N eigenvalues, but $b(\xi)$ is not zero. Equation (5) with non-vanishing $b(\xi)$ is complicated to be solved. As is shown in §4, however, the long-time asymptotic solutions have relatively simple structure, consisting of only the solitons.

In view of Eq. (70) the norm of the j-th soliton is obtained,

$$\|S_j\|=4\eta_j=4(N+a-j+1/2).$$

Substituting this into Eq. (64), we get

$$\|u\|_\infty=2N(N+2a).$$

The initial norm is given by

$$\|u\|_0=\int_{-\infty}^{\infty}|A\operatorname{sech}x|^2dx=2(N+a)^2.$$

The ratio of the norms is then obtained,

$$\|u\|_\infty/\|u\|_0=1-a^2/(N+a)^2. \tag{73}$$

We may say from Eq. (73) that the norm associated with the non-soliton part may be $a^2/(N+a)^2$ and that the long-time asymptotic solutions can be well described only by the soliton terms in the limit $N\gg1$. Such a circumstance is illustrated in Fig. 2, where the time evolutions of the maximum value of $|u|$ are shown for $A=0.8\sim1.4$ which are the cases only one soliton exists. All cases tend to steady levels oscillating around them.

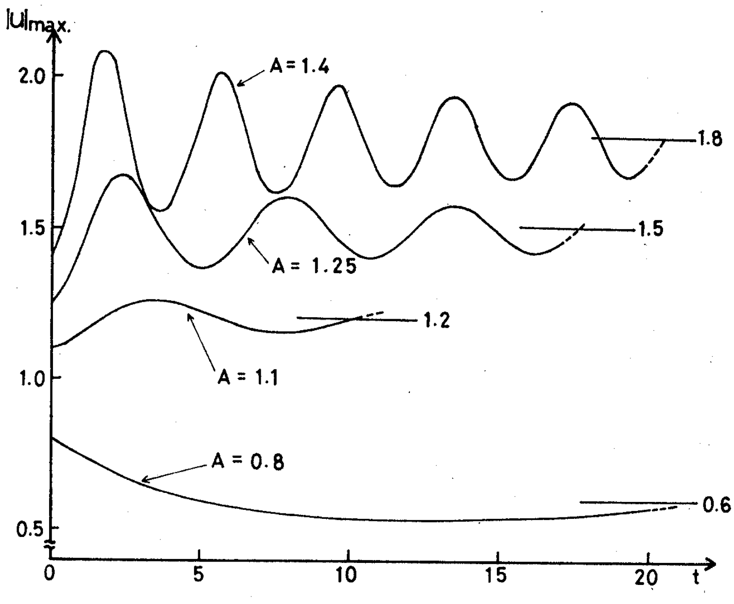

Fig. 2. Time development of the maximum value of $|u(x)|$ for $u(x, t=0)=A\times\operatorname{sech}x$. Straight lines denote the steady level the solutions converge. The relative errors are estimated; $|\Delta I_1/I_1|=2.52\%$ and $|\Delta I_2/I_2|=1.90\%$ for $A=1.4$, $|\Delta I_1/I_1|=0.25\%$ and $|\Delta I_2/I_2|=0.38\%$ for $A=1.25$, $|\Delta I_1/I_1|=0.41\%$ and $|\Delta I_2/I_2|=0.62\%$ for $A=1.1$, $|\Delta I_1/I_1|=0.10\%$ and $|\Delta I_2/I_2|=0.83\%$ for $A=0.8$. In these cases, the maximum of $|u|$ occurs at $x=0$.

5-3 *Numerical solutions*

First, we deal with the initial values given by

$$u(x)=\operatorname{sech}(x-x_0)+\exp(i\alpha)\operatorname{sech}(x+x_0), \tag{74}$$

where x_0 and α are both real. If $x_0 \gg 1$, we can apply the perturbation analysis in §3-2. Substituting the eigenfunction in Table I into Eq. (51), we obtain after the lengthy calculations

$$\begin{aligned}\zeta_\pm = &-\sin\alpha[1-2x_0\coth(2x_0)]/\sinh(2x_0)\\ &\pm\sin(\alpha/2)[1-2x_0/\sinh(2x_0)]/(2\sinh(x_0))\\ &+i\{1/2+\cos\alpha[2x_0/\sinh(2x_0)]\\ &\pm\cos(\alpha/2)[1+2x_0/\sinh(2x_0)]/(2\cosh(x_0))\}.\end{aligned} \tag{75}$$

