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# *B*-polarization induction in two generalized thin sheets at the surface of a conducting half-space

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**Summary.** A new closed-form solution is obtained analytically for a $B$-polarization induction problem of geophysical interest, in which a local region of the Earth is represented by a generalized thin sheet at the surface of and in electrical contact with a uniformly conducting half-space. The generalized sheet, first introduced by Ranganayaki & Madden, is a mathematical idealization of a double layer which consists, in this problem, of two adjacent half-planes with distinct conductances representing a surface conductivity discontinuity such as an ocean–coast boundary, underlain by a uniform sheet of finite integrated resistivity representing the lower crust. The resistive sheet exerts a considerable mathematical influence on the solution causing, under certain conditions, an additional pole to appear in one of the forms of contour integral by which the solution can be expressed; it also weakens or eliminates field singularities that would otherwise occur at the conductance discontinuity. A numerical calculation is made for model parameters typifying an ocean–coast boundary underlain by a highly resistive crust. It is found that the residue of the pole associated with the resistive sheet dominates the solution for this example, the main consequence of which is a huge increase in the horizontal range over which the induced currents adjust themselves between the different 'skin-effect' distributions at infinity on either side of the model. Moreover the solution shows that this 'adjustment distance' has a more complicated dependence on the conductance and integrated resistivity of the sheet than that given simply by the square root of their product which was the length parameter proposed by Ranganayaki & Madden.

## 1 Introduction

In a previous paper (Dawson & Weaver 1979), hereafter referred to as DW, we analysed the problem of $B$-polarization induction in two adjacent half-sheets of different conductances

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at the surface of, and in electrical contact with, a uniformly conducting half-space. This model is of geophysical interest since it can be regarded as a mathematical idealization of a region of the Earth where the surface layer possesses a lateral change in electrical conductivity such as that occurring at an ocean–coast boundary. Since the $B$-polarization solution necessarily gives a constant magnetic field above and on the surface of the Earth it is possibly of less immediate practical interest than the corresponding $E$-polarization solution (Weidelt 1971) in which there appears the anomalous vertical magnetic field that is characteristic of the coast effect. Nevertheless, it does provide information on the behaviour of the apparent resistivity as measured by the magneto-telluric method with the telluric component in a direction normal to the conductivity boundary, and if one of the half-sheets represents an ocean, the solution on the underside of the sheet gives the variation of the electromagnetic field along the ocean floor. Moreover, in two-dimensional geometry it is only the $B$-polarization mode of induction that produces a poloidal current flow between the surface sheet and the underlying half-space and the importance of such currents is currently a matter of some interest (see Parkinson & Jones 1979, section 6 for a recent review).

One difficulty we encountered in attempting to make sensible calculations on such a simplified model of the coast effect was the choice of an appropriate value for the conductivity of the lower half-space. The integrated conductivity of the half-sheet representing the ocean was taken to be $1.6 \times 10^4$ S which is about right for a layer of sea-water 4 km deep, and the other half-sheet representing the surface rocks of the upper crust on the landward side of the coastal boundary was assumed to have an integrated conductivity of 400 S which is also a reasonable figure. However, the underlying half-space has two roles to play. It must have a sufficiently high conductivity to reproduce the attenuating effect of the conductive part of the mantle, and yet it must also serve as the electrical contact with the surface sheets. In the real Earth this contact is provided by a layer of the much more resistive material in the lower crust, and the influence it exerts is considerable since it virtually controls the flow of the poloidal currents. In DW the half-space was assigned a conductivity of $0.1\ \mathrm{S\,m^{-1}}$ which seems a reasonable average value for the mantle but is almost certainly too large for the lower crust. It is probable, therefore, that in the model calculations presented by DW the flow of electrical currents between the ocean and the mantle was exaggerated, and this in turn would have reduced the width of the zone in which field variations associated with the coast effect were important.

In a recent investigation Ranganayaki & Madden (1980) have recognized the significance of the resistive lower crust and its influence on the magneto-telluric response of the Earth in regions where the surface conductivity is changing laterally. For the numerical modelling of such regions they proposed an ingenious method of incorporating the effect of the resistive layer directly into a thin sheet theory, by making the sheet anisotropic with a vertical resistivity that was different from the reciprocal of the horizontal conductivity, and by modifying the thin sheet boundary conditions accordingly. On the basis of some approximate calculations in two-dimensional geometry they deduced that an important parameter for determining the 'adjustment distance' – the distance required by electric currents induced in the crust to adjust to changes in crustal conductance – was the square root of the conductivity-thickness product multiplied by the resistivity-thickness product for the crustal layer. Thus high values of the resistivity-thickness product were seen to be responsible for enormous increases in the adjustment distance.

The anisotropic (or generalized) thin sheet can also be regarded as a thin double layer (with the conductive layer on top and the resistive one underneath as in the real Earth), and the boundary conditions stated by Ranganayaki & Madden can be slightly simplified if they are reduced to the form obtained in the mathematical limit as the thickness of each layer approaches zero under the assumption that the conductivity-thickness and resistivity-thickness products both remain finite in the limit. It is then possible to obtain an *exact* analytical solution of the problem considered in DW extended to include a uniform resistive sheet beneath the two conductive half-sheets.

The extended model is clearly a much better representation of the real Earth than the model considered in DW, and the analytical solution will reveal more precisely than the approximate calculations by Ranganayaki & Madden exactly how the response of the Earth depends on the various parameters in the problem, especially on the integrated resistivity of the crust. Ideally it would be desirable to have the resistive sheet composed of two different half-sheets as well, since the resitivity of the sub-oceanic crust is thought to be smaller than that of the lower crust under the continents. However, it does not appear possible to solve the problem analytically unless the integrated resistivity of the lower sheet is uniform.

It is our main purpose in this paper to develop the analytical solution in full, and only one numerical calculation will be presented to illustrate the effect of the resistive sheet on the induced field. In a subsequent paper we hope, with the aid of further calculations, to examine the solution in more detail, and in particular to determine more precisely how the adjustment distance varies with changing crustal conditions.

## 2 Basic equations and boundary conditions

In Cartesian coordinates $(x, y, z)$ the model under consideration consists of a uniform half-space $z > 2\epsilon$ of conductivity $\sigma_0$ separated from a non-conducting half-space $z < 0$ by a thin region $0 < z < 2\epsilon$ composed of a conducting layer $0 < z < \epsilon$ of conductivity $\kappa(y, z)$ over a resistive layer $\epsilon < z < 2\epsilon$ of resistivity $\rho(z)$.

It is assumed that $\kappa$ and $\rho$ are smooth functions of $z$ and that $\kappa$ is piecewise smooth in $y$. In the particular model to be investigated here, which is shown in Fig. 1, the upper layer is divided by $y = 0$ into two regions of different conductivities so that

$$\kappa(y, z) = \begin{cases} \kappa_1(z) & (y > 0), \\ \kappa_2(z) & (y < 0). \end{cases} \tag{2.1}$$

Ultimately we shall treat this double layer as a mathematical thin sheet in the plane $z = 0$ by

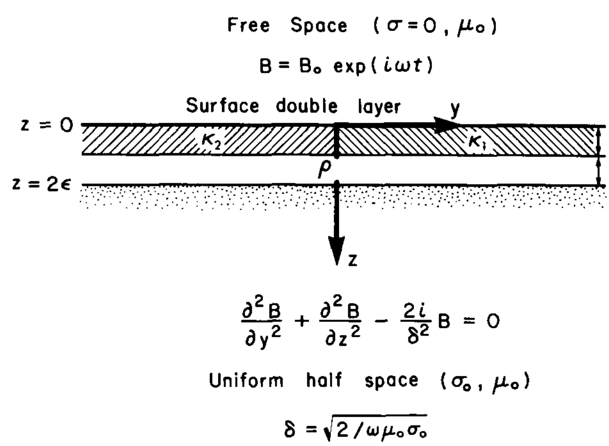


**Figure 1.** The mathematical model: the generalized thin sheet in the plane $z = 0$ is obtained by letting $\epsilon \to 0$.

assuming that the integrated conductivity and the integrated resistivity

$$\tilde{\kappa}_\epsilon(y) = \int_0^\epsilon \kappa(y, z)\, dz, \qquad \tilde{\rho}_\epsilon = \int_\epsilon^{2\epsilon} \rho(z)\, dz \tag{2.2}$$

both tend to finite, non-vanishing values, $\tilde{\kappa}(y)$ and $\tilde{\rho}$ respectively, as $\epsilon \to 0$. (It would be possible, of course, to assume that the two layers have different thicknesses initially and to let each tend to zero separately, but the final results are the same.) Vacuum permeability $\mu_0$ is assumed everywhere and all quantities are measured in SI units.

The inducing field is assumed independent of $x$ and harmonic in time $t$ with angular frequency $\omega$. The resulting problem is strictly two-dimensional in the coordinates $y$ and $z$. The $B$-polarization field to be considered here has the single total magnetic field component $B(y, z) \exp(i\omega t)$ in the $x$-direction, and the electric field components $V(y, z) \exp(i\omega t)$ in the $y$-direction and $W(y, z) \exp(i\omega t)$ in the (vertically downwards) $z$-direction. In the quasi-static approximation, the spatial parts of the field components are related by the simplified Maxwell equations

$$i\omega B(y, z) = \partial V(y, z)/\partial z - \partial W(y, z)/\partial y, \tag{2.3}$$

$$\mu_0 \sigma(y, z) V(y, z) = \partial B(y, z)/\partial z, \tag{2.4}$$

$$\mu_0 \sigma(y, z) W(y, z) = -\partial B(y, z)/\partial y, \tag{2.5}$$

where

$$\sigma(y, z) = \begin{cases} 0 & (z < 0), \\ \kappa(y, z) & (0 < z < \epsilon), \\ 1/\rho(z) & (\epsilon < z < 2\epsilon), \\ \sigma_0 & (z > 2\epsilon). \end{cases} \tag{2.6}$$

Equations (2.3) to (2.6) show that above the double layer the magnetic field has the constant value

$$B(y, z) = B_0 \qquad (z < 0), \tag{2.7}$$

and that below the layer it satisfies

$$(\partial^2/\partial y^2 + \partial^2/\partial z^2 - 2i/\delta^2) B(y, z) = 0 \qquad (z > 2\epsilon), \tag{2.8}$$

where

$$\delta = (2/\omega\mu_0\sigma_0)^{1/2} \tag{2.9}$$

is the skin depth in the lower half-space.

Within the conducting layer $0 < z < \epsilon$ it follows from a vertical integration of (2.3) and substitution of equation (2.5) that

$$V(y, z) - V(y, 0-) = \int_0^z \left\{ i\omega B(y, u) - \frac{1}{\mu_0} \frac{\partial}{\partial y} \left( \frac{\partial B(y, u)/\partial y}{\kappa(y, u)} \right) \right\} du \tag{2.10}$$

where we have noted that $V(y, 0-) = V(y, 0+)$ by the continuity of the tangential electrical field across the boundary $z = 0$. Now the magnetic field $B$ is continuous and bounded in the layer and the second term in the integrand above is also continuous except where $\kappa$ or $\partial\kappa/\partial y$ has a horizontal discontinuity. Elsewhere the term is clearly bounded and it remains so, with the bound diminishing, as the layer approaches a thin sheet since $1/\kappa$ must tend to zero as $\epsilon \to 0$ in order that the integrated conductivity may have a non-vanishing value in the limit. For the model shown in Fig. 1 the only discontinuity in $\kappa$ is at

$y = 0$ and the modulus of the integrand in (2.9) is therefore bounded for $y < 0$ and $y > 0$. It follows that we may estimate the integral in (2.10) as $O(z)$, and since $z < \epsilon$ we deduce from (2.10)

$$V(y, z) - V(y, 0-) = O(\epsilon) \qquad (0 < z < \epsilon). \tag{2.11}$$

When $\epsilon \to 0$, this gives the familiar boundary condition that the tangential electric field is continuous across a conducting sheet. This condition was the basic assumption made by Price (1949) when he first developed the theory of induction in thin sheets.

