## B

Wave Propagations in Non-Uniform Media

Naruyoshi Asano<br>Department of Engineering Mathematics Utsunomiya University, Utsunomiya

(Received January 29, 1974


#### Abstract

The reductive perturbation methods for the wave propagation in weakly inhomogeneous media and also spatially homogeneous but weakly unstable media are developed in virtue of appropriate strained variables for the waves and the media. For each case, low dispersive long wave and modulated amplitude of the self-interacting nearly monochromatic wave can be described by relatively simple scalar equations, many of which have one linear extra term with a variable coefficient in comparison with the equation for the constant media. Modulation of nearly monochromatic wave which has a complex frequency with a small imaginary part, is also considered in an unsteady medium and a similar governing equation is obtained. These theories are applied, directly or in extended forms, to the illustrative examples from fluid mechanics, plasma physics and astrophysics.


## § 1. Introduction and strained variables

During the past decade or so, various aspects of weakly nonlinear waves in homogeneous medium have been investigated by singular perturbation methods. Especially, remarkable properties of the nonlinear wave, i.e., soliton, ${ }^{1)}$ infinitely many conserved quantities, ${ }^{2)}$ amplitude dispersion ${ }^{3)}$ and so on, have been revealed out. When the medium is not uniform, as in the most real physical systems, interaction of the wave with the non-uniformity becomes very important. In respect to the interaction of the nonlinear wave with the non-uniformity, one has reached the important concept of nonlinear stability criterion, which, some cases, yields the inverse of the prediction of the linear theory. ${ }^{4}$

In this paper, the theory presented in part I, i.e., reductive perturbation method, is extended to the wave propagation in weakly non-uniform media. Here, the nomenculture "non-uniform" is used to mean "spatially inhomogeneous and time independent" or "spatially homogeneous and varying in time." Typical examples of the interaction of the nonlinear wave with the non-uniformity are the steepening of the sound wave propagating upward in the atmosphere, decay of the water solitary wave in the interaction with the bottom irregularities, growing or damping of the wave in a homogeneously
expanding or contracting or also in the homogeneous radiative or reacting gas and so on. Interaction of the heighly dispersive wave, such as electron plasma wave, with the non-uniformity is also important.

Before going to the detailed discussions of each problem, let us classify several cases, for which the strained variables have different forms. ${ }^{5)}$ This classification is useful for the understanding of the physical meaning of the strained variables and also for the later descriptions. Since the strained variables can describe the asymptotic properties of the nonlinear, wave, they have a close relation to the coupling of the nonlinearity and the dissipative or dispersive effect of the wave. Let us consider, as an example, the equation ${ }^{6}$ )

$$
\frac{\partial U}{\partial t}+A \frac{\partial U}{\partial x}+\sum_{\beta=1}^{s} \prod_{\alpha=1}^{p}\left(H_{\alpha}^{\beta} \frac{\partial}{\partial t}+K_{\alpha}^{\beta} \frac{\partial}{\partial x}\right) U=0
$$

with $p \geq 2$, where $U$ is a column vector with $n$ unknown components $u_{1}, u_{2}, \cdots$, $u_{n} ; A, H_{\alpha}^{\beta}$ and $K_{\alpha}^{\beta}$ are $n \times n$ matrices, the elements of which are functions of $u_{i} ; x$ and $t$ are the space and time coordinates respectively. Dispersion relation for long wave is obtained by linearizing Eq. ( $1 \cdot 1$ ) about a constant state $U_{0}$ and using a successive approximation. Thus, simple calculation gives the phase velocity

$$
\frac{\omega}{k}=\lambda_{0}+c_{1} k^{p-1}+c_{2} k^{2(p-1)}+\cdots
$$

where $\omega$ and $k$ are the frequency and the wave number; $\lambda_{0}$ is an eigenvalue of $A_{0}=A\left(U_{0}\right)$ and $c_{i}$ s $(i=1,2, \ldots)$ are constants given in terms of the right and left eigenvectors $r, l$ of $A_{0}$ for $\lambda_{0}$. In fact, $c_{1}$ takes the form

$$
c_{1}=i^{p-1} l \sum_{\beta=1}^{s} \prod_{\alpha=1}^{p}\left(K_{0 \alpha}^{\beta}-\lambda_{0} H_{0 \alpha}^{\beta}\right) r / l r,
$$

in which $K_{0^{\alpha}}^{\beta}$ and $H_{0^{\alpha}}^{\beta}$ are the respective values of $K_{\alpha}^{\beta}$ and $H_{\alpha}^{\beta}$ at $U=U_{\mathbf{0}}$. On the other hand, the typical nonlinear character of Eq. (l-1) may be seen most easily by neglecting the last term. Expanding $U$ about $U_{0}$ in powers of small but finite parameter $\varepsilon$, i.e., $U=U_{0}+\varepsilon U_{1}+\varepsilon^{2} U_{2}+\cdots$, and substituting it into Eq. (l-l) with the last term neglected, one has the characteristic curve in the form

$$
\frac{d x}{d t}=\lambda_{0}+\varepsilon \lambda_{1}+\varepsilon^{2} \lambda_{2}+\cdots
$$

where $\lambda_{1}$ is proportional to $l(\nabla A)_{0} \cdot U_{1} r ; \nabla A \cdot U_{1}=\sum_{i=1}^{n}\left(\partial A / \partial u_{i}\right)_{U=U_{0}} \cdot u_{1 i}$. Comparison of the two velocities $(1 \cdot 2)$ and $(1 \cdot 4)$ shows that the interaction time of the dispersion and nonlinearity, which usualy act in the directions opposite to each other, becomes the longest if the equality

$$
O(k)=\varepsilon^{a}
$$

holds along the characteristics, where $a=1 /(p-1)$ and the conditions $c_{1} \neq 0$ and $\lambda_{1} \neq 0$ are assumed. Since the relation ( $1 \cdot 5$ ) implies that $\varepsilon^{a} \times$ (wave length) ₹order of unity in the wave frame, the strained variable

$$
\xi=\varepsilon^{a}\left(x-\lambda_{0} t\right)
$$

becomes an independent variable of the coefficients in the expansion of $U$. Another variable may be chosen as

$$
\begin{equation*}
\eta=\varepsilon^{a+1} x \quad \text { or } \quad \tau=\varepsilon^{a+1} t \tag{1.7a,b}
\end{equation*}
$$

by which the characteristics (1-4) is transferred to $\lambda_{1}=d \xi / d \eta$ or $\lambda_{1}=d \xi / d \tau$ respectively in the first order of $\varepsilon$. When the unperturbed state is non-uniform, $\lambda_{0}$ depends on $x$ or $t$ and should be presented in the integrated form in Eq.(1.6) in $x$ or $t$ depending on the nature of the non-uniformity.

So far, we have considered the case $c_{1} \neq 0$ and $\lambda_{1} \neq 0$. If $c_{1}=0, c_{2} \neq 0$ and $\lambda_{1} \neq 0$, the coupling of the nonlinearity and the dispersive effect is made via the order relation $k^{2(p-1)} \approx \varepsilon$ and leads to the variable $(1 \cdot 6)$ with $a=1 / 2(p-1)$. In this way, discussions similar to the above hold also for other values of $c_{i}$ and $\lambda_{i}$ and can be summarized as in Table I.

Table I. Strained variables.

| non-uniformity <br> Case | (a) inhomogeneous | (b) unsteady |
| :---: | :---: | :---: |
| I $\begin{aligned} & c_{1} \neq 0, \quad \lambda_{1} \neq 0, \\ & a=\frac{1}{p-1} \end{aligned}$ | $\begin{aligned} & \xi=\varepsilon^{a}\left(\int \frac{d x}{\lambda_{0}}-t\right) \\ & \eta=\varepsilon^{a+1} x \end{aligned}$ | $\begin{aligned} & \xi=\varepsilon^{a}\left(x-\int \lambda_{0} d t\right) \\ & \tau=\varepsilon^{a+1 t} \end{aligned}$ |
| $\text { II } \quad \begin{aligned} & c_{1}=0, \quad c_{2} \neq 0 \\ & \lambda_{1} \neq 0 \\ & a=\frac{1}{2(p-1)} \end{aligned}$ | same as I (a) | same as I (b) |
| $\text { III } \begin{aligned} & c_{1} \neq 0, \quad \lambda_{\mathbf{l}}=0 \\ & \lambda_{2} \neq 0 \\ & a=\frac{2}{p-1} \end{aligned}$ | $\begin{aligned} & \xi=\varepsilon^{a}\left(\int \frac{d x}{\lambda_{0}}-t\right) \\ & \eta=\varepsilon^{a+2} x \end{aligned}$ | $\begin{aligned} & \xi=\varepsilon^{a}\left(x-\int \lambda_{0} d t\right) \\ & \tau=\varepsilon^{a+2} t \end{aligned}$ |
| IV otherwise | otherwise | otherwise |

Since the envelope of nearly monochromatic wave is considered as a long wave, strained variables for the envelope wave can also be obtained in a similar way. ${ }^{7)}$ As easily verified, the phase velocity of the envelope comprising two plane waves with the characteristics ( $k, \omega$ ) and ( $k^{\prime}, \omega^{\prime}$ ) is

$$
\Lambda=\lambda_{0}+\frac{1}{2} \frac{\partial^{2} \omega}{\partial k^{2}} K+\frac{1}{6} \frac{\partial^{3} \omega}{\partial k^{3}} K^{2}+\cdots
$$

where $\lambda_{0}$ is the group velocity of the carrier wave, and the wave number of the envelope $K\left(=k^{\prime}-k\right)$ is assumed to be small. For the wave number such as $\partial^{2} \omega / \partial k^{2} \neq 0$, the phase velocity ( $1 \cdot 8$ ) yields the variables corresponding to Case I with $a=1$ and, as verified later, $\lambda_{0}$ given by

$$
\begin{array}{ll}
\frac{1}{\lambda_{0}}=\frac{\partial k}{\partial \omega}+\frac{\partial^{2} k}{\partial \omega \partial \eta} \eta & \text { for (a), } \\
\lambda_{0}=\frac{\partial \omega}{\partial k}+\frac{\partial^{2} \omega}{\partial k \partial \tau} \tau & \text { for (b). } \tag{1.9b}
\end{array}
$$

Here and in this paper, the non-uniformity of the unperturbed state is assumed to be described by one of the variables $(1 \cdot 7)$ for Cases I and II. In the reductive perturbation method for the constant unperturbed state, the parameter $\varepsilon$ can be chosen freely but, for the non-uniform case, should be determined by the assumption, i.e., by the order of the non-uniformity. It may be worthwhile to note that various extensions and modifications are possible for the model equations and hence the strained variables, as some of those will be presented in the examples at the end of each chapter.

In §2, wave propagations in weakly inhomogeneous media are studied. For considerably wide, complex physical systems, we obtain relatively simple scalar equation which have one extra term with a variable coefficient, in comparison with the equations for the constant media. Propagation in the media varying slowly in time, which includes unstable case, are considered in §3. Equations similar with those in $\S 2$ are obtained both for long waves and modulated waves. Finally, modulations of the self-interacting nearly monochromatic wave with a complex frequency of a small imaginary part, are investigated for an unsteady medium in $\S 4$. For many cases, the amplitude equation of the modulation can be reduced to the nonlinear Schrödinger type equation in an external potential. Each chapter has one or several illustrative examples of the applications from fluid mechanics, plasma physics or astrophysics. Some of the applications are presented in extended or modified forms of the general theory, including the multi-dimensional extensions of the wave propagations in $\S 2$ and 4 and the propagation in an inhomogeneous, unsteady medium in $\S 3$.

## § 2. Propagation in weakly inhomogeneous media

### 2.1 Long waves ${ }^{5)}$

Inhomogeneous physical systems are usually described by the set of the equations which do not have constant state solution. Hence, as a model equation, we consider

$$
\frac{\partial U}{\partial t}+A \frac{\partial U}{\partial x}+\sum_{\beta}^{s} \prod_{\alpha}^{p}\left(H_{\alpha}^{\beta} \frac{\partial}{\partial t}+K_{\alpha}^{\beta} \frac{\partial}{\partial x}\right) U+B \frac{d S}{d x}=0
$$

where $B$ is an $n \times n$ matrix whose components are functions of $U$ and $S ; S$ is a vector valued function of $x$ only and other symbols have the same meanings with those in Eq. $(1 \cdot 1)$ but depend also on $S$. As will be shown in $\S 2.3$, equations of the flow throw a duct of varying cross section, shallow water motion on the variable depth bottom and so on have the form given by Eq. $(2 \cdot 1)$. We consider, at first, Case I (a) defined in §l. The unperturbed state is denoted by the index zero and, from Eq. $(2 \cdot 1)$, determined by

$$
A_{0} \frac{d U_{0}}{d \eta}+B_{0} \frac{d S}{d \eta}=0
$$

where $S$ is assumed to be a function of the slow variable $\eta$ and the order of $\varepsilon^{p(a+1)}$ is neglected. Our aim is to study the wave propagation in the inhomogeneous medium determined by Eq. (2-2).

