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β-Stability Line and Liquid-Drop Mass Formulas

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An empirical analysis of the masses of odd-mass nuclei is made, putting emphasis on the gross structure of the nuclear mass surface, especially on the β-stability line. As a consequence of the analysis, three macroscopic (liquid-drop) mass formulas are derived. It is pointed out that large volume and surface symmetry-energies are necessary for the reproduction of the experimental β-stability line.

§ 1. Introduction

Nuclear mass formulas have been very useful and important in nuclear physics. They give us estimates of masses of unknown nuclei as well as information about nuclear interactions.

It is possible to consider that the mass surface has two characters: One is the gross and smooth character which may originate from the classical liquid-drop nature of the nucleus, and the other is the fine structure which arises from the shell effects, nuclear deformation, etc. It is often necessary to use mass formulas with a large extrapolation in the study of, e.g. the nuclear fission, superheavy nuclei and nuclei far from the β-stability line. When we make such extrapolations, a knowledge of gross properties of the mass surface will be indispensable.

A number of liquid-drop-type mass formulas with rather phenomenological shell corrections have been published till now,[1] and many of them represent the nuclear masses fairly well. The liquid-drop parts include effects such as surface symmetry energy, nuclear compressibility and surface diffuseness in addition to the ordinary Weizsäcker-Bethe terms.[2),3)] However the magnitudes of these liquid-drop correction terms are considerably scattered. Different sets of parameters often do not show any significant differences in the χ^2-fit, provided they are chosen in an appropriately correlated way. Thus it is rather difficult to single out the correct set of parameters only from the χ^2-fit of overall mass data. This difficulty is increased by the shell effects, because the differences among the effects of various parameter sets are often hidden in the uncertainty of the shell effects which cannot accurately be calculated at present. Therefore, it will be worthwhile to make a further study of gross properties of the nuclear mass surface and of liquid-drop correction terms. Once a good gross mass formula is obtained, the fine structure of the mass surface may be added with the aid of, e.g. the binding energy systematics.

In order to avoid the above-mentioned difficulty we put emphasis on the β-stability line rather than treat all masses equivalently. As has been pointed out by several authors,[4)~7)] the β-stability line shows a somewhat peculiar behavior. The β-stability line is a line on the N-Z plane that shows the most β-stable isobars as a function of mass number A. We express the experimental β-stability line by $I_{0\mathrm{exp}}(A)$, which is, for a given A, the neutron number minus the proton number giving the minimum of the mass parabola. In order to be free from the pairing effects, we deal only with odd-A nuclides in this paper. Mass excesses of even-A nuclides can be represented by adding the usual even-odd term to the odd-A mass formula. For $A\leq 47$, the effect of the V-shaped valley[8)] of the mass surface at $N=Z$ is taken into account. For the sake of graphical representation we use the reference mass formula of Weizsäcker-Bethe type (^{12}C standard, in MeV),

$$M_{E\mathrm{ref}}c^2=(7.68004-15.88485)A+0.39131I$$
$$+18.32695A^{2/3}+23.64332I^2/A+0.71994\frac{Z^2}{A^{1/3}}, \quad (1)$$

where the coefficients are taken from Ref. 6). The reference β-stability line is then given by

$$I_{0\mathrm{ref}}=\frac{0.35997A^{2/3}-0.39131}{0.35997A^{2/3}+47.28664}A\,. \quad (2)$$

The experimental β-stability line measured from the reference line, together with its gross tendency in which the shell effects are supposed to be eliminated, is