By integrating equation (2.4) across the thickness of the layer, substituting from (2.11) and (2.2), and noting that $B(y, 0+) = B(y, 0-)$, and $B(y, \epsilon+) = B(y, \epsilon-)$ by continuity of the tangential magnetic field across the surfaces $z = 0$ and $z = \epsilon$, we obtain

$$B(y, \epsilon+) - B(y, 0-) = \mu_0 \tilde{\kappa}_\epsilon(y) V(y, 0-) + O(\epsilon). \tag{2.12}$$

A similar integration within the second layer $\epsilon < z < 2\epsilon$ together with the fact that $B$ is continuous across $z = 2\epsilon$ gives

$$B(y, 2\epsilon+) - B(y, z) = \mu_0 \int_z^{2\epsilon} \frac{V(y, u)}{\rho(u)}\, du.$$

Here the integrand is just the current density which must be bounded because the layer is resistive. Thus we may write

$$B(y, 2\epsilon+) - B(y, z) = O(\epsilon) \qquad (\epsilon < z < 2\epsilon), \tag{2.13}$$

and since the horizontal second derivative of the current density is also bounded,

$$\partial^2 B(y, 2\epsilon+)/\partial y^2 - \partial^2 B(y, z)/\partial y^2 = O(\epsilon), \qquad (\epsilon < z < 2\epsilon). \tag{2.14}$$

Putting $z = \epsilon+$ in (2.13) and adding it to (2.12) we obtain

$$B(y, 2\epsilon+) - B(y, 0-) = \mu_0 \tilde{\kappa}_\epsilon(y) V(y, 0-) + O(\epsilon). \tag{2.15}$$

Finally, we integrate (2.3) across the second layer and substitute from equation (2.5) to get

$$V(y, 2\epsilon+) - V(y, \epsilon-) = \int_\epsilon^{2\epsilon} \left\{ i\omega B(y, z) - \frac{\rho(z)}{\mu_0} \frac{\partial^2 B(y, z)}{\partial y^2} \right\} dz$$

where this time we have noted that $V(y, \epsilon-) = V(y, \epsilon+)$ and $V(y, 2\epsilon-) = V(y, 2\epsilon+)$. Substituting from (2.13), (2.14) and the second definition (2.2), and adding the resulting equation to (2.11) evaluated at $z = \epsilon-$, we obtain

$$V(y, 2\epsilon+) - V(y, 0-) = -(\tilde{\rho}_\epsilon/\mu_0)\partial^2 B(y, 2\epsilon+)/\partial y^2 + O(\epsilon). \tag{2.16}$$

Equations (2.15) and (2.16) give the change in $B$ and $V$ across the double layer. In the limit as $\epsilon \to 0$ they reduce to the boundary conditions satisfied by the tangential magnetic and electric fields across the generalized thin sheet comprising a non-uniform conducting sheet of integrated conductivity $\tilde{\kappa}(y)$ over a uniform resistive sheet of integrated resistivity $\tilde{\rho}$. For this mathematical idealization to apply to the real Earth it will be necessary for the two layers to be sufficiently thin that the horizontal electric field remains fairly uniform in the conducting layer and likewise for the horizontal magnetic field through the resistive layer. This will ensure that the basic conditions (2.11) and (2.13) of the thin sheet idealization are approximately satisfied, and will clearly impose limitations on the actual thicknesses permitted for a given frequency of the inducing field. The appropriate values of $\tilde{\kappa}$ and $\tilde{\rho}$ in a generalized thin sheet representing actual layers of thicknesses $d_1$ and $d_2$ composed of materials whose conductivities are $\sigma_1$ and $\sigma_2$ respectively would be $\sigma_1 d_1$ and $d_2/\sigma_2$ as indicated by the definitions (2.2).

For the particular model shown in Fig. 1 equations (2.15) and (2.16) can be simplified by noting that $B(y, 0-)$ and $B(y, 2\epsilon+)$ satisfy equations (2.7) and (2.8) respectively. In addition $V(y, 2\epsilon+)$ can be related to the vertical derivative of $B$ in the underlying half-space through equation (2.4). Making these substitutions, eliminating $V(y, 0-)$ from (2.15) and (2.16), and finally taking the limit as $\epsilon \to 0$ we obtain a single boundary condition on $B$ at the lower surface of the generalized thin sheet in the form

$$\{1 - 2ir\lambda(y)/\delta^2\}B(y, 0+) = B_0 + \lambda(y)\{B'(y, 0+) - rB''(y, 0+)\} \qquad (y \neq 0), \tag{2.17}$$

where

$$r = \sigma_0\tilde{\rho}, \qquad \lambda(y) = \frac{\tilde{\kappa}(y)}{\sigma_0} = \begin{cases} \lambda_1 & (y > 0), \\ \lambda_2 & (y < 0). \end{cases} \tag{2.18}$$

Here, and throughout the rest of this paper, we denote derivatives in $z$ by primes on the function symbols. The parameters $\lambda_1$ and $\lambda_2$ have been introduced as the integrated conductivities, normalized by the conductivity of the underlying half-space, of the half-sheets $y > 0$ and $y < 0$ that form the conducting layer in the generalized thin sheet. They have the dimensions of length, as does the normalized integrated resistivity $r$.

The boundary condition (2.17) is valid at all points $y$ in the regions $y > 0$ and $y < 0$. The form it takes and the method of obtaining it are rather different from those given by Ranganayaki & Madden (1980). They proceeded by approximating the vertical derivatives in Maxwell's equations by finite differences and by using impedance boundary conditions on the top and bottom of the sheet. They also retained in their analysis some, but not all, of the (small) terms involving the thicknesses of the layers rather than taking the limit as these thicknesses tend to zero. (The integrated conductivity and resistivity of the two layers were simply taken as the conductivity-thickness and resistivity-thickness products of the respective layers.) The outcome was a general condition, involving several different terms, giving the tangential electric field on the top surface of a generalized thin sheet in terms of the inducing magnetic field and the impedances (expressed in terms of the model parameters and the wavenumbers of the field) of the upper and lower (layered) half-spaces. Although their result appears quite complicated, a number of terms drop out under the simplifying conditions pertaining to the model being discussed here, i.e. a strictly two-dimensional field in the $B$-polarization mode, a lower half-space of uniform conductivity, and a sheet of negligible thickness. If the simplified equation is then expressed in terms of the tangential magnetic field on the bottom of the sheet, rather than the electric field on the top, it is easy to show that their condition reduces to our (2.17).

In our notation the tangential electric field $V(y, 0-)$ on top of the sheet is given, through (2.4) and the limit of (2.16) as $\epsilon \to 0$, by

$$\mu_0\sigma_0 V(y, 0-) = B'(y, 0+) + r\partial^2 B(y, 0+)/\partial y^2. \tag{2.19}$$

As $y \to \pm\infty$ the model approaches distinct, strictly one-dimensional problems in $z$ alone. All derivatives with respect to $y$ vanish at infinity, and the limiting values of $B(y, z)$ are identical to those of DW, namely

$$B_j(z) = B_0 \exp[-(1+i)z/\delta]/\{1 + (1+i)\lambda_j/\delta\} \qquad (z > 0) \tag{2.20}$$

for $j = 1, 2$ where $B_1(z) \equiv B(+\infty, z)$ and $B_2(z) \equiv B(-\infty, z)$. The average and difference fields

$$\bar{B}(z) = \tfrac{1}{2}\{B_1(z) + B_2(z)\}, \qquad \Delta B(z) = B_1(z) - B_2(z) \tag{2.21}$$

will prove useful later.

## 3 Solution by the Wiener–Hopf technique

The functions $f_+$ and $f_-$ are defined by

$$f_+(y, z) + f_-(y, z) \equiv f(y, z) = B(y, z) - B_1(z) \tag{3.1}$$

and $f_+ = 0$ $(y < 0)$; $f_- = 0$ $(y > 0)$. It is clear from equations (3.1), (2.20) and (2.8) that $f$ satisfies the differential equation

$$(\partial^2/\partial y^2 + \partial^2/\partial z^2 - 2i/\delta^2) f(y, z) = 0, \qquad (z > 0). \tag{3.2}$$

The boundary conditions on $f_\pm$ as $y \to \pm\infty$ can be deduced from equations (3.1) and (2.20). The anomalous field $f_+$ on the right side of the model will decay exponentially within the conductor and we may therefore assert that

$$f_+(y, z) = O[\exp(-cy)] \qquad (y \to +\infty), \tag{3.3}$$

where $c > 0$. The exact value of $c$ depends on the skin depth $\delta$, and on the integrated conductivities $\lambda_1$, $\lambda_2$ and resistivity $r$, and is discussed in more detail in the Appendix. On the left side of the model, the limiting value is

$$f_-(y, z) \sim -\Delta B(z), \qquad (y \to -\infty). \tag{3.4}$$

It is clear from (3.1) that $f_\pm \to 0$ as $z \to +\infty$.

The conditions on $f$ at the surface can be derived by eliminating $B$ in (2.17) in favour of $f_\pm$. It follows that

$$(1 - 2ir\lambda_1/\delta^2) f_+(y, 0+) = \lambda_1 \{f_+'(y, 0+) - r f_+''(y, 0+)\} \tag{3.5}$$

$$(1 - 2ir\lambda_2/\delta^2) f_-(y, 0-) = \lambda_2 \{f_-'(y, 0+) - r f_-''(y, 0+)\} + iA\Delta\lambda\sqrt{2\pi} \tag{3.6}$$

where

$$A = (i - 1) B_1(0)/(\delta\sqrt{2\pi}), \qquad \Delta\lambda = \lambda_1 - \lambda_2, \tag{3.7}$$

and where primes again denote differentiation in $z$.

The above set of equations can be solved by introducing the Fourier transform in $y$ defined by

$$F(\zeta, z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(y, z) \exp(i\zeta y)\, dy = F_+(\zeta, z) + F_-(\zeta, z) \tag{3.8}$$

where $\zeta = \xi + i\eta$ is complex. The second equality expresses $F$ as the sum of the Fourier transforms $F_\pm$ of $f_\pm$ where $F_\pm$ and $F_-$ are analytic in the respective half-planes $\eta > -c$ and $\eta < 0$. The function $F$ itself is analytic in the strip $-c < \eta < 0$ of the complex $\zeta$-plane.

The solution that vanishes as $z \to +\infty$ of the differential equation (3.2), Fourier transformed by (3.8), is

$$F(\zeta, z) = F(\zeta, 0+) \exp[-z\gamma(\zeta)] \qquad (-c < \eta < 0), \tag{3.9}$$

where

$$\{\gamma(\zeta)\}^2 = \zeta^2 + 2i/\delta^2, \tag{3.10}$$

with $\mathrm{Re}\,\gamma \geqslant 0$ in $-c < \eta < 0$.

From (3.9) we obtain the results

$$F'(\zeta, 0+) = -\gamma(\zeta)F(\zeta, 0+), \qquad F''(\zeta, 0+) = \{\gamma(\zeta)\}^2 F(\zeta, 0+). \tag{3.11}$$

The boundary conditions (3.5) and (3.6) can be Fourier-transformed into

$$\begin{aligned}(1-2ir\lambda_1/\delta^2)F_+(\zeta, 0+) &= \lambda_1\{F_+'(\zeta, 0+) - rF_+''(\zeta, 0+)\}\\ (1-2ir\lambda_2/\delta^2)F_-(\zeta, 0+) &= \lambda_2\{F_-'(\zeta, 0+) - rF_-''(\zeta, 0+)\} + A\,\Delta\lambda/\zeta.\end{aligned} \tag{3.12}$$

Equations (3.12) can be combined so that the terms $F_\pm' - rF_\pm''$ add to give $F' - rF''$, which in turn can be expressed in terms of $F = F_+ + F_-$ by (3.11). The resulting equation is

$$F_+(\zeta, 0+)K_1(\zeta)/\lambda_1 + F_-(\zeta, 0+)K_2(\zeta)/\lambda_2 = A\,\Delta\lambda/(\lambda_2\zeta) \tag{3.13}$$

where

$$K_j(\zeta) \equiv 1 + \lambda_j\gamma(\zeta) + r\lambda_j\zeta^2 \qquad (j = 1, 2). \tag{3.14}$$

It should be noted that $K_1$ and $K_2$ are the appropriate generalizations to this problem of the functions $P$ and $N$ respectively of DW, and they reduce to them when $r \to 0$. From equation (3.14) and definition (3.7) we obtain the important result

$$\lambda_1 K_2(\zeta) - \lambda_2 K_1(\zeta) = \Delta\lambda. \tag{3.15}$$

The factorizations

$$K_j(\zeta) = K_j^+(\zeta)K_j^-(\zeta) \qquad (j = 1, 2), \tag{3.16}$$

of the functions $K_j$ into parts $K_j^+$ and $K_j^-$ analytic and zero-free in the half-planes $\eta > -c$ and $\eta < c$ respectively are considered in the Appendix. The ratio functions

$$R(\zeta) \equiv K_1(\zeta)/K_2(\zeta), \qquad R_\pm(\zeta) = K_1^\pm(\zeta)/K_2^\pm(\zeta) \tag{3.17}$$

are introduced for convenience in the algebra to follow.

