

Background Risk and Trading in a Full-Information Rational Expectations Economy

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April 20, 2010

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Abstract

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In this paper we assume that investors have the same information, but trade due to the evolution of their non-market wealth. In our formulation, investors rebalance their portfolios in response to changes in their expected non-market wealth, and hence trade. We assume an incomplete market in which risky non-market wealth is non-hedgeable and independent of the market risk and thus represents an additive background risk. Investors who experience positive shocks to their expected wealth buy more stocks from those who experience less positive shocks.

1 Introduction

It has long been a challenge for financial economists to explain trading in the context of rational expectations asset pricing models. For example, in the complete markets Arrow-Debreu model, agents choose state-contingent claims on the initial date, but do not trade at subsequent dates, since they have already purchased claims that hedge against various future outcomes; thus, there is no need for them to adjust their portfolio holdings as the state of the world is revealed. This inability to explain trading in a rational model flies in the face of evidence that there is a large volume of trading in various securities: bonds, stocks, and increasingly in various types of contingent claims, such as options and futures contracts.

Several attempts have been made in the literature in the past to explain trading by relaxing some of the assumptions of completeness of markets and information available to agents in the economy. One possibility is that when investors have asymmetric information, this gives them an incentive to trade in order to profit from that information. However, as Grossman and Stiglitz (1980) point out, the mere act of trading reveals the information possessed by a particular agent and this gets reflected in market prices. While there may be some “sand in the gears” introduced if the process of expectations formation is noisy, the central intuition that prices reflect private information still prevails, reducing the motivation to trade substantially.

This argument was taken one step further by Milgrom and Stokey (1982) who argue that when the agents begin with a Pareto optimal allocation relative to their prior beliefs, they do not trade upon receiving private information, even at equilibria that are less than fully revealing, since “the information conveyed by price changes swamps each traders private information.” This surprisingly general result arises because if the initial allocation is Pareto optimal, there is no valid insurance motive for trading. The willingness of other traders to take the opposite side implies at least to one trader that his own bet is unfavorable. Hence no trade is acceptable to all traders. The Milgrom and Stokey propositions rely on two crucial assumptions: a) that it is common knowledge that when a trade occurs it is feasible and acceptable to all agents, and b) the agents beliefs are concordant, i.e., that they agree about how the information should be interpreted.

Another strand of the literature that has provided a motivation for trading is on market micro-structure, most prominently by Kyle (1985) and Glosten and Milgrom (1985). These models try to explain the bid-offer spread in markets by appealing to asymmetric information. However, a crucial assumption in such models is the existence of noise traders, who trade for liquidity reasons, and these are not explicitly modeled. Furthermore, it is unclear why in such models, investors trade for liquidity reasons in risky securities such as stocks,

rather than trading bonds, unless some market imperfection is assumed. In the Milgrom and Stokey sense, it must be the case that the allocation in these models is not ex-ante Pareto optimal, and/or that the beliefs are not concordant.

The broad conclusion from the information-based literature on trading is that the Milgrom and Stokey “no-trade” result will obtain, unless there is some market imperfection, significant deviation from rational expectations equilibria or an exogenous reason to trade, such as liquidity motivations.

In this paper, we explore an alternative motivation for trading, which is the existence of non-marketable wealth. Non-marketable wealth may take many forms, but the most obvious example is wealth arising from labor income. Human capital, which is the value of future labor income, has been shown in many studies, both theoretical and empirical, to have an influence on portfolio demand. Another example is housing wealth, which is a significant component of the portfolios of households. Again, there is an extensive literature documenting how housing wealth affects portfolio choice and, in turn, feeds back on to the equilibrium prices of traded assets. The effect of non-market wealth is that it alters the agents’ demand for the traded assets. An early example of this distortion is the work of Bodie, Merton and Samuelson (1992) in the context of non-stochastic, positive non-marketable wealth for an agent with constant relative risk aversion. They show that this agent acts much like another agent with a lower, but increasing relative risk aversion.

The problem gets more complex when the non-marketable wealth has stochastic properties. There is an extensive literature on background risk that studies the portfolio behavior of agents with such non-marketable wealth, whose future cash flows are also stochastic. For most common utility functions, the existence of background risk makes agents more risk averse and hence reduces their demand for risky securities. [See, for example, Gollier and Pratt (1996), Kimball (1993) and Eekhoudt, Gollier and Schlesinger (1996).] The natural question is how the changes in the agents’ portfolio decisions affect the portfolio demand and sharing rules of the marketable securities in equilibrium, a problem first analyzed by Franke, Stapleton and Subrahmanyam (1998) [FSS].

We extend this framework to consider a multi-period version of the FSS framework. Following the outcome of the background risk in the intermediate period, agents adjust their holdings of the marketable securities, to be in line with their new level of derived risk aversion in the presence of the updated distribution of background wealth. If the outcomes of the background risk are heterogeneous across agents, it creates a motivation for trading, as different agents may wish to adjust their portfolio holdings in opposite directions. We explore this simple intuition formally for investors with constant relative risk aversion in our analysis.

Section 2 presents the set up of the model and derives the portfolio demand for traded state-contingent claims. Section 3 describes the evolution of the background risk over time. Section 4 derives optimal demand in the special case where all uncertainty of background risk is resolved at time 1. Section 5 generalizes the results using an approximation. Section 6 presents our conclusions.

2 A Single-Period Model

In this section we derive the optimal demand for contingent claims for agents in a single-period equilibrium economy. The results will provide the basic building block for our multi-period trading model in later sections. The set-up of the model is similar to that in Franke, Stapleton and Subrahmanyam (1998) [FSS].¹ As in FSS, we assume that all agents maximize the expected utility of wealth, w at the end of a single period. For agent i , $w_i = x_i + y_i$, where x_i is a set of claims on a single aggregate market cash flow X_a and y_i is the nonmarketable income, e.g. labor income. In general, the non-marketable income $y_i = a_i + e_i$, where a_i is a constant representing the expected value of non-market income, and e_i is an independent, zero-mean background risk. Each agent solves the following maximization problem:

$$\max_{x_i} E_{X_a}[E_e[u_i(w_i)]], \text{ s.t. } E[\phi(X_a)x_i] = E[\phi(X_a)\hat{x}_{i0}], \quad (1)$$

given an initial endowment of x , \hat{x}_{i0} . In (1), $\phi(X_a)$ is the forward pricing kernel. The budget constraint states that the forward price of the chosen portfolio of claims has to equal the forward value of the endowed claims. In FSS, agents have utility functions $u_i(w_i)$ which belong to the HARA class, excluding the exponential function. Here, we assume essentially the same setup with

$$u_i(w_i) = \frac{w_i^{1-\gamma_i}}{1-\gamma_i}. \quad (2)$$

where γ_i is the coefficient of relative risk aversion. Utility for wealth is a power function, exhibiting constant relative risk aversion, but the derived utility for x_i is of the HARA form, when the background risk e_i does not exist.²

¹However, we cannot simply use the results in FSS since in that paper they do not solve for the Lagrangian multipliers, see λ_i below. Hence their results show that some investors buy and some sell contingent claims but do not show how many are bought.