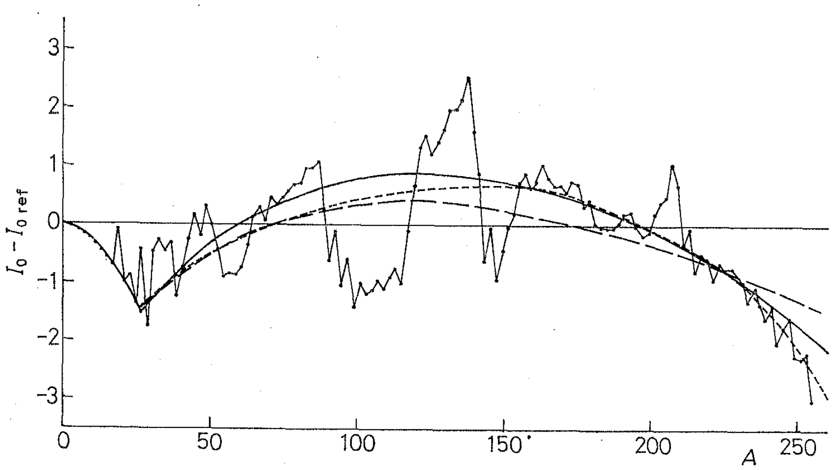

Fig. 1. β-stability lines measured from the reference line $I_{0\mathrm{exp}}$(Eq. (2)). The zigzag line is the experimental one. The dotted curve is drawn as the "smoothed" experimental line, in which the shell effects are supposed to be eliminated. The dashed curve is the one calculated from Eq. (9a), and the solid line is for both Eqs. (9b) and (9c) which give two indistinguishable curves on this figure.

given in Ref. 6), and we reproduce them in Fig. 1. This smoothed curve (a dotted line in Fig. 1) cannot be obtained by adjusting the constants in Eq. (1). Thus the Weizsäcker-Bethe formula is insufficient for explaining the experimental β-stability line.

Most of the liquid-drop mass formulas without shell corrections failed to represent the above-mentioned behavior of the β-stability line. If shell correction terms are added the situation may be improved, but the convex feature of the smoothed β-stability line in Fig. 1 would not be explained by the shell effects only.

Yamada proposed a mass formula for compressible nuclei and succeeded in explaining the behavior of the β-stability line.[6] However, the magnitude of the compressibility obtained in Ref. 6) is much larger than usually thought.[9] Furthermore, still there remains a question whether the nuclear compressibility is essential for explaining the property of the experimental β-stability line. In the heaviest region the mass formula derived by Viola and Wilkins[10] represents the β-stability line much better than the usual formulas, but it does not in the other mass region. Thus a further investigation of gross properties of the nuclear mass surface, especially of the β-stability line, is necessary.

In this paper, we assume that the nucleus is incompressible, and investigate how the incompressible model can represent the gross property of the mass surface, putting emphasis on the β-stability line.

§ 2. Analysis of experimental data and liquid-drop mass formulas

For each fixed value of A, we consider the mass excesses of the isobars as a function of $I(=N-Z)$. Since I/A is not a large quantity for all A, we may express the mass excesses minus apparent Coulomb energies in a power series in I/A. On the assumption that the nuclear forces are charge symmetric, we may neglect the odd-power terms in I except for the neutron-hydrogen mass difference which is linear in I. Among even powers in I, we take only the first term, i.e. the I^2 term. The contributions from the higher powers are negligibly small unless their coefficients are abnormally large. In addition, a term proportional to $|I|$ was suggested by Wigner,[11] and its existence is apparent from the experimental evidence.[8),12] Thus we write the mass excesses in the form (in MeV);

$$M_E(A, I)c^2 = 7.68004A + 0.39131I + a(A)\cdot A + b(A)\cdot|I| + c(A)\cdot I^2/A + E_c(A, Z). \quad (3)$$

The first two terms arise from the rest masses of neutron and proton. The term E_c stands for the Coulomb energies. This expression for nuclear masses was proposed previously by Ayres et al.[13] They examined A-dependences of the coefficients by simple many-body caculations and determined them semi-empirically, but the results are not satisfactory as to the β-stability line.

In this section, we first investigate the behavior of the coefficients by analyzing experimental masses without making any theoretical assumption. Then we derive gross formulas for these coefficients by smoothing out apparent shell corrections. In this process we put emphasis on the β-stability line as stated in § 1.

We calculate the the Coulomb energies based on a simple charge distribution. The trapezoidal charge distribution is applied for this purpose (see Fig. 2).

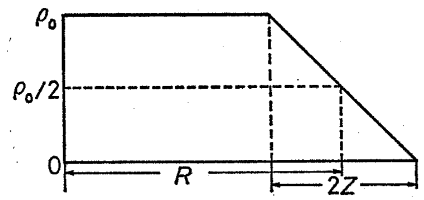

Fig. 2. Trapezoidal model of nuclear charge distribution.