The elimination of $\Delta\lambda$ from equations (3.13) and (3.15), and a division of both sides of the resulting equation by $K_1^- K_2^+$ leads to the expression

$$F_+(\zeta, 0+)R_+(\zeta)/\lambda_1 + A\{R_+(\zeta) - R_+(0)\}/\zeta = G(\zeta) \tag{3.18}$$

where

$$G(\zeta) = \frac{A\lambda_1}{\zeta}\left\{\frac{1}{\lambda_2 R_-(\zeta)} - \frac{R_+(0)}{\lambda_1}\right\} - \frac{F_-(\zeta, 0+)}{\lambda_2 R_-(\zeta)}. \tag{3.19}$$

Arguments analogous to those used in DW show that equations (3.18) and (3.19) together define $G$ as a bounded entire function in the $\zeta$-plane, and that $G$ vanishes as $\zeta \to \infty$ since $F_\pm(\zeta, 0+) = O(1/\zeta)$ for a magnetic field $B(y, 0+)$ that is bounded at the origin. Hence $G(\zeta) \equiv 0$ and we may then solve (3.19) for $F_-(\zeta, 0+)$, (3.18) for $F_+(\zeta, 0+)$, and add the results to obtain, with the help of (3.15),

$$F(\zeta, 0+) = \frac{A\,\Delta\lambda\,K_1^+(0)K_2^-(0)}{\zeta K_2(0)K_1^+(\zeta)K_2^-(\zeta)}. \tag{3.20}$$

Equations (3.9) and (3.20) constitute the solution to the problem. Defining the functions

$$k_j(\zeta) = K_j(\zeta)/K_j(0), \qquad k_j^\pm(\zeta) = K_j^\pm(\zeta)/K_j^\pm(0) \qquad (j = 1, 2), \tag{3.21}$$

we can simplify equation (3.20) to

$$F(\zeta, 0+) = i\Delta B(0)/\{\zeta k_1^+(\zeta)k_2^-(\zeta)\sqrt{2\pi}\} \tag{3.22}$$

by liberal use of (3.21), (3.7), (2.20) and (2.21). The fundamental solution for $B(y, z)$ in $z > 0$ follows by substitution of (3.22) into (3.9), Fourier inversion and subsequent elimination of $f$ with equation (3.1). We find that

$$B(y,z) = B_1(z) - \frac{\Delta B(0)}{2\pi i} \int_{-\infty - ib}^{\infty - ib} \frac{\exp[-iy\zeta - z\gamma(\zeta)]}{k_1^+(\zeta)k_2^-(\zeta)} \frac{d\zeta}{\zeta}, \tag{3.23}$$

where $0 < b < c$.

## 4 Evaluation of the integral

As in DW, the integral (3.23) can be transformed to two useful forms, one with exponential decay in $z$ and the other with exponential decay in $|y|$. The most direct form is obtained by letting $b \to 0+$ in equation (3.23). The contribution from the pole at $\zeta = 0$ is $\pi i \exp[-(1+i)z/\delta]$, and there remains an integral along the real axis evaluated as a Cauchy principal value. This integral can be simplified with equation (A29), and when account is taken of equations (2.20) and (2.21), the result can be expressed as

$$B(y,z) = \bar{B}(z) - L_{02}^-(y,z) \tag{4.1}$$

where we have defined

$$L_{mn}^{\pm}(y,z) = \frac{\Delta B(0)}{2\pi i} \int_0^{\infty} \left\{ \frac{\exp(-iy\xi)}{k_1^+(\xi)k_2^-(\xi)} \pm \frac{\exp(iy\xi)}{k_1^-(\xi)k_2^+(\xi)} \right\} \frac{\{\gamma(\xi)\}^m}{\xi^{n-1}} \exp[-z\gamma(\xi)]\, d\xi. \tag{4.2}$$

This equation is identical in form to equation (4.3) of DW; it differs only in the nature of the functions $k_j^{\pm}$.

Alternative forms are obtained when the integral in (3.23) is evaluated by closing the contour outside the strip of analyticity $-c < \eta < 0$. This can be done by analytic continuation of the function $\gamma(\zeta)$, defined by (3.10), into the whole complex plane in such a manner that its real part remains non-negative thereby preserving the convergence of the integral (3.23) for $z > 0$. If the principal value of the square root function in (3.10) is taken, it is necessary to cut the $\zeta$-plane from the branch points $\zeta = \pm(1-i)/\delta$ to infinity along the hyperbolic arcs $\xi\eta = -1/\delta^2$, $|\eta| > |\xi|$, as shown in Fig. 2.

For $y < 0$ the contour must be closed in the upper half-plane. If we substitute $\exp[h(\zeta)] \equiv k_1^+(\zeta)/k_2^+(\zeta)$ and $K_2(\zeta) \equiv M_2(\zeta)N_2(\zeta)$ in the integrand of (3.23), where $M_2$, $N_2$ and $h$ are defined in equations (A18), (A19) and (A30) of the Appendix, then the integral to be considered is

$$I = \frac{K_2(0)\Delta B(0)}{2\pi i} \int_{\Gamma_2} \frac{\exp[-iy\zeta - z\gamma(\zeta) - h(\zeta)]}{\zeta M_2(\zeta) N_2(\zeta)}\, d\zeta. \tag{4.3}$$

The closed contour $\Gamma_2$ is shown in Fig. 2. It encloses a simple pole at $\zeta = 0$ and possibly one other pole located in the upper half-plane, since it is shown in the Appendix (equation A19 *et seq.*) that the factor

$$N_2(\zeta) \equiv \gamma(\zeta) + \chi_2^-$$

in the denominator may have zeros at the points $\zeta = \pm\nu_2^-$ defined by

$$\nu_2^- \equiv \left(\frac{\chi_2^-}{r} - \frac{1}{\lambda_2 r}\right)^{1/2}, \quad \chi_2^- \equiv \frac{1}{2r} - \left(\frac{1}{4r^2} - \frac{1}{\lambda_2 r} + \frac{2i}{\delta^2}\right)^{1/2} = \pm\gamma(\nu_2^-).$$

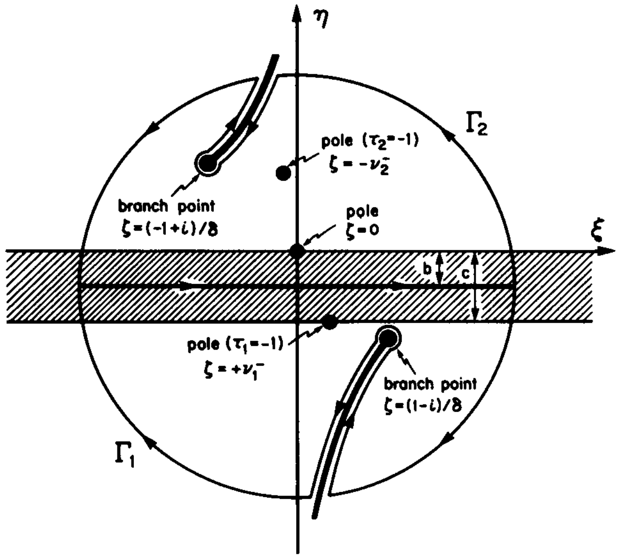


**Figure 2.** The complex $\zeta$-plane showing the hyperbolic branch cuts, the contours $\Gamma_1$ and $\Gamma_2$, and the poles enclosed by them.

Thus, if we define $\tau_2 \equiv \chi_2^-/\gamma(\nu_2^-) = \pm 1$, as in (A20) we may write

$$\frac{1}{N_2(\zeta)} = \frac{\gamma(\zeta) - \tau_2\gamma(\nu_2^-)}{(\zeta - \nu_2^-)(\zeta + \nu_2^-)}$$

and it follows from (A15) that there is also a pole in the upper half-plane at $\zeta = -\nu_2^-$, except when $\tau_2 = +1$ in which case the numerator above also vanishes at $\zeta = -\nu_2^-$ since $\gamma$ is an even function.

A straightforward evaluation of the integral $I$ by residues with the aid of (2.20) and (2.21) and the expressions (A12), (A19) and (A31) gives

$$I = \Delta B(z) + \tfrac{1}{2}(1-\tau_2)\,C_2(y, z) \tag{4.4}$$

where

$$C_2(y, z) = \frac{r\chi_2^- K_2(0)\Delta B(0)}{(1-\lambda_2\chi_2^-)(1-2r\chi_2^-)} \exp\left[iy\nu_2^- + z\chi_2^- - h(-\nu_2^-)\right]. \tag{4.5}$$

The factor $\tfrac{1}{2}(1-\tau_2)$ in (4.4), which emerges quite naturally as part of the residue, ensures that the second term vanishes when $\tau_2 = +1$.

The contribution to $I$ from the semicircular portion of $\Gamma_2$ clearly vanishes as the radius of the semicircle tends to infinity. Thus the integral in (3.23) is simply the value of $I$ given by (4.4) less the contribution to $I$ arising from that part of $\Gamma_2$ which runs alongside the branch cut in the upper half-plane. The integrals along these two hyperbolic paths can be transformed into new integrals, as in DW, by the substitution $u = \{\eta^2 - 1/(\delta^4\eta^2)\}^{1/2}$, which amounts to putting $\zeta = i\gamma(u)$ and replacing $\gamma(\zeta)$ by $+iu$ on the inside (facing the imaginary axis) of the branch cut, and by $-iu$ on the outside of the cut. With these simplifications, substitution in (3.23) yields the final expression for the magnetic field in $y < 0$ as

$$B(y, z) = B_2(z) - \tfrac{1}{2}(1-\tau_2)\,C_2(y, z) + P_2(y, z), \tag{4.6}$$

where

$$P_2(y,z) = \frac{K_2(0)\Delta B(0)}{\pi\lambda_2} \int_0^\infty \frac{u^2 \cos zu + ua_2(u)\sin zu}{[u^2 + \{a_2(u)\}^2]\{\gamma(u)\}^2} \exp[y\gamma(u) - g(u)]\, du \tag{4.7}$$

with

$$a_2(u) = 1/\lambda_2 - r\{\gamma(u)\}^2 \tag{4.8}$$

and $g(u) \equiv h[i\gamma(u)]$ as defined in (A31).

