Çankaya University Journal of Science and Engineering
Volume 12, No. 1 (2015) 001–013

B-spline Differential Quadrature Method for the Modified Burgers' Equation

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Abstract: In this study, the Quintic B-spline Differential Quadrature method (QBDQM) is applied to find the numerical solution of the modified Burgers' equation (MBE). The efficiency and accuracy of the method are measured by calculating the maximum error norm L_∞ and the discrete root mean square error L_2. The obtained numerical results are compared with published numerical results and the comparison shows that the method is an effective numerical scheme to solve the MBE. A rate of convergence analysis is also given.

Keywords: Partial differential equations, Differential quadrature method, Modified Burgers' equation, Quintic B-Splines, Fourth order Runge-Kutta method.

1. Introduction

The one-dimensional Burgers' equation, which is a nonlinear partial differential equation of second order, was first introduced by Bateman [1] and later treated by Burgers' [2]. It has the form

$$U_t + UU_x - vU_{xx} = 0, \tag{1}$$

where v is a positive parameter and the subscripts x and t denote space and time derivatives, respectively. This equation is very important in fluid dynamics especially for turbulence problems, gas dynamics, heat conduction, continuous stochastic processes and the theory of shock waves [3]. Analytical solutions for the equation were found for restricted values of v which represent the kinematics viscosity of fluid motion. So the numerical solution of the Burgers' equation has been subject of many papers. Various numerical methods have been studied based on finite difference [4, 5], the Runge-Kutta-Chebyshev method [6, 7], group-theoretic methods [8], finite element methods including Galerkin, Petrov-Galerkin, least squares and collocation $[9-17]$. The modified Burgers' equation (MBE) which we discuss in this study is based upon the Burgers' equation (BE) of the form

$$U_t + U^2U_x - vU_{xx} = 0. \tag{2}$$

The equation has strong non-linear aspects and has been used in many practical transport problems such as non-linear waves in a medium with low-frequency pumping or absorption, turbulence transport, wave processes in thermoelastic medium, transport and dispersion of pollutants in rivers and sediment transport, ion reflection at quasi-perpendicular shocks. Recently, numerical studies of the equation have been presented. M. A. Ramadan et al. [18] obtained numerical solutions of the MBE using the quintic B-spline collocation finite element method. A special lattice Boltzmann model has been developed by Y. Duan et al. [19]. B. Saka et al. [20] have developed a Galerkin finite element solution of the equation using quintic B-splines and the time-split technique. A solution based on the sextic B-spline collocation method has been proposed by D. Irk [21]. T. Roshan et al. [22] applied a Petrov-Galerkin method using a linear hat function as the trial function and a cubic B-spline function as the test function. A discontinuous Galerkin method has been presented by Zhang Rong-Pei et al. [23]. A. G. Bratsos [24] has used a finite difference scheme based on fourth-order rational approximates to the matrix-exponential term in a two-time level recurrence relation to calculate the numerical solution of the equation.

Bellman et al. [26, 27] first introduced the Differential Quadrature Method (DQM) in 1972 to solve partial differential equations. The method has widely become popular in recent years thanks to its simplicity of application. The fundamental idea behind the method is to find the weighting coefficients of the functional values at the nodal points by using base functions, derivatives of which are already known at the same nodal points over the entire region. Numerous researchers have developed different types of DQMs by utilizing various test functions. For example, Bellman et al. [26, 27] have used Legendre polynomials and spline functions in order to obtain weighting coefficients. Quan and Chang [28, 29] have introduced an explicit formulation to determine the weighting coefficients using Lagrange interpolation polynomials. Shu and Richards [30] have presented explicit formulae including both Lagrange interpolation polynomials. Moreover, Shu and Xue [31] have used the Lagrange interpolated trigonometric polynomials to determine weighting coefficients in an explicit manner. Zhong [32], Guo and Zhong [33] and Zhong and Lan [34] have introduced another efficient DQM as a spline based DQM and applied it to numerous problems. Cheng et al. [35] have used Hermite polynomials to find the weighting coefficients required for DQM. Shu and Wu [36] have introduced some of the implicit formulations of weighting coefficients with the help of radial basis functions. The weighting coefficients have also been found by Striz et al. [37] using harmonic functions implicitly. Sinc functions have been used as basis functions in order to find the weighting coefficients by Bonzani [38]. In the past decades, DQM has come to be a very efficient and effective method to obtain the numerical solutions of various types of partial differential equations due to its simplicity of application. The DQM has many advantages over the classical techniques. It prevents linearization and perturbation in order to find better solutions of given nonlinear equations. Since QBDQM do not need transforming to solve the equation, this method has been preferred.