Let us expand $U$ about $U_{0}$ in powers of $\varepsilon$ as

$$
U=U_{0}+\varepsilon U_{1}+\varepsilon^{2} U_{2}+\cdots,
$$

then, $A, B, H_{\alpha}^{\beta}$ and $K_{\alpha}^{\beta}$ can be expanded as

$$
\begin{aligned}
& A=A_{0}+\varepsilon A_{1}+\cdots, \\
& B=B_{0}+\varepsilon B_{1}+\cdots,
\end{aligned}
$$

and likewise for $H_{\alpha}^{\beta}$ and $K_{\alpha}^{\beta}$, where $A_{1}, B_{1}$ and so on are given in terms of $U_{1}$;

$$
A_{1}=U_{1} \cdot(\nabla A)_{0}=(\nabla A)_{0} \cdot U_{1}=\sum_{i=1}^{n}\left(\partial A / \partial u_{i}\right)_{U=U_{0}} u_{1 i}, \quad \text { etc. }
$$

Substituting these expansions and appropriate strained variables given in the table in $\S 1$, one has for the order of $\varepsilon$,

$$
\left(-I+\frac{1}{\lambda_{0}} A_{0}\right) \frac{\partial U_{1}}{\partial \xi}=0
$$

and for the order of $\varepsilon^{2}$,

$$
\begin{align*}
& \left(-I+\frac{1}{\lambda_{0}} A_{0}\right) \frac{\partial U_{2}}{\partial \xi}+A_{0} \frac{\partial U_{1}}{\partial \eta}+\frac{1}{\lambda_{0}} A_{1} \frac{\partial U_{1}}{\partial \xi}+A_{1} \frac{d U_{0}}{d \eta} \\
& \quad+\sum_{\beta}^{s} \prod_{\alpha}^{p}\left(-H_{0 \alpha}^{\beta}+\frac{1}{\lambda_{0}} K_{0 \alpha}^{\beta}\right) \frac{\partial^{p} U_{1}}{\partial \xi^{p}}+B_{1} \frac{d S}{d \eta}=0 .
\end{align*}
$$

General solution of Eq. (2.3) is given in terms of the right eigenvector $r$ of $A_{0}$ for $\lambda_{0}$ as

$$
U_{1}=r(\eta) \varphi(\xi, \eta)+V(\eta)
$$

where $\varphi$ is a scalar function of $\xi$ and $\eta$, while $V$ is an arbitrary vector valued function of $\eta$ only. Multiplying Eq. $(2 \cdot 4)$ by a left eigenvector $l$ of $A_{0}$ for $\lambda_{0}$ and substituting Eq. $(2 \cdot 5)$ into the resulting equation, we have the equation that determines $\varphi$,

$$
\frac{\partial \varphi}{\partial \eta}+\left(\alpha \varphi+\alpha^{\prime}\right) \frac{\partial \varphi}{\partial \xi}+\beta \frac{\partial^{p} \varphi}{\partial \xi^{p}}+\gamma \varphi+\gamma^{\prime}=0
$$

where the coefficients are given by

$$
\begin{align*}
& \alpha=\frac{l r \cdot(\nabla A)_{0} r}{\lambda_{0}^{l r}}, \\
& \alpha^{\prime}=\frac{l V \cdot(\nabla A)_{0} r}{\lambda_{0}^{2} l r}, \\
& \beta=\frac{l \Sigma_{\beta}^{s} \prod_{\alpha}^{p}\left(-H_{0 \alpha}^{\beta}+K_{0 \alpha}^{\beta} / \lambda_{0}\right) r}{\lambda_{0} l r} \\
& \gamma=\frac{\lambda_{0} l(d r / d \eta)+l r \cdot\left[(\nabla A)_{0}\left(d U_{0} / d \eta\right)+(\nabla B)_{0}(d S / d \eta)\right]}{\lambda_{0} l r}
\end{align*}
$$

and

$$
\begin{equation*}
r^{\prime}=\frac{\lambda_{0} l(d V / d \eta)+l V \cdot\left[(\nabla A)_{0}\left(d U_{0} / d \eta\right)+(\nabla B)_{0}(d S / d \eta)\right]}{\lambda_{0} l r} \tag{2.7e}
\end{equation*}
$$

The vector $V$ in the coefficients is determined if a boundary condition of $U_{1}$ and $\varphi$ is given, say, at $\xi=\xi_{c}$. Thus, the components $u_{1 i}$ can satisfy each boundary condition as shown in the example in $\S 3.3$. It is to be noted that a little more simple forms of Eq. $(2 \cdot 6)$ are obtained by the transformation

$$
\left.\begin{array}{l}
\phi=e^{\int \gamma d \eta} \varphi+\int \gamma^{\prime} e^{\int \gamma d \eta} d \eta \\
\zeta=\xi-\int \alpha^{\prime} d \eta+\int d \eta\left[\alpha e^{-\int \gamma d \eta} \int \gamma^{\prime} e^{\int \gamma d \eta} d \eta\right] \\
\theta=\eta
\end{array}\right\}
$$

to lead

$$
\frac{\partial \phi}{\partial \theta}+\alpha e^{-\int \tau d \theta} \phi \frac{d \phi}{d \zeta}+\beta \frac{d^{p} \phi}{d \zeta^{p}}=0
$$

or, by the further transformation

$$
\left.\begin{array}{l}
\psi=\frac{a}{\beta} e^{-\int \gamma d \theta} \phi \\
\sigma=\int \beta d \theta
\end{array}\right\}
$$

to lead

$$
\frac{\partial \psi}{\partial \sigma}+\psi \frac{\partial \psi}{\partial \zeta}+\frac{\partial^{p} \psi}{\partial \zeta^{p}}+\left[\frac{d}{d \sigma} \ln \left(\frac{\beta}{a}\right)+\frac{\gamma}{\beta}\right] \psi=0 .
$$

The illustrative applications of the theory are presented in §2.3.
Next, let us consider Case II (a), which is specified by the condition $c_{1}=0$. In this case, the interaction of the nonlinearity of the order of $\varepsilon$ and the dispersive effect of the order $k^{2(p-1)}$ plays the dominant role. Since the expansion of $U$ in powers of $\varepsilon$ corresponds to the expansion in powers of $k^{2(p-1)}$ and excludes the contributions from the orders of $k^{p-1}, k^{3(p-1)}, \cdots$, we must expand $U$ in powers of $\varepsilon^{1 / 2}$,

$$
U=U_{0}+\varepsilon U_{1}+\varepsilon^{3 / 2} U_{2}+\varepsilon^{2} U_{3}+\cdots,
$$

where the term of the order of $\varepsilon^{1 / 2}$ is dropped out because it turns out to be identically zero. Furthermore, the unperturbed state is subject to the condition $c_{1}=0$, i.e.,

$$
l \sum_{\beta}^{s} \prod_{\alpha}^{p}\left(-H_{0 \alpha}^{\beta}+\frac{1}{\lambda_{0}} K_{0 \alpha}^{\beta}\right) r=0,
$$

which is a direct consequence of Eq. (1.3). Substitution of Eq. (2•10) and corresponding expansions of $A, B, H_{\alpha}^{\beta}$ and $K_{\alpha}^{\beta}$ into Eq. (2•1) yields the set of equations

$$
\begin{gather*}
A_{0} \frac{d U_{0}}{d \eta}+B_{0} \frac{d S}{d \eta}=0 \\
\left(-I+\frac{1}{\lambda_{0}} A_{0}\right) \frac{\partial U_{1}}{\partial \xi}=0 \\
\left(-I+\frac{1}{\lambda_{0}} A_{0}\right) \frac{\partial U_{2}}{\partial \xi}+\sum_{\beta}^{s} \prod_{\alpha}^{p}\left(-H_{0 \alpha}^{\beta}+\frac{1}{\lambda_{0}} K_{0 \alpha}^{\beta}\right) \frac{\partial^{p} U_{1}}{\partial \xi^{p}}=0
\end{gather*}
$$

for the order of $\varepsilon^{a}, \varepsilon^{a+1 / 2}, \varepsilon^{a+1}$ respectively and

$$
\begin{align*}
&\left(-I+\frac{1}{\lambda_{0}} A_{0}\right) \frac{\partial U_{3}}{\partial \xi}+A_{0} \frac{\partial U_{1}}{\partial \eta}+\frac{1}{\lambda_{0}} A_{1} \frac{\partial U_{1}}{d \xi} \\
&+\sum_{\beta}^{s} \prod_{\alpha}^{p}\left(-H_{0 \alpha}^{\beta}+\frac{1}{\lambda_{0}} K_{0 \alpha}^{\beta}\right) \frac{\partial^{p} U_{2}}{\partial \xi^{p}}+B_{1} \frac{d S}{d \eta}=0
\end{align*}
$$

for the order of $\varepsilon^{a+3 / 2}$. From Eq. (2•13), we obtain $U_{1}$ in terms of the right eigenvector $r$ of $A_{0}$ for $\lambda_{0}$, a scalar function $\varphi(\xi, \eta)$ and a vector $V(\eta)$ as

$$
U_{1}=r(\eta) \varphi(\xi, \eta)+V(\eta) .
$$

In virtue of Eq. (2•11), the solution of Eq. (2-14) becomes

$$
U_{2}=r(\eta) \tilde{\psi}(\xi, \eta)+R \frac{\partial^{p-1} \varphi}{\partial \xi^{p-1}}
$$

where $\tilde{\psi}$ is an arbitrary scalar function of $\xi$ and $\eta$ and $R$ is the column vector satisfying the equation

$$
\left(-I+\frac{1}{\lambda_{0}} A_{0}\right) R=-\sum_{\beta}^{s} \prod_{\alpha}^{p}\left(-H_{0 \alpha}^{\beta}+\frac{1}{\lambda_{0}} K_{0 \alpha}^{\beta}\right) r .
$$

Then, multiplying Eq. $(2 \cdot 15)$ by the left eigenvector $l$ of $A_{0}$ for $\lambda_{0}$ and substituting Eqs. $(2 \cdot 16)$ and ( $2 \cdot 17$ ) result in the equation for $\varphi$

$$
\frac{\partial \varphi}{\partial \eta}+\left(\alpha \varphi+\alpha^{\prime}\right) \frac{\partial \varphi}{\partial \xi}+\beta \frac{\partial^{2 p-1} \varphi}{\partial \xi^{2 p-1}}+\gamma \varphi+\gamma^{\prime}=0,
$$

where the coefficients are given by

$$
\begin{align*}
& \alpha=\frac{l r \cdot(\nabla A)_{0} r}{\lambda_{0}^{2} l r}, \\
& a^{\prime}=\frac{l V \cdot(\nabla A)_{0} r}{\lambda_{0}^{2} l r}, \\
& \beta=\frac{l \Sigma_{\beta}^{s} \prod_{\alpha}^{p}\left(-H_{0 \alpha}^{\beta}+K_{0 \alpha}^{\beta} / \lambda_{0}\right) R}{\lambda_{0} l r}, \\
& r=\frac{\lambda_{0} l(d r \mid d \eta)+l r \cdot\left[(\nabla A)_{0}\left(d U_{0} / d \eta\right)+(\nabla B)_{0}(d S / d \eta)\right]}{\lambda_{0} l r}, \\
& r^{\prime}=\frac{\lambda_{0} l(d V / d \eta)+l V \cdot\left[(\nabla A)_{0}\left(d U_{0} / d \eta\right)+(\nabla B)_{0}(d S / d \eta)\right]}{\lambda_{0} l r}
\end{align*}
$$

It is to be noted that the systems belonging to Case II are always dispersive irrespective of $p$ being odd or even and the transformations ( $2 \cdot 8$ ) reduce Eq. (2-19) into

$$
\frac{\partial \phi}{\partial \theta}+\alpha e^{-\int \gamma d \theta} \phi \frac{\partial \phi}{\partial \zeta}+\beta \frac{\partial^{2 p-1} \phi}{\partial \zeta^{2 p-1}}=0
$$

or

$$
\frac{\partial \psi}{\partial \sigma}+\psi \frac{\partial \psi}{\partial \zeta}+\frac{\partial^{2 p-1} \psi}{\partial \zeta^{2 p-1}}+\left[\frac{d}{d \sigma} \ln \left(\frac{\beta}{a}\right)+\frac{\gamma}{\beta}\right] \psi=0
$$

An example of Case II (a) is the oblique hydromagnetic wave in a cold collision-free plasma which will be discussed in Part III.