A similar procedure is used to evaluate the integral in (3.23) when $y > 0$. The integrand is rearranged so that $M_1(\zeta)N_1(\zeta)$ appears in the denominator, and this gives, by (A29), a factor $\exp[h(-\zeta)]$ in the numerator. The closed contour $\Gamma_1$ in the lower half-plane is used. It encloses just one simple pole at $\zeta = +\nu_1^-$ if $\tau_1 = -1$, but none if $\tau_1 = +1$. On the branch cut we now have $\zeta = -i\gamma(u)$ and $\gamma(\zeta)$ again takes the value $+iu$ on the inside of the cut and $-iu$ on the outside. The final solution obtained from (3.23) for the magnetic field in the region $y > 0$ is very similar in form to the solution in $y < 0$ represented by (4.6), (4.7) and (4.8). In fact, with the understanding that $j = 1$ when $y > 0$ and $j = 2$ when $y < 0$, i.e.

$$j = 2 - H(y) \tag{4.9}$$

where $H$ is the Heaviside function, and that upper signs go with $j = 1$ and lower signs with $j = 2$, the solutions for $y > 0$ and $y < 0$ can be combined into the single set of formulae

$$B(y,z) = B_j(z) \pm \{\tfrac{1}{2}(1-\tau_j)C_j(y,z) - P_j^2(y,z)\} \tag{4.10}$$

where

$$C_j(y,z) = \frac{\lambda_j r\chi_j^- D_j}{(1-\lambda_j\chi_j^-)(1-2r\chi_j^-)} \exp[z\chi_j^- \pm \{h(-\nu_j^-) - iy\nu_j^-\}] \tag{4.11}$$

$$D_j = \Delta B(0)K_j(0)/\lambda_j = \Delta B(0)\{1/\lambda_j + (1+i)/\delta\} \tag{4.12}$$

$$P_j^m(y,z) = \frac{D_j}{\pi} \int_0^\infty \frac{u^2 \cos zu + ua_j(u)\sin zu}{[u^2 + \{a_j(u)\}^2]\{\gamma(u)\}^m} \exp[\pm\{g(u) - y\gamma(u)\}]\, du \tag{4.13}$$

$$a_j(u) = 1/\lambda_j - r\{\gamma(u)\}^2. \tag{4.14}$$

The corresponding expressions for the electric field components in the half-space $z > 0$ are readily found from the solutions above by means of the relations (2.4) and (2.5). From (4.1) and (4.2) we have

$$\mu_0\sigma_0 V(y,z) = -(1+i)\bar{B}(z)/\delta + L_{12}^-(y,z), \tag{4.15}$$

$$\mu_0\sigma_0 W(y,z) = -L_{01}^+(y,z). \tag{4.16}$$

The alternative forms of solutions based on equations (4.10) to (4.14) are

$$\mu_0\sigma_0 V(y,z) = -(1+i)B_j(z)/\delta \pm \{\tfrac{1}{2}(1-\tau_j)\chi_j^- C_j(y,z) - Q_j(y,z)\} \tag{4.17}$$

where

$$Q_j(y,z) = \frac{D_j}{\pi} \int_0^\infty \frac{u^2 a_j(u)\cos zu - u^3 \sin zu}{[u^2 + \{a_j(u)\}^2]\{\gamma(u)\}^2} \exp[\pm\{g(u) - y\gamma(u)\}]\, du \tag{4.18}$$

and

$$\mu_0\sigma_0 W(y,z) = \tfrac{1}{2}i(1-\tau_j)\nu_j^- C_j(y,z) - P_j^1(y,z). \tag{4.19}$$

In all of the above formulae the relation (4.9) and the associated sign convention is understood.

Evaluated at $z = 0+$ these solutions give the fields on the lower surface of the generalized thin sheet, or just underneath the resistive lower crust in the real Earth – hardly a region where actual field measurements are likely to be made! However, since the boundary condition (2.13) shows that the magnetic field is approximately constant across the resistive layer, the solutions (4.1) or (4.10) evaluated at $z = 0+$ may also be regarded as giving the horizontal magnetic field on the lower surface of the conductive layer, which could be of practical interest if the layer represents an ocean and magnetometers located on the ocean floor are used. Otherwise the only component that corresponds to a field that can be recorded in practice is the horizontal electrical field $V(y, 0-)$ on the surface of the Earth. Expressions for this component are obtained either by substituting (4.1) in (2.19) to obtain

$$\mu_0 \sigma_0 V(y, 0-) = -(1+i)\bar{B}(0)/\delta + L_{12}^-(y, 0) + rL_{00}^-(y, 0) \tag{4.20}$$

or by substituting (4.10) in (2.19) and using (4.14), (A12) and (A13) to obtain the alternative form

$$\mu_0 \sigma_0 V(y, 0-) = -(1+i)B_j(0)/\delta \pm (1/\lambda_j)\{\tfrac{1}{2}(1-\tau_j)C_j(y, 0) - P_j^2(y, 0)\}. \tag{4.21}$$

The boundary condition (2.11) shows that equations (4.20) and (4.21) also give approximately the horizontal electrical field at the bottom of the conductive layer, e.g. on the ocean floor.

## 5 Special cases

In this section we shall examine the behaviour of the solutions for the limiting cases $r \to \infty$, $r \to 0$, $\lambda_1 \to \infty$ and $\lambda_2 \to 0$.

Assume first that $\lambda_1$ and $\lambda_2$ are non-zero and finite and that $r \to \infty$. In this limit the model consists of two conducting half-sheets electrically insulated from the underlying uniform half-space. With no vertical flow of current into or out of the sheet allowed, the problem becomes trivial. A line charge density will be formed on the boundary $y = 0$, $z = 0$ between the two half-sheets to ensure the continuity of the normal surface current density across the boundary and hence there will be a uniform surface current flow, harmonic in time, in the whole sheet. It follows from the boundary condition (2.15) that the magnetic field discontinuity across the sheet is everywhere the same, and since the magnetic field is uniform above the sheet it must therefore be also uniform in the plane $z = +0$ on the other side of the sheet. This conclusion is confirmed by taking the limit as $r \to \infty$ in the solution (4.10). We note first from (A13) and (A14) that $\chi_j^\pm \to \pm(1+i)/\delta$, $\nu_j^\pm \to 0$, and hence by (A20) that $\tau_j = -1$ which shows that poles at $\nu_1^-$ and $-\nu_2^-$ contribute through the second term in (4.10) to the solution for the magnetic field. Moreover, since the neighbourhood of the origin belongs to the set $\bar{\mathscr{S}}$ defined in the Appendix (equation A4 *et seq*. and Fig. A1) it follows from the definition (A24) that $\tilde{\tau}_j = \tau_j = -1$. This means that the leading term in the factorization formula (A28) does not vanish, but because this formula takes the same limiting form for both $j = 1$ and $j = 2$ we may deduce that $\phi_1^+ \to \phi_2^+$, and hence in (A30) that $h(\zeta) \to 0$, as $r \to \infty$. Furthermore, the integral $P_j^2$ defined by (4.13) vanishes as $r \to \infty$ since the factor $a_j$ is $O(r)$, while the limiting value of the expression $C_j(y, z)$ defined by (4.11) can be written as $-\tfrac{1}{2}\Delta B(z)$ by virtue of equations (2.20) and (2.21). Substituting these results in (4.10) and using (2.21) we have

$$B(y, z) = \bar{B}(z) \qquad (r \to \infty), \tag{5.1}$$

and the uniform field on the surface $z = 0+$ is just $\bar{B}(0)$. The solution (5.1) represents the 'skin effect' diffusion of this surface field into the conducting half-space below.

When $r = 0$, there is in effect no resistive part of the sheet at all and the problem reduces to that considered by DW. As we let $r \to 0$ in (A12) and (A13) we obtain

$$\chi_j^+ = 1/r - 1/\lambda_j + O(r), \qquad \chi_j^- = 1/\lambda_j + O(r) \tag{5.2}$$

$$\nu_j^+ = 1/r - 1/\lambda_j + O(r), \qquad \nu_j^- = (1/\lambda_j^2 - 2i/\delta^2)^{1/2} + O(r) \tag{5.3}$$

so that when $r = 0$, it follows from (A19) that

$$M_j(\zeta) = \lambda_j, \qquad N_j(\zeta) = \gamma(\zeta) + 1/\lambda_j.$$

Their product

$$K_j(\zeta) = M_j(\zeta) N_j(\zeta) = 1 + \lambda_j \gamma(\zeta) \tag{5.4}$$

is precisely of the form considered by DW, and indeed equation (5.4) can be obtained directly by letting $r = 0$ in the expression for $K_j$ quoted in (3.14). It also follows from (5.2) and (5.3) that if $r = 0$ then

$$\chi_j^- = \gamma(\nu_j^-) = 1/\lambda_j, \qquad -\tfrac{1}{4}\pi < \arg \nu_j^- < 0.$$

From the first of these statements we deduce that $\tau_j = +1$, and from the second that $\nu_j^-$ lies in the region $\mathscr{S}$ so that $\tilde{\tau}_j = \tau_j = +1$. Thus in this example there is no pole at $\zeta = \nu_1^-$ or $\zeta = -\nu_2^-$ contributing to the second term in the solution (4.10), nor is the first term in the factorization formula (A28) present. Finally, since $a_j(u) \to 1/\lambda_j$ as $r \to 0$ in (4.14), the integral $P_j^2$ given by (4.13) reduces to exactly the same form appearing in the solution given by DW.

For the remainder of this section we shall assume that $r$ is finite and non-vanishing. Consider first the limiting case $\lambda_1 \to \infty$ implying that the topside of the generalized thin sheet in $y > 0$ is perfectly conducting. From (A12) and (A13) we obtain

$$\chi_1^\pm \to 1/(2r) \pm \gamma[1/(2r)], \qquad \nu_1^\pm \to (\chi_1^\pm/r)^{1/2} \tag{5.5}$$

and it can be seen immediately from the first of these limits that $\mathrm{Re}\,\chi_1^- < 0$ and $\mathrm{Im}\,\chi_1^- < 0$. Since, by definition, $\mathrm{Re}\,\gamma(\nu_1^-)$ must be non-negative it follows from (A20) that $\tau_1 = -1$ in this example. It can also be seen from (5.5) that $-\tfrac{1}{2}\pi < \arg \nu_1^- < -\tfrac{1}{4}\pi$ so that $|\mathrm{Re}\,\nu_1^-| < |\mathrm{Im}\,\nu_1^-|$, but it is not immediately apparent which of the regions $\mathscr{S}$ and $\bar{\mathscr{S}}$ the pole $\nu_1^-$ lies in. However, it is a straightforward exercise to show that

$$(\mathrm{Re}\,\nu_1^-)(\mathrm{Im}\,\nu_1^-) = -(\sqrt{2}/\delta^2)\,[(1 + 64 r^4/\delta^4)^{1/2} + 1]^{-1/2} > -1/\delta^2$$

which confirms that $\nu_1^- \in \bar{\mathscr{S}}$ according to the definition following equation (A4). Hence by (A24) we have $\tilde{\tau}_1 = \tau_1 = -1$ when $\lambda_1 \to \infty$, which shows that for $j = 1$ there is a contribution from the pole to the field solution and also a non-vanishing leading term in the factorization formula (A28). The terms outside the integral in the solution (4.10) are much simplified in this case, for $B_1(z)$ vanishes as $\lambda_1 \to \infty$ and we obtain with the help of (5.5), (4.11) and (4.12)

$$B_1(z) + \tfrac{1}{2}(1 - \tau_1) C_1(y, z) \to \frac{(1 + i) B_0 \exp[\nu_1^-(z r \nu_1^- - iy) + h(-\nu_1^-)]}{\{1 + (1 + i)\lambda_2/\delta\}\,(\delta^2/r^2 + 8i)^{1/2}}.$$

The integrand of $P_1^2$, the remaining part of the solution, is also simplified by the fact that $a_1(u) \to -r(u^2 + 2i/\delta^2)$ when $\lambda_1 \to \infty$ in (4.14). It is of further interest to note that $V(y, 0-) \to 0$ for $y > 0$ (i.e. $j = 1$) as $\lambda_1 \to \infty$ in equation (4.20). This is the required result that the tangential electric field at the surface of a perfect conductor vanishes.