When x_0 becomes $\lesssim 1$, the perturbation analysis cannot be applied. For this case, the behavior of solutions can be studied by analyzing numerically computed solutions of Eq. (1). In Fig. 3 the case of $x_0=0.6$ and $\alpha=\pi/2$ is illustrated. We can observe there the soliton of the amplitude ~ 1.70 moving with the velocity ~ 0.35. It is expected from Eq. (75) that there exist two eigenvalues, $\zeta_1 \cong 0.41+1.0i$ and $\zeta_2 \simeq 0.18-0.03i$. The observed soliton may correspond to the eigenvalue ζ_1, which implies the velocity of soliton to be $2\times 0.41 \sim 0.8$ and the amplitude $2\times 1.0 \sim 2.0$. Another eigenvalue, ζ_2, has negative imaginary part and, therefore, corresponds to no real soliton. These values have only rough meaning since the perturbation analysis is less reliable in this case.

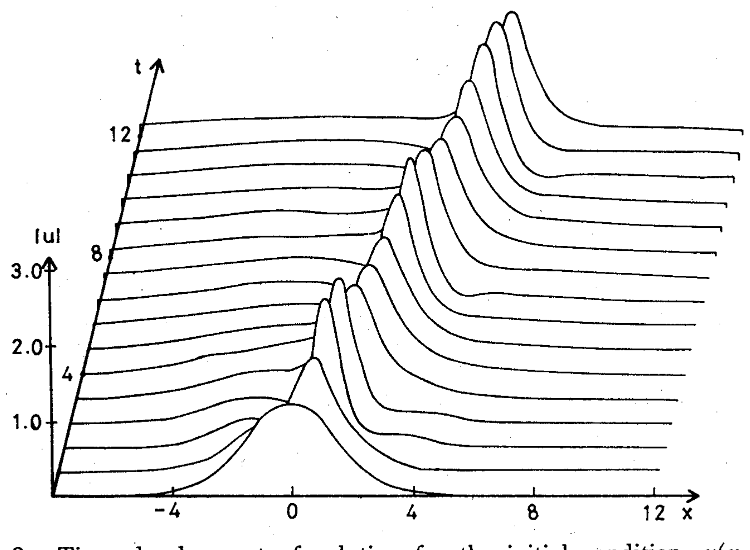

Fig. 3. Time development of solution for the initial condition, $u(x, t=0)= \operatorname{sech}(x-0.6)+i\operatorname{sech}(x+0.6)$. The amplitude converges to 1.70 with mean velocity 0.35. The relative errors are $|\Delta I_1/I_1|=0.30\%$ and $|\Delta I_2/I_2|=2.73\%$.

Figure 4 illustrates the case of the antisymmetric $u(x)$, i.e., $\alpha=\pi$ and $x_0=0.6$. In this case, the eigenvalues are estimated from Eq. (75) as $\zeta^{\pm}=\pm 0.16-0.29i$ so that no real solitons are expected to appear. The numerical solution recognizes such a tendency.

In Fig. 5, the numerical computation for the antisymmetric case, $u(x)=2\text{sech}(x-0.6)-2\text{sech}(x+0.6)$, is presented. The perturbation analysis (52) is applied to yield the eigenvalues $\zeta_1^{\pm}=\pm 0.14+0.17i$ and $\zeta_2^{\pm}=\pm 0.02-1.35i$, which imply the solitons of the amplitude ~ 0.34 and the velocity $\sim\pm 0.28$ to

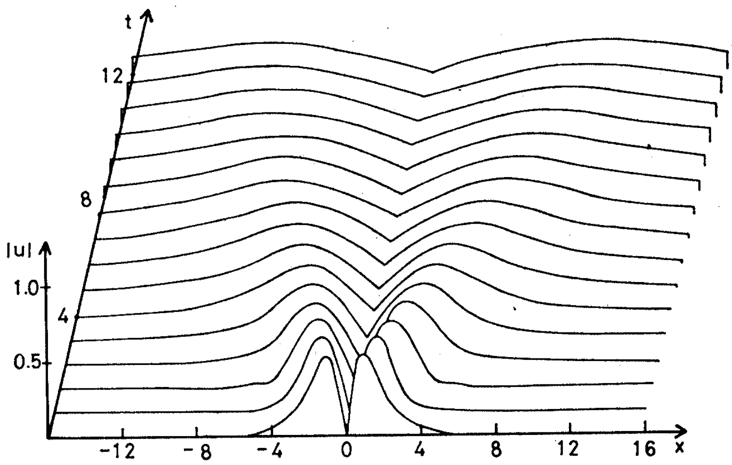

Fig. 4. Time development of solution for the initial condition, $u(x, t=0)=\text{sech}(x-0.6)-\text{sech}(x+0.6)$. The two peaks spread out to decay monotonously. The relative errors are $|\Delta I_1/I_1|=0.14\%$ and $|\Delta I_2/I_2|=0.07\%$.