We determine the parameters of this distribution referring to electron scattering data[14] as

$$z = 1.5\text{fm},$$

$$r_0 = 1.1\ \text{fm},$$

where $2z$ is the surface thickness and r_0 is related to the central density as

$$r_0 = \left(\frac{4}{3}\pi\rho_0\right)^{-1/3}.$$

We assume no difference between the neutron and proton distributions, and then the central charge density is given by

$$\rho_c = \frac{eZ}{A}\rho_0\,,$$

where $-e$ is the electron charge. The half-density radius R is calculated to be

$$R = r_0 \cdot \left[\left\{\frac{A}{2} + \sqrt{\left(\frac{A}{2}\right)^2 + \frac{1}{27}\left(\frac{z}{r_0}\right)^6}\right\}^{1/3} - \left\{\sqrt{\left(\frac{A}{2}\right)^2 + \frac{1}{27}\left(\frac{z}{r_0}\right)^6} - \frac{A}{2}\right\}^{1/3}\right]. \qquad (4)$$

The Coulomb energy of the trapezoidal charge distribution is approximately given by (in MeV)

$$E_c(A, Z) = (4\pi\rho_c)^2 \frac{R^5}{15}\left\{1 + \frac{5}{6}\left(\frac{z}{R}\right)^2 + \frac{1}{2}\left(\frac{z}{R}\right)^3 + \frac{1}{6}\left(\frac{z}{R}\right)^4 - \frac{1}{42}\left(\frac{z}{R}\right)^5\right\}$$

$$- \frac{0.66}{r_0}\left(\frac{Z}{A}\right)^{4/3} \cdot A\,, \qquad (5)$$

where the last term is the Coulomb exchange energy calculated on the Fermi-gas model. Since the exchange term is not so large compared to the direct term,