²Utility is of the Hypobolic Absolute Risk Averse (HARA) class if

$$u_i(w_i) = \frac{(w_i + a_i)^{1-\gamma_i}}{1-\gamma_i}$$

Let λ_i be the Lagrangian multiplier associated with the budget constraint of investor i . The Lagrangian multiplier is then:

$$L = u_i(w_i) + \lambda_i(E[\phi(X_a)\hat{x}_{i0}] - E[\phi(X_a)x_i]). \quad (3)$$

It follows that the first order condition of the optimization problem is:

$$E_e[(x_i + a_i + e_i)^{-\gamma_i}] = \lambda_i\phi(X_a). \quad (4)$$

Following Kimball (1990), we can define the precautionary premium $\psi_i(x_i)$ by the relation

$$E_e[(x_i + a_i + e_i)^{-\gamma_i}] \equiv [x_i + a_i - \psi_i(x_i)]^{-\gamma_i} \quad (5)$$

Hence $(x_i + a_i - \psi_i)^{-\gamma_i}$ is the certainty equivalent of $E_e(x_i + a_i + e_i)^{-\gamma_i}$. Note that ψ_i itself will be a function of x_i and also depends on the distribution of e_i . More specifically, the function $\psi(\cdot)$ is decreasing and convex. The above result differs slightly from FSS in that we allow the mean of the background risk to be non-zero. This difference is essential for our setting because in the dynamic case, analyzed in sections 3 and 4, the mean of the background risk will be non-zero after the initial date.

Substituting the above certainty equivalence into the first order condition:

$$[x_i + a_i - \psi_i]^{-\gamma} = \lambda_i\phi(X_a). \quad (6)$$

and it follows that the demand for contingent claims is given by:

$$x_i = (\lambda_i)^{-1/\gamma_i}\phi(X_a)^{-1/\gamma_i} - a_i + \psi_i. \quad (7)$$

The optimal demand consists of three separate parts. The first term is the demand if the expected non-marketable income is zero and the precautionary premium is also zero (i.e. the background risk is zero). When the expected non-marketable income is positive (negative) the demand is reduced (increased) in each state to compensate. This explains the second term. The third term adjusts for the effect of the background risk.

To obtain the optimal demand, we need to solve for λ_i and the pricing kernel $\phi(X_a)$. It turns out that it is more convenient to use the per capita term X , instead of the aggregate X_a . Using the market clearing condition $\frac{1}{I}\sum_i x_i = X$, where I is the number of agents and assuming $\gamma_i = \gamma$ for all i , we have:³

$$X = \lambda^{-1/\gamma}\phi(X)^{-1/\gamma} - A + \psi, \quad (8)$$

³One could still keep the general form of different γ_i at this stage, but the resulting expression will be quite complicated.

where

$$\psi = \frac{1}{I} \sum_i \psi_i, \quad (9)$$

$$A = \frac{1}{I} \sum_i a_i, \quad (10)$$

$$\lambda^{-1/\gamma} = \frac{1}{I} \sum_i \lambda_i^{-1/\gamma}. \quad (11)$$

Note that the aggregate ψ is a function of the state indexed by X and depends also on the distribution $\{e_i\}_{i=1,\dots,n}$. This is essentially a representative agent version of equation (7), assuming that all the γ_i 's are the same. note however that we do *not* assume that the background risks are identical across all agents. Indeed, in the subsequent analysis we will use the fact that a_i and ψ_i vary across agents to create an incentive to trade. Initially, the agents are all identical in terms of their original risk aversion. However, the realization of the background risks can differ and consequently the *derived* risk aversion can be different. This is the basic intuition behind the trading in our model.

It then follows, solving (8) for ϕ we find

$$\phi(X) = (X + A - \psi)^{-\gamma} \lambda^{-1}. \quad (12)$$

now, substituting the solution of x_i in (7) above back into the individual budget constraint

$$E[\phi(X)x_i] = E[\phi(X)\hat{x}_{i0}],$$

it follows that:

$$\begin{aligned} E[\phi(X)\hat{x}_{i0}] &= E\{\phi(X)[\lambda_i^{-1/\gamma} \phi(X)^{-1/\gamma} - a_i + \psi_i]\} \\ &= \lambda_i^{-1/\gamma} E[\phi(X)^{1-\frac{1}{\gamma}}] - E[\phi(X)a_i] + E[\phi(X)\psi_i] \end{aligned}$$

Then we obtain the following:

$$\lambda_i^{-1/\gamma} = \frac{E[\phi(X)(\hat{x}_{i0} + a_i - \psi_i)]}{E[\phi(X)^{1-\frac{1}{\gamma}}]} \quad (13)$$

$$\text{or } \lambda_i = \left\{ \frac{E[\phi(X)(\hat{x}_{i0} + a_i - \psi_i)]}{E[\phi(X)^{1-\frac{1}{\gamma}}]} \right\}^{-\gamma} \quad (14)$$

Hence, the optimal individual investor demand is (using equation (12)):

$$x_i = \frac{E[\phi(X)(\hat{x}_{i0} + a_i - \psi_i)]}{E[\phi(X)^{1-\frac{1}{\gamma}}]} \phi(X)^{-\frac{1}{\gamma}} - a_i + \psi_i(x_i) \quad (15)$$

$$= \frac{E[(X + A - \psi)^{-\gamma}(\hat{x}_{i0} + a_i - \psi_i)]}{E[(X + A - \psi)^{1-\gamma}} (X + A - \psi) - a_i + \psi_i. \quad (16)$$

The expression for the demand for contingent claims in (16) is complex. If there were no background risk for all investors, ψ would be zero and x_i would be linear in X . However, in general both ψ and ψ_i are convex functions implying a non-linear demand function. Also, the optimal demand is implicit since ψ_i is a function of x_i for each i .

Again, the optimal individual demand consists of three parts. The first term is linear in per capita market cash flow. The coefficient depends on the expectation of the individual precautionary premium. The second term is the adjustment for the non-zero expected background risk, a_i . The third term is the adjustment for individual precautionary premium.

3 The Evolution of Background Risk Over Time

So far, we have assumed that agents face a background risk e_i which is resolved at the end of a single period. As in FSS, e_i has a zero mean and is independent of the market cash flow, X . We now introduce a multiperiod model in which the risk, e_i , evolves over time. This is required to study trading volume in the following sections, since trading is essentially an intertemporal issue.

There are three dates, $t = 0, 1, 2$ in the model. These are represented in Figure 1 below.

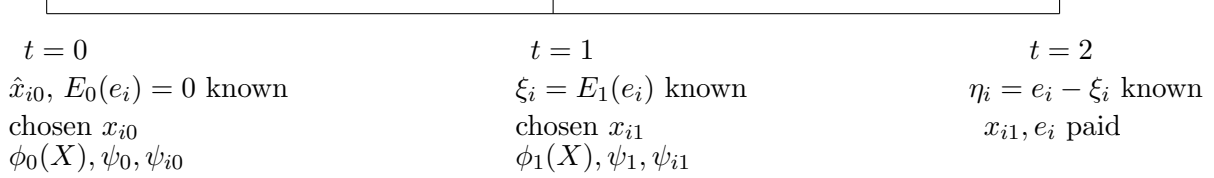


Fig 1. The timeline

At time $t = 0$, each agent is endowed with \hat{x}_{i0} , which is a portfolio of marketable contingent claims. Also, at $t = 0$ each agent knows about the distribution of the background risk e_i , which will be fully revealed at $t = 2$. The agent chooses a portfolio of marketable contingent claims, at time 0, x_{i0} , to maximize the expected utility of the wealth at time $t = 2$. The maximization is given the pricing kernel $\phi_0(X)$ and the precautionary premium ψ_{i0} . Note that, all the payoffs, which include the payoff from the marketable contingent claims and the background risk e_i , are at $t = 2$.

At time $t = 1$, the agent receives information about her background risk, ξ_i , and revises her expectation of e_i to $E_1(e_i) = \xi_i$. Given this information and the revised distribution of the background risk e_i , she chooses a new portfolio of contingent claims given an updated pricing kernel $\phi_1(X)$ and a revised precautionary premium, ψ_{i1} . Then at time $t = 2$, the agent receives more information about her background risk, η_i and both payments x_{i1} and e_i are paid to the agent.

Note that, in this model, the agent knows about part of the final payoff from the background risk at $t = 1$. Thus, ξ_i is the conditional expectation at $t = 1$ of the background risk at $t = 2$. However we should emphasize that even though the agent knows about ξ_i , she cannot use it directly to trade the contingent claims because it is non-marketable. However, the agent does change her optimal portfolio holdings of marketable claims at $t = 1$, given the new information.

We now assume that there are two groups of agents, which are indexed as $i = m, n$. Without loss of generality, we assume that the two groups are of equal size. For simplicity, we also assume that ex ante at time $t = 0$, the distributions of e_i are the same for the two groups $i = m, n$.

The trading that takes place at $t = 1$ depends on the cross-sectional realization of ξ_i across the agents. If it happens that the outcome ξ_i is the same for both groups of investors, there will be no trade. However, if the realization of ξ_i 's are different for the two groups, then there will be trade. We proceed by first considering a special case where the precautionary premia at $t = 1$ are zero for all investors. This is the case where there is full resolution of uncertainty about e_i at $t = 1$.

4 A Special Case: Full Resolution of Background Risk at Time 1

In this section, we investigate the case where all the uncertainty of ε_i is resolved at $t = 1$. As discussed above, in the general case the demand for contingent claims is an implicit function. This is due to the fact that the demand is a function of the precautionary premium, but the precautionary premium is a function of the demand itself. However, in the special case where all the uncertainty of the background risk ε_i is resolved at $t = 1$, the precautionary premium, $\psi_i(x_i)$, is zero at time 1. So, in this case, there is an explicit solution for the optimal demand at time 1.

At time 0, all the investors are identical, and only differ in the resolution of the uncertainty of e_i . Since the investors are identical at $t = 0$ and since ε_i has the same distribution for all i , the investors must hold the same portfolios at $t = 0$. That implies that the initial demand $x_i = X$, since X is the average allocation of claims across investors.

We now generate this to multi-state case. namely the realization of ξ_i can take multi-values: $\xi_i = a_k, k = 1, \dots, K$. The only difference is when the realization of ξ_i across the two agents are heterogeneous. Specifically, we assume $(\xi_m, \xi_n) = (a_m, a_n)$, where $a_m, a_n \in \{a_1, a_2, \dots, a_K\}$. The average per capita realization of background risk is $A \equiv \frac{1}{2}(a_m + a_n)$.

The demand of two types at time $t = 1$ are:

$$x_{m1}^* = \frac{E_1[(X+A)^{-\gamma}(X+a_m)]}{E_1[(X+A)^{1-\gamma}]}(X+A) - a_m \quad (17)$$

$$= X + \frac{\Delta_{mn}E_1[(X+A)^{-\gamma}]}{E_1[(X+A)^{1-\gamma}]}(X+A) - \Delta_{mn}, \quad (18)$$

$$x_{n1}^* = \frac{E_1[(X+A)^{-\gamma}(X+a_n)]}{E_1[(X+A)^{1-\gamma}]}(X+A) - a_n \quad (19)$$

$$= X - \frac{\Delta_{mn}E_1[(X+A)^{-\gamma}]}{E_1[(X+A)^{1-\gamma}]}(X+A) + \Delta_{mn}, \quad (20)$$

$$(21)$$

where

$$\Delta_{mn} \equiv \frac{1}{2}(a_m - a_n) = (a_m - A) = -(a_n - A). \quad (22)$$

So from the above expression, we can see clearly that the deviation of the holdings of each agent from their initial holding ($x_{i,0} = X$, $i = m, n$) is due to the difference between the realization of two ξ_i . Specifically, for the two states $(\xi_m, \xi_n) = (a_m, a_n)$, $\Delta_{mn} = a_m - a_n$. Note that the demand for each agent is linear in X . This is true regardless of the size of the aggregate shock A . In other words, the contingent claims can be obtained solely through stock trading, where the stock pays the average X .

Some special cases now illustrate the nature of the trading result. First, it is quite possible that ξ_m and ξ_n are positively correlated. If so, a positive realization a_m is likely to be associated with a positive realization a_n . In the very special case that $a_m = a_n$, $\Delta_{mn} = 0$ and neither agent changes their holding, with $x_{m,1} = x_{n,1} = X$. The amount of trading depends not only on $\Delta_{mn} = 0$ but also on the size of the aggregate shock, A . This is due to the fact that the size of the coefficient $\frac{E_1[(X+A)^{-\gamma}]}{E_1[(X+A)^{1-\gamma}]}$ depends on A , and in fact declines as A increases.

[Qi, Marti look at spreadsheet Trading 1A Try changing the input A in cell B13]

4.1 Unequal M and N

In this section, we generalize the results to the case in which the number of agents in the two groups are different. Specifically, let M and N denote the total number of the two agents respectively, and define:

$$\rho \equiv \frac{M}{N}. \quad (23)$$

It is easy to see that $\rho \in (0, 1)$. With this definition, the aggregate variables are then:

$$\begin{aligned} A &= \frac{1}{M+N}(Ma_m + Na_n) \\ &= \frac{1}{1+\rho}(\rho a_m + a_n) \end{aligned} \quad (24)$$

$$\begin{aligned} X &= \frac{1}{M+N}(Mx_{m1} + Nx_{n1}) \\ &= \frac{1}{1+\rho}(\rho x_{m1} + x_{n1}). \end{aligned} \quad (25)$$

Given this definition, the optimal demand becomes:

$$x_{m1}^* = \frac{E_1[(X+A)^{-\gamma}(X+a_m)]}{E_1[(X+A)^{1-\gamma}]}(X+A) - a_m \quad (26)$$

$$\begin{aligned} &= \frac{E_1[(X+A)^{-\gamma}(X+A-A+a_m)]}{E_1[(X+A)^{1-\gamma}]}(X+A) - a_m \\ &= X+A + \frac{E_1[(X+A)^{-\gamma}(a_m-A)]}{E_1[(X+A)^{1-\gamma}]}(X+A) - a_m \\ &= X + \frac{\Delta_a E_1[(X+A)^{-\gamma}]}{E_1[(X+A)^{1-\gamma}]}(X+A) - \Delta_a, \end{aligned} \quad (27)$$

$$x_{n1}^* = \frac{E_1[(X+A)^{-\gamma}(X+a_n)]}{E_1[(X+A)^{1-\gamma}]}(X+A) - a_n \quad (28)$$

$$= X - \frac{\rho \Delta_a E_1[(X+A)^{-\gamma}]}{E_1[(X+A)^{1-\gamma}]}(X+A) + \rho \Delta_a, \quad (29)$$

$$(30)$$

where

$$\Delta_a \equiv \frac{1}{1+\rho}(a_m - a_n) = (a_m - A) = -\frac{1}{\rho}(a_n - A). \quad (31)$$

With the above results, we have the following observations:

- Suppose $a_m > a_n$. If there are a lot more m agents than n agents, namely $\rho = M/N \rightarrow \infty$, it follows that:

$$\Delta_a \rightarrow 0, \quad \rho \Delta_a \rightarrow a_m - a_n.$$

Thus $x_{m1}^* \rightarrow X$ and

$$x_{n1}^* \rightarrow X - \frac{(a_m - a_n)E_1[(X+A)^{-\gamma}]}{E_1[(X+A)^{1-\gamma}]}(X+A) + (a_m - a_n).$$

This says that each m agent doesn't change her holdings much from initial holdings ($x_{i0} = X$), while each n agent does. The reason for this is very straight forward: Any change in holdings of *all* m agents will need *all* n agents to take the other side of the trade. Since there are a lot more m agents than n agents, each m agent does not need to change much while each n agent need to change a lot.

In the extreme case, suppose that are only one n agent and many m agent, then the total trading volume is proportional to $a_m - a_n$.

- When $\rho \rightarrow \infty$, $A \rightarrow a_m$. As a result, the pricing kernel which is proportional to $X + A$, will be only a function of a_m . This is also understandable: The population is dominated by m agents, thus the prices are dominated by the realization of a_m .
- The above two observations imply that in an economy with mostly homogeneous agents, a small amount heterogenous agent has not much effect on the prices, but the trading volume effect is always finite.

5 The General Case

As we saw earlier, in the general case where there is unresolved background risk at time 1 the optimal demand cannot be solved analytically. The demand x_i depends on the precautionary premium, which in turn depends on the demand. We solved the problem above by considering a special case. We now analyze the general case, but with the use of an approximation.

5.1 An Approximation for ψ_i .

We start with an approximation for ψ_i . From equation (5)

$$E_e(x_i + a_i + e_i)^{-\gamma} \equiv (x_i + a_i - \psi_i)^{-\gamma} \quad (32)$$

Taking the Taylor expansion for the left hand side we have

$$\begin{aligned} E_e(x_i + a_i + e_i)^{-\gamma} &= (x_i + a_i)^{-\gamma} E_e \left(1 + \frac{e_i}{x_i + a_i} \right)^{-\gamma} \\ &\approx (x_i + a_i)^{-\gamma} E_e \left[1 - \gamma \frac{e_i}{x_i + a_i} + \frac{\gamma(\gamma + 1)}{2} \left(\frac{e_i}{x_i + a_i} \right)^2 \right] \\ &= (x_i + a_i)^{-\gamma} \left[1 + \frac{\gamma(\gamma + 1)\sigma_{e_i}^2}{2(x_i + a_i)^2} \right]. \end{aligned}$$

where, in the last step, we use the assumption that $Ee_i = 0$. Now, it follows that

$$(x_i + a_i - \psi_i)^{-\gamma} = (x_i + a_i)^{-\gamma} \left[1 + \frac{\gamma(\gamma + 1)\sigma_{\epsilon_i}^2}{2(x_i + a_i)^2} \right]$$

and hence

$$\begin{aligned} x_i + a_i - \psi_i &= (x_i + a_i) \left[1 + \frac{\gamma(\gamma + 1)\sigma_{\epsilon_i}^2}{2(x_i + a_i)^2} \right]^{-1/\gamma} \\ &\approx (x_i + a_i) \left[1 - \frac{(\gamma + 1)\sigma_{\epsilon_i}^2}{2(x_i + a_i)^2} \right] \\ &= x_i + a_i - \frac{(1 + \gamma)\sigma_{\epsilon_i}^2}{2(x_i + a_i)}. \end{aligned}$$

Which yields the approximate result for ψ_i :

$$\psi \approx \frac{(1 + \gamma)\sigma_{\epsilon_i}^2}{2(x_i + a_i)}. \quad (33)$$

Thus we have an approximate solution for ψ_i as a function of x_i . As we can see, it satisfies all the properties for the precautionary premium, ψ_i , as stated in FSS:

$$\psi_i > 0, \quad \frac{\partial \psi_i}{\partial x} < 0, \quad \frac{\partial^2 \psi_i}{\partial x^2} > 0, \quad (34)$$

$$\frac{\partial \psi_i}{\partial \sigma} > 0, \quad \frac{\partial^2 \psi_i}{\partial \sigma \partial x} < 0, \quad \frac{\partial^3 \psi_i}{\partial \sigma \partial x^2} > 0. \quad (35)$$

Also the approximation has additional implications with respect to the constant mean change in a_i :

$$\frac{\partial \psi_i}{\partial a_i} < 0, \quad \frac{\partial^2 \psi_i}{\partial a_i^2} > 0 \quad (36)$$

Finally, the cross relationships on σ and a_i are similar to those on σ and x_i .

5.2 Optimal demand given the approximation for ψ_i

Recall the holdings of contingent claims by agents m or n are all $x_{i0} = X, i = m, n$ at $t = 0$. With a realization of $(\xi_m, \xi_n) = (a_m, a_n)$ at time $t = 1$, the average $A = \frac{1}{2}(a_m + a_n)$. Applying the demand equation the optimal demand of an agent in, say m , is:

$$x_{m1} = \frac{E_1[(X + A - \psi_1)^{-\gamma}(X + a_m - \psi_{m1})]}{E_1[(X + A - \psi_1)^{1-\gamma}]}(X + A - \psi_1) - a_m + \psi_{m1},$$

where

$$\begin{aligned}\psi_{m1} &= \frac{(1 + \gamma)\sigma_\eta^2}{2(x_{m1} + a_m)} \\ \psi_{n1} &= \frac{(1 + \gamma)\sigma_\eta^2}{2(x_{n1} + a_n)} \\ \psi_1 &= \frac{1}{2}(\psi_{m1} + \psi_{n1}).\end{aligned}$$

This follows since all agents inherit the average allocation X at $t = 1$. Also, ψ_{m1} depends only on the remaining (unresolved) background risk σ_η . For convenience, define:

$$p \equiv x_{m1} + a_m \tag{37}$$

$$q \equiv x_{n1} + a_n. \tag{38}$$

From these definitions we can rewrite the optimal demand for agent m as:

$$p = \frac{E_1[(X + A - \psi_1)^{-\gamma}(X + a_m - \psi_{m1})]}{E_1[(X + A - \psi_1)^{1-\gamma}]}(X + A - \psi_1) + \psi_{m1}, \tag{39}$$

where

$$\psi_{m1} = \frac{(1 + \gamma)\sigma_\eta^2}{2p} \tag{40}$$

$$\psi_{n1} = \frac{(1 + \gamma)\sigma_\eta^2}{2q} \tag{41}$$

$$\psi_1 = \frac{(1 + \gamma)\sigma_\eta^2(X + A)}{2pq}. \tag{42}$$

ψ_1 in (42) follows from the fact that $X = \frac{1}{2}(x_{m1} + x_{n1})$ and $A = \frac{1}{2}(a_m + a_n)$.

Similarly, the optimal demand for agent n is:

$$q = \frac{E_1[(X + A - \psi_1)^{-\gamma}(X + a_n - \psi_{n1})]}{E_1[(X + A - \psi_1)^{1-\gamma}]}(X + A - \psi_1) + \psi_{n1}. \tag{43}$$

The optimal demands of the two types of agent are implicit in equations (39) and (43). However, in the appendix we show, using approximations, that the following proposition holds in the general case:

Proposition 1

$$p = p^* + \frac{(1+\gamma)\sigma_\eta^2}{2} \left(B_{1p}(X+A) + B_{2p}\frac{1}{(X+A)} \right) \quad (44)$$

$$q = q^* + \frac{(1+\gamma)\sigma_\eta^2}{2} \left(B_{1q}(X+A) + B_{2q}\frac{1}{(X+A)} \right), \quad (45)$$

where

$$\begin{aligned} B_{1p} &= \frac{1}{E_1[(X+A)^{1-\gamma}]} \left[E_1 \left(\frac{\gamma(X+A)^{-\gamma}(X+a_m)}{p^*q^*} \right) - E_1 \left(\frac{(X+A)^{-\gamma}}{p^*} \right) \right. \\ &\quad \left. + E_1 [(X+A)^{-\gamma}(X+a_m)] \frac{E_1 \left(\frac{(1-\gamma)(X+A)^{1-\gamma}}{p^*q^*} \right)}{E_1[(X+A)^{1-\gamma}]} \right], \\ B_{2p} &= \frac{E_1[(X+A)^{1-\gamma}]}{E_1[(X+A)^{-\gamma}(X+a_m)]} - \frac{E_1[(X+A)^{1-\gamma}]}{E_1[(X+A)^{-\gamma}(X+a_m)]}. \end{aligned}$$

and $B_{1q} = -B_{1p}$ and $B_{2q} = -B_{2p}$.

Proof See Appendix

5.3 Unequal M and N Agents

In this section, we again consider the case in which there are unequal number of agents in two groups. Applying the demand equation the optimal demand of an agent in, say m , is:

$$x_{m1} = \frac{E_1[(X+A-\psi_1)^{-\gamma}(X+a_m-\psi_{m1})]}{E_1[(X+A-\psi_1)^{1-\gamma}]}(X+A-\psi_1) - a_m + \psi_{m1},$$

where

$$\begin{aligned} \psi_{m1} &= \frac{(1+\gamma)\sigma_\eta^2}{2(x_{m1}+a_m)} \\ \psi_{n1} &= \frac{(1+\gamma)\sigma_\eta^2}{2(x_{n1}+a_n)} \\ \psi_1 &= \frac{1}{M+N}(M\psi_{m1}+N\psi_{n1}) \\ &= \frac{1}{1+\rho}(\rho\psi_{m1}+\psi_{n1}) \\ A &= \frac{1}{1+\rho}(\rho a_m + a_n), \end{aligned}$$

where $\rho = M/N$.

Again define:

$$\begin{aligned} p &\equiv x_{m1} + a_m \\ q &\equiv x_{n1} + a_n. \end{aligned}$$

Thus

$$X + A = \frac{1}{1 + \rho}(\rho p + q). \quad (46)$$

The optimal demand can thus be written as:

$$p = \frac{E_1[(X + A - \psi_1)^{-\gamma}(X + a_m - \psi_{m1})]}{E_1[(X + A - \psi_1)^{1-\gamma}]}(X + A - \psi_1) + \psi_{m1}, \quad (47)$$

where

$$\begin{aligned} \psi_{m1} &= \frac{(1 + \gamma)\sigma_\eta^2}{2p} \\ \psi_{n1} &= \frac{(1 + \gamma)\sigma_\eta^2}{2q} \\ \psi_1 &= \frac{(1 + \gamma)\sigma_\eta^2}{2(1 + \rho)}\left(\frac{\rho}{p} + \frac{1}{q}\right) \\ &= \frac{(1 + \gamma)\sigma_\eta^2}{2pq} \left[\frac{1}{1 + \rho}(\rho q + p) \right] \\ &= \frac{(1 + \gamma)\sigma_\eta^2}{2pq} \left[X + A + \frac{1 - \rho}{1 + \rho}(p - q) \right] \\ &= \frac{(1 + \gamma)\sigma_\eta^2}{2pq} (X + A)(1 + \Delta_{pq}), \end{aligned} \quad (48)$$

and

$$\Delta_{pq} \equiv \frac{1 - \rho}{1 + \rho} \cdot \frac{p - q}{X + A} \quad (49)$$

$$= \frac{1 - \rho}{1 + \rho} \cdot \frac{(x_{m1} - x_{n1}) + (a_m - a_n)}{X + A} \quad (50)$$

Thus the approximations are:

$$(X + A - \psi_1)^{-\gamma} = \left[X + A - \frac{(1 + \gamma)\sigma_\eta^2}{2} \frac{X + A}{pq} (1 + \Delta_{pq}) \right]^{-\gamma} \quad (51)$$

$$= (X + A)^{-\gamma} \left[1 - \frac{(1 + \gamma)\sigma_\eta^2}{2pq}(1 + \Delta_{pq}) \right]^{-\gamma} \quad (52)$$

$$\approx (X + A)^{-\gamma} \left[1 + \frac{\gamma(\gamma + 1)\sigma_\eta^2}{2pq}(1 + \Delta_{pq}) \right], \quad (53)$$

where the last step we use the approximation that $\sigma_\eta^2/(pq)$ is small.

Similarly we obtain the approximation:

$$(X + A - \psi_1)^{1-\gamma} \approx (X + A)^{1-\gamma} \left[1 - \frac{(1 - \gamma^2)\sigma_\eta^2}{2pq}(1 + \Delta_{pq}) \right]. \quad (54)$$

Thus:

$$\frac{1}{E_1 \left\{ (X + A)^{1-\gamma} \left[1 - \frac{(1-\gamma^2)\sigma_\eta^2}{2pq}(1 + \Delta_{pq}) \right] \right\}} \approx \frac{1}{E_1[(X + A)^{1-\gamma}]} \left\{ 1 + \frac{E_1 \left[\frac{(1-\gamma^2)\sigma_\eta^2(X+A)^{1-\gamma}}{2pq}(1 + \Delta_{pq}) \right]}{E_1[(X + A)^{1-\gamma}]} \right\} \quad (55)$$

Substituting these into the optimal demand function, it follows:

$$\begin{aligned} p \approx & E_1 \left[(X + A)^{-\gamma} \left(1 + \frac{\gamma(\gamma + 1)\sigma_\eta^2}{2pq}(1 + \Delta_{pq}) \right) \left(X + a_m - \frac{(1 + \gamma)\sigma_\eta^2}{2p} \right) \right] \\ & \frac{1}{E_1[(X + A)^{1-\gamma}]} \left[1 + \frac{E_1 \left(\frac{(1-\gamma^2)\sigma_\eta^2(X+A)^{1-\gamma}}{2pq}(1 + \Delta_{pq}) \right)}{E_1[(X + A)^{1-\gamma}]} \right] \left(X + A - \frac{(1 + \gamma)\sigma_\eta^2(X + A)}{2pq}(1 + \Delta_{pq}) \right) \\ & + \frac{(1 + \gamma)\sigma_\eta^2}{2p}. \end{aligned} \quad (56)$$

Then, under our assumption the terms $\sigma^4/p^4, \sigma^4/p^3q, \sigma^4/p^2q^2 \rightarrow 0$. Thus we have:

$$\begin{aligned} p \approx & E_1 \left[(X + A)^{-\gamma} \left(X + a_m + \frac{\gamma(1 + \gamma)\sigma_\eta^2(X + a_m)}{2pq}(1 + \Delta_{pq}) - \frac{(\gamma + 1)\sigma_\eta^2}{2p} \right) \right] \frac{(X + A)}{E_1[(X + A)^{1-\gamma}]} \\ & \left[1 - \frac{(1 + \gamma)\sigma_\eta^2}{2pq}(1 + \Delta_{pq}) + \frac{E_1 \left(\frac{(1-\gamma^2)\sigma_\eta^2(X+A)^{1-\gamma}}{2pq}(1 + \Delta_{pq}) \right)}{E_1[(X + A)^{1-\gamma}]} \right] + \frac{(1 + \gamma)\sigma_\eta^2}{2p} \\ = & \left\{ \frac{E_1[(X + A)^{-\gamma}(X + a_m)]}{E_1[(X + A)^{1-\gamma}]} (X + A) \right. \end{aligned}$$

$$\begin{aligned}
& + \frac{(1+\gamma)\sigma_\eta^2(X+A)}{2E_1[(X+A)^{1-\gamma}]} \left[E_1 \left(\frac{\gamma(X+A)^{-\gamma}(X+a_m)}{pq} (1+\Delta_{pq}) \right) - E_1 \left(\frac{(X+A)^{-\gamma}}{p} \right) \right] \Big\} \\
& \left[1 - \frac{(1+\gamma)\sigma_\eta^2}{2pq} (1+\Delta_{pq}) + \frac{(1+\gamma)\sigma_\eta^2}{2} \frac{E_1 \left(\frac{(1-\gamma)(X+A)^{1-\gamma}}{pq} (1+\Delta_{pq}) \right)}{E_1[(X+A)^{1-\gamma}]} \right] + \frac{(1+\gamma)\sigma_\eta^2}{2p}
\end{aligned}$$

Further multiplying out the above expression, it follows:

$$\begin{aligned}
p & \approx \frac{E_1[(X+A)^{-\gamma}(X+a_m)]}{E_1[(X+A)^{1-\gamma}]} (X+A) \\
& + \frac{(1+\gamma)\sigma_\eta^2}{2E_1[(X+A)^{1-\gamma}]} (X+A) \left[E_1 \left(\frac{\gamma(X+A)^{-\gamma}(X+a_m)}{pq} (1+\Delta_{pq}) \right) - E_1 \left(\frac{(X+A)^{-\gamma}}{p} \right) \right] \\
& - \frac{(1+\gamma)\sigma_\eta^2}{2pq} \frac{E_1[(X+A)^{-\gamma}(X+a_m)]}{E_1[(X+A)^{1-\gamma}]} (X+A)(1+\Delta_{pq}) \\
& + \frac{(1+\gamma)\sigma_\eta^2}{2} \frac{E_1 \left(\frac{(1-\gamma)(X+A)^{1-\gamma}}{pq} (1+\Delta_{pq}) \right)}{E_1[(X+A)^{1-\gamma}]} \frac{E_1[(X+A)^{-\gamma}(X+a_m)]}{E_1[(X+A)^{1-\gamma}]} (X+A) + \frac{(1+\gamma)\sigma_\eta^2}{2p} \quad (57) \\
& = \frac{E_1[(X+A)^{-\gamma}(X+a_m)]}{E_1[(X+A)^{1-\gamma}]} (X+A) \\
& + \frac{(1+\gamma)\sigma_\eta^2}{2E_1[(X+A)^{1-\gamma}]} \left\{ \left[E_1 \left(\frac{\gamma(X+A)^{-\gamma}(X+a_m)}{pq} (1+\Delta_{pq}) \right) \right. \right. \\
& \left. \left. - E_1 \left(\frac{(X+A)^{-\gamma}}{p} \right) + E_1 [(X+A)^{-\gamma}(X+a_m)] \frac{E_1 \left(\frac{(1-\gamma)(X+A)^{1-\gamma}}{pq} (1+\Delta_{pq}) \right)}{E_1[(X+A)^{1-\gamma}]} \right] (X+A) \right. \\
& \left. - E_1[(X+A)^{-\gamma}(X+a_m)] \frac{(X+A)}{pq} (1+\Delta_{pq}) + \frac{E_1((X+A)^{1-\gamma})}{p} \right\} \quad (58)
\end{aligned}$$

Finally, the approximate explicit solution is found by substituting $p = p^*$, $q = q^*$ to obtain

$$\begin{aligned}
p & \approx p^* + \frac{(1+\gamma)\sigma_\eta^2}{2E_1[(X+A)^{1-\gamma}]} \left\{ \left[E_1 \left(\frac{\gamma(X+A)^{-\gamma}(X+a_m)}{p^*q^*} (1+\Delta_{p^*q^*}) \right) - E_1 \left(\frac{(X+A)^{-\gamma}}{p^*} \right) \right. \right. \\
& \left. \left. + E_1 [(X+A)^{-\gamma}(X+a_m)] \frac{E_1 \left(\frac{(1-\gamma)(X+A)^{1-\gamma}}{p^*q^*} (1+\Delta_{p^*q^*}) \right)}{E_1[(X+A)^{1-\gamma}]} \right] (X+A) \right. \\
& \left. - E_1[(X+A)^{-\gamma}(X+a_m)] \frac{(X+A)}{p^*q^*} (1+\Delta_{p^*q^*}) + \frac{E_1((X+A)^{1-\gamma})}{p^*} \right\} \\
& = p^* + \frac{(1+\gamma)\sigma_\eta^2}{2} \left\{ B_{1p}(X+A) - \frac{E_1[(X+A)^{-\gamma}(X+a_m)](X+A)}{E_1((X+A)^{1-\gamma})p^*q^*} (1+\Delta_{p^*q^*}) + \frac{1}{p^*} \right\}
\end{aligned}$$

$$= p^* + \frac{(1+\gamma)\sigma_\eta^2}{2} \left[B_{1p}(X+A) + B_{2p} \frac{1}{(X+A)} \right],$$

where

$$\begin{aligned} B_{1p} &= \frac{1}{E_1[(X+A)^{1-\gamma}]} \left[E_1 \left(\frac{\gamma(X+A)^{-\gamma}(X+a_m)}{p^*q^*} (1 + \Delta_{p^*q^*}) \right) - E_1 \left(\frac{(X+A)^{-\gamma}}{p^*} \right) \right. \\ &\quad \left. + E_1 [(X+A)^{-\gamma}(X+a_m)] \frac{E_1 \left(\frac{(1-\gamma)(X+A)^{1-\gamma}}{p^*q^*} (1 + \Delta_{p^*q^*}) \right)}{E_1[(X+A)^{1-\gamma}]} \right], \\ B_{2p} &= \frac{E_1[(X+A)^{1-\gamma}]}{E_1[(X+A)^{-\gamma}(X+a_m)]} - \frac{E_1[(X+A)^{1-\gamma}]}{E_1[(X+A)^{-\gamma}(X+a_n)]} (1 + \Delta_{p^*q^*}). \end{aligned}$$

Using the expression for x_{m1}^*, x_{n1}^* , we can obtain explicit expression for $\Delta_{p^*q^*}$:

$$\begin{aligned} \Delta_{p^*q^*} &= \frac{1 - \rho (x_{m1}^* + a_m) - (x_{n1}^* + a_n)}{1 + \rho} \frac{X + A}{X + A} \\ &= \frac{1 - \rho \frac{E_1[(X+A)^{-\gamma}(X+a_m)]}{E_1[(X+A)^{1-\gamma}]} (X + A) - \frac{E_1[(X+A)^{-\gamma}(X+a_n)]}{E_1[(X+A)^{1-\gamma}]} (X + A)}{1 + \rho} \frac{X + A}{X + A} \\ &= \frac{1 - \rho (a_m - a_n) E_1[(X+A)^{-\gamma}]}{1 + \rho E_1[(X+A)^{1-\gamma}]} \\ &= \frac{(1 - \rho) \Delta_a E_1[(X+A)^{-\gamma}]}{E_1[(X+A)^{1-\gamma}]} \end{aligned}$$

$$p = p^* + \frac{(1+\gamma)\sigma_\eta^2}{2} \left(B_{1p}(X+A) + B_{2p} \frac{1}{(X+A)} \right) \quad (59)$$

$$q = q^* + \frac{(1+\gamma)\sigma_\eta^2}{2} \left(B_{1q}(X+A) + B_{2q} \frac{1}{(X+A)} \right), \quad (60)$$

6 Numerical Examples

In this section, we will numerically analyze the effects of background risk on the trading the prices using the demand function we derived in the previous section.

6.1 The Effects of Heterogeneous Agents on Prices and Trading

At time $t = 1$, with a representative agent with per capita supply of contingent claims X and expected mean of the future background risk of A , the pricing kernel is :

$$\phi_1 = \frac{(X + A - \psi_{1a})^{-\gamma}}{E_1[(X + A - \psi_{1a})^{-\gamma}]}, \quad (61)$$

where

$$\psi_{1a} = \frac{(1 + \gamma)\sigma_\eta^2}{2(X + A)}.$$

Comparing this with the pricing kernel of the two types of agents case, the latter has the pricing kernel:

$$\begin{aligned} \psi_1 &= \frac{(1 + \gamma)\sigma_\eta^2}{2pq}(X + A)(1 + \Delta_{pq}) \\ &\approx \frac{(1 + \gamma)\sigma_\eta^2}{2p^*q^*}(X + A)(1 + \Delta_{p^*q^*}) \\ &= \frac{(1 + \gamma)\sigma_\eta^2}{2} \frac{(E_1[(X + A)^{1-\gamma}])^2}{E_1[(X + A)^{-\gamma}(X + a_m)]E_1[(X + A)^{-\gamma}(X + a_n)]} \frac{1}{X + A} (1 + \Delta_{p^*q^*}) \\ &= \frac{(1 + \gamma)\sigma_\eta^2}{2} \frac{(E_1[(X + A)^{1-\gamma}])^2}{(E_1[(X + A)^{1-\gamma}])^2 + (1 - \rho)\Delta_a E_1[(X + A)^{-\gamma}]E_1[(X + A)^{1-\gamma}] - \rho\Delta_a^2 (E_1[(X + A)^{-\gamma}])^2} \\ &\quad \frac{1}{X + A} (1 + \Delta_{p^*q^*}), \end{aligned}$$

where the last step we use the fact that $A = (\rho a_m + a_n)/(1 + \rho)$.

A special case of the above expression is when $\rho = 1$. Then it becomes:

$$\psi_1 = \frac{(1 + \gamma)\sigma_\eta^2}{2} \frac{(E_1[(X + A)^{1-\gamma}])^2}{(E_1[(X + A)^{1-\gamma}])^2 - \Delta_a^2 (E_1[(X + A)^{-\gamma}])^2} \frac{1}{X + A}.$$

So the heterogeneity of the agents has two effect on the pricing kernel:

1. It causes precautionary saving motive to be larger than an otherwise homogeneous agent system. The larger the difference between the shocks to background risk, the larger the effect.
2. The overall effect of heterogeneity is unclear, since A, ρ, Δ_a are not independent. When we discuss this in the following numerical examples, we need to make sure that the correct comparative static results are used.

So ψ_1 determines the price effect of heterogeneous agent, while $\psi_M - \psi_1$ (and $\psi_N - \psi_1$) determines the actual demand (trading) of each agent.

6.2 Full Resolution at time $t = 1$

This is the case in which the optimal demand of agents are linear in the average per capita supply X . We will explore the following results:

6.2.1 $M = N$

When there are equal number of two types of agents, $\rho = M/N = 1$. We want to study the effects of different A . Also as noted before, the demand (thus the trade) is a function of $\Delta_a = \frac{1}{2}(a_m - a_n)$.

The following three figures illustrate the function values of $\psi_1, \psi_{M1}, \psi_{M1} - \psi_1, \phi, x_{M1}, x_{M1} - X$. The common parameters are: There are 12 states of X , which starts at 2 and increases by one each. $\gamma = 2$.

Figure 1-3 illustrate the effect of Δ_a while holding A fixed. As we can see that there is not much change in pricing kernel, while there are still tradings between two agents. Figure 4 illustrate the effect of A while holding Δ_a fixed. Here both pricing kernel and trading change.

6.2.2 $M \neq N$

Figure 4a is the case for different ρ : here we fix $A = 0.5$ and change ρ (and a_n as well, while holding a_m fixed). Without the effect from A , changing ρ has no effect on the demand.

6.3 General Case

In the general case, we again want to study the effect of different A . First we study the case in which $M = N$ or $\rho = 1$, then we study the general case of different *rho*. Note there are four parameters here: (a_m, a_n, A, ρ) , which are related by

$$A = \frac{\rho a_m + a_n}{\rho + 1}.$$

We also have:

$$\Delta_a = \frac{a_m - a_n}{1 + \rho}.$$

So we can study different comparative static results.

6.3.1 $M = N$

When we fix $\rho = 1$, we can study two effects: The effect of A while fixing Δ_a and the effect of Δ_a while fixing A . The common parameters are: There are 12 states of X with initial value of 2 and increasing each by one. $\sigma_\eta = .5$ and $\gamma = 2$.

First let us look the case in which we fix $a_m = -3$, and we change $A = .5, 0, -.5$. This is achieved by changing $a_n = 4, 3, 2$, with corresponding $\Delta_a = -3.5, -3, -2.5$. And this is Figure 5.

6.3.2 $M \neq N$

This is the most general case. Let us start with a case that we fix $a_m = -3, a_n = 1$, and we change ρ . As a result, A changes as well. This is the case in which individual agents' shock are fixed while the number of the agents are different. Figure 6 illustrate this. In the following we hold $a_m = -3$ fixed.

The effect of above is a mixed effect from two fronts: one from changing in A and the other from changing in ρ . So we