So far, we have considered only the cases that $\lambda_{1} \neq 0$, i.e., Cases I and II. The formulation for Case III is quite similar to that for Case II and we do not discuss it in detail here. The final equation which one obtains is a modified nonlinear equation in the sense that the nonlinear term of Eq. (2.6) is replaced by $\alpha \varphi^{2} \partial \varphi / \partial \xi$. The example of Case III is the Alfvén wave in a cold collision-less plasma, which gives modified Korteweg-de Vries (K-d-V) equation in the sense above.

### 2.2 Modulated waves

Modulation of the quasi-monochromatic wave due to the self-interaction in constant unperturbed state are known for various examples. If the unperturbed state has inhomogeneity, the characteristic scale of which is larger than the wave length of the quasi plane wave by a degree, the reductive perturbation method can apply to include its effects. In such a system, quasi monochromatic wave is modulated by the inhomogeneity as well as the selfinteraction. In this section, we consider the modulation of the quasi monochromatic wave with a constant frequency $\omega$ and slowly varying wave number $k$, which depends on the slow variable and, via dispersion relation, on $\omega$. Here, we limit ourselves only to the case $\partial^{2} \omega / \partial k^{2} \neq 0$, i.e., Case I (a) with $a=1$, the appropriate strained variables and $\lambda_{0}$ for which, are given in the table and by Eq. ( 1.9 a) in §l, respectively.

The model equation for the present formulation is

$$
\frac{\partial U}{\partial t}+A \frac{\partial U}{\partial x}+B=0
$$

where $U$ is a column vector with $n$ components, $u_{1}, u_{2}, \cdots, u_{n} ; A$ an $n \times n$ matrix, and $B$ a column vector ( $b_{1}, b_{2}, \cdots, b_{n}$ ), the elements of which are functions of $u_{i}$. The unperturbed state $U_{0}$ is assumed to depend on $\eta$ and to satisfy the equation $B^{(0)} \equiv B\left(U_{0}\right)=0$. As easily verified, the dispersion relation of the quasi plane wave

$$
\sim \exp \{ \pm i[k(\eta) x-\omega t]\}
$$

becomes

$$
\operatorname{det}\left|\mp i \omega I \pm i k^{\prime} A^{(0)}+\delta B^{(0)}\right|=0,
$$

where $I$ is the unit matrix, $\delta B^{(0)}$ is an $n \times n$ matrix with ( $i, j$ ) components $\left(\partial b_{i} / \partial u_{j}\right)_{U=U^{(0)}}$, and $k^{\prime}=\partial(\eta k) / \partial \eta$. It is to be noted that the dispersion relation (2.23) is a differential equation for $k$ and, if Eq. (2.23) is algebraically solved for $k^{\prime}(\omega, \eta), k$ is given by the equation

$$
k=\frac{1}{\eta}\left[\eta_{c} k_{c}+\int_{\eta_{c}}^{\eta} k^{\prime}(\omega, \eta) d \eta\right],
$$

where $k_{c}$ is the wave number at $\eta=\eta_{c}$.
Expanding $U$ in powers of the parameter $\varepsilon$ and of the harmonics $\exp [i l(k x-\omega t)]$ around the state $U^{(0)}(\eta)$, i.e.,

$$
U=U^{(0)}(\eta)+\sum_{\alpha=1}^{\infty} \sum_{l=-\infty}^{\infty} \varepsilon^{\alpha} U_{l}^{(\alpha)}(\xi, \eta) \exp [i l(k x-\omega t)]
$$

where the reality condition of $U$ is $U_{l}^{(\alpha)}=U_{-l}^{(\alpha) *}$, and substituting it into Eq. (2.22), one has for the first order of $\varepsilon$ and $l$ th harmonics,

$$
W_{l} U_{l}^{(1)}=0,
$$

where the matrix $W_{l}$ is given by

$$
W_{l}=-i l \omega I+i l k^{\prime} A^{(0)}+\delta B^{(0)}
$$

Let us restrict ourselves to the modulation of the fundamental mode with $l=1$ for $U^{(1)}$. Then, the dispersion relation of the carrier wave is given by $\operatorname{det} W_{ \pm 1}=0$ or Eq. $(2 \cdot 23)$ and we may assume that $\operatorname{det} W_{l} \neq 0$ for $|l| \neq 1$, which gives the solution of Eq. $(2 \cdot 25)$ as

$$
\begin{align*}
& U_{1}^{(1)}=R(\eta) \varphi(\xi, \eta), \\
& U_{l}^{(1)}=0 \quad \text { for } \quad|l| \neq 1,
\end{align*}
$$

where $R$ is a right eigenvector of $W_{l}$ and $\varphi$ is a scalar function to be determined later. The second order, $l$ th equation takes the form

$$
\begin{align*}
W_{l} U_{l}^{(2)}+ & \left(-I+\frac{1}{\lambda_{0}} A^{(0)}\right) \frac{\partial U_{l}^{(1)}}{\partial \xi}+\sum_{l^{\prime}=-\infty}^{\infty} i l^{\prime} k^{\prime}\left(\nabla A^{(0)} \cdot U_{l-l^{\prime}}^{(1)}\right) U_{l^{\prime}}^{(1)} \\
& +A^{(0)} \frac{d U^{(0)}}{d \eta} \delta_{l 0}+\frac{1}{2} \sum_{l^{\prime}=-\infty}^{\infty}\left(\nabla \delta B^{(0)} \cdot U_{l-l^{\prime}}^{(1)}\right) U_{l^{\prime}}^{(1)}=0
\end{align*}
$$

where $\delta_{l 0}$ is Kronecker's delta, $\nabla A^{(0)} \cdot U^{(1)}$ and $\nabla \delta B^{(0)} \cdot U^{(1)}$ are the matrices defined by $\nabla A^{(0)} \cdot U^{(1)}=\sum_{i=1}^{n}\left(\partial A / \partial u_{i}\right)_{U=U^{(0)}} u_{i}^{(1)}$ and $\nabla \delta B^{(0)} \cdot U^{(1)}=\sum_{i=1}^{n}(\partial \delta B \mid$ $\left.\partial u_{i}\right)_{U=U^{(0)}} u_{i}^{(1)}$. In virtue of Eqs. (2-26), the component of Eq. (2•27) for $l=1$ becomes

$$
W_{1} U_{1}^{(2)}+\left(-I+\frac{1}{\lambda_{0}} A^{(0)}\right) \frac{\partial U_{1}^{(1)}}{\partial \xi}=0
$$

which can be solved for $U_{1}^{(2)}$ in terms of an arbitrary scalar function $\tilde{\psi}$ as

$$
U_{1}^{(2)}=R(\eta) \tilde{\psi}(\xi, \eta)-i \frac{\partial R}{\partial \omega} \frac{\partial \varphi}{\partial \xi} .
$$

In the derivation of Eq. (2-28a), we have used two identities $-I+A^{(0)} / \lambda_{0}=$ $-i\left(\partial W_{1} / \partial \omega\right)$ and $\left(\partial W_{1} / \partial \omega\right) R=-W_{1}(\partial R / \partial \omega)$. For $l=0$ and $l=2$, Eq. (2•27) yields

$$
U_{0}^{(2)}=R_{0}^{(2)}|\varphi|^{2}+V(\eta)
$$

and

$$
U_{2}^{(2)}=R_{2}^{(2)} \varphi^{2}
$$

where the vectors $R_{0}^{(2)}, R_{2}^{(2)}$ and $V$ are given by

$$
R_{0}^{(2)}=-W_{0}^{-1}\left\{i k^{\prime}\left[\left(\nabla A^{(0)} \cdot R^{*}\right) R-\mathrm{c} . \mathrm{c} .\right]+\frac{1}{2}\left[\left(\nabla \delta B^{(0)} \cdot R\right) R^{*}+\mathrm{c} . \mathrm{c} .\right]\right\},
$$

$$
R_{2}^{(2)}=-W_{2}^{-1}\left[i k^{\prime}\left(\nabla A^{(0)} \cdot R\right) R+\frac{1}{2}\left(\nabla \delta B^{(0)} \cdot R\right) R\right]
$$

and

$$
V=-W_{0}^{-1} A^{(0)} \frac{d U^{(0)}}{d \eta} .
$$

It is easy to see that the components $U_{l}^{(2)}$ with $|l| \geq 3$ are identically vanishing. In Eq.(2.28b), we assumed, of course, that $\operatorname{det} W_{0} \neq 0$. However, for systems with $\operatorname{det} W_{0}=0$, one could obtain $U_{0}^{(2)}$ in the form (2.28b) from Eq. (2.27) and the third order equation $(2 \cdot 29)$ for $l=0$ and so on. Finally, the third order equation is

$$
\begin{align*}
& W_{l} U_{l}^{(3)}+A(0) \frac{\partial U_{l}^{(1)}}{\partial \eta}+\nabla A^{(0)} \cdot U_{l}^{(1)} \frac{d U^{(0)}}{d \eta}+\left(-I+\frac{1}{\lambda_{0}} A^{(0)}\right) \frac{\partial U_{l}^{(2)}}{\partial \xi} \\
+ & \frac{1}{\lambda_{0}} \sum_{l^{\prime}}\left(\nabla A^{(0)} \cdot U_{l-l^{\prime}}^{(1)}\right) \frac{\partial U_{l^{\prime}}^{(1)}}{\partial \xi}+i k^{\prime}\left[\sum_{l^{\prime}} l^{\prime} \nabla A^{(0)} \cdot\left(U_{l-l^{\prime}}^{(1)} U_{l^{\prime}}^{(2)}+U_{l-l^{\prime}}^{(2)} U_{l^{\prime}}^{(1)}\right)\right. \\
+ & \left.\frac{1}{2} \sum_{l^{\prime}} \sum_{l^{\prime \prime}}\left(\nabla \nabla A^{(0)}: U_{l-l^{\prime}-l^{\prime \prime}}^{(1)} U_{l^{\prime}}^{(1)}\right) U_{l^{\prime}}^{(1)}\right]+\sum_{l^{\prime}}\left(\nabla \delta B^{(0)} \cdot U_{l-l^{\prime}}^{(1)}\right) U_{l^{\prime}}^{(2)} \\
+ & \frac{1}{6} \sum_{l^{\prime}} \sum_{l^{\prime \prime}}\left(\nabla \nabla \delta B^{(0)}: U_{l-l^{\prime}-l^{\prime}}^{(1)} U_{l^{\prime}}^{(1)}\right) U_{l^{\prime \prime}}^{(1)}=0,
\end{align*}
$$

where the matrices $\nabla \nabla A^{(0)}: U^{(1)} U^{(1)}$ and $\nabla \nabla \delta B^{(0)}: U^{(1)} U^{(1)}$ are defined by $\sum_{i=1}^{n} \sum_{j=1}^{n}\left(\partial^{2} A / \partial u_{i} \partial u_{j}\right)_{U=U^{(0)}}^{(1)} u_{i}^{(1)} u_{j}^{(1)}$ and so on. Multiplying Eq. (2-29) for $l=1$, by the left eigenvector $L$ of $W_{l}$ and substituting Eqs. $(2 \cdot 26)$ and (2-28) into the equation obtained, we have the equation for $\varphi$.

$$
i \frac{\partial \varphi}{\partial \eta}-\frac{1}{2} \frac{\partial^{2} k^{\prime}}{\partial \omega^{2}} \frac{\partial^{2} \varphi}{\partial \xi^{2}}+\mu|\varphi|^{2} \varphi+\kappa \varphi=0
$$

where

$$
\begin{aligned}
\mu=i L & \left\{i k ^ { \prime } \left[\left(2 \nabla A^{(0)} \cdot R^{*}\right) R_{2}^{(2)}+\left(\nabla A(0) \cdot R_{0}^{(2)}\right) R-\left(\nabla A^{(0)} \cdot R_{2}^{(2)}\right) R^{*}\right.\right. \\
& \left.+\left(\nabla \nabla A^{(0)}: R R^{*}\right) R-\frac{1}{2}\left(\nabla \nabla A^{(0)}: R R\right) R^{*}\right]+\left(\nabla \delta B^{(0)} \cdot R\right) R_{0}^{(2)} \\
& \left.+\left(\nabla \delta B^{(0)} \cdot R^{*}\right) R_{2}^{(2)}+\frac{1}{2}\left(\nabla \nabla \delta B^{(0)}: R R\right) R^{*}\right\} / L A^{(0)} R
\end{aligned}
$$

and

$$
\kappa=i L\left[A^{(0)} \frac{d R}{d \eta}+\left(\nabla A^{(0)} \cdot R\right) \frac{d U^{(0)}}{d \eta}+i k\left(\nabla A^{(0)} \cdot V\right) R+\left(\nabla \delta B^{(0)} \cdot R\right) V\right] / L A^{(0)} R .
$$

The coefficients of Eq.(2.30) are, in general, the complex valued functions of $\eta$. However, it is to be noted that if $\mu$ is real, the coefficients of the second and the third terms of Eq. $(2 \cdot 30)$ may be put to constants, by the transformation

$$
\left.\begin{array}{l}
\phi=\left(2 \left\lvert\, \frac{\mu}{\partial^{2} k^{\prime} / \partial \omega^{2}}\right.\right)^{1 / 2} e^{-i \int \kappa_{r} d \eta} \varphi, \\
\sigma=-\frac{1}{2} \int\left(\frac{\partial^{2} k^{\prime}}{\partial \omega^{2}}\right) d \eta,
\end{array}\right\}
$$

which reduces Eq. (2-30) into

$$
i \frac{\partial \phi}{\partial \sigma}+\frac{\partial^{2} \phi}{\partial \xi^{2}} \pm|\phi|^{2} \phi+i\left(\frac{d}{d \sigma} \ln \left|\frac{\partial^{2} k^{\prime} / \partial \omega^{2}}{\mu}\right|^{1 / 2}+\kappa i\right) \phi=0,
$$

where the signs $\pm$ of the term $|\phi|^{2} \phi$ correspond to $\mu\left(\partial^{2} \omega^{\prime} \mid \partial k^{2}\right) \gtrless 0$ respectively, and suffixes to $\kappa$ denote the real or the imaginary part. One may call Eq.(2-32) as the nonlinear Schrödinger type equation in an external potential. In the next section, Eq. $(2 \cdot 32)$ for the electron plasma wave will be given.