In the present work, we have applied a quintic B-spline differential quadrature method to the MBE. To show the performance and accuracy of the method and to make a comparison of numerical solutions, we have taken different values of v. A rate of convergence analysis is also given.

2. Quintic B-spline Differential Quadrature Method

We will consider (2) with the boundary conditions chosen from:

$$U(a,t) = g_1(t), \qquad U(b,t) = g_2(t), \qquad t \geq 0, \tag{3}$$

with the initial condition

$$U(x,0) = f(x), \qquad a \leq x \leq b, \tag{4}$$

where β_1 and β_2 are constants. DQM can be defined as an approximation to a derivative of a given function by using the linear summation of its values at specific discrete nodal points over the solution domain of the problem. If we take the grid distribution $a = x_1 < x_2 < \cdots < x_N = b$ of a finite interval $[a,b]$ into consideration and provided that any given function $U(x)$ is smooth enough over the solution domain, its derivatives with respect to x at a nodal point x_i can be approximated by a linear summation of all the functional values in the solution domain, namely,

$$U_x^{(r)}(x_i) = \frac{d^{(r)}U}{dx^{(r)}}\Big|_{x_i} = \sum_{j=1}^{N} w_{ij}^{(r)} U(x_j), \quad i = 1,2,...,N, \quad r = 1,2,...,N-1 \tag{5}$$

where r denotes the order of derivative, $w_{ij}^{(r)}$ represent the weighting coefficients of the $r-th$ order derivative approximation, and N denotes the number of nodal points in the solution domain. Here, the index j represents the fact that $w_{ij}^{(r)}$ is the corresponding weighting coefficient of the functional value $U(x_j)$.

In this work, we need first and second order derivatives of the function $U(x)$. Therefore, we will find the value of (5) for $r = 1,2$. If we consider (5) carefully, then it is seen that the fundamental process for approximating the derivatives of any given function through DQM is to find the corresponding weighting coefficients $w_{ij}^{(r)}$. The main idea behind DQM approximation is to find the corresponding weighting coefficients $w_{ij}^{(r)}$ by means of a set of base functions spanning the problem domain. While determining the corresponding weighting coefficients, a different basis may be used. In the present study, we will attempt to compute the weighting coefficients with the quintic B-spline basis.

Let $Q_m(x)$, be the quintic B-splines with knots at the points x_i where the uniformly distributed N nodal points are taken as $a = x_1 < x_2 < \cdots < x_N = b$ on the ordinary real axis. Then, the B-splines

$\{Q_{-1}, Q_0, \ldots, Q_{N+2}\}$ form a basis for functions defined over $[a,b]$. The quintic B-splines $Q_m(x)$ are defined by the relationships:

$$Q_m(x) = \frac{1}{h^5}\begin{cases} (x-x_{m-3})^5, & x \in [x_{m-3}, x_{m-2}], \\ (x-x_{m-3})^5 - 6(x-x_{m-2})^5, & x \in [x_{m-2}, x_{m-1}], \\ (x-x_{m-3})^5 - 6(x-x_{m-2})^5 + 15(x-x_{m-1})^5, & x \in [x_{m-1}, x_m], \\ (x-x_{m-3})^5 - 6(x-x_{m-2})^5 + 15(x-x_{m-1})^5 - \\ 20(x-x_m)^5, & x \in [x_m, x_{m+1}], \\ (x-x_{m-3})^5 - 6(x-x_{m-2})^5 + 15(x-x_{m-1})^5 - \\ 20(x-x_m)^5 + 15(x-x_{m+1})^5, & x \in [x_{m+1}, x_{m+2}], \\ (x-x_{m-3})^5 - 6(x-x_{m-2})^5 + 15(x-x_{m-1})^5 - \\ 20(x-x_m)^5 + 15(x-x_{m+1})^5 - 6(x-x_{m+2})^5, & x \in [x_{m+2}, x_{m+3}], \\ 0, & otherwise. \end{cases}$$

where $h = x_m - x_{m-1}$ for all m.