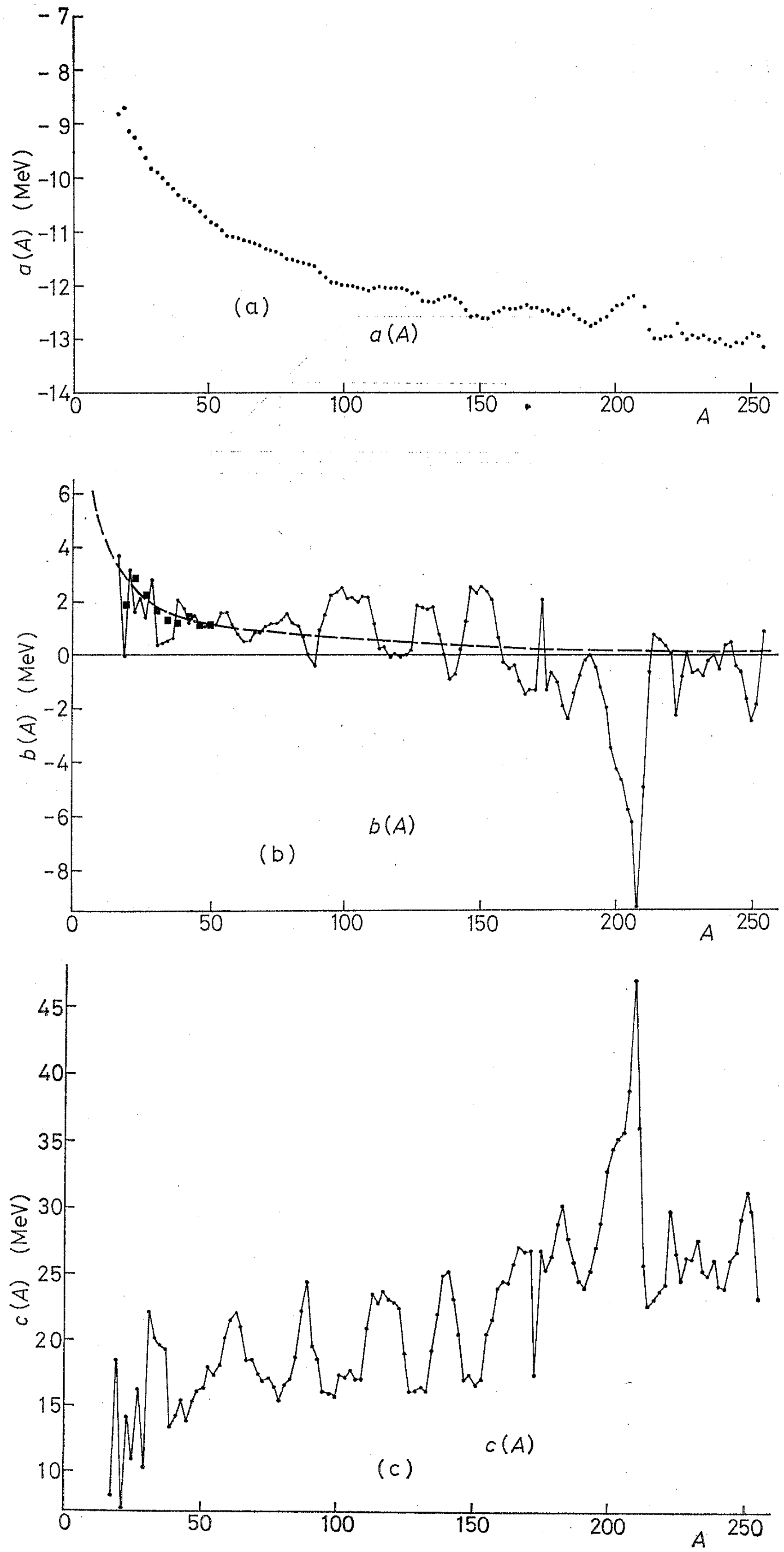

Fig. 3. Results of the first fitting. Also shown in 3(b) with black squares are the estimates from the separation energy systematics. The dashed line in 3(b) is the line calculated from Eq. (6).

the Fermi-gas approximation will not cause a serious error.

For $A \geq 5$, mass excesses of more than 3 isobars are known experimentally or by the systematics.[15)~17)] However, more than 2 mass excesses for different $|I|$ values are necessary in order to determine the coefficients $a(A)$, $b(A)$ and $c(A)$. This requirement is met for $A \geq 17$, and then the coefficients can be determined by the least-squares fits. These coefficients are shown in Figs. 3 (a), 3 (b) and 3 (c). They fluctuate due to shell effects, and these fluctuations are amplified by the correlation among the parameters; e.g. even if we modify $b(A)$ into a smooth curve, the fit does not become much worse provided the other coefficients are adjusted appropriately. In spite of these shell effects, it is possible to extract gross tendencies from these figures. We express these coefficients by analytic functions of A in order to get a gross mass formula. The analytic representations of these coefficients are done one by one, because there exists a strong correlation among them and separate determinations of these coefficients will lead to inaccurate results. We first determine $b(A)$ since the effects of this term is not so large as the other two terms. Secondly, we determine $c(A)$ in order to preserve the property of the β-stability line as far as possible. The coefficient $a(A)$ is determined in the last by fitting the mass excesses on the β-stability line.

The Wigner-term coefficient $b(A)$ can also be determined from the separation energy systematics[17)] for $A \leq 55$. For infinitely large A values, we assume that $b(A)$ vanishes.[12)] Then it is approximated by a power series in $A^{-1/3}$ as (in MeV)

$$b(A) = -3.9A^{-1/3} + 30.6A^{-2/3}, \tag{6}$$

where we have determined the numerical coefficients somewhat arbitrarily by looking into Fig. 3 (b). For very large A values, Eq. (6) gives negative values, but their absolute magnitudes are so small that no serious difficulty will arise.

Substituting Eq. (6) into Eq. (3), we redetermine the coefficients $a(A)$ and $c(A)$ by the least-squares fits. This time we can determine them also for $A < 17$, sine the mass excesses for two different $|I|$ values are sufficient to determine $a(A)$ and $c(A)$ for each A value. The results of this second fitting are shown in Figs. 4 (a) and 4 (b). The A-dependent symmetry energy coefficient $c(A)$ becomes fairly smooth in this case compared to the first fitting, Fig. 3 (c). Figure 4 (b) clearly shows the gross tendency that $c(A)$ increases with A; this is one of the best evidence for the surface symmetry energy.

We approximate $c(A)$ by a power series in $A^{-1/3}$. As seen in Fig. 4 (b), the coefficient increases rather rapidly in the heaviest region. This property of the symmetry energy reflects the convex nature of the β-stability line in Fig. 1. Because of this strong A-dependence, it is somewhat difficult to determine the best-fit curve uniquely, so that we give three different expressions for $c(A)$.

If we use the first three terms in the power series and put emphasis on the

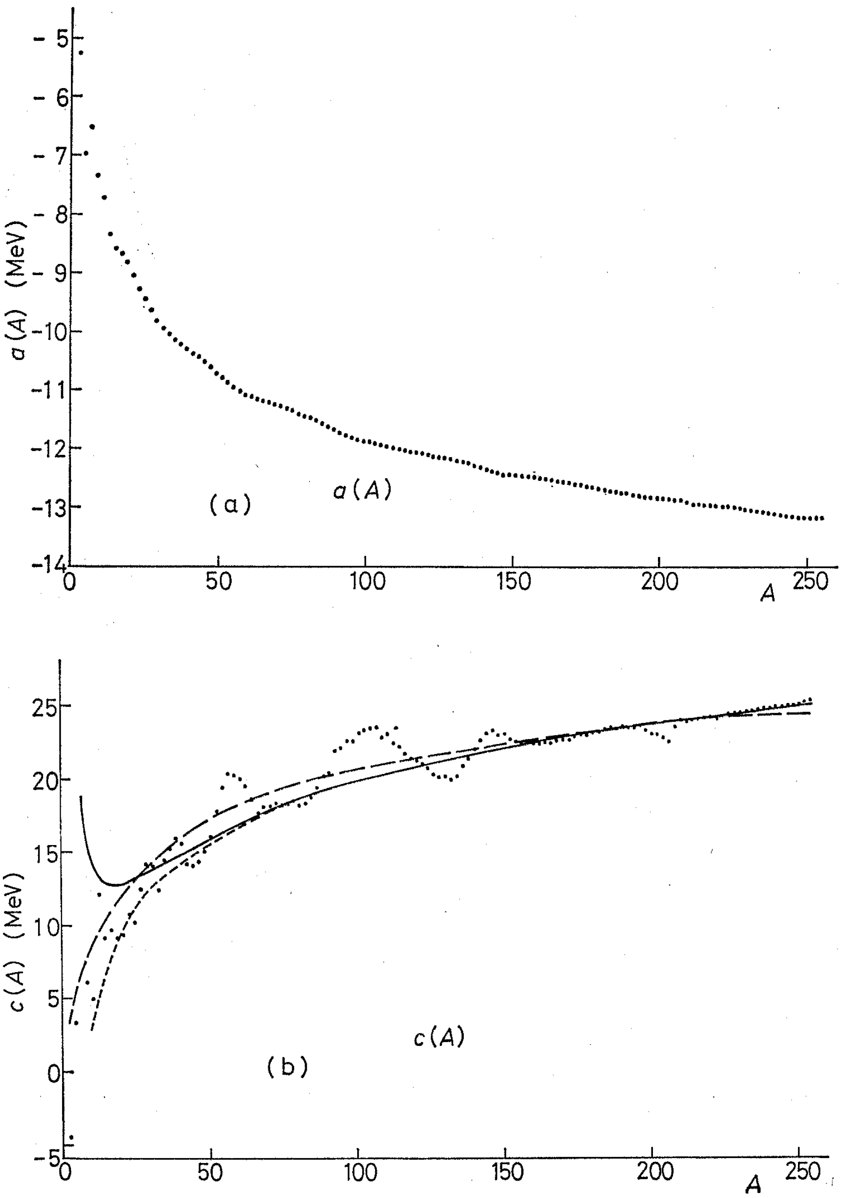

Fig. 4. Results of the second fitting. The dashed line in 4(b) is the curve calculated from Eq. (7a), the solid curve from Eq. (7b), and the dotted curve from Eq. (7c).

behavior in light and medium-weight nuclear regions, we get the formula (in MeV),

$$c_1(A) = 37.61 - 92.83A^{-1/3} + 65A^{-2/3}. \tag{7a}$$

On the other hand, if we fit the data mainly in the heavier region, we get (in MeV)

$$c_2(A) = 48.1 - 184A^{-1/3} + 240A^{-2/3}. \tag{7b}$$

Inclusion of the A^{-1} term improves the fit of $c_2(A)$ at $A<30$. In this case we obtain (in MeV)

$$c_3(A) = 57.25 - 321.25A^{-1/3} + 917.1A^{-2/3} - 1098A^{-1}. \qquad (7c)$$

As far as we do not concern with the lightest region $A<20$, it is difficult to give a preference among these three expressions. For $A>150$, Eqs. (7b) and (7c) are better than Eq. (7a), whereas for $40<A<100$, the situation is reversed.

With the aid of these analytic expressions and Eqs. (5) and (6), we can calculate the β-stability line; the calculated lines are shown in Fig. 1. The calculated lines for Eqs. (7b) and (7c) are almost the same, and they are represented by one curve in Fig. 1 (the solid curve), while that for Eq. (7a) is drawn in dashed line. These curves should be compared to the smoothed experimental β-stability line (the dotted line). As expected, Eq. (7a) is better than Eqs. (7b) and (7c) for $A<100$, and for $A>150$, Eqs. (7b) and (7c) are better.