In the final special case $\lambda_2 = 0$, the surface sheet in the half-plane $y < 0, z = 0$ is purely resistive. When $\lambda_2 \to 0$ equations (A12) and (A13) give

$$\chi_2^{\pm} = \pm i(\lambda_2 r)^{-1/2} + 1/2r + O(\lambda_2^{1/2}), \qquad \nu_2^{\pm} = \pm i(\lambda_2 r)^{-1/2} + 1/2r + O(\lambda_2^{1/2}) \tag{5.6}$$

so that $\operatorname{Re}\chi_2^- > 0$ and $\tau_2 = +1$. On the other hand for sufficiently small $\lambda_2$ the point $\nu_2^-$ clearly lies in the region $\mathscr{S}$ and remains there as $\lambda_2 \to 0$. It follows, therefore, from (A24) that $\tilde{\tau}_2 = -\tau_2 = -1$, which shows, incidentally, that the parameters $\tau_j$ and $\tilde{\tau}_j$ may differ in sign in some circumstances. Thus although the first term in the factorization expression will be retained it nevertheless vanishes as $\lambda_2 \to 0$ along with the rest of (A28) because $\nu_2^{\pm} = O(\lambda_2^{-1/2})$ by (5.6), and $\chi_2^{\pm}\Psi(w, \nu_2^{\pm}) = O(\lambda_2^{1/2}\log\lambda_2)$ by (5.6), (A32), (A33) and (A6). This is expected for clearly $K_2(\zeta) \equiv 1$ when $\lambda_2 = 0$ in (3.14). With the aid of these results applied to equations (4.12)–(4.14), (A31), (2.20) and (2.21), the solution (4.10) for the magnetic field in $y < 0$ becomes, when $\lambda_2 = 0$,

$$B(y, z) = B_0 \exp[-(1+i)z/\delta] - \frac{(1+i)\lambda_1 B_1(0)}{\pi\delta} \int_0^\infty \frac{u \sin zu \exp[y\gamma(u)]}{(u^2 + 2i/\delta^2)k_1^+[i\gamma(u)]}\, du. \tag{5.7}$$

On $z = 0+$ this reduces to $B(y, 0+) = B_0$, $(y < 0)$, which satisfies the requirement that the magnetic field must be continuous across the surface layer when there is no conducting sheet present to support a surface current. Substitution of this result in equation (2.5), and also in equation (2.19) combined with (2.4), shows that $W(y, 0+) = 0$ and $V(y, 0-) = V(y, 0+)$ for $y < 0$. The first result is quite expected since there can be no flow of current into a purely resistive sheet. The second, which states that the horizontal electric field is continuous across a purely resistive sheet, is contrary to the boundary condition applying in general but is nonetheless plausible on physical grounds. When an overlying conducting sheet is present in the plane $z = 0-$ the distribution of surface charge in it will, in general, be different from that on the surface $z = 0+$ of the conducting half-space, in order that it may govern the flow of current across the intervening resistive sheet. This is consistent with the horizontal electric field being discontinuous across the resistive part of the generalized thin sheet. However, in the absence of an overlying conducting sheet capable of supporting a distribution of surface charge, the only accumulation of charge will be on the surface $z = 0+$ and it will then serve merely to prevent a normal flow of current from the half-space into the purely resistive sheet. In this event the horizontal electric field will be uniform across the resistive sheet in accordance with the revised boundary condition for $\lambda_2 = 0$.

## 6 Behaviour of the field at the origin

The behaviour of the field components near the origin is governed by the convergence properties of the integrals as $|y| \to 0$ on $z = 0\pm$, as discussed by DW for the special case $r = 0$. For the present, we shall assume that $\lambda_1$, $\lambda_2$ and $r$ are all non-zero and finite. By virtue of the estimate (A48) we may assert that as $\xi \to \infty$

$$\frac{\exp(-iy\xi)}{k_1^+(\xi)k_2^-(\xi)} - \frac{\exp(iy\xi)}{k_2^+(\xi)k_1^-(\xi)} = \frac{2i \sin y\xi}{\beta_1\beta_2\xi^2} + O\left(\frac{\log\xi}{\xi^3}\right), \tag{6.1}$$

$$\frac{\exp(-iy\xi)}{k_1^+(\xi)k_2^-(\xi)} + \frac{\exp(iy\xi)}{k_2^+(\xi)k_1^-(\xi)} = -\frac{2\cos y\xi}{\beta_1\beta_2\xi^2} + O\left(\frac{\log\xi}{\xi^3}\right), \tag{6.2}$$

where the constants $\beta_1$ and $\beta_2$ are defined as in equation (A45). The integrands of $L_{02}^-(y, 0)$, $L_{12}^-(y, 0)$ and $L_{01}^+(y, 0)$ defined by (4.2) and appearing in the solutions (4.1), (4.15) and

(4.16) evaluated at $z = 0+$, are seen by (6.1) and (6.2) to be $O(\xi^{-3})$, $O(\xi^{-2})$ and $O(\xi^{-2})$ respectively. These estimates are sufficient to ensure that each integral converges uniformly in $y$, and as a result each field component is continuous and bounded everywhere, including the origin.

The behaviour of the horizontal electric field above the surface sheet is different, and can be deduced from (4.20). The integrand of $L_{12}^-$ is again $O(\xi^{-2})$ so that this part of the solution remains uniformly convergent for all $y$, but the result (6.1) shows that the integrand of $rL_{00}^-$ has the asymptotic form $(2ir\sin y\xi)/(\beta_1\beta_2\xi) + O(\xi^{-2}\log\xi)$. Hence if the term $(2ir\sin y\xi)/(\beta_1\beta_2\xi)$ is subtracted from the integrand of $rL_{00}^-$ the remainder will be $O(\xi^{-2}\log\xi)$ and the resulting integral will be uniformly convergent. Thus it becomes possible to separate out that part, $V^*(y)$ say, of the solution (4.20) for $V(y, 0-)$ that is bounded and continuous for all $y$, as follows

$$\mu_0\sigma_0\{V(y, 0-) - V^*(y)\} = \frac{r\Delta B(0)}{\pi\beta_1\beta_2}\int_0^\infty \frac{\sin y\xi}{\xi}\,d\xi = \frac{r\Delta B(0)}{2\beta_1\beta_2}\,\mathrm{sgn}\,y. \tag{6.3}$$

This result shows that $V(y, 0-)$ has a finite jump discontinuity at the origin, the jump being $r\Delta B(0)/(\beta_1\beta_2)$.

The behaviour described above is unaltered when $\lambda_1 \to \infty$ since it was pointed out in the Appendix that the asymptotic forms of $k_j^\pm$ do not change in this limit, provided that $r$ and $\lambda_2$ are finite and non-vanishing.

There are significant changes in the behaviour of the field near the origin when $\lambda_2 = 0$. In this case, $k_2^\pm(\xi) = 1$ and hence by (A48) we find that as $\xi \to \infty$,

$$\frac{\exp(-iy\xi)}{k_1^+(\xi)} - \frac{\exp(iy\xi)}{k_1^-(\xi)} = \frac{2\cos y\xi}{\beta_1\xi} + O\left(\frac{\log\xi}{\xi^2}\right) \tag{6.4}$$

$$\frac{\exp(-iy\xi)}{k_1^+(\xi)} + \frac{\exp(iy\xi)}{k_1^-(\xi)} = \frac{-2i\sin y\xi}{\beta_1\xi} + O\left(\frac{\log\xi}{\xi^2}\right) \tag{6.5}$$

in place of (6.1) and (6.2). The asymptotic behaviour (6.4) is still sufficient for the integral $L_{02}^-$ in (4.2) to converge uniformly in $y$ when $z = 0+$, and so the magnetic field defined by (4.1) remains bounded and continuous at the origin. The integrand of $L_{01}^+$ in the solution (4.16) for the vertical electric field on $z = 0+$ is given by (6.5) when $\lambda_2 = 0$ so that by analogy with (6.3), $W(y, 0+)$ is finite but discontinuous at $y = 0$ (in fact $W(y, 0+) \equiv 0$ for $y < 0$ as shown in Section 5). The integrand of $L_{12}^-$ in the solution (4.15) for the horizontal electric field is, by (6.4), $O(\xi^{-1}\cos y\xi)$ when $z = 0+$, and as shown in DW, this is indicative of a $\log|y|$ singularity. This can also be seen, somewhat differently, by examining the alternative form of solution given by (4.17) with $\lambda_2 = 0$. In fact, for $y < 0$ the solution is most easily obtained by differentiation of (5.7) according to the relation (2.4), and it can be seen that the integrand becomes

$$\frac{u^2\exp[y\gamma(u)]}{k_1^+[i\gamma(u)]\{\gamma(u)\}^2} = \frac{-i\exp[y\gamma(u)]}{\beta_1\gamma(u)} + O\left(\frac{\log u}{u^2}\right) \tag{6.6}$$

by (A48). A standard integral representation and expansion of the modified Bessel function of the second kind and order zero (see, e.g. Gradshteyn & Ryzhik 1980, sections 8.432 and 8.447) gives

$$\int_0^\infty \frac{\exp[-y\gamma(u)]}{\gamma(u)}\,du = K_0[(1+i)|y|/\delta] = -\log|y| + O(1) \tag{6.7}$$

as $|y| \to 0$. This result confirms the logarithmic nature of the singularity in $V(y, 0+)$ as $y \to 0-$. To determine the behaviour of $V(y, 0+)$ as $y \to 0+$, we return to the solution (4.17) and note that when $\lambda_2 \to 0$ the integrand of $Q_1(y, 0+)$, defined in (4.18), has the asymptotic form

$$\frac{u^2 a_1(u) k_1^+[i\gamma(u)] \exp[-y\gamma(u)]}{[u^2 + \{a_1(u)\}^2] \{\gamma(u)\}^2} = -\frac{i\beta_1}{r\gamma(u)} \exp[-y\gamma(u)] + O\left(\frac{\log u}{u^2}\right) \tag{6.8}$$

for large $u$, by (A31), (A48) and (4.14). The dominant term here will again give a logarithmic singularity at the origin by analogy with (6.6). On the upper surface of the sheet the electric field $V(y, 0-)$ also has a logarithmic singularity as $y \to 0-$ because, as we saw in Section 5, it is continuous across the sheet when $\lambda_2 = 0$. On the half-plane $y > 0$, however, $V(y, 0-)$ is given by equation (4.21) and its behaviour near the origin is determined by the convergence properties of the integral $P_1^2(y, 0)$. Using the same substitutions and estimates as before we can express its integrand in the form

$$\frac{u^2 k_1^+[i\gamma(u)] \exp[-y\gamma(u)]}{[u^2 + \{a_1(u)\}^2] \{\gamma(u)\}^2} = \frac{i\beta_1}{r^2 u^3} \exp[-y\gamma(u)] + O\left(\frac{\log u}{u^4}\right) \tag{6.9}$$

which shows that the integral is uniformly convergent for $y > 0$ and hence that $V(y, 0-)$ remains finite as $y \to 0+$. This is a reasonable result since the conducting part of the generalized thin sheet in $y > 0$ will carry a surface current of finite density.

If $\lambda_2 = 0$ and $\lambda_1 \to +\infty$ simultaneously, the behaviour of the field described in the last paragraph is preserved except that $V(y, 0-)$ is now not only finite, but in fact identically zero when $y > 0$.

In the discussion above, it has been assumed that $0 < r < \infty$. As we have seen in Section 5 the limiting case $r \to \infty$ is trivial, and the case $r \to 0$ reduces the problem to the one considered by DW. It is of interest to recall here the conclusions reached by DW in order to see how the generalized sheet considered in this paper modifies the behaviour of the electromagnetic field components near the origin. DW found that when the thin sheet is purely conducting with $\lambda_1$ and $\lambda_2$ non-vanishing, the magnetic field $B(y, 0+)$ is continuous, the horizontal electric field $V(y, 0+)$ has a finite jump discontinuity, and the vertical electric field $W(y, 0+)$ has a logarithmic singularity, as $|y| \to 0$. When $r$ is non-vanishing, however, we have seen that all three of these components are bounded and continuous at the origin. By contrast, the behaviour of the horizontal electrical field on the upper side of the sheet (the surface of the Earth) remains unchanged whether $r = 0$ or $r > 0$; in both cases $V(y, 0-)$ has a finite jump discontinuity at the origin. The statements above continue to hold when $\lambda_1 \to \infty$, but must be modified for $\lambda_2 = 0$. In this case DW showed that $B(y, 0+)$ is bounded and continuous at the origin, $V(y, 0+)$ is bounded as $y \to 0+$ but has an algebraic singularity of the form $O(|y|^{-1/2})$ as $y \to 0-$, and $W(y, 0+)$ has the same algebraic singularity as $y \to 0+$ but vanishes on $y < 0$. As we have seen, the corresponding behaviour for a generalized thin sheet with non-vanishing $r$ is that $B(y, 0+)$ remains continuous, but $V(y, 0+)$ is now singular as $y \to 0$ from both directions (although the severity of the singularity is reduced from algebraic to logarithmic) and $W(y, 0+)$, which still vanishes for $y < 0$, has only a jump discontinuity at the origin. The surface electric field $V(y, 0-)$ behaves in much the same manner in both cases; it is bounded as $y \to 0+$ and singular as $y \to 0-$, with a singularity of type $O(|y|^{-1/2})$ when $r = 0$ and $O(\log y)$ when $r > 0$.