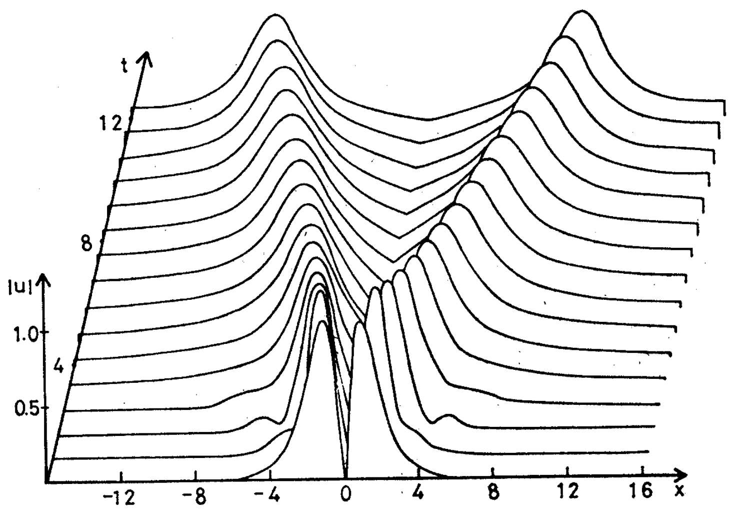

Fig. 5. Time development of solution for the initial condition, $u(x, t=0)=2\text{sech}(x-0.6)-2\text{sech}(x+0.6)$. Two solitons with the same amplitude ~ 0.75 and the opposite velocities with the same magnitude $\sim\pm 0.65$ are observed. The relative errors are $|\Delta I_1/I_1|=0.07\%$ and $|\Delta I_2/I_2|=0.75\%$.

be expected. The numerical computation indicates such a tendency, the symmetric decay of the initially modulated wave into the series of solitons.

Finally another example is shown in Fig. 6, in which the initial value is expressed by $u(x)=2\text{sech}(x)+i\,\text{sech}(x)\tanh(x)$. One can see that the solution decays into the two solitons, one of which has the larger amplitude and velocity than the other. This is also qualitatively expected from the perturbation analysis in §3-1.

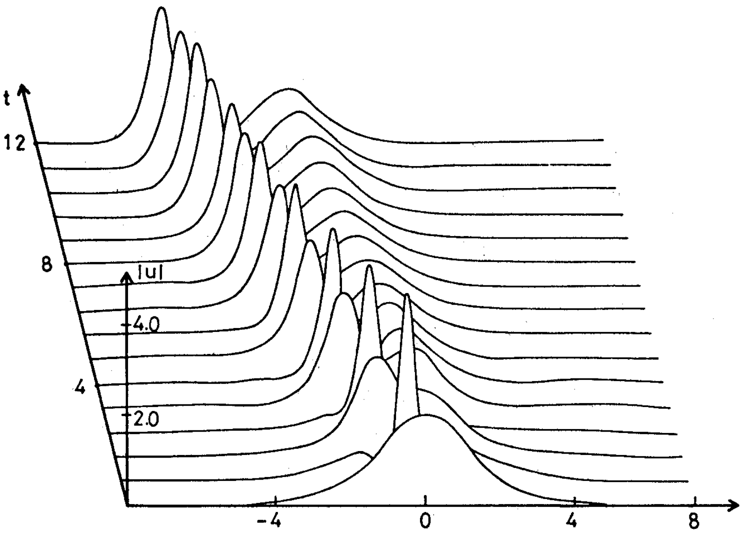

Fig. 6. Time development of solution for the initial condition, $u(x, t=0)=2\,\text{sech}\,(x)+i\,\text{sech}\,(x)\tanh\,(x)$. Two solitons emerge; one moves with the velocity ~-0.38 and has the amplitude ~3.1, and the other with the velocity ~0.16 and the amplitude ~1.2. The relative errors are $|\Delta I_1/I_1|=0.22\%$ and $|\Delta I_2/I_2|=14.65\%$.