TABLE 1. The value of quintic B-splines and derivatives functions at the grid points.

x	x_{m-3}	x_{m-2}	x_{m-1}	x_m	x_{m+1}	x_{m+2}	x_{m+3}
Q	0	1	26	66	26	1	0
Q'	0	$\frac{5}{h}$	$\frac{50}{h}$	0	$-\frac{50}{h}$	$-\frac{5}{h}$	0
Q''	0	$\frac{20}{h^2}$	$\frac{40}{h^2}$	$-\frac{120}{h^2}$	$\frac{40}{h^2}$	$\frac{20}{h^2}$	0
Q'''	0	$\frac{60}{h^3}$	$-\frac{120}{h^3}$	0	$\frac{120}{h^3}$	$-\frac{60}{h^3}$	0
$Q^{(4)}$	0	$\frac{120}{h^4}$	$\frac{480}{h^4}$	$\frac{720}{h^4}$	$-\frac{480}{h^4}$	$\frac{120}{h^4}$	0

Using the quintic B-splines as test functions in the fundamental DQM equation (5) leads to the equation

$$\frac{\partial^{(r)} Q_m(x_i)}{\partial x^{(r)}} = \sum_{j=m-2}^{m+2} w_{i,j}^{(r)} Q_m(x_j), \quad m = -1, 0, \ldots, N+2,\ i = 1, 2, ..., N. \tag{6}$$

An arbitrary choice of i leads to an algebraic equation system

$$\begin{bmatrix} Q_{-1,-3} & Q_{-1,-2} & Q_{-1,-1} & Q_{-1,0} & Q_{-1,1} & & & \\ & Q_{0,-2} & Q_{0,-1} & Q_{0,0} & Q_{0,1} & Q_{0,2} & & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & & \\ & & Q_{N+1,N-1} & Q_{N+1,N} & Q_{N+1,N+1} & Q_{N+1,N+2} & Q_{N+1,N+3} & \\ & & & Q_{N+2,N} & Q_{N+2,N+1} & Q_{N+2,N+2} & Q_{N+2,N+3} & Q_{N+2,N+4} \end{bmatrix} W_1 = \Phi_1 \tag{7}$$

where $Q_{i,j}$ denotes $Q_i(x_j)$,

$$W_1 = \left[w_{i,-3}^{(r)} \quad w_{i,-2}^{(r)} \quad \cdots \quad w_{i,N+3}^{(r)} \quad w_{i,N+4}^{(r)} \right]^T \tag{8}$$

and

$$\Phi_1 = \left[\frac{\partial^{(r)} Q_{-1}(x_i)}{\partial x^{(r)}} \quad \frac{\partial^{(r)} Q_0(x_i)}{\partial x^{(r)}} \quad \cdots \quad \frac{\partial^{(r)} Q_{N+1}(x_i)}{\partial x^{(r)}} \qquad \frac{\partial^{(r)} Q_{N+2}(x_i)}{\partial x^{(r)}} \right]^T. \tag{9}$$

The weighting coefficients $w_{i,j}^{(r)}$ related to the $i-th$ grid point are determined by solving equation system (7). Equation system (7) consists of $N+8$ unknowns and $N+4$ equations. For this system to have a unique solution, it is required to add four additional equations to the system. By the addition of the equations

$$\frac{\partial^{(r+1)} Q_{-1}(x_i)}{\partial x^{(r+1)}} = \sum_{j=-3}^{1} w_{i,j}^{(r)} Q'_{-1}(x_j), \tag{10}$$

$$\frac{\partial^{(r+1)} Q_{0}(x_i)}{\partial x^{(r+1)}} = \sum_{j=-2}^{2} w_{i,j}^{(r)} Q'_{0}(x_j), \tag{11}$$