In general, then, the inclusion of a resistive sheet beneath the conducting sheet tends either to remove or reduce the severity of the singularities in the induced electromagnetic field at the origin.

## 7 A numerical calculation

The field solutions on the surface $z = 0\pm$ have been evaluated for the dimensionless parameter values $\lambda_1/\delta = 8\pi/15$, $\lambda_2/\delta = \pi/75$ and $r/\delta = 1000\pi/3$. For an inducing field of period 1 h and an underlying half-space of conductivity $0.1\,\mathrm{S\,m^{-1}}$, the skin depth is $\delta = (3/\pi) \times 10^5$ m and the dimensionless values above correspond to actual conductances of $1.6 \times 10^4$ S and 400 S respectively, and an integrated resistivity of $10^9\,\Omega\mathrm{m}^2$. This value of $r$ represents, for example, a resistive layer 10 km thick of resistivity $10^5\,\Omega$m, while as we mentioned in Section 1 the values chosen for the conductances are very reasonable ones for an ocean–continent model. The model calculations made by Ranganayaki & Madden (1980) were for crustal resistivities ranging from $10^2$–$10^6\,\Omega$ and they cited several references to support their assumption that a resistivity-thickness product of $10^9\,\Omega\mathrm{m}^2$ is quite plausible for the continental crust. This figure is probably too large, however, for the oceanic crust, and Drury (1981) in particular has questioned Ranganayaki & Madden's conclusions based on such high resistivity values. He draws on his own and other measurements of laboratory samples, and on other evidence as well, to estimate a resistivity-thickness product of $10^6$–$10^7\,\Omega\mathrm{m}^2$ for the oceanic crust. Nevertheless, the example we have chosen for numerical evaluation is an appropriate one, since it represents the other extreme case, at the opposite end of the spectrum of $r$ values to the one investigated by DW in which it was implicit that $r = 0$. It is of interest to compare the calculations with the corresponding ones for the DW model since the true behaviour probably lies somewhere between these two extremes.

For these particular values of the model parameters it was found that

$$|\operatorname{Im}\nu_1^-|\delta = 4.076 \times 10^{-2}, \qquad |\operatorname{Im}\nu_2^-|\delta = 1.541 \times 10^{-1}$$

to four significant figures, and that $\tau_1 = \tau_2 = \tilde{\tau}_1 = \tilde{\tau}_2 = -1$. Thus in this example the width of the strip of analyticity shown in Fig. 2 and defined by (A21) is $c = 0.04076/\delta$ and the possible poles at $\nu_1^-$ and $-\nu_2^-$ do indeed exist. Thus the terms arising from the residues of these poles are present in the form of the solution expressed by equations (4.10), (4.17) and (4.19). In fact the calculations showed that these residue terms completely dominate the

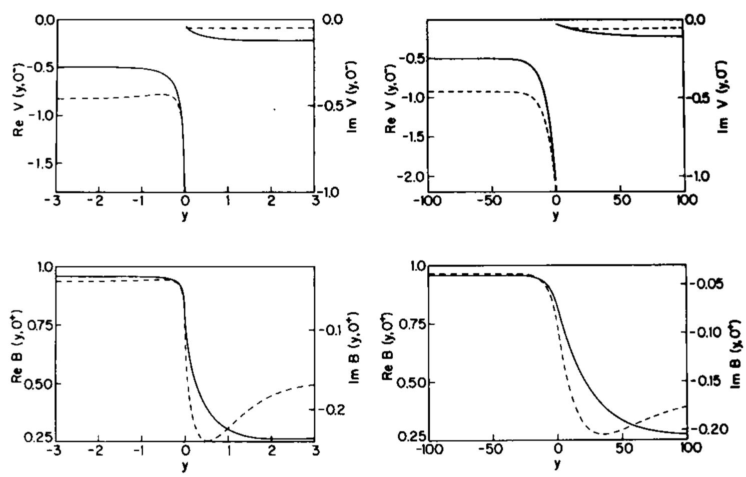


**Figure 3.** Variation of the real part (solid line and left vertical scale) and imaginary part (broken line and right vertical scale) of the horizontal electric field $V(y, 0-)$ on the surface $z = 0-$, and of the magnetic field $B(y, 0+)$ on the surface $z = 0+$, for a coast-effect model in which $\lambda_1 = 40\ \lambda_2 = (8\pi/15)\delta$. $V$ is given in units of $\omega\delta B_0$, $B$ in units of $B_0$, and the horizontal distance $y$ in units of the skin-depth $\delta$. The diagrams on the left are for the model investigated by Dawson & Weaver (1979) in which $r = 0$; those on the right are for the generalized thin sheet with $r = (1000\,\pi/3)\delta$.

solution in this example. The contributions from the integrals $P_j^2$, $Q_j$ and $P_j^1$ affected only the fourth decimal place in the numerical results.

In Fig. 3 we have plotted the variation of the horizontal electric field along the surface of the Earth ($z = 0-$) and the magnetic field along the ocean floor (the same as the field on $z = 0+$ for $y > 0$). Of less practical interest are the variations shown in Fig. 4 of the horizontal and vertical electric fields along the surface $z = 0+$, i.e. beneath the resistive sheet. Both real and imaginary parts of the field components are plotted with the left and right vertical scales referring to the real and imaginary parts respectively. Note that the variations of the horizontal electric field $V$ for the DW model ($r = 0$) are the same in Figs 3 and 4, i.e. for both $z = 0+$ and $z = 0-$. This is of course expected since $V$ is continuous across a purely conductive sheet.

The most striking feature apparent from Figs 3 and 4 is the enormous increase in adjustment distance caused by the resistive sheet, as indicated by the very different horizontal scales required for plotting the graphs in the two models. For example, in the DW model the effect of the ocean on land-based magneto-telluric measurements is felt only at distances less than a skin-depth (i.e. less than 100 km) from the coastal boundary, whereas in the presence of a highly resistive lower crust this effect appears to spread out to about 10 skin-depths (Fig. 3). The influence of the coastline on seafloor magnetic measurements is even more spectacularly extended by a resistive suboceanic crust to a distance of over 50 skin-depths. Otherwise the corresponding field variations depicted by the curves in Fig. 3 for the two different models are very similar in shape and magnitude. It appears that the increase in adjustment distance is the only significant effect of the resistive sheet.

The same cannot be said about its influence on the fields below the generalized thin sheet on the surface of the uniform half-space. In addition to the enlargement of the adjustment distance the graphs in Fig. 4 show that the variation of the horizontal electric field $V(y, 0+)$ is completely different for the two models (as expected, of course, because $V$ is discontinuous across the resistive sheet) and the singularity at the origin in the vertical electric field in the DW model has been reduced to a finite cusp of very small magnitude when the resistive sheet is present.

Finally, lines of constant magnetic field in the conducting half-space $z > 0$, which are the same as the streamlines of the induced currents, have been plotted in Fig. 5 at intervals of

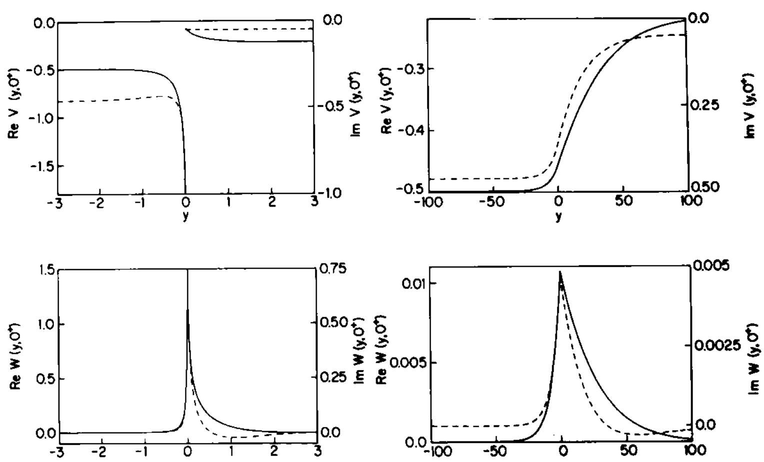


**Figure 4.** As in Fig. 3, but depicting the variations of the horizontal and vertical electric field components, $V(y, 0+)$ and $W(y, 0+)$, on the surface $z = 0+$.

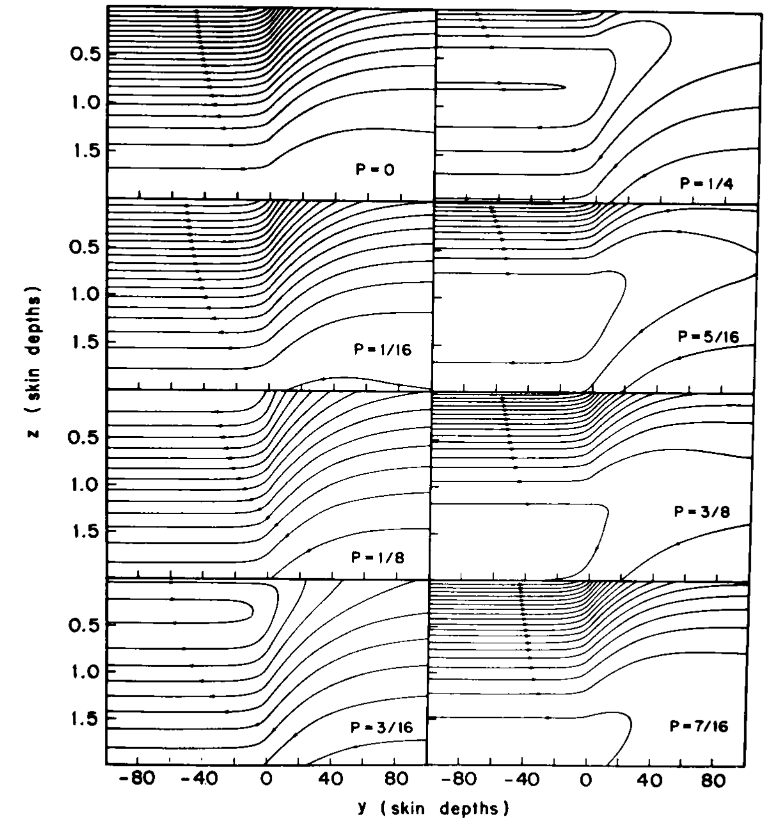


**Figure 5.** Streamlines of the electric currents (i.e. lines of constant magnetic field) induced in the half-space $z > 0$ beneath the generalized thin sheet, plotted at fractional intervals $P$ of one period of the oscillation. The line separation corresponds to a change in magnetic field of $0.049 B_0$.

$T/16$, from $P = 0$ to $P = 7/16$, where $T$ is the period of the inducing field and $P = t/T$ is a dimensionless time parameter. These diagrams show very clearly how the pattern of induced current flow changes during a complete half-cycle, and may be compared with similar diagrams in DW, and also in an earlier paper by Bailey (1977) in which the ocean was taken to be perfectly conducting. The main feature to observe is again the very great width of the region needed by the currents in order to adjust themselves between the different purely horizontal flows that must obtain as $y \to \pm\infty$. For $y > 0$ the out-of-phase currents ($P = 1/4$) appear to be largely confined to the ocean.

## 8 Concluding remarks

The rather lengthy mathematical discourse presented in this paper has led to a new exact solution of the coast-effect problem in the $B$-polarization mode for a mathematical model which we believe represents a region of the real Earth near a conductivity boundary more faithfully than any previous model that has been solved analytically. This is because the model has included, for the first time, the effect of both the conductive and resistive components of the Earth's crust combined together in a single generalized thin sheet of the type first suggested by Ranganayaki & Madden (1980). A numerical calculation has confirmed the general conclusions reached by Ranganayaki & Madden concerning the effect of the resistive lower crust on the magneto-telluric response of the Earth near a coastline or other lateral change in the conductivity of the surface layer.