The coefficient $a(A)$ is determined by fitting the masses on the β-stability line. The experimental mass excesses on the β-stability line given in Ref. 6) are reproduced in Figs. 5(a), 5(b) and 5(c). The analytic expression for $a(A)$ is so determined that the mass formula including it represents the gross feature of the experimental mass excesses on the β-stability line. For each $c_i(A)$, we obtain (in MeV)

$$a_1(A) = -18.0007 + 39.84616A^{-1/3} - 68.0739A^{-2/3} + 71.685A^{-1}, \qquad (8a)$$

$$a_2(A) = -18.6115 + 48.4575A^{-1/3} - 107.409A^{-2/3} + 129.42A^{-1}, \qquad (8b)$$

$$a_3(A) = -18.64314 + 48.8946A^{-1/3} - 109.3385A^{-2/3} + 132.093A^{-1}. \qquad (8c)$$

Thus we get three gross mass formulas (in MeV)

$$\begin{aligned} M_{E1}(A, I)c^2 = 7.68004A + 0.39131I + (&-18.0007 + 39.84616A^{-1/3} \\ &- 68.0739A^{-2/3} + 71.685A^{-1}) \cdot A + (-3.9A^{-1/3} + 30.6A^{-2/3})|I| \\ &+ (37.61 - 92.83A^{-1/3} + 65A^{-2/3}) I^2/A + E_c(A, Z), \end{aligned} \qquad (9a)$$

$$\begin{aligned} M_{E2}(A, I)c^2 = 7.68004A + 0.39131I + (&-18.6115 + 48.4575A^{-1/3} \\ &- 107.409A^{-2/3} + 129.42A^{-1})A + (-3.9A^{-1/3} + 30.6A^{-2/3})|I| \\ &+ (48.1 - 184A^{-1/3} + 240A^{-2/3}) I^2/A + E_c(A, Z), \end{aligned} \qquad (9b)$$

$$\begin{aligned} M_{E3}(A, I)c^2 = 7.68004A + 0.39131I + (&-18.64314 + 48.8946A^{-1/3} \\ &- 109.3385A^{-2/3} + 132.093A^{-1})A + (-3 \cdot 9A^{-1/3} + 30.6A^{-2/3})|I| \\ &+ (57.25 - 321.25A^{-1/3} + 917.1A^{-2/3} - 1098A^{-1}) I^2/A + E_c(A, Z), \end{aligned} \qquad (9c)$$

where E_c is given by Eqs. (4) and (5).

The mass excesses on the β-stability line calculated from the new mass formulas (9a), (9b) and (9c) are shown in Figs. (5a), (5b) and (5c) respectively. In the heaviest region, the calculated mass excesses seem to increase somewhat too rapid, compared to the experimental data.

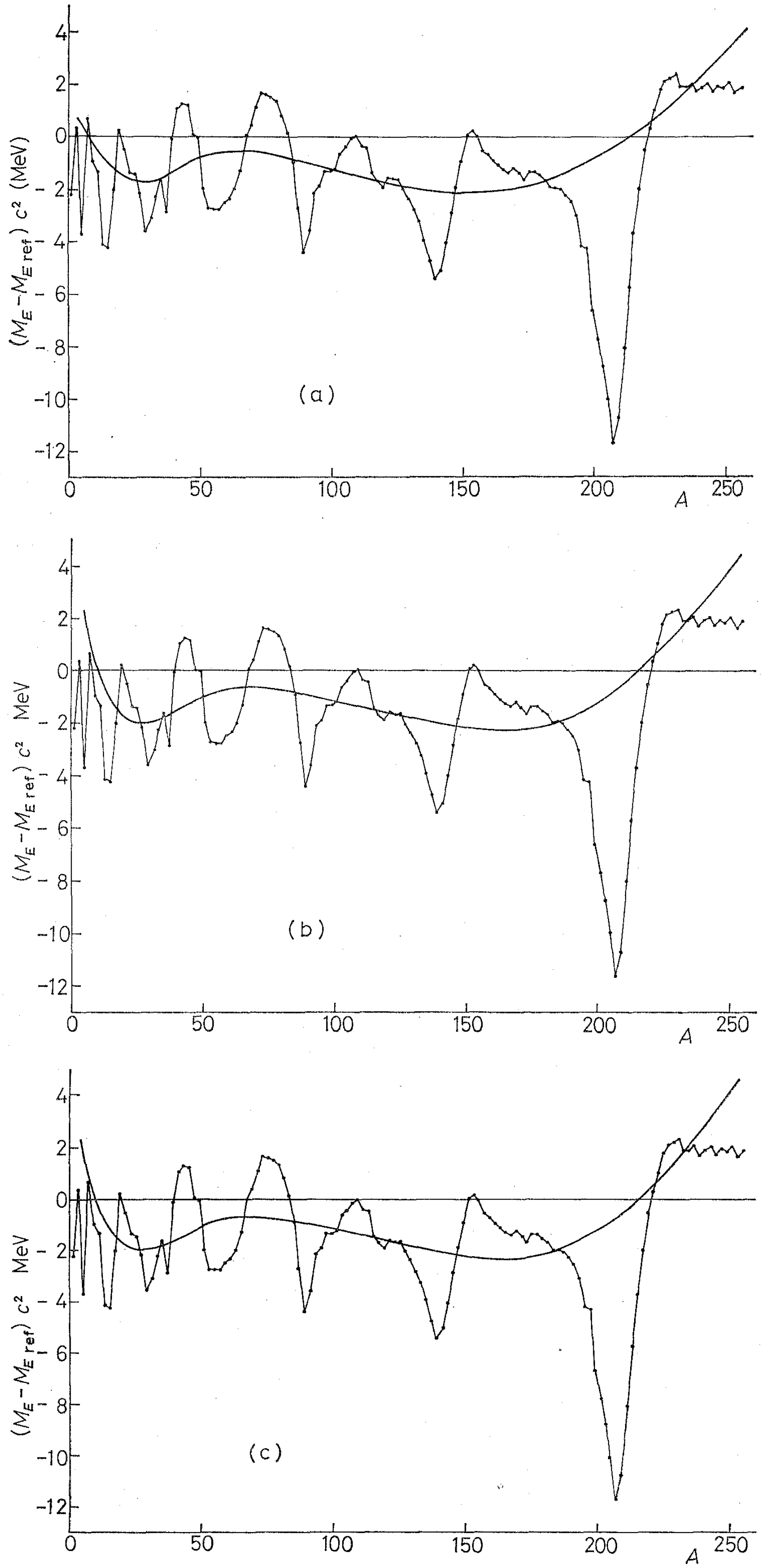

Fig. 5. Figure captions a printed on the next page below.

§ 3. Discussion

In § 2, we have derived three gross mass formulas based on an incompressible liquid-drop model. They exhibit the gross tendency of the experimental β-stability line and the masses on it fairly well. They are of the best formulas for the gross features of masses, in particular for the β-stability line. The values of the parameters in these formulas are significantly larger than those of the usual mass formulas. Our volume energy is about 18 MeV which is 1~3 MeV greater than the usual values. Our surface coefficients are very large; even the smallest one among the three is about 39 MeV, which should be compared with the ordinary mass-formula values of 18~20 MeV and recent theoretical estimates of ~20 MeV.[18]

Equation (9a) has a symmetry term $c(A)\cdot I^2/A$ just comparable to that of Myers and Swiatecki.[19] The surface symmetry energy obtained by Viola and Wilkins[11] in their analysis of the trans-radium nuclei, is about 311 MeV and close to our value 321.3 MeV in Eq. (9c).

In Ref. 6), it was shown that if one introduces the effect of nuclear compressibility the behavior of the β-stability line, in particular that in the heaviest region, can be explained. Also it can be known from Fig. 3 of Ref. 6) that the compressibility causes an energy decrease in the heaviest region. Therefore, introduction of the nuclear compressibility is expected to improve Eq. (9a), in regard both to the β-stability line and to the masses on it in the heaviest region. However, the deviations of Eq. (9a) from the experimental data are rather small and the magnitude of the compressibility to be introduced will be considerably smaller than that of Ref. 6). Thus it is probable that Eq. (9a) becomes the best one among the three by the introduction of the effect of nuclear compressibility.

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Fig. 5. Mass excesses on the β-stability line measured from the reference formula. The zigzag line represents the experimental mass excesses, namely the minimum values of the mass parabolas. The solid curves in (a), (b), (c) represent the mass excesses on the β-stability lines calculated from Eqs. (9a), (9b) and (9c), respectively.

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