$$\frac{\partial^{(r+1)} Q_{N+1}(x_i)}{\partial x^{(r+1)}} = \sum_{j=N-1}^{N+3} w_{i,j}^{(r)} Q'_{N+1}(x_j), \tag{12}$$

$$\frac{\partial^{(r+1)} Q_{N+2}(x_i)}{\partial x^{(r+1)}} = \sum_{j=N}^{N+4} w_{i,j}^{(r)} Q'_{N+2}(x_j), \tag{13}$$

equation system (7) becomes

$$M_1 W_1 = \Phi_2, \tag{14}$$

where

$$M_1 = \begin{bmatrix}
Q_{-1,-3} & Q_{-1,-2} & Q_{-1,-1} & Q_{-1,0} & Q_{-1,1} & & & \\
Q'_{-1,-3} & Q'_{-1,-2} & Q'_{-1,-1} & Q'_{-1,0} & Q'_{-1,1} & & & \\
 & Q_{0,-2} & Q_{0,-1} & Q_{0,0} & Q_{0,1} & Q_{0,2} & & \\
 & Q'_{0,-2} & Q'_{0,-1} & Q'_{0,0} & Q'_{0,1} & Q'_{0,2} & & \\
 & & Q_{1,-1} & Q_{1,0} & Q_{1,1} & Q_{1,2} & Q_{1,3} & \\
 & & \ddots & \ddots & \ddots & \ddots & \ddots & \\
 & & Q_{N+1,N-1} & Q_{N+1,N} & Q_{N+1,N+1} & Q_{N+1,N+2} & Q_{N+1,N+3} & \\
 & & Q'_{N+1,N-1} & Q'_{N+1,N} & Q'_{N+1,N+1} & Q'_{N+1,N+2} & Q'_{N+1,N+3} & \\
 & & & Q_{N+2,N} & Q_{N+2,N+1} & Q_{N+2,N+2} & Q_{N+2,N+3} & Q_{N+2,N+4} \\
 & & & Q'_{N+2,N} & Q'_{N+2,N+1} & Q'_{N+2,N+2} & Q'_{N+2,N+3} & Q'_{N+2,N+4}
\end{bmatrix}$$

and

$$W_1 = \left[w_{i,-3}^{(r)} \quad w_{i,-2}^{(r)} \quad \cdots \quad w_{i,N+3}^{(r)} \quad w_{i,N+4}^{(r)} \right]^T$$

and

$$\Phi_2 = [\tfrac{\partial^{(r)} Q_{-1}(x_i)}{\partial x^{(r)}} \quad \tfrac{\partial^{(r+1)} Q_{-1}(x_i)}{\partial x^{(r+1)}} \quad \tfrac{\partial^{(r)} Q_{0}(x_i)}{\partial x^{(r)}} \quad \tfrac{\partial^{(r+1)} Q_{0}(x_i)}{\partial x^{(r+1)}} \quad \tfrac{\partial^{(r)} Q_{1}(x_i)}{\partial x^{(r)}}$$
$$\cdots \quad \tfrac{\partial^{(r)} Q_{N+1}(x_i)}{\partial x^{(r)}} \quad \tfrac{\partial^{(r+1)} Q_{N+1}(x_i)}{\partial x^{(r+1)}} \quad \tfrac{\partial^{(r)} Q_{N+2}(x_i)}{\partial x^{(r)}} \quad \tfrac{\partial^{(r+1)} Q_{N+2}(x_i)}{\partial x^{(r+1)}}]^T.$$