The numerical analysis were made by replacing Eq. (1) with the difference equation, $i[u(x, t+\Delta t)-u(x, t-\Delta t)]/(2\Delta t)=[u(x+\Delta x, t)-2u(x, t)+u(x-\Delta x, t)]/(2\Delta x^2)+|u(x, t)|^2u(x, t)$. We used the periodic boundary condition, $u(x, t)=u(x+2L, t)$, $-L<x<L$. The value of L should be large enough not to influence on the behaviors of solutions. In our case $L=40$ was chosen. The mesh size was taken as $\Delta x=0.08$ and $\Delta t=0.0016$. The runs were inspected at every step by using the conserved quantities, $I_1=\int_{-L}^{L}|u(x, t)|^2dx$ and $I_2=\int_{-L}^{L}[|du(x, t)/dx|^2-|u(x, t)|^4/2]dx$. The maximum relative errors, $|\Delta I_1/I_1|$ and $|\Delta I_2/I_2|$ are written in each figure.

Appendix

Let us consider the integral

$$I=\int_{-\infty}^{\infty}\Delta(\xi)\exp[2it(\xi^2-\xi x/t)]d\xi, \tag{A·1}$$

where $\Delta(\xi)$ is analytic without the pole-singularities. Introducing the new variable η by $\eta=\xi-x/(2t)$, we get

$$I=\exp[-ix^2/(2t)]\int_{-\infty}^{\infty}\Delta(x/2t+\eta)\exp(2it\eta^2)d\eta. \tag{A·2}$$

By making use of the method of complex integral, one has

$$\begin{aligned}I=&\exp[-ix^2/(2t)]\{(i/(2t))^{1/2}\\&\times\int_{-\infty}^{\infty}\Delta(x/(2t)+i^{1/2}\rho/(2t)^{1/2})\exp(-\rho^2)d\rho\\&+2\pi i\textstyle\sum_j\Delta(\zeta_j)\exp[2it(\zeta_j-x/(2t))^2]\},\end{aligned} \tag{A·3}$$

where ζ_j is the pole of Δ involved in the fan-shape domain,

$$\begin{aligned}&\mathrm{Re}(\zeta_j-x/(2t))\mathrm{Im}(\zeta_j)>0,\\&[\mathrm{Re}(\zeta_j-x/(2t))]^2-[\mathrm{Im}(\zeta_j)]^2>0.\end{aligned} \tag{A·4}$$

In view of Eq. (A·4) the pole contributions in Eq. (A·3) decay exponentially as $t\to\infty$. Therefore, we obtain

$$I\simeq(i\pi/2t)^{1/2}\Delta(x/2t)\exp(-ix^2/2t)=O(t^{-1/2}) \tag{A·5}$$

for $t\to\infty$. Similarly, one can obtain

$$\begin{aligned}I^*=&\int_{-\infty}^{\infty}\Delta^*(\xi)\exp[-2it(\xi^2-\xi x/t)]d\xi\\&\simeq(-i\pi/2t)^{1/2}\Delta^*(x/2t)\exp(ix^2/2t)=O(t^{-1/2}).\end{aligned} \tag{A·6}$$

Integrating Eq. (17a) with respect to ξ over $(-\infty,\infty)$, using

$$(1\pm T)F(\xi)=(i\pi)^{-1}\int_{-\infty}^{\infty}F(\xi')/(\xi'-\xi\mp i\varepsilon)d\xi' \tag{A·7}$$

and applying Eqs. (A·5) and (A·6), one can show

$$\int_{-\infty}^{\infty}\Phi_1(\xi)d\xi=O(t^{-1/2}). \tag{A·8}$$

Multiplying Eq. (17a) by $(\xi-\zeta_j^*)^{-1}$ and integrating with respect to ξ over $(-\infty,\infty)$, one obtain, after similar calculations as above,

$$\int_{-\infty}^{\infty}\Phi_1(\xi)/(\xi-\zeta_j^*)d\xi=O(t^{-1/2}). \tag{A·9}$$

Similar procedure yields

$$\int_{-\infty}^{\infty}\Phi_2^*(\xi)d\xi,\quad\int_{-\infty}^{\infty}\Phi_2^*(\xi)/(\xi-\zeta_j)d\xi=O(t^{-1/2}). \tag{A·10}$$

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