The advantage of having available a solution to this problem in closed analytical form is not simply that it gives the induced field to any desired degree of accuracy but rather that it shows the exact mathematical dependence of the solution on the various parameters that define the model. In this regard we recall that the solution calculated for the particular set of model parameters representing a coastal boundary with a highly resistive lower crust considered in Section 7 was given to a high degree of accuracy by the first two terms in equations (4.10), (4.17) and (4.19) alone. Thus very approximately the solution can be expressed in an entirely algebraic form and the dependence on $y$ is given by the factor $\exp(-iy\nu_1^-)$ when $y > 0$ and $\exp(iy\nu_2^-)$ when $y < 0$ appearing in the expression (4.11) for $C_j(y, z)$, $(j = 1, 2)$. This shows that the exponential decay of the anomalous field on either side of the conductivity discontinuity is governed by the attenuation constants $-\mathrm{Im}\,\nu_1^-$ for $y > 0$ and $-\mathrm{Im}\,\nu_2^-$ for $y < 0$, both positive by virtue of (A15), and their reciprocals should be compared with the corresponding adjustment distances, $\sqrt{\lambda_1 r}$ and $\sqrt{\lambda_2 r}$ respectively, defined by Ranganayaki & Madden (1980). It seems that the adjustment distance depends on the integrated conductivity and resistivity of the generalized thin sheet in a more complicated manner than is indicated by the simple parameter introduced by Ranganayaki & Madden. In fact it can be seen that $(-\mathrm{Im}\,\nu_j^-)^{-1}$, $(j = 1, 2)$, reduces to $\sqrt{\lambda_j r}$ only if the term $\chi_j^-/r$ in the definition (A13) is negligible compared with $1/(\lambda_j r)$.

However, a more detailed examination of adjustment distance and of other points of interest arising from the analytical solution must be deferred to a subsequent paper.

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## Appendix

In order to obtain the factorizations of $K_1$ and $K_2$ introduced in equation (3.16) it is helpful to review first the factorization, discussed by DW and in more detail by Dawson (1979), of the function

$$K(\zeta) = \lambda\{s + \gamma(\zeta)\}, \qquad |\eta| < 1/\delta \tag{A1}$$

into a product of functions $K_+$ and $K_-$ analytic and zero-free in the regions $\eta > -1/\delta$ and $\eta < 1/\delta$ respectively of the complex $\zeta$-plane where $\zeta = \xi + i\eta$. The quantities $\lambda$ and $s$ are real and positive (in fact $\lambda s = 1$ in DW but we prefer to retain the more general form (A2) here). Since $K$ is obviously analytic and non-zero within the strip $|\eta| < 1/\delta$, the function

$$K'(\zeta)/K(\zeta) = K_+'(\zeta)/K_+(\zeta) + K_-'(\zeta)/K_-(\zeta) \tag{A2}$$

must also be analytic in this strip, and by the definitions of $K_+$ and $K_-$ the first and second terms in the right side of (A2) are analytic in the regions $\eta > -1/\delta$ and $\eta < 1/\delta$ respectively. The general formula (Noble 1958, p. 13) for the separation of an analytic function in this manner gives

$$\frac{K_\pm'(\zeta)}{K_\pm(\zeta)} = \int_{-\infty + ib_\pm}^{\infty + ib_\pm} \frac{K'(w)/K(w)}{w - \zeta}\, dw \tag{A3}$$

where $-1/\delta < b_+ < b_- < 1/\delta$. The integrals can be evaluated by closing the contours in the upper and lower half-planes respectively (Fig. A1) and using Cauchy's theorem of residues.

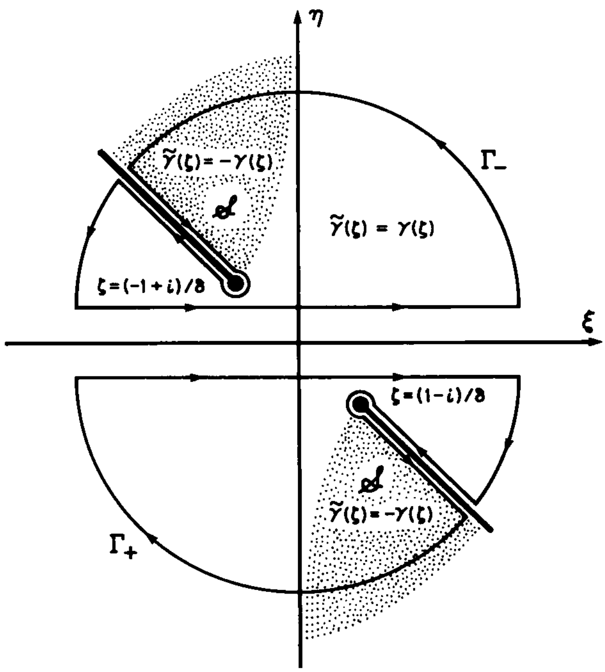


Figure A1. The complex $\zeta$-plane showing the diagonal branch cuts and the contours $\Gamma_+$ and $\Gamma_-$ used in the factorization of $K_j(\zeta)$. The shaded area is the region $\mathscr{S}$ where $\tilde\gamma$ has a negative real part.

It is necessary, therefore, to find an appropriate analytic continuation of $K(\zeta)$ out of the strip $|\eta| < 1/\delta$ into the whole complex $\zeta$-plane.

It turns out that the analytic continuation of $\gamma(\zeta)$ as the principal square root function $(\zeta^2 + 2i/\delta^2)^{1/2}$ defined in the cut plane illustrated in Fig. 2 is not the most suitable one to choose for the factorization integrals. As in DW, it is more convenient to define a second branch, denoted by $\tilde{\gamma}(\zeta)$, which agrees with $\gamma(\zeta)$ within the strip $|\eta| < 1/\delta$ and which is continued into the complex $\zeta$-plane now cut from the branch points $\zeta = \pm(1-i)/\delta$ to infinity along the rays $\xi = -\eta$, $|\eta| > 1/\delta$ (see Fig. A1).

This new branch is analytic in the cut plane, is a solution of equation (3.10), and is related to the original branch by the equation

$$\tilde{\gamma}(\zeta) = \begin{cases} -\gamma(\zeta) & (\zeta \in \mathscr{S}), \\ +\gamma(\zeta) & (\zeta \in \bar{\mathscr{S}}), \end{cases} \tag{A4}$$

where $\mathscr{S} = \{\xi + i\eta \,\|\, |\xi| < |\eta|,\ \xi\eta < -1/\delta^2\}$ defines the set of points $\zeta$ in the regions of the complex plane shown shaded in Fig. A1. The set $\bar{\mathscr{S}}$ is the complement of $\mathscr{S}$ in the cut plane. Writing $\nu = (s^2 - 2i/\delta^2)^{1/2}$, with the principal value of the square root understood, we see immediately that $s = \gamma(\nu) = \tilde{\gamma}(\nu)$, the values of the two branches being equal at $\zeta = \nu$ since $\nu \in \bar{\mathscr{S}}$. Thus $K$ may also be written as

$$K(\zeta) = \lambda\{\tilde{\gamma}(\zeta) + \tilde{\gamma}(\nu)\} \tag{A5}$$

and in this form it continues analytically into the complex $\zeta$-plane cut as in Fig. A1. It is also clear from (A5) that $K$ has no zeros anywhere in the $\zeta$-plane in contrast to $K_1$ and $K_2$ of (3.14), as we shall see later. This means that there are no poles of $K'/K$ whose residues contribute to the integrations around the closed contours $\Gamma_+$ and $\Gamma_-$ shown in Fig. A1. Thus the integrals (A3) are transformed by Cauchy's theorem into integrals along the branch cuts in Fig. A1, and these integrals can be evaluated by a change of variable in terms of the function

$$\psi(\zeta) \equiv \frac{1}{2\tilde{\gamma}(\zeta)} \left\{\pi - i \log\left[\frac{\tilde{\gamma}(\zeta) - \zeta}{\tilde{\gamma}(\zeta) + \zeta}\right]\right\} \tag{A6}$$

which is discussed in DW (see also Dawson 1979). The function $\psi$ is analytic in the entire $\zeta$-plane except for a simple pole at the lower branch point $\zeta = (1-i)/\delta$ of $\tilde{\gamma}$, and satisfies

$$\psi[(i-1)/\delta] = \tfrac{1}{2}(1-i)\delta, \qquad \psi(\zeta) + \psi(-\zeta) = \pi/\tilde{\gamma}(\zeta). \tag{A7}$$

The final results are then conveniently expressed as

$$K_\pm'(\zeta)/K_\pm(\zeta) = \mp\,\tilde{\gamma}(\nu)\Psi(\pm\zeta, \nu) \tag{A8}$$

where

$$2\pi\Psi(\zeta, \nu) \equiv [\psi(\zeta) - \psi(\nu)]/(\zeta - \nu) + [\psi(\zeta) - \psi(-\nu)]/(\zeta + \nu). \tag{A9}$$

An integration of (A8) gives the required factorization of (A1) in the form

$$k_\pm(\zeta) \equiv \frac{K_\pm(\zeta)}{K_\pm(0)} = \exp\left(\mp\tilde{\gamma}(\nu)\int_0^\zeta \Psi(\pm w, \nu)\, dw\right). \tag{A10}$$

We next consider the functions $K_j$ ($j = 1, 2$) defined, as in equation (3.14) by

$$K_j(\zeta) = 1 + \lambda_j\gamma(\zeta) + \lambda_j r\zeta^2. \tag{A11}$$

We define also the auxiliary parameters

$$\chi_j^{\pm} = \frac{1}{2r} \pm \left( \frac{1}{4r^2} - \frac{1}{\lambda_j r} + \frac{2i}{\delta^2} \right)^{1/2}, \tag{A12}$$

$$\nu_j^{\pm} = \left( \frac{\chi_j^{\pm}}{r} - \frac{1}{\lambda_j r} \right)^{1/2}, \tag{A13}$$

where principal values of the square roots are understood, so that

$$\operatorname{Re}\chi_j^+ > 0, \quad \operatorname{Im}\chi_j^+ > 0, \quad \operatorname{Im}\chi_j^- < 0 \tag{A14}$$

$$\operatorname{Re}\nu_j^+ > 0, \quad \operatorname{Im}\nu_j^+ > 0, \quad \operatorname{Re}\nu_j^- > 0, \quad \operatorname{Im}\nu_j^- < 0. \tag{A15}$$

Furthermore, it is easily shown that

$$(\chi_j^{\pm})^2 = (\nu_j^{\pm})^2 + 2i/\delta^2 \equiv \{\gamma(\nu_j^{\pm})\}^2 \tag{A16}$$

and also that

$$\chi_j^+ \chi_j^- = 1/(\lambda_j r) - 2i/\delta^2, \quad \chi_j^+ + \chi_j^- = 1/r. \tag{A17}$$

With the aid of the results (A17) it is easy to show that $K_j$ may be written as the product

$$K_j(\zeta) = M_j(\zeta) N_j(\zeta) \tag{A18}$$

where

$$M_j(\zeta) = \lambda_j r \{\gamma(\zeta) + \chi_j^+\}, \quad N_j(\zeta) = \gamma(\zeta) + \chi_j^-. \tag{A19}$$

Now from (A14) and (A16) it is readily seen that

$$\chi_j^+/\gamma(\nu_j^+) = +1, \quad \tau_j \equiv \chi_j^-/\gamma(\nu_j^-) = \pm 1. \tag{A20}$$

Here the uncertainty in the sign of $\tau_j$ is a consequence of the fact that $\operatorname{Re}\chi_j^-$, conspicuously absent from (A14), may be positive or negative depending on the values of $\lambda_j$ and $r$. Equations (A19) and (A20) show that $M_j$, and also $N_j$ when $\tau_j = +1$, are non-vanishing but that $N_j$ has zeros at the points $\zeta = \pm\nu_j^-$ when $\tau_j = -1$. Thus the strip parallel to the real axis in which the functions $K_j'/K_j$ ($j = 1, 2$) are analytic may (depending on where the possible zeros of $N_j$, and hence of $K_j$, lie) be narrower than the strip $|\eta| < 1/\delta$ in which the functions $K_j$ are analytic. Specifically we can state that both $K_1'/K_1$ and $K_2'/K_2$ are analytic in the strip $|\eta| < c$ where

$$c = \min(1/\delta, c_1, c_2) \tag{A21}$$

with $c_j$ ($j = 1, 2$) defined by $c_j = \tfrac{1}{2}(1 - \tau_j)|\operatorname{Im}\nu_j^-| + \tfrac{1}{2}(1 + \tau_j)/\delta$. It follows that $K_j^+$ obtained by the factorization procedure described earlier will be analytic and zero-free in the half-plane $\eta > -c$, and therefore, by a well-known property of the Fourier integral (Noble 1958, p. 25), the value of $c$ defined by equation (A21) also provides an estimate of the attenuation of the anomalous field as $y \to +\infty$ in the manner anticipated by the assertion (3.3).