### 2.3 Examples

The theories presented in the preceding sections are applied to (1) shallow water wave, and, with some extensions to (2) acoustic wave under a weak external force and energy supply, (3) two dimensional propagation of drift wave and (4) modulation of the electron plasma wave in an inhomogeneous medium. The details of the calculations are not shown since many of them have been given in the references cited.
(1) Shallow water wave ${ }^{5}$ )

For long waves on a beach, Peregrine ${ }^{8)}$ obtained the equations

$$
\begin{aligned}
& \frac{\partial \boldsymbol{u}}{\partial t}+(\boldsymbol{u} \cdot \boldsymbol{\nabla}) \boldsymbol{u}+\boldsymbol{\nabla} h=\frac{1}{2} H \frac{\partial}{\partial t} \boldsymbol{\nabla}[\boldsymbol{\nabla} \cdot(H \boldsymbol{u})]-\frac{1}{6} H^{2} \frac{\partial}{\partial t} \boldsymbol{\nabla}(\boldsymbol{\nabla} \boldsymbol{u}), \\
& \frac{\partial h}{\partial t}+\boldsymbol{\nabla}[(H+h) \boldsymbol{u}]=0
\end{aligned}
$$

where $\boldsymbol{u}$ is the horizontal velocity averaged over the vertical direction, $H$ the depth of the unperturbed water layer and $h$ the wave amplitude. If only the motion along an $x$ axis are considered, these equations are easily reduced to the form (2.1) for the column vector $U=(h, u)$ and $S=H$. For the constant state $U_{0}=0$ and the right eigenvector $r=\left(H, H^{1 / 2}\right)$, we obtain the Kortweg-de Vries equation of the variable coefficients

$$
\frac{\partial \phi}{\partial \eta}+\frac{3}{2} H^{-7 / 4} \phi \frac{\partial \phi}{\partial \xi}+\frac{1}{6} H^{1 / 2} \frac{\partial^{3} \phi}{\partial \xi^{3}}=0,
$$

where $\phi=H^{5 / 4} \varphi$ and $V$ is neglected in Eq. (2.5) or, further,

$$
\frac{\partial \psi}{\partial \sigma}+\psi \frac{\partial \psi}{\partial \xi}+\frac{\partial^{3} \psi}{\partial \xi^{3}}+\frac{9}{4}\left(\frac{d}{d \sigma} \ln H\right) \psi=0
$$

for $\psi=H^{-9 / 4} \phi / 4=\varphi / 4 H$ and $\sigma=\int H^{1 / 2} \mathrm{~d} \eta / 6$. From these equations, one can see
the decay of a solitary wave and so on.
(2) Acoustic wave under a weak external force and small energy supply9)

As easily seen, the theory of the long wave presented in $\S 2.1$ can be extended to include a small additive vector $\varepsilon^{a+1} F(U, S)$ to Eq. $(2 \cdot 1)$. As an example, let us consider the flow of an inviscid fluid in a duct of varying cross section under an external force $\varepsilon f$ and an energy source $\varepsilon q$. Note that for inviscid fluid we may put $a=0$. The equations to be studied are

$$
\begin{aligned}
& \frac{\partial \rho}{\partial t}+u \frac{\partial \rho}{\partial x}+\rho \frac{\partial u}{\partial x}+u \rho \frac{d}{d x} \ln s=0 \\
& \frac{\partial u}{\partial t}+u \frac{\partial u}{\partial x}+\frac{1}{\rho} \frac{\partial p}{\partial x}=\varepsilon f \\
& \frac{\partial E}{\partial t}+u \frac{\partial E}{\partial x}-\frac{p}{\rho^{2}}\left(\frac{\partial \rho}{\partial t}+u \frac{\partial \rho}{\partial x}\right)=\varepsilon q
\end{aligned}
$$

where $s$ is the cross section of the duct and the internal energy $E$ is given by $E=p /(\Gamma-1) \rho$ with $\Gamma$, the adiabatic constant. If $s$ has the form $s \propto x^{n}$ one can always put $d \ln s / d x=\varepsilon d \ln \sigma / d \eta$ where $\sigma \propto \eta^{n}$ or, otherwise, we assume that $s$ is a function of $\eta$. Equation (2.9a) for $U=(\rho, u, E)$ and $R=\left(\rho_{0}, \lambda_{0}-u_{0}\right.$, $\left.(\Gamma-1) E_{0}\right)$ with $\lambda_{0}=u_{0} \pm a_{0}\left(a_{0}^{2}=\Gamma p_{0} / \rho_{0}\right)$ has the coefficients $\alpha=[(\Gamma+1) / 2]$ $\times\left(\lambda_{0}-u_{0}\right) / \lambda_{0}^{2}, \beta=0$ and $\gamma$ given by

$$
\begin{aligned}
\gamma= & \left\{\left[\Gamma \lambda_{0}-(\Gamma-1) u_{0}\right] \frac{1}{\rho_{0}} \frac{d \rho_{0}}{d \eta}+\left(\Gamma^{2}-\Gamma+2\right) \frac{d u_{0}}{d \eta}+\left(\frac{3}{2} \Gamma \lambda_{0}-u_{0}\right) \frac{1}{E_{0}} \frac{d E_{0}}{d \eta}\right. \\
& +\left[\Gamma \lambda_{0}+(\Gamma-1)(\Gamma-2) u_{0}\right] \frac{1}{s} \frac{d s}{d \eta}-\frac{\Gamma}{a_{0}^{2}}\left(\lambda_{0}-u_{0}\right)(R \cdot \nabla f)_{0} \\
& \left.-\frac{\Gamma(\Gamma-1)}{a_{0}^{2}}(R \cdot \nabla q)_{0}\right\} / 2 \Gamma \lambda_{0}
\end{aligned}
$$

where we put $V$ to be zero and used the notations $(R \cdot \nabla f)_{0}$ and $(R \cdot \nabla q)_{0}$ defined in §2.1. Formally, at least, Eq. $(2 \cdot 9 \mathrm{a})$ with $\beta=0$ can be solved analytically.
(3) Two-dimensional propagation of drift wave ${ }^{10}$ )

So far, only one dimensional wave motions are discussed. However, as presented in Part I, $\S 7$, the reductive perturbation method applies to the multidimensional, inhomogeneous system, if the suitable ordering and strained variables are chosen. In this example, drift waves in a fluid model is considered. Plasma of cold ions and hot electrons with a Boltzman distribution is put in the strong magnetic field applied in the $z$-direction. If the density gradient exists in $x$-direction, drift wave propagates in the $y-z$ plane. Neglecting the magnetic field and temporal induction due to the electron drift, one may use, as the basic equations of the ion fluid

$$
\begin{aligned}
& \frac{\partial n}{\partial t}+\nabla(n \boldsymbol{V})=0, \\
& \frac{\partial \boldsymbol{V}}{\partial t}+\boldsymbol{V} \cdot \boldsymbol{\nabla} \boldsymbol{V}=\omega_{c i}\left[\boldsymbol{V} \times \boldsymbol{e}_{z}\right]-\frac{e}{m_{i}} \nabla \phi, \\
& \nabla^{2} \phi=-4 \pi e\left[n-n_{0} e^{\left.e \phi / T_{\ell}\right],}\right.
\end{aligned}
$$

where $\boldsymbol{e}_{z}$ is the unit vector in $z$-direction. In equilibrium, the ions are at rest and electrons drift in the $y$-direction with a velocity $V_{0}=-\left(c T_{e} / e B\right)\left(d n_{0} / d x\right) / n_{0}$. The strained variables for the wave are given in terms of $c_{s}=\left(T_{e} / m_{i}\right)^{1 / 2}$,

$$
\zeta=\varepsilon^{1 / 2}\left(z-c_{s} t\right), \quad \eta=\varepsilon^{1 / 2} y, \quad \tau=\varepsilon^{3 / 2} t,
$$

while the variable for the inhomogeneity is

$$
\xi=\varepsilon x .
$$

The vector $U=\left(n, V_{y}, V_{z}, \phi\right)$ is expanded in powers of $\varepsilon$ around $U^{(0)}=\left(n_{0}\right.$, $0,0,0)$ as $U=U^{(0)}+\varepsilon U^{(1)}+\varepsilon^{2} U^{(2)}+\cdots$ but $V_{x}$ must be put to $V_{x}=\varepsilon^{1 / 2}\left(\varepsilon V_{x}^{(1)}+\right.$ $\left.\varepsilon^{2} V_{x}^{(2)}+\cdots\right)$. Then, one has

$$
V_{x}^{(1)}=-\frac{c_{s}^{2}}{n_{0} \omega_{c i}} \frac{\partial n^{(1)}}{\partial \eta}, \quad V_{y}^{(1)}=0, \quad V_{z}^{(1)}=\frac{c_{s}}{n_{0}} n^{(1)}, \quad \phi^{(1)}=\frac{T_{e}}{e n_{0}} n^{(1)}
$$

and

$$
V_{y}^{(2)}=\frac{c_{s}^{2}}{n_{0} \omega_{c i}}\left(\frac{\partial n^{(1)}}{\partial \xi}+\frac{c_{s}}{\omega_{c i}} \frac{\partial^{2} n^{(1)}}{\partial \eta \partial \zeta}\right) .
$$

The density $n^{(1)} \equiv n_{0} \varphi$ is determined from the K-d-V equation in the two dimensional space;

$$
\begin{aligned}
& \frac{\partial \varphi}{\partial \tau}+c_{s} \varphi \frac{\partial \varphi}{\partial \zeta}+\frac{1}{2} \frac{c_{s}^{3}}{\omega_{c i}^{2}} \frac{\partial^{3} \varphi}{\partial \eta^{2} \partial \zeta}+\frac{1}{2} c_{s} D^{2}\left(\frac{\partial^{3}}{\partial \eta^{2} \partial \zeta}+\frac{\partial^{3}}{\partial \zeta^{3}}\right) \varphi \\
&+\frac{1}{2} \frac{c_{s}^{2}}{\omega_{c i}} \kappa \frac{\partial \varphi}{\partial \eta}=0
\end{aligned}
$$

where $\kappa=\left|d n_{0}\right| d \xi \mid / n_{0}$ and $D$ is the Debye length. At present, it is not possible to see general behavior of solution but special solution, including the solitary wave solution, can be obtained easily.
(4) Modulation of the electron plasma wave

As an example of the theory in $\S 2.2$, the interaction of the modulated electron plasma wave with the inhomogeneity is investigated. The hydrodynamic equation are coupled with Poisson's equation and the first Maxwell equation

$$
\frac{\partial n}{\partial t}+\frac{\partial}{\partial x}(n u)=0,
$$

$$
\begin{aligned}
& \frac{\partial u}{\partial t}+u \frac{\partial u}{\partial x}+\frac{a^{2}}{n} \frac{\partial n}{\partial x}+\frac{e}{m} E=0 \\
& \frac{\partial E}{\partial t}+u \frac{\partial E}{\partial x}-4 \pi e n_{0} u=0
\end{aligned}
$$

where $a$ is the electron sound velocity and $n_{0}$ is the unperturbed density which depends on $\eta=\varepsilon^{2} x$. Because of $\operatorname{det} W_{0}=0$ for the present case, Eq.(2-29) for $l=0$ must be used to determine $U_{0}^{(2)}$. After some calculations for $U=(n, u, E)$, one has the dispersion relation $\omega^{2}=\omega_{0}^{2}+a^{2} k^{2}$, where $\omega_{0}^{2}=4 \pi n_{0} e^{2} / m$, the right eigenvector $R=(1, \omega / k n, 4 \pi i e / k)$ and

$$
U_{0}^{(2)}=-\frac{2 \omega}{n_{0}^{2} k^{\prime}}\left(\begin{array}{l}
0 \\
1 \\
0
\end{array}\right)|\varphi|^{2}+\frac{a^{2} m}{e n_{0}} \frac{d n_{0}}{d \eta}\left(\begin{array}{c}
v_{1} \\
0 \\
1
\end{array}\right)
$$

here, the explicit form of $v_{1}$ is not necessary to obtain the equation for $\varphi$;

$$
i \frac{\partial \varphi}{\partial \eta}-\frac{1}{2} \frac{\partial^{2} k^{\prime}}{\partial \omega^{2}} \frac{\partial^{2} \varphi}{\partial \xi^{2}}-k^{\prime} \frac{8 \omega^{2}+\omega_{0}^{2}}{6 n_{0}^{2} \omega_{0}^{2}}|\varphi|^{2} \varphi-\frac{i}{2}\left[\frac{d}{d \eta} \ln \left(k^{\prime} n_{0}\right)\right] \varphi=0 .
$$

The dispersion relation is solved for $k^{\prime}$, as $k^{\prime}=\left(\omega^{2}-\omega_{0}^{2}\right)^{1 / 2} / a$ but it is to be emphasized that the wave number of the carrier wave is not $k^{\prime}$ but $k$ given by

$$
k=\frac{1}{\eta_{c}}\left[\eta_{c} k_{c}+\frac{1}{a} \int_{\eta_{c}}^{\eta}\left(\omega^{2}-\frac{4 \pi e^{2}}{m} n_{0}(\eta)\right)^{1 / 2} d \eta\right]
$$

The transformation (2-31) brings the equation for $\varphi$ into that for $\phi=\left[\left(a k^{\prime}\right)^{2}\left(8 \omega^{2}\right.\right.$ $\left.\left.+\omega_{0}^{2}\right)^{1 / 2} / 3^{1 / 2} n_{0} \omega_{0}^{2}\right] \varphi$ with $\sigma=\int\left(\omega_{0}^{2} / k^{\prime 3}\right) d \eta / 2 a^{4} ;$

$$
\begin{aligned}
& i \frac{\partial \phi}{\partial \sigma}+\frac{\partial^{2} \phi}{\partial \xi^{2}}-|\phi|^{2} \phi+i\left[\frac{d}{d \sigma} \ln \left[\frac{n_{0} \omega_{0}^{2}}{k^{\prime 2}\left(8 \omega^{2}+\omega_{0}^{2}\right)^{1 / 2}}\right]\right. \\
&\left.-\frac{\omega_{0}^{2}}{4 a\left(a k^{\prime}\right)^{3}} \frac{d}{d \sigma} \ln \left(k^{\prime} n_{0}\right)\right] \phi=0
\end{aligned}
$$

## § 3. Propagation in the media slowly varying in time

### 3.1 Long waves

The method of the reduction for the wave propagation in unsteady medium is similar to that in inhomogeneous medium. The physical systems, the unperturbed state of which depend on the time, can often be represented by the equations of the form

$$
\frac{\partial U}{\partial t}+A \frac{\partial U}{\partial x}+\sum_{\beta}^{s} \prod_{\alpha}^{p}\left(H_{\alpha}^{\beta} \frac{\partial}{\partial t}+K_{\alpha}^{\beta} \frac{\partial}{\partial x}\right) U+\kappa B=0
$$

where $\kappa$ is a constant to be specified below, $B$ is a vector valued function of $U$ and other symbols have the same meaning with those in Eq. $(1 \cdot 1)$. Evidently, Eq. (3•1) has not constant unperturbed solution unless $B$ vanishes. Let us consider only Case I and assume that the unperturbed state $U_{0}$ depends only on the slow variable $\tau$ defined in §l. In this case, $\kappa$ should be small such as $\kappa=\varepsilon^{a+1}$, which then, specifies the value of $\varepsilon$, and $U_{0}$ by the equation

$$
\frac{d U_{0}}{d \tau}+B_{0}=0
$$

Expanding $U$ around $U_{0}$ in powers of $\varepsilon$, i.e., $U=U_{0}+\varepsilon U_{1}+\varepsilon^{2} U_{2}+\cdots$, and substituting it and strained variables for Case I (b) into Eq. (3•1) yield the lowest order equation

$$
\left(A_{0}-\lambda_{0} I\right) \frac{\partial U_{1}}{\partial \xi}=0
$$

in virtue of Eq. (3.2), and the second order equation

$$
\left(A_{0}-\lambda_{0} I\right) \frac{\partial U_{2}}{\partial \xi}+\frac{\partial U_{1}}{\partial \tau}+A_{1} \frac{\partial U_{1}}{\partial \xi}+\sum_{\beta}^{s} \prod_{\alpha}^{p}\left(K_{0 \alpha}^{\beta}-\lambda_{0} H_{0 \alpha}^{\beta}\right) \frac{\partial^{p} U_{1}}{\partial \xi^{p}}+\delta B_{0} U_{1}=0
$$

where $A_{1}$ and $\delta B$ are the matrices given by $A_{1}=(\nabla A)_{0} \cdot U_{1}=\sum_{i=1}^{n}\left(\partial A / \partial u_{i}\right)_{U=U_{0}}$ $\cdot u_{1 i}$ and $\delta B=\left\{\delta b_{i j}\right\}=\left\{\partial b_{i} / \partial u_{j}\right\}$. The solution of Eq. (3.3) is

$$
U_{1}=r(\tau) \varphi(\xi, \tau)+V(\tau)
$$

where $r$ is a right eigenvector of $A_{0}$ for $\lambda_{0}, \varphi$ a scalar function and $V$ is a vector determined by the boundary codition on $U_{1}$ and $\varphi$ at $\xi=\xi c$, say. Substituting Eq. (3.5) into Eq. $(3 \cdot 4)$ and multiplying the left eigenvector $l$ of $A_{0}$ for $\lambda_{0}$, we have an equation to determine $\varphi$,

$$
\frac{\partial \varphi}{\partial \tau}+\left(\alpha \varphi+a^{\prime}\right) \frac{\partial \varphi}{\partial \xi}+\beta \frac{\partial^{p} \varphi}{\partial \xi^{p}}+\gamma \varphi+\gamma^{\prime}=0
$$

where the coefficients are given by

$$
\begin{align*}
& \alpha=\frac{l r \cdot(\nabla A)_{0} r}{l r}, \\
& a^{\prime}=\frac{l V \cdot(\nabla A)_{0} r}{l r},  \tag{3.7b}\\
& \beta=\frac{l \Sigma_{\beta}^{s} \prod_{\alpha}^{p}\left(-\lambda_{0} H_{0 \alpha}^{\beta}+K_{0 \alpha}^{\beta}\right) r}{l r}, \\
& r=\frac{l(d r \mid d \tau)+l \delta B r}{l r}
\end{align*}
$$

and

$$
\gamma^{\prime}=\frac{l(d V \mid d \tau)+l \delta B V}{l r} .
$$

The transformation corresponding to Eq. (2.8a) takes the form

$$
\left.\begin{array}{l}
\phi=e^{\int \gamma d \tau} \varphi+\int \gamma^{\prime} e^{\int \gamma d \tau} d \tau \\
\zeta=\xi-\int \alpha^{\prime} d \tau+\int d \tau\left[a e^{-\int \gamma d \tau} \int \gamma^{\prime} e^{\int \gamma d \tau} d \tau\right] \\
\theta=\tau
\end{array}\right\}
$$

and reduces Eq. (3.6) into the equation for $\phi$

$$
\frac{\partial \phi}{\partial \theta}+\alpha e^{-\int \gamma d \theta} \phi \frac{\partial \phi}{\partial \zeta}+\beta \frac{\partial^{p} \phi}{\partial \zeta^{p}}=0
$$

and the transformation (2.8b) brings Eq. (3.9) into Eq. (2.9b). As the direct applications of the present formulation for Eq. (3•1), we can treat the nonlinear wave in the slowly reacting gas, if the spatial structure of the unperturbed state can be neglected. Wave propagations in a homogeneously expanding or contracting medium may also be analyzed, which is an important problem in some branches of gas dynamics and astrophysics. In §3.3, an example of the propagation in the time dependent medium is given for Eq. (2•1) with $H_{\alpha}^{\beta}=K_{\alpha}^{\beta}=0$, that is, the propagation in the inhomogeneous, unsteady medium is considered. In this example, an initial and boundary value problem is solved by means of $V(\tau)$, for a flow in a duct.

### 3.2 Modulated waves

In this section, the modulation of nearly monochromatic wave with a constant wave number is discussed in the medium which varies slowly with time $\tau=\varepsilon^{2} t$, i.e., Case I (b) with $a=1$ and $\lambda_{0}$ given by Eq. (1.9b). We treat the same equation with Eq. (2.22), that is,

$$
\frac{\partial U}{\partial t}+A \frac{\partial U}{\partial x}+B=0
$$

where $A$ is the $n \times n$ matrix function of $U$ and $B$ is a column vector valued function of $U$. Since the frequency $\omega$ of the quasi monochromatic wave depends on $\tau$, the dispersion relation corresponding to Eq. $(2 \cdot 23)$ becomes

$$
\operatorname{det}\left|\mp i \omega^{\prime} I \pm i k A^{(0)}+\delta B^{(0)}\right|=0
$$

where $I$ is the unit matrix, $\delta B^{(0)}=\left\{\left(\partial b_{i} / \partial u_{j}\right) U=U^{(0)}\right\}$ and suffix ( 0 ) denotes the unperturbed values. The frequency $\omega^{\prime}$ is defined by $\omega^{\prime}=\partial(\tau \omega) / \partial \tau$, hence $\omega$ is given by the algebraic solution $\omega^{\prime}(k, \tau)$ of Eq. (3•11), in the form

$$
\omega=\frac{1}{\tau_{c}}\left[\tau_{c} \omega_{c}+\int_{\tau_{c}}^{\tau} \omega^{\prime}(k, \tau) d \tau\right],
$$

in which $\omega_{c}$ is a constant frequency at a given time $\tau_{c}$.
Manipulation of the reduction of the amplitude equation goes in parallel to that for the inhomogeneous medium: Expanding $U$ in powers of $\varepsilon$ and of the harmonics, $\exp [i l(k x-\omega t)]$ around the state $U^{(0)}(\tau)$, i.e.,

$$
U=U^{(0)}(\tau)+\sum_{\alpha=1}^{\infty} \sum_{l=-\infty}^{\infty} \epsilon^{\alpha} U_{l}^{\langle\alpha)}(\xi, \tau) \exp [i l(k x-\omega t)]
$$

where the reality condition is $U_{l}^{(\alpha)}=U_{-l}^{(\alpha)^{*}}$, and substituting it into Eq. (3•10), one get the first, the second and the third order equations of $l$ th component. They are,

$$
\begin{align*}
& W_{l} U_{l}^{(1)}=0 \\
& W_{l} U_{l}^{(2)}+\left(-\lambda_{0} I+A^{(0)}\right) \frac{\partial U_{l}^{(1)}}{\partial \xi}+\sum_{l^{\prime}=-\infty}^{\infty} i l^{\prime} k \nabla A^{(0)} \cdot U_{l-l^{\prime}}^{(1)} U_{l^{\prime}}^{(1)} \\
& \quad+\frac{d U^{(0)}}{d \tau} \delta_{l 0}+\frac{1}{2} \sum_{l^{\prime}=-\infty}^{\infty}\left(\nabla \delta B^{(0)} \cdot U_{l-l^{\prime}}^{(1)} U_{l^{\prime}}^{(1)}=0\right.
\end{align*}
$$

and

$$
\begin{align*}
& W_{l} U_{l}^{(3)}+\frac{\partial U_{l}^{(1)}}{\partial \tau}+\left(-\lambda_{0} I+A^{(0)}\right)-\frac{\partial U_{l}^{(2)}}{\partial \xi}+\sum_{l^{\prime}=-\infty}^{\infty}\left(\nabla A(0) \cdot U_{l-l^{\prime}}^{(1)}\right) \frac{\partial U_{l^{\prime}}^{(1)}}{\partial \xi} \\
& +i k\left[\sum_{l^{\prime}} l^{\prime} \nabla A^{(0)} \cdot\left(U_{l-l^{\prime}}^{(2)} U_{l^{\prime}}^{(1)}+U_{l-l^{\prime}}^{(1)} U_{l^{\prime}}^{(2)}\right)\right. \\
& \\
& \left.\quad+\frac{1}{2} \sum_{l^{\prime}} \sum_{l^{\prime \prime}} l^{\prime \prime} \nabla \nabla A^{(0)}: U_{l-l^{\prime}-l^{\prime \prime}}^{(1)} U_{l^{\prime}}^{(1)} U_{l^{\prime \prime}}^{(1)}\right] \\
& \quad+\sum_{l^{\prime}} \nabla \delta B^{(0) \cdot} \cdot U_{l-l^{\prime}}^{(1)} U_{l^{\prime}}^{(2)}+\frac{1}{6} \sum_{l^{\prime}} \sum_{l^{\prime \prime}} \nabla \nabla \delta B^{(0)}: U_{l-l^{\prime}-l^{\prime \prime}}^{(1)} U_{l^{\prime}}^{(1)} U_{l^{\prime \prime}}^{(1)} \\
& =0
\end{align*}
$$

respectively, where $\nabla A^{(0)} \cdot U^{(1)}=\sum_{i=1}^{n}\left(\partial A / \partial u_{i}\right)_{U=U^{(0)}}^{u_{i}^{(1)}, ~} \nabla \nabla A^{(0)}: U^{(1)} U^{(2)}=$


$$
W_{l}=-i l \omega^{\prime} I+i l k A^{(0)}+\delta B^{(0)}
$$

If we restrict ourselves to the case that $\operatorname{det} W_{ \pm 1}=0$ and $\operatorname{det} W_{l} \neq 0$ for $|z| \neq 1$, the solution of Eq. (3•13) becomes

$$
\begin{align*}
& U_{1}^{(1)}=R(\tau) \varphi(\xi, \tau), \\
& U_{l}^{(1)}=0 \quad \text { for } \quad|l| \neq 1
\end{align*}
$$

where $R$ is a right eigenvector of $W_{1}$ and $\varphi$ is a scalar function. By means of Eqs. (3•17), Eq. (3•14) for $l=1$ takes the form

$$
W_{1} U_{1}^{(2)}+\left(-\lambda_{0} I+A^{(0)}\right) \frac{\partial U_{1}^{(1)}}{\partial \xi}=0,
$$

the solution of which is yielded by the same way as in §2.2;
where $\tilde{\psi}$ is an arbitrary scalar function. Other components of $U_{l}^{(2)}$ are likewise determined as in $\S 2.2$, to lead

$$
\begin{align*}
& U_{0}^{(2)}=R_{0}^{(2)}|\varphi|^{2}+V(\tau), \\
& U_{2}^{(2)}=R_{2}^{(2)} \varphi^{2}, \\
& U_{l}^{(2)}=0 \quad \text { for } \quad|l| \geq 3,
\end{align*}
$$

where

$$
\begin{aligned}
& R_{0}^{(2)}=-W_{0}^{-1}\left[i k\left(\nabla A^{(0)} \cdot R^{*} R-\text { c.c. }\right)+\frac{1}{2}\left(\nabla \delta B^{(0)} \cdot R R^{*}+\text { c.c. }\right)\right], \\
& R_{2}^{(2)}=-W_{2}^{-1}\left[i k \nabla A^{(0)} \cdot R R+\frac{1}{2} \nabla \delta B^{(0)} \cdot R R\right]
\end{aligned}
$$

and

$$
V=-W_{0}^{-1} \frac{d U^{(0)}}{d \tau}
$$

Finally, the equation for $\varphi$ is obtained from Eq. (3•15) with $l=1$, by substituting the above solutions and multiplying the left eigenvector $L$ of $W_{1}$. After some rearrangements, we have

$$
i \frac{\partial \varphi}{\partial \tau}+\frac{1}{2} \frac{\partial^{2} \omega^{\prime}}{\partial k^{2}} \frac{\partial^{2} \varphi}{\partial \xi^{2}}+\mu|\varphi|^{2} \varphi+\kappa \varphi=0,
$$

in which

$$
\begin{aligned}
\mu= & i\left[i k L \left(2 \nabla A^{(0)} \cdot R^{*} R_{2}^{(2)}+\nabla A^{(0)} \cdot R_{0}^{(2)} R-\nabla A^{(0)} \cdot R_{2}^{(2)} R^{*}\right.\right. \\
& \left.+\nabla \nabla A^{(0)}: R R^{*} R-\frac{1}{2} \nabla \nabla A^{(0)}: R R R^{*}\right) \\
& \left.+L\left(\nabla \delta B^{(0)} \cdot R R_{0}^{(2)}+\nabla \delta B^{(0)} \cdot R^{*} R_{2}^{(2)}+\frac{1}{2} \nabla \nabla \delta B^{(0)}: R R R^{*}\right)\right] / L R
\end{aligned}
$$

and

$$
\kappa=i L\left(\frac{d R}{d \tau}+i k \nabla A^{(0)} \cdot V R+\nabla \delta B^{(0)} \cdot R V\right) / L R .
$$

When $\mu$ is real, the transformation corresponding to Eq. (2.31) takes the form

$$
\left.\begin{array}{l}
\phi=\left(2 \frac{\mu}{\partial^{2} \omega^{\prime} / \partial k^{2}}\right)^{1 / 2} e^{-i \int \kappa_{r} d \tau} \varphi, \\
\sigma=\frac{1}{2} \int\left(\frac{\partial^{2} \omega^{\prime}}{\partial k^{2}}\right) d \tau
\end{array}\right\}
$$

and the equation for $\phi$ becomes nonlinear Schrödinger type equation in an external potential,

$$
i \frac{\partial \phi}{\partial \sigma}+\frac{\partial^{2} \phi}{\partial \xi^{2}} \pm|\phi|^{2} \phi+i\left(\frac{d}{d \sigma} \ln \left|\frac{\partial^{2} \omega^{\prime} \mid \partial k^{2}}{\mu}\right|^{1 / 2}+\kappa i\right) \phi=0 .
$$

A slight modification and application of the general theory is given in the following section for the Jean's wave in a self gravitating gas in an expanding universe.

### 3.3 Examples

As an example of the wave propagation in an unsteady medium, we consider, at first, the receding acoustic wave in a duct of varying cross section. In the flow of a fluid with the velocity $u$ the sound waves are constituted of the advancing wave with the velocity $u+a$ and the receding wave with the velocity $u-a$, where $a$ is the sound velocity relative to the medium. If the flow velocity is near the sound velocity, i.e., $u-a=0(\varepsilon)$, and only one-dimensional motion along the axis of the duct is considered, the time variation of the medium seen by the receding wave is in phase with the given variation at a point on the axis, that is, the time variation of the flow can be described by Eq. $(3 \cdot 2)$ or by the term $V(\tau)$ in Eq. $(3 \cdot 5)$ for the case $B_{0} \equiv 0$. In this case, we use Eq. $(2 \cdot 1)$ with $H_{\alpha}^{\beta}=K_{\alpha}^{\beta}=0$ to treat the inhomogeneity of the duct and express the small variation of the flow state, by $V(\tau)$. The second example is the modulation of the nearly monochromatic wave in a self gravitating gas with a homogeneous temperature in an expanding or contracting universe, the scale parameter $a(t)$ of which is assumed to be a function of $\tau$ and may be determined by Friedman type equations. Since we consider only real $\omega$, the wave number $k$ must be larger than the Jean's wave number $k_{J}$ but the amplitude equation shows that some kinds of condensations are possible to exist.
(1) Receding wave in non-stationary flow through a duct of varying cross section ${ }^{11)}$

The motion of an ideal fluid in a duct of varying cross section $s$ is governed by the set of equations

$$
\begin{aligned}
& \frac{\partial \rho}{\partial t}+\frac{\partial(u \rho)}{\partial x}+\frac{1}{s} \frac{d s}{d x} \rho u=0, \\
& \frac{\partial u}{\partial t}+u \frac{\partial u}{\partial x}+\frac{1}{\rho} \frac{\partial p}{\partial x}=0,
\end{aligned}
$$

$$
\frac{\partial p}{\partial t}+u \frac{\partial p}{\partial x}+\gamma p \frac{\partial u}{\partial x}+\frac{1}{s} \frac{d s}{d x} \gamma p u=0
$$

where $\gamma$ is the adiabatic constant. The variation of $s$ is assumed of the order of $\varepsilon^{2}$, i.e., $s=s_{0}+\varepsilon^{2} s_{1}$, then the flow quantities $U=(\rho, u, p)$ can be expanded in powers of $\varepsilon$, if the unperturbed constant velocity $u_{0}$ is equal to $a_{0}=\left(\gamma p_{0} / \rho_{0}\right)^{1 / 2}$. For the receding wave in inviscid fluid, the strained variables are given by those of Case I (b) with $\lambda_{0}=u_{0}-a_{0}=0$ and $a=0$. As the boundary conditions on $U_{1}$ and $\varphi$, we give the value of $U_{1}$ at $x=x_{c}$ and put $a_{0} \varphi=u_{1}$ there. Then, in virtue of the vector $R=\left(-\rho_{0}, a_{0},-\gamma p_{0}\right)$, the components of $V(\tau)$ in Eq. (3.5) become $v_{1}(\tau)=\rho_{0} \varphi\left(x_{c}, \tau\right)+\rho_{1}\left(x_{c}, \tau\right), v_{2}(\tau)=0$ and $v_{3}(\tau)=\gamma p_{0} \varphi\left(x_{c}, \tau\right)+p_{1}\left(x_{c}, \tau\right)$. Since, the flow variables can be expressed in terms of the Mach number $M$, it is convenient to give the final equation in terms of $M_{1}$ where $M=1+\varepsilon M_{1}+\cdots$; Noting the equation $M_{1}=\left(u_{1} / a_{0}\right)-\left(p_{1} / p_{0}-\rho_{1} / \rho_{0}\right) / 2$, we obtain

$$
\frac{\partial M_{1}}{\partial \tau}+a_{0} M_{1} \frac{\partial M_{1}}{\partial x}-F(\tau)-\frac{\gamma+1}{4} \frac{a_{0}}{s_{0}} \frac{d s_{1}}{d x}=0
$$

where $F(\tau)$ is a function of $\tau$ only, given at $x=x_{\boldsymbol{c}}$

$$
\begin{aligned}
F(\tau) & =\frac{d M_{1}\left(x_{c}, \tau\right)}{d \tau}+a_{0} M_{1}\left(x_{c}, \tau\right) \frac{d M_{1}\left(x_{c}, \tau\right)}{d x}-\frac{\gamma+1}{4} \frac{a_{0}}{s_{0}} \frac{d s_{1}\left(x_{c}\right)}{d x} \\
& =\frac{1}{2}\left[\frac{1}{\rho_{0}} \frac{d \rho_{1}\left(x_{c}, \tau\right)}{d \tau}-\frac{\gamma-3}{2} \frac{1}{a_{0}} \frac{d u_{1}\left(x_{c}, \tau\right)}{d \tau}-\frac{\gamma-1}{2 \gamma} \frac{1}{p_{0}} \frac{d p_{1}\left(x_{c}, \tau\right)}{d \tau}\right] .
\end{aligned}
$$

From the equation for $M_{1}$, one can obtain, of course, Whitham's rule of the shock wave propagation in the steady flow and also shock trajectory, strength and so on, in the unsteady flow through the duct of varying cross section.
(2) Self gravitating waves in an expanding or contracting universe

In this sub-section, the wave which is stable in the linear theory is studied in an expanding or contracting universe. The inviscid hydrodynamic equations for the matter-dominant self-gravitating medium which is globally homogeneous and isotropic, are, in the coordinate system comoving with the mean motion, ${ }^{12)}$

$$
\begin{aligned}
& \frac{\partial \rho}{\partial t}+\frac{\partial u \rho}{\partial x}+3 H \rho=0 \\
& \frac{\partial u}{\partial t}+u \frac{\partial u}{\partial x}+\left(\frac{c}{a}\right)^{2} \frac{1}{\rho} \frac{\partial \rho}{\partial x}-\frac{1}{a^{2}} f+2 H u=0 \\
& \frac{\partial f}{\partial x}+4 \pi G a^{2}\left(\rho-\rho^{(0)}\right)=0
\end{aligned}
$$

where $H=(d a \mid d t) \mid a$ the Hubble constant, - $f$ the gravitational force, $c$ the sound velocity, $a$ the scale parameter of the expansion determined by the

Friedman type equation and $\rho^{(0)}$ the unperturbed density; these are the functions of $t$. If the period of the carrier fundamental wave is much smaller than the lifetime of the universe, the unperturbed universe may be assumed to evolve with $\tau$ and hence the Hubble constant can be put to $H=\varepsilon^{2} \tilde{H}$ where $\tilde{H}=$ $(d a / d \tau) / a$. Then, introducing the variable $\sigma=\ln \rho$ and the column vector $U=(\sigma, u, f)$ the basic equations can be written in the matrix form

$$
E \frac{\partial U}{\partial t}+A \frac{\partial U}{\partial x}+B+\varepsilon^{2} C=0
$$

where

$$
E=\left(\begin{array}{lll}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 0
\end{array}\right), \quad A=\left(\begin{array}{ccc}
u & 1 & 0 \\
\left(\frac{c}{a}\right)^{2} & u & 0 \\
0 & 0 & 1
\end{array}\right), \quad B=\left(\begin{array}{c}
0 \\
-\frac{f}{a^{2}} \\
4 \pi G\left(e^{\sigma}-\rho^{(0)}\right) a^{2}
\end{array}\right)
$$

and $C$ is the column vector $\tilde{H}(3,2 u, 0)$. The manipulation in $\S 3.2$ is available also for the present system, if the terms $C^{(0)} \delta_{l 0}$ and $\delta C^{(0)} U_{l}^{(0)}$ are added to the left-hand side of Eqs. $(3 \cdot 14)$ and (3.15) respectively. For the unperturbed state $U^{(0)}=\left(\sigma^{(0)}, 0,0\right)$ the dispersion relation of the fundamental mode takes the form

$$
\omega^{\prime 2}=\left(\frac{c}{a}\right)^{2} k^{2}-\omega_{0}^{2}
$$

where $\omega_{0}^{2}=4 \pi G \rho^{(0)}$ and the right eigenvector $R=\left(k, \omega^{\prime}, i\left(a \omega_{0}\right)^{2}\right)$ is used. Since $\operatorname{det} W_{0}$ vanishes also for the present example, $U_{0}^{(2)}$ is determined from the equation for $U_{0}^{(3)}$ or the first equation of the basic equations for $\alpha=2$, and $l=0$. The amplitude equation thus obtained is

$$
i \frac{\partial \varphi}{\partial \tau}+\frac{1}{2} \frac{\partial^{2} \omega^{\prime}}{\partial k^{2}} \frac{\partial^{2} \varphi}{\partial \xi^{2}}+\mu|\varphi|^{2} \varphi+\kappa \varphi=0
$$

where $\partial^{2} \omega^{\prime} / \partial k^{2}$ is given from the dispersion relation by

$$
\frac{\partial^{2} \omega^{\prime}}{\partial k^{2}}=-\left(\frac{c}{a}\right)^{2}\left(\frac{\omega_{0}}{\omega^{\prime}}\right) \frac{1}{\omega^{\prime}},
$$

$\mu=k^{2}\left(8 \omega^{\prime 4}+7 \omega_{0}^{2} \omega^{\prime 2}+5 \omega_{0}^{4}\right) / 6 \omega_{0}^{2} \omega^{\prime}$, and $\kappa=i\left[d \ln \omega^{\prime 1 / 2} / d \tau+\tilde{H}\right]=(i / 2) d \ln \left(\omega^{\prime} a^{2}\right) / \mathrm{d} \tau$. The unperturbed density is governed by the equation $d \sigma^{(0)} / \mathrm{d} \tau=-3 \tilde{H}$, which is integrated to lead $\rho^{(0)} a^{3}=$ constant. The transformation (3.20) takes the forms

$$
\phi=\frac{a}{c} \frac{\omega^{\prime}}{\omega_{0}}\left(2 \mu \omega^{\prime}\right)^{1 / 2} \varphi, \quad \sigma=-\frac{1}{2} \int\left(\frac{c}{a}\right)^{2}\left(\frac{\omega_{0}}{\omega^{\prime}}\right)^{2} \frac{d \tau}{\omega^{\prime}},
$$

for the present case and reduces the amplitude equation into the nonlinear Schrödinger type equation for $\phi$,

$$
i \frac{\partial \phi}{\partial \sigma}+\frac{\partial^{2} \phi}{\partial \xi^{2}}-|\phi|^{2} \phi+i \delta \phi=0
$$

in which $\delta$ is a real function of $\tau$ given by

$$
\delta=\frac{1}{2}\left\{\frac{d}{d \sigma} \ln \left[\left(\frac{c}{a}\right)^{2}\left(\frac{\omega_{0}}{\omega^{\prime}}\right) \frac{1}{\mu \omega^{\prime}}\right]-\frac{1}{2}\left(\frac{c}{a}\right)^{2}\left(\frac{\omega_{0}}{\omega^{\prime}}\right) \frac{1}{\omega^{\prime}} \frac{d}{d \sigma} \ln \left(\omega^{\prime} a^{2}\right)\right\} .
$$

For a steady universe, $\delta$ vanishes identically, so, the equation of $\phi$ reduces to the K-d-V equation in an asymptotic sense ${ }^{13)}$ and hence has the solitary wave solution. In the unsteady universe, the sign of $\delta$ may affect the stability of the solitons.

## § 4. Modulation of weakly unstable quasi monochromatic wave

### 4.1 Modulated waves with complex frequency?

In the dissipative or unsteady media, linear dispersion relation has usually complex roots, so, the theory of the amplitude modulation presented so far cannot be applied directly to such system. In this section, the perturbation method is extended to a system of the equations which, when linearized, has a monochromatic wave solution with complex frequency $\bar{\omega}$ of a small imaginary part $\omega_{i}$. Since the complex frequency is often connected with the dissipative or unsteady medium, the unperturbed state is assumed unsteady and the strained variables for Case I (b) with $a=1$ are used, while the model equation is

$$
\frac{\partial U}{\partial t}+A \frac{\partial U}{\partial x}+B+C \frac{\partial^{2} U}{\partial x^{2}}=0
$$

where $A=A(U, \nu), B=B(U, \nu)$ and $C=C(U, \nu)$ are the $n \times n$ matrix functions of $U$ and a parameter $\nu$ characterizing the imaginary part of the frequency. Let us assume that $\omega_{i}$ is of the order of $\varepsilon^{2}$, then one may put $\nu=\varepsilon^{2} \bar{\nu}$ where $\bar{\nu}$ is of the order of unity and expand $\omega_{i}$ in powers of $\nu$ as $\omega_{i}=\omega_{i}\left(k, U^{(0)}, \nu\right)=$ $\varepsilon^{2}\left(\partial \omega_{i} / \partial \nu\right)_{\nu=0} \bar{\nu}+O\left(\varepsilon^{4}\right)$. Reductive perturbation method for the case $\omega_{i}=O(\varepsilon)$ seems difficult to obtain self-consistent solution. The expansions of $A, B$ and $C$ must be carried out with respect to $U$ and $\nu$;

$$
\begin{aligned}
A= & A^{(0)}+\varepsilon \nabla A^{(0)} \cdot U^{(1)}+\varepsilon^{2}\left(\nabla A^{(0)} \cdot U^{(2)}+\frac{1}{2} \nabla \nabla A^{(0)}: U^{(1)} U^{(1)}+A_{\nu}^{(0)}\right)+\cdots, \\
B= & \varepsilon \delta B^{(0)} U^{(1)}+\varepsilon^{2}\left(\delta B^{(0)} U^{(2)}+\frac{1}{2} \nabla \delta B^{(0)} \cdot U^{(1)} U^{(1)}+B_{\nu}^{(0)}\right) \\
& +\varepsilon^{3}\left(\delta B^{(0)} U^{(3)}+\nabla \delta B^{(0)} \cdot U^{(1)} U^{(2)}\right. \\
& \left.\quad+\frac{1}{6} \nabla \nabla \delta B^{(0)}: U^{(1)} U^{(1)} U^{(1)}+\delta B_{\nu}^{(0)} U^{(1)}\right)+\cdots,
\end{aligned}
$$

and the expansion of $C$ similar to $A$, here notations $\nabla A^{(0)} \cdot U^{(1)}=\sum_{i=1}^{n}(\partial A \mid$
 and so on are employed. The harmonics for the expansion of $U^{(\alpha)}$ are defined in terms of the real part $\omega$ of $\bar{\omega} ; U^{(\alpha)}=\sum_{l=-\infty}^{\infty} U_{l}^{(\alpha)} \exp [i l(k x-\omega t)]$. Substituting these expansions and the strained variables into Eq. (4.1) and equating each power of $\varepsilon$ of the $l$ th harmonics to zero, we have

$$
\begin{align*}
& W_{l} U_{l}^{(1)}=0, \\
& W_{l} U_{l}^{(2)}+\left(-\lambda_{0} I+A^{(0)}+2 i l k C^{(0)}\right) \frac{\partial U_{l}^{(1)}}{\partial \xi}+\sum_{l^{\prime}}\left(i l^{\prime} k \nabla A^{(0)} \cdot U_{l-l^{\prime}}^{(1)}\right. \\
& \left.+\frac{1}{2} \delta B^{(0)} \cdot U_{l-l^{\prime}}^{(1)}-l^{\prime 2} k^{2} \nabla C^{(0)} \cdot U_{l-l^{\prime}}^{(1)}\right) U_{l^{\prime}}^{(1)}+\left(B_{l}^{(0)}+\frac{d U^{(0)}}{d \tau}\right) \delta_{l 0}=0, \\
& W_{l} U_{l}^{(3)}+\left(-\lambda_{0} I+A^{(0)}+2 i l k C^{(0)}\right) \frac{\partial U_{l}^{(2)}}{\partial \xi}+\sum_{l^{\prime}}\left(\nabla A^{(0)}+2 i l^{\prime} k \nabla C^{(0)}\right) \\
& \times U_{l-l^{\prime}} \frac{\partial U_{l}^{(1)}}{\partial \xi}+i k\left[\sum_{l^{\prime}} l^{\prime} \nabla A^{(0)} \cdot\left(U_{l-l^{\prime}}^{(1)} U_{l^{\prime}}^{(2)}+U_{l-l^{\prime}}^{(2)} U_{l^{\prime}}^{(1)}\right)\right. \\
& \left.+\frac{1}{2} \sum_{l^{\prime}} \sum_{l^{\prime \prime}} l^{\prime \prime} \nabla \nabla A^{(0)}: U_{l-l^{\prime}-l^{\prime \prime}}^{(1)} U_{l^{\prime}}^{(1)} U_{l^{\prime}}^{(1)}\right]+\sum_{l^{\prime}} \nabla \delta B^{(0)} \cdot U_{l-l^{\prime}}^{(1)} U_{l^{\prime}}^{(2)} \\
& +\frac{1}{6} \sum_{l^{\prime}} \sum_{l^{\prime \prime}} \nabla \nabla \delta B^{(0)}: U_{l-l^{\prime}-l^{\prime \prime}}^{(1)} U_{l^{\prime}}^{(1)} U_{l^{\prime \prime}}^{(1)}-k^{2}\left[\sum _ { l ^ { \prime } } l ^ { \prime } \nabla C ^ { ( 0 ) } \cdot \left(U_{l-l^{\prime}}^{(1)} U_{l^{\prime}}^{(2)}\right.\right. \\
& \left.\left.+U_{l^{\prime}-l^{\prime}}^{(2)} U_{l^{\prime}}^{(1)}\right)-\frac{1}{2} \sum_{l^{\prime}} \sum_{l^{\prime \prime}} l^{\prime \prime 2} \nabla \nabla C^{(0)}: U_{l-l^{\prime}-l^{\prime \prime}}^{(0)} U_{l^{\prime}}^{(1)} U_{l^{\prime \prime}}^{(1)}\right]+\frac{\partial U_{l}^{(1)}}{\partial \tau} \\
& +C^{(0)} \frac{\partial^{2} U_{l}^{(1)}}{\partial \xi^{2}}+\left(i l k A_{\nu}^{(0)}+\delta B_{l^{(0)}}^{(0)}-l^{2} k^{2} C_{\left.l^{(0)}\right)}^{(0)} U_{l}^{(1)}=0,\right.
\end{align*}
$$

where the matrix $W_{l}$ is defined by

$$
W_{l}=-i l \omega^{\prime} I+i l k A^{(0)}+\delta B^{(0)}-l^{2} k^{2} C^{(0)},
$$

and $\omega^{\prime}=\partial(\tau \omega) / \partial \tau$. Let us consider the modulation of the fundamental mode with $l=1$ and assume that $\operatorname{det} W_{ \pm 1}=0$, and $\operatorname{det} W_{l} \neq 0$ for $|l| \neq 1$, then, Eq. (4•2) is solved for $U^{(1)}$ as,

$$
\begin{align*}
& U_{1}^{(1)}=R(\tau) \varphi(\xi, \eta), \\
& U_{l}^{(1)}=0 \quad \text { for } \quad|l| \neq 1,
\end{align*}
$$

where $R$ is a right eigenvector of $W_{1}$ and $\varphi$ is a scalar function. In virtue of these solutions and an identity $-\lambda_{0} I+A^{(0)}+2 i k C^{(0)}=-i\left(\partial W_{1} / \partial k\right)$, the component with $l=1$ of Eq. (4.3) yields

$$
W_{1} U_{1}^{(2)}+\left(-\lambda_{0} I+A^{(0)}+2 i k C^{(0)}\right) \frac{\partial U_{1}^{(1)}}{\partial \xi}=W_{1}\left(U_{1}^{(2)}+i \frac{\partial R}{\partial k} \frac{\partial \varphi}{\partial \xi}\right)=0,
$$

the solution $U_{1}^{(2)}$ of which is, in terms of an arbitrary function $\tilde{\psi}(\xi, \tau)$,

$$
U_{1}^{(2)}=R \tilde{\psi}-i \frac{\partial R}{\partial k} \frac{\partial \varphi}{\partial \xi} .
$$

Further, the component with $l=0$ of Eq. (4•3) becomes

$$
\begin{aligned}
& W_{0} U_{0}^{(2)}+i k\left(\nabla A^{(0)} \cdot U_{-1}^{(1)} U_{1}^{(1)}-\text { c.c. }\right)+\frac{1}{2}\left(\delta B^{(0)} \cdot U_{-1}^{(1)} U_{1}^{(1)}+\text { c.c. }\right) \\
& \\
& \quad-k^{2}\left(\nabla C^{(0)} \cdot U_{-1}^{(1)} U_{1}^{(1)}+\text { c.c. }\right)+B_{\Sigma}^{(0)}+\frac{d U^{(0)}}{d \tau}=0
\end{aligned}
$$

and gives $U_{0}^{(2)}$ in the form

$$
U_{0}^{(2)}=R_{0}^{(2)}|\varphi|^{2}+V,
$$

where $R_{0}^{(2)}$ and $V$ are vectors satisfying

$$
\begin{aligned}
& W_{0} R_{0}^{(2)}+i k\left(\nabla A^{(0)} \cdot R^{*} R-\text { c.c. }\right)+\frac{1}{2}\left(\delta B^{(0)} \cdot R^{*} R+\text { c.c. }\right) \\
& \quad-k^{2}\left(\nabla C^{(0)} \cdot R^{*} R+\text { c.c. }\right)=0
\end{aligned}
$$

and

$$
W_{0} V+B_{\nu}^{(0)}+\frac{d U^{(0)}}{d \tau}=0
$$

respectively. Other components of $U^{(2)}$ are likewise obtained from Eqs.(4.3) and (4.6);

$$
\begin{align*}
& U_{2}^{(2)}=R_{2}^{(2)} \varphi^{2}, \\
& U_{l}^{(2)}=0 \quad \text { for } \quad|l| \geq 3
\end{align*}
$$

with

$$
R_{2}^{(2)}=-W_{2}^{-1}\left[i k \nabla A^{(0)} \cdot R R+\frac{1}{2} \nabla \delta B^{(0)} \cdot R R-k^{2} \nabla C^{(0)} \cdot R R\right] .
$$

Finally, Eq. $(4 \cdot 4)$ for $l=1$ gives the equation of $\varphi$ as the condition for the existence of the solution $U^{(3)}$ : Multiplying the left eigenvector $L$ of $W_{1}$ and substituting the above solutions into Eq. (4.4) with $l=1$, we get, after some rearrangement,

$$
i \frac{\partial \varphi}{\partial \tau}+\frac{1}{2} \frac{\partial^{2} \omega^{\prime}}{\partial k^{2}} \frac{\partial^{2} \varphi}{\partial \xi^{2}}+\mu|\varphi|^{2} \varphi+\kappa \varphi=0
$$

here $\mu=i\left(\mu_{a}+\mu_{b}+\mu_{c}\right) / L R$;

$$
\mu_{a}=i k L\left[2\left(\nabla A^{(0)} \cdot R^{*}\right) R_{2}^{(2)}-\left(\nabla A^{(0)} \cdot R_{2}^{(2)}\right) R^{*}+\left(\nabla A^{(0)} \cdot R_{0}^{(2)}\right) R\right.
$$

$$
\begin{aligned}
& \left.+\left(\nabla \nabla A^{(0)}: R R^{*}\right) R-\frac{1}{2}\left(\nabla \nabla A^{(0)}: R R\right) R^{*}\right], \\
\mu_{b}= & L\left[\nabla \delta B^{(0)} \cdot R R_{0}^{(2)}+\nabla \delta B^{(0)} \cdot R^{*} R_{2}^{(2)}+\frac{1}{2} \nabla \nabla \delta B^{(0)}: R R^{*} R\right], \\
\mu_{c}= & -k^{2} L\left[4\left(\nabla C^{(0)} \cdot R^{*}\right) R_{2}^{(2)}+\left(\nabla C^{(0)} \cdot R_{2}^{(2)}\right) R+\left(\nabla C^{(0)} \cdot R_{0}^{(2)}\right) R\right. \\
& \left.+\left(\nabla \nabla C^{(0)}: R R^{*}\right) R+\frac{1}{2}\left(\nabla \nabla C^{(0)}: R R\right) R^{*}\right]
\end{aligned}
$$

and

$$
\begin{aligned}
\kappa= & i L\left[i k A_{\nu}^{(0)} R+\delta B_{\nu}^{(0)} R-k^{2} C_{\nu}^{(0)} R+\frac{d R}{d \tau}\right. \\
& \left.+i k\left(\nabla A^{(0)} \cdot V\right) R+\left(\nabla \delta B^{(0)} \cdot R\right) V-k^{2}\left(\nabla C^{(0)} \cdot V\right) R\right] / L R .
\end{aligned}
$$

By means of the transformation (3•20), we again have a nonlinear Schrödinger type equation in an external potential. In the next section, the convective wave, that is, the wave in a fluid heated from below is considered. Since the motion of the fluid is essentially two dimensional, the theory in this section must be extended to the multi-spatial dimensions. In $m$-dimensional space, the vectors $k, x, \lambda_{0}$ and $\xi$ have $m$ components and the matrices $A$ and $C$ are replaced by the vector valued matrix $A_{i}$ and the tensor valued matrix $C_{i j}$ respectively, where suffices $i$ and $j$ run from $l$ to $m$.

### 4.2 Example: Thermal convective mode7)

In the Boussinesq approximation, the motion of an incompressible, viscous fluid heated from below is governed by the set of equations

$$
\begin{aligned}
& \frac{\partial \boldsymbol{V}}{\partial t}+(\boldsymbol{V} \cdot \boldsymbol{\nabla}) \boldsymbol{V}+\frac{1}{\rho_{0}} \nabla p+a \boldsymbol{g} T-\pi \nabla^{2} \boldsymbol{V}=0, \\
& \frac{\partial T}{\partial t}+(\boldsymbol{V} \cdot \boldsymbol{\nabla}) T+d w-\kappa \nabla^{2} T=0, \\
& \boldsymbol{\nabla} \cdot \boldsymbol{V}=0
\end{aligned}
$$

where $\boldsymbol{V}=(u, v, w)$ is the velocity, $p$ the pressure, $\rho_{0}$ the constant density, $a$ the thermal expansion coefficient, $\boldsymbol{g}=(0,0,-g)$ the gravitational acceleration in the negative $z$-direction, $T$ the deviation of the temperature from the linear profile and, $\pi$ and $\kappa$ are respectively the viscosity and diffusivity constants. The constant $d$ is defined by $d=T_{0}\left(d \ln T_{0} / d \ln p_{0}-\nabla_{a}\right) d \ln p_{0} / d z$, which is a measure of the excess of the temperature gradient over the adiabatic one $\nabla_{a} \equiv$ ㅡ $(\partial \ln s / \partial \ln p)_{0} /(\partial \ln s / \partial \ln T)_{0}$. In the present calculation, we assume that $d$ is positive hence the convective mode is stable in the linear theory. The pressure is eliminated by the equation of incompressibility $\nabla \cdot \boldsymbol{V}=0$. In fact, the
dispersion relation for $\bar{\omega}$ and $k$ takes the form, if one put $\nabla p=\rho_{0} \beta \nabla T$,

$$
\left(\pi k^{2}-i \bar{\omega}\right)\left(\kappa k^{2}-i \bar{\omega}\right)-d\left(i \beta k_{z}-\alpha g\right)=0
$$

and the corresponding right eigenvector $R \propto\left(\beta k_{x}, \beta k_{y}, \beta k_{z}+i a g, \bar{\omega}+i \pi k^{2}\right)$ for $U=(u, v, w, T)$, are substituted in the equation $\nabla \cdot V=0$ to give $\beta=-i a g k_{z} / k^{2}$, where $k^{2}=k_{\perp}^{2}+k_{z}^{2}$ and $k_{\perp}^{2}=k_{x}^{2}+k_{y}^{2}$. When $\pi$ and $\kappa$ are of the order of $\varepsilon^{2}$, one can put $\pi=\varepsilon^{2} \pi^{\prime}$ and $\kappa=\varepsilon^{2} \kappa^{\prime}$ which yields the real and the imaginary part of $\bar{\omega}$ as

$$
\omega=(\alpha d g)^{1 / 2}-\frac{k_{\perp}}{k}, \quad \omega_{i}=-\varepsilon^{2} \frac{1}{2}\left(\pi^{\prime}+\kappa^{\prime}\right) k^{2}
$$

respectively, if the terms of $O\left(\varepsilon^{4}\right)$ are neglected. Because of the equation $\operatorname{det} W_{0}=0$, the components of $U_{0}^{(2)}$ cannot completely be determined from the equation of $O\left(\varepsilon^{2}\right)$ but they are completely obtained if Eq. $(4 \cdot 4)$ for $l=0$ is supplemented. The amplitude equation thus obtained for $R=\left(k_{x} k_{z}, k_{y} k_{z},-k_{\perp}^{2}\right.$, $\left.i(d / a g)^{1 / 2} k k_{\perp}\right)$ takes the form

$$
i \frac{\partial \varphi}{\partial \tau}+\frac{1}{2} \Sigma_{i, j} \frac{\partial^{2} \omega}{\partial k_{i} \partial k_{j}} \frac{\partial^{2} \varphi}{\partial \xi_{i} \partial \xi_{j}}-\frac{\left(k k_{\perp}\right)^{3}}{(a d g)^{1 / 2}}|\varphi|^{2} \varphi+\frac{i}{2}\left(\pi^{\prime}+\kappa^{\prime}\right) k^{2} \varphi=0
$$

where $\partial^{2} \omega / \partial k_{i} \partial k_{j}=\partial \lambda_{i} / \partial k_{j}$ can be presented by a $2 \times 2$ tensor

$$
\begin{aligned}
& \frac{\partial \lambda_{\perp}}{\partial k_{\perp}}=-3(\alpha d g)^{1 / 2} \frac{k_{z}^{2} k_{\perp}}{k^{5}}, \quad \frac{\partial \lambda_{z}}{\partial k_{z}}=(\alpha d g)^{1 / 2} \frac{k_{\perp}}{k^{5}}\left(2 k_{z}^{2}-k_{\perp}^{2}\right) \\
& \frac{\partial \lambda_{\perp}}{\partial k_{z}}=\frac{\partial \lambda_{z}}{\partial k_{\perp}}=(\alpha d g)^{1 / 2} \frac{k_{z}}{k^{5}}\left(2 k_{\perp}^{2}-k_{z}^{2}\right)
\end{aligned}
$$

As easily verified, the tensor $\partial^{2} \omega / \partial k_{i} \partial k_{j}$ is negative definite and hence has the same sign with the coefficient of the nonlinear term. Thus, the finite amplitude plane wave solution is unstable and decays into solitons.

In concluding this paper, we may emphasize the importance of the extensive studies of the equations like Eqs. $(2 \cdot 9 \mathrm{~b})$ and $(2 \cdot 32)$ because there are a lot of problems which can be treated by the reductive perturbation method for non-uniform media and provide fruitful knowledge on the nature and applications.

## References

1) J. K. Perring and T. H. R. Skyme, Nucl. Phys. 31 (1962), 550.
N. J. Zabusky and M. D. Kruskal, Phys. Rev. Letters 15 (1965), 240.
2) R. M. Miura, C. S. Gardner and M. D. Kruskal, J. Math. Phys. 9 (1968), 1204.
V. E. Zakharov and A. B. Shabat, Soviet Phys.-JETP 34 (1972), 62.
3) G. B. Whitham, Proc. Roy. Soc. 299A (1967), 2.
4) K. Stewartson and J. T. Stuart, J. Fluid Mech. 48 (1971), 529.
L. M. Houking and K. Stewartson, Proc. Roy. Soc. 326A (1972), 289.
5) N. Asano and H. Ono, J. Phys. Soc. Japan 31 (1971), 1830.
6) T. Taniuti and C. C. Wei, J. Phys. Soc. Japan 24 (1968), 941.
7) N. Asano, J. Phys. Soc. Japan 36 (1974), 861.
8) D. H. Peregrine, J. Fluid Mech. 27 (1967), 815.
9) N. Asano, J. Phys. Soc. Japan 29 (1969), 220.
10) K. Nozaki and T. Taniuti, DPNU-14 June (1973).
11) N. Asano, J. Fluid Mech. 46 (1971), 111.
12) T. Kihara, Publ. Astron. Soc. Japan 19 (1967), 121.
13) N. Asano, T. Taniuti and N. Yajima, J. Math. Phys. 10 (1969), 2020.