After using the values of the quintic B-splines at the grid points and eliminating $w_{i,-3}^{(r)}, w_{i,-2}^{(r)}, w_{i,N+3}^{(r)}$ and $w_{i,N+4}^{(r)}$ from the equation system, we obtain an algebraic equation system having 5-banded coefficient matrix of the form

$$M_2 W_2 = \Phi_3, \tag{15}$$

where

$$M_2 = \begin{bmatrix} 37 & 82 & 21 & & & & & & \\ 8 & 33 & 18 & 1 & & & & & \\ 1 & 26 & 66 & 26 & 1 & & & & \\ & 1 & 26 & 66 & 26 & 1 & & & \\ & & \ddots & \ddots & \ddots & \ddots & \ddots & & \\ & & & 1 & 26 & 66 & 26 & 1 & \\ & & & & 1 & 26 & 66 & 26 & 1 \\ & & & & & 1 & 18 & 33 & 8 \\ & & & & & & 21 & 82 & 37 \end{bmatrix} \quad \text{and} \quad W_2 = \begin{bmatrix} w_{i,-1}^{(r)} \\ w_{i,0}^{(r)} \\ \vdots \\ w_{i,i-2}^{(r)} \\ w_{i,i-1}^{(r)} \\ w_{i,i}^{(r)} \\ w_{i,i+1}^{(r)} \\ w_{i,i+2}^{(r)} \\ \vdots \\ w_{i,N+1}^{(r)} \\ w_{i,N+2}^{(r)} \end{bmatrix}.$$

The nonzero entries of the load vector Φ_3 are given as,

$$\Phi_{-1} = \frac{1}{30}\left[-5Q_{-1}^{(p)}(x_i) + hQ_{-1}^{(p+1)}(x_i) + 40Q_0^{(p)}(x_i) + 8hQ_0^{(p+1)}(x_i)\right],$$

$$\Phi_0 = \frac{1}{10}\left[5Q_0^{(p)}(x_i) - hQ_0^{(p+1)}(x_i)\right],$$

$$\Phi_{i-2} = Q_{i-2}^{(p)}(x_i),$$

$$\Phi_{i-1} = Q_{i-1}^{(p)}(x_i),$$

$$\Phi_i = Q_i^{(p)}(x_i),$$

$$\Phi_{i+1} = Q_{i+1}^{(p)}(x_i),$$

$$\Phi_{i+2} = Q_{i-2}^{(p)}(x_i),$$

$$\Phi_{N+1} = \frac{1}{10}\left[5Q_{N+1}^{(p)}(x_i) + hQ_{N+1}^{(p+1)}(x_i)\right],$$

$$\Phi_{N+2} = \frac{-1}{30}\left[-40Q_{N+1}^{(p)}(x_i) + 8hQ_{N+1}^{(p+1)}(x_i) + 5Q_{N+2}^{(p)}(x_i) + hQ_{N+2}^{(p+1)}(x_i)\right].$$

Equation system (15) is solved by the 5-banded Thomas algorithm.

3. Numerical discretization

The modified Burgers' equation of the form

$$U_t + U^2 U_x - \nu U_{xx} = 0,$$

with boundary conditions (3) and initial condition (4) is rewritten as

$$U_t = -U^2 U_x + \upsilon U_{xx}. \tag{16}$$

Then, the differential quadrature derivative approximations given in (5), for the value of $r = 1,2$ are used in (16). The application of the boundary conditions results in

$$\frac{dU(x_i)}{dt} = -U^2(x_i,t)\sum_{j=2}^{N-1} w_{i,j}^{(1)} U(x_j,t) + \upsilon \sum_{j=2}^{N-1} w_{i,j}^{(2)} U(x_j,t) + B(U),\ i = 2,3,...,N-1 \tag{17}$$

where

$$B(U) \quad = \quad -U^2(x_i,t)\left[w_{i,1}^{(1)} g_1(t) + w_{i,N}^{(1)} g_2(t)\right] + \upsilon \left[w_{i,1}^{(2)} g_1(t) + w_{i,N}^{(2)} g_2(t)\right].$$

Then, the ordinary differential equation given by (17) is integrated in time by means of any appropriate method. Here, we have selected the fourth-order Runge-Kutta method thanks to its advantages such as accuracy, stability and memory allocation properties.

4. Numerical examples and results

In this section, we obtain the numerical solutions of the MBE by the QBDQM. The accuracy of the numerical method is checked using the error norms L_2 and L_∞ respectively:

$$\begin{aligned} L_2 &= \left\|U^{exact} - U_N\right\|_2 \simeq \sqrt{h\sum_{J=1}^{N}\left|U_j^{exact} - (U_N)_j\right|^2}, \\ L_\infty &= \left\|U^{exact} - U_N\right\|_\infty \simeq \max_j \left|U_j^{exact} - (U_N)_j\right|, j = 1,2,...,N-1. \end{aligned}$$

Stability analysis of a numerical method for a nonlinear differential equation requires the determination of eigenvalues of coefficient matrices. With the numerical discretization of partial differential equation MBE, it turns into an ordinary differential equation. These necessary operations cause enormous difficulties for the stability and convergence analysis. Many times, instead of stability analysis, numerical rate of convergence (ROC) analysis is preferred. Therefore, in order to overcome the difficulties, we calculate the ROC with the help of following formula

$$ROC \approx \frac{\ln\left(E(N_2)/E(N_1)\right)}{\ln\left(N_1/N_2\right)}.$$

Here $E(N_j)$ denotes either the L_2 error norm or the L_∞ error norm in case of using N_j grid points. Therefore, some further numerical runs for different numbers of space steps are performed. Ultimately, some computations are made about the ROC by assuming that these methods are algebraically convergent in space. In particular, we suppose that $E(N) \sim N^p$ for some $p < 0$ when $E(N)$ denotes the L_2 or the L_∞ error norms by using N subintervals. The analytical solution of the MBE is given in [25] as:

$$U(x,t) = \frac{(x/t)}{1 + (\sqrt{t}/c_0)\exp(x^2/4vt)}, \tag{18}$$

where c_0 is a constant and $0 < c_0 < 1$. For our numerical calculation, we take $c_0 = 0.5$. We use the initial condition for (18) evaluating at $t = 1$ and the boundary conditions are taken as $U(0,t) = 0$ and $U(1,t) = 0$.

For the numerical simulation, we have chosen the various viscosity parameters $v = 0.01, 0.001$ and time step $\Delta t = 0.01$ over the interval $0 \leq x \leq 1$ and $0 \leq x \leq 1.3$. As seen from Figure 1, when we select the solution domain $0 \leq x \leq 1$, the right part of the shock wave cannot be seen clearly. By using a larger domain such as $0 \leq x \leq 1.3$ as seen in Figure 2, the solution is better than for the narrow domain $0 \leq x \leq 1$, as shown in Figure 1.

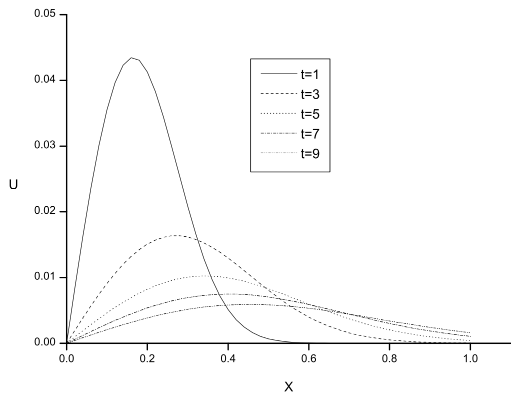

FIGURE 1. Solutions for $v = 0.01$, $h = 0.02$, $\Delta t = 0.01$, $0 \leq x \leq 1$.

The computed values of the error norms L_2 and L_∞ are presented at some selected times up to $t = 10$. The obtained results are tabulated in Tables 2 and 3.

As seen from Tables 2 and 3, the error norms L_2 and L_∞ are sufficiently small and satisfactorily acceptable. Furthermore, it is clear from these tables that if the value of viscosity v decreases, the value of the error norms will decrease. We obtain better results if the value of the viscosity parameter is smaller. The behaviors of the numerical solutions for viscosity $v = 0.01$ and 0.001 and time step $\Delta t = 0.01$ at times $t = 1, 3, 5, 7$ and 9 are shown in Figures 1 to 3.

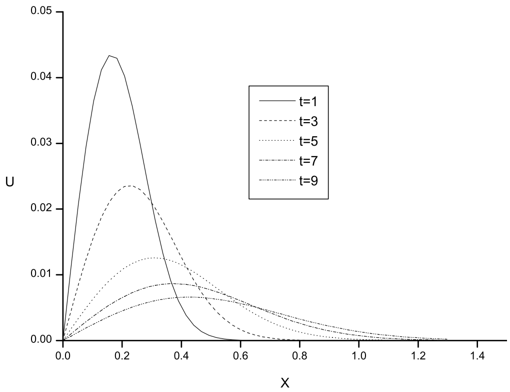

FIGURE 2. Solutions for $v = 0.01$, $h = 0.02$, $\Delta t = 0.01$, $0 \leq x \leq 1.3$.

TABLE 2. L_2 and L_∞ error norms for $v = 0.01$, $h = 0.02$, $\Delta t = 0.01$.

	QBDQM		*QBDQM* $[0,1.3]$		Ramadan et al.[16]	
Time	$L_2 \times 10^3$	$L_\infty \times 10^3$	$L_2 \times 10^3$	$L_\infty \times 10^3$	$L_2 \times 10^3$	$L_\infty \times 10^3$
2	0.6883159313	1.4061155014	0.6475135665	1.4186922574	0.7904296620	1.7030921188
3	0.6111942976	1.2284699151	0.6038312099	1.2481612622	0.6551928290	1.1832698216
4	0.5518907404	1.0470408075	0.5597368097	1.0744317361	0.5576794264	0.9964523368
5	0.5243679591	0.9114703246	0.5248818857	0.9393940240	0.5105617536	0.8561342445
6	0.5360036465	0.8147368174	0.4962790307	0.8341732147	0.5167229575	0.7610530060
7	0.5837932334	1.0140945729	0.4729494376	0.7511005752	0.5677438614	1.0654548090
8	0.6527370179	1.3014950978	0.4556226380	0.6853562194	0.6427542266	1.3581113635
9	0.7279265681	1.5456068136	0.4457470777	0.6313503003	0.7236430257	1.6048306653
10	0.8001311820	1.7425840423	0.4443904541	0.5873008192	0.8002564201	1.8023938553

TABLE 3. L_2 and L_∞ error norms for $v = 0.001$, $\Delta t = 0.01$, $N = 166$.

	QBDQM		Ramadan et al.[16]	
Time	$L_2 \times 10^3$	$L_\infty \times 10^3$	$L_2 \times 10^3$	$L_\infty \times 10^3$
2	0.1272271131	0.4571371972	0.1835491190	0.8185211112
3	0.1108493122	0.3892325088	0.1441424335	0.5234833346
4	0.0985692037	0.3332002275	0.1144110783	0.3563537207
5	0.0902342480	0.2885116847	0.0947865272	0.2549790058
6	0.0840729951	0.2546793589	0.0814174677	0.2134847835
7	0.0791869199	0.2283464335	0.0718977757	0.1880048432
8	0.0751261273	0.2071234782	0.0648368942	0.1682601770
9	0.0716455900	0.1900234319	0.0594114970	0.1524074966
10	0.0685991848	0.1759277031	0.0551151456	0.1394312127

It is observed from the figures that the top curve is at $t = 1$ and the bottom curve is at $t = 9$. It is obvious that a smaller viscosity value v in the shock wave with a smaller amplitude and propagation front becomes smoother. Moreover, we have seen from the figures that as the time increases, the curve of the the numerical solution decays. With smaller viscosity values, the numerical solution decay gets faster. These numerical solution graphs also agree with earlier published work [16]. Table 2 presents a comparison of the values of the error norms obtained by the QBDQM with those which were obtained by the other method [16]. It is clearly seen from Table 2 that the error

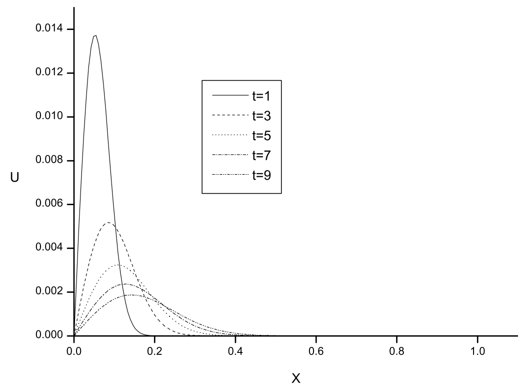

FIGURE 3. Solutions for $v = 0.001$, $\Delta t = 0.01$, $N = 166$, $0 \leq x \leq 1$.

norms L_2 and L_∞ obtained by the present method are smaller than at the beginning and at the end of the run of these given in [16]. Additionally, in Table 3, the error norms of L_2 and L_∞ obtained by the present method are acceptably small. Error variations are drawn at time $t = 10$ in Figures 4 and 5 from which the maximum error occurs at the right hand boundary when the greater value of viscosity $v = 0.01$ is used and with the smaller value of viscosity $v = 0.001$, the maximum error is recorded around the location where the shock wave has the highest amplitude. The L_2 and L_∞ error norms and the numerical rate of convergence analysis for $v = 0.001$ and $\Delta t = 0.01$ and different numbers of grid points are tabulated in Table 4.

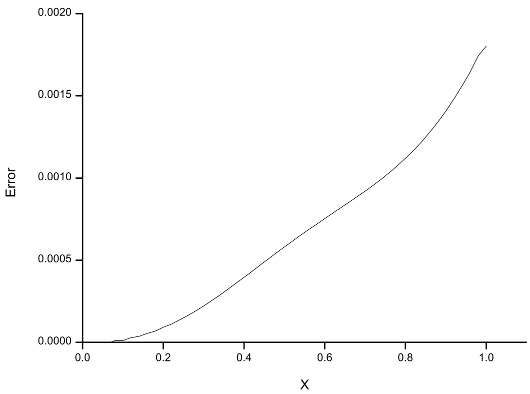

FIGURE 4. Error for $v = 0.01$, $\Delta t = 0.01$, $h = 0.02$, $0 \leq x \leq 1$.

As seen in Table 4, when the number of grid points is increased, the error norms decrease and both of $ROC\,(L_2)$ and $ROC\,(L_\infty)$ change similarly. The values of $ROC\,(L_2)$ and $ROC\,(L_\infty)$ change in the region of $[0.27, 1.07]$ and $[0.17, 1.17]$, respectively.

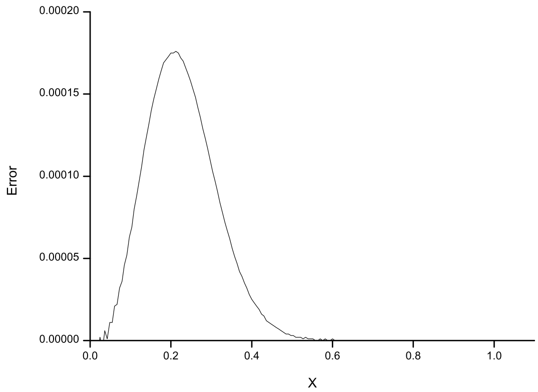

FIGURE 5. Error for $v = 0.001$, $\Delta t = 0.01$, $N = 166$, $0 \le x \le 1$.

TABLE 4. Error norms and rate of convergence for various numbers of grid points at $t = 10$.

N	$L_2 \times 10^3$	$ROC\,(L_2)$	$L_\infty \times 10^3$	$ROC\,(L_\infty)$
11	0.26	-	0.64	-
21	0.13	1.07	0.30	1.17
31	0.11	0.43	0.25	0.47
41	0.10	0.34	0.23	0.30
61	0.09	0.27	0.21	0.23
81	0.08	0.42	0.20	0.17

5. Conclusions

In this study, we have implemented DQM based on quintic B-splines for a numerical solution of modified Burgers' equation. The performance and accuracy of the method were shown by calculating the L_2 and L_∞ error norms. Numerical rate of convergence analysis of the numerical approximation was also obtained. It is observed that by comparing between the obtained values of the L_2 and L_∞ error norms by the present method and earlier works, QBDQM results were considered acceptable. The obtained results show that QBDQM can be used to produce reasonably accurate numerical solutions of the modified Burgers' equation. Therefore, QBDQM is a reliable method to obtain the numerical solutions of some physically important nonlinear problems.

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