It is now possible to obtain the required factorization of $K_j$ by inspection. Following the reasoning that led to $K$ being written in the form (A5) we can analytically continue the functions $K_j$ into the cut plane of Fig. A1 by replacing $\gamma(\zeta)$ by $\tilde{\gamma}(\zeta)$ everywhere. Moreover, since it is obvious from (A15) that $\nu_j^+ \in \mathscr{S}$, we deduce from (A4) and (A20) that

$$\chi_j^+ = \tilde{\gamma}(\nu_j^+). \tag{A22}$$

Thus the function $M_j$ in (A19) becomes

$$M_j(\zeta) = \lambda_j r[\tilde{\gamma}(\zeta) + \tilde{\gamma}(\nu_j^+)], \tag{A23}$$

and defining

$$\tilde{\tau}_j \equiv \frac{\chi_j^-}{\tilde{\gamma}(\nu_j^-)} = \begin{cases} -\tau_j & (\nu_j^- \in \mathscr{S}), \\ \tau_j & (\nu_j^- \in \overline{\mathscr{S}}), \end{cases} \tag{A24}$$

where the equality with $\pm\tau_j$ follows from (A4) and (A20), we can also rewrite $N_j$ in (A19) as

$$N_j(\zeta) = \tilde{\gamma}(\zeta) + \tilde{\tau}_j\tilde{\gamma}(\nu_j^-). \tag{A25}$$

The expressions (A23) and (A25) are analytic in the cut plane and agree, of course, with the original definitions of $M_j$ and $N_j$ within the strip $|\eta| < 1/\delta$. They are also written precisely in the form of (A5) (unless $\tilde{\tau}_j = -1$ in the case of $N_j$) so that the required factorizations are given immediately by the formula (A10). Even when $\tilde{\tau}_j = -1$, a simple algebraic rearrangement of (A25) gives

$$N_j(\zeta) = (\zeta - \nu_j^-)(\zeta + \nu_j^-)/\{\tilde{\gamma}(\zeta) + \tilde{\gamma}(\nu_j)\}$$

in which the divisor is again of the form (A5) and the rest of the expression is already factored in the appropriate manner since it is clear from (A15) and (A21) that the zeros of $\zeta - \nu_j^-$ and $\zeta + \nu_j^-$ are in the half-planes $\eta < -c$ and $\eta > c$ respectively.

Combining the separate factorizations of $M_j$ and $N_j$ in equation (A18) and defining

$$k_j^{\pm}(\zeta) = K_j^{\pm}(\zeta)/K_j^{\pm}(0) \tag{A26}$$

as in (3.21), we obtain

$$k_j^{\pm}(\zeta) = (\zeta \mp \nu_j^-)^{(1-\tilde{\tau}_j)/2} \exp\left(\mp \int_0^{\zeta} \{\tilde{\gamma}(\nu_j^+)\Psi(\pm w, \nu_j^+) + \tilde{\tau}_j\tilde{\gamma}(\nu_j)\Psi(\pm w, \nu_j^-)\}\, dw\right).$$

Expressing the first factor above as an exponential function, and simplifying with the aid of (A22) and (A24), we can write this result in the more compact form

$$k_j^{\pm}(\zeta) = \exp\left(\int_0^{\zeta} \phi_j^{\pm}(w)\, dw\right) \tag{A27}$$

where

$$\phi_j^+(w) = -\phi_j^-(-w) = \tfrac{1}{2}(1-\tilde{\tau}_j)(w-\nu_j^-)^{-1} - \chi_j^+\Psi(w, \nu_j^+) - \chi_j^-\Psi(w, \nu_j^-). \tag{A28}$$

An immediate deduction from (A27) and (A28) is

$$k_j^+(-\zeta) = k_j^-(\zeta). \tag{A29}$$

A useful function for the computation of the field integrals is $h$ defined by

$$\exp[h(\zeta)] \equiv R_+(\zeta)/R_+(0) \equiv k_1^+(\zeta)/k_2^+(\zeta)$$

the second identity resulting from (3.17) and (A26). It follows from (A27) that

$$h(\zeta) = \int_0^{\zeta} \{\phi_1^+(w) - \phi_2^+(w)\}\, dw. \tag{A30}$$

In particular it is convenient to write

$$g(u) = h[i\gamma(u)], \tag{A31}$$

where $u$ is a real variable.

The asymptotic behaviour of the functions $k_j^{\pm}(\zeta)$ for large $|\zeta|$ was required in the discussion of the convergence properties of the field integrals. We shall show here that they are both $O(\zeta)$, provided that $r$ is non-zero and finite. (We may restrict ourselves to such values of $r$ since it was shown in Section 5 that $r = 0$ reduces the problem to that of DW, while $r \to \infty$ is a trivial problem.) In addition we require, for the present, that both $\lambda_1$ and $\lambda_2$ are non-zero and finite.

With the help of equation (A7) we can write (A9) in the useful alternative form

$$\tilde{\gamma}(a)\Psi(\zeta, a) = -\Theta(\zeta, a) - \tfrac{1}{2}\zeta/(\zeta^2 - a^2) \tag{A32}$$

where we have defined

$$2\pi\Theta(\zeta, a) = -\tilde{\gamma}(a)\,\{2\zeta\psi(\zeta) - a\psi(a) + a\psi(-a)\}/(\zeta^2 - a^2), \tag{A33}$$

and where $a$ is a complex parameter with $\mathrm{Im}\, a \neq 0$. It is obvious from (A6) that $\zeta\psi(\zeta) \equiv O(\log\zeta)$ as $|\zeta| \to \infty$, whence

$$\Theta(\zeta, a) = O(\zeta^{-2}\log\zeta). \tag{A34}$$

Thus we can define

$$J(a) \equiv \int_0^\infty \Theta(\xi, a)\, d\xi, \tag{A35}$$

the path of integration being the real axis. We now let $\xi$ be a large positive real number, where for convenience we assume $\xi \gg |a|$, and consider the equation

$$-\tilde{\gamma}(a)\int_0^\xi \Psi(u, a)\, du = J(a) - \int_\xi^\infty \theta(u, a)\, du + \tfrac{1}{2}\log\xi - \tfrac{1}{4}\log(-a^2) + \tfrac{1}{4}\log\left(1 - \frac{a^2}{\xi^2}\right) \tag{A36}$$

obtained by integration of (A32) and some manipulation of the logarithm. From the estimate (A34), it is evident that the integral in (A36) is $O(\xi^{-1}\log\xi)$, so that we can write

$$-\tilde{\gamma}(a)\int_0^\xi \Psi(u, a)\, du = \tfrac{1}{2}\log\xi + \alpha(a) + O\left(\frac{\log\xi}{\xi}\right) \tag{A37}$$

where $\alpha(a) = J(a) - \tfrac{1}{4}\log(-a^2)$ is a complex constant.

We now consider the behaviour of the integral (A36) for complex $\zeta$ of large modulus. We put $\zeta = R\exp(i\phi)$ where $R \gg |a|$ and we consider only the upper half-plane $0 \leqslant \phi \leqslant \pi$. The function $\Psi(\zeta, a)$ is analytic in the upper half-plane, and so by Cauchy's theorem we may write

$$\int_0^\zeta \Psi(w, a)\, dw = \int_0^R \Psi(u, a)\, du + iR\int_0^\phi \Psi[R\exp(i\theta), a]\exp(i\theta)\, d\theta. \tag{A38}$$

The asymptotic form of the first term follows immediately from equation (A37). In the second term we substitute from (A32) and note that

$$R\int_0^\phi \Theta[R\exp(i\theta), a]\exp(i\theta)\, d\theta = O\left(\frac{\log R}{R}\right) \tag{A39}$$

by (A34). There remains only the integral

$$\frac{-\tfrac{1}{2}i}{\tilde{\gamma}(a)}\int_0^\phi \frac{R^2\exp(2i\theta)\, d\theta}{R^2\exp(2i\theta) - a^2} = \frac{-\tfrac{1}{4}}{\tilde{\gamma}(a)}\log\frac{\exp(2i\phi) - a^2/R^2}{1 - a^2/R^2} = \frac{-\tfrac{1}{2}i\phi}{\tilde{\gamma}(a)} + O(1/R^2). \tag{A40}$$

Combining (A37), (A38), (A39) and (A40) we obtain

$$-\tilde{\gamma}(a)\int_0^{\zeta}\Psi(w,a)\,dw = \tfrac{1}{2}\log|\zeta| + \tfrac{1}{2}i\arg\zeta + \alpha(a) + O\left(\frac{\log\zeta}{\zeta}\right) \qquad (0 < \arg\zeta < \pi). \tag{A41}$$

The asymptotic form of $k_j^{\pm}(\zeta)$ is now easily found with the aid of equation (A41). With reference to (A28), (A22) and (A24) we write

$$\int_0^{\zeta}\phi_j^+(w)\,dw = \frac{1-\tilde{\tau}_j}{2}\int_0^{\zeta}\frac{dw}{w-\nu_j^-} - \tilde{\gamma}(\nu_j^+)\int_0^{\zeta}\Psi(w,\nu_j^+)\,dw - \tilde{\tau}_j\tilde{\gamma}(\nu_j^-)\int_0^{\zeta}\Psi(w,\nu_j^-)\,dw, \tag{A42}$$

where as above, $\zeta$ is in the upper half-plane, and we assume for convenience that $|\zeta| \gg \max\{|\nu_j^-|, |\nu_j^+|\}$. It is readily shown that

$$\int_0^{\zeta}\frac{dw}{w-\nu_j^-} = \log\zeta - \log(-\nu_j^-) + O(1/\zeta). \tag{A43}$$

Substitution of equations (A41) and (A43) in (A42) yields

$$\int_0^{\zeta}\phi_j^+(w)\,dw = \log\zeta + \log\beta_j^+ + O\left(\frac{\log\zeta}{\zeta}\right), \tag{A44}$$

where we have defined

$$\log\beta_j^+ = \alpha(\nu_j^+) + \tilde{\tau}_j\alpha(\nu_j^-) - \tfrac{1}{2}(1-\tilde{\tau}_j)\log(-\nu_j^-), \tag{A45}$$

a constant that depends only on the parameters of the problem. From (A44) and (A37) it follows that

$$k_j^+(\zeta) = \beta_j^+\zeta\{1 + O(\zeta^{-1}\log\zeta)\} \tag{A46}$$

as $|\zeta| \to \infty$ with $0 \leqslant \arg\zeta \leqslant \pi$. The analogous result

$$k_j^-(\zeta) = \beta_j^-\zeta\{1 + O(\zeta^{-1}\log\zeta)\}, \tag{A47}$$

as $|\zeta| \to \infty$ with $-\pi \leqslant \arg\zeta \leqslant 0$ can be derived in a similar manner. Now $k_j^+$ and $k_j^-$ are related by equation (A29) and since $\beta_j^{\pm}$ depend only on the model parameters we must have $\beta_j^+ = -\beta_j^- = \beta_j$ (say). Therefore we may write the required asymptotic forms in the single expression

$$k_j^{\pm}(\zeta) = \pm\beta_j\zeta\{1 + O(\zeta^{-1}\log\zeta)\} \tag{A48}$$

when $|\zeta| \to \infty$ in the appropriate half-plane.

Equation (A48) holds provided that $r$ and $\lambda_j$ are non-zero and finite. From the discussion of Section 5 it is clear that it continues to hold as $\lambda_j \to \infty$. However, when $\lambda_j = 0$, we have $K_j(\zeta) \equiv 1$ by (A11), and hence $k_j^{\pm}(\zeta) = 1$. Finally, for $\lambda_j > 0$ it follows from (A44) and the definitions (A30) and (A31) that

$$h(\zeta) = O(1), \qquad g(u) = O(1), \tag{A49}$$

as $|\zeta| \to \infty$ and $u \to \infty$ respectively. In the special case $\lambda_2 = 0$ (say), the corresponding asymptotic behaviour is

$$h(\zeta) = O(\log\zeta), \qquad g(u) = O(\log u). \tag{A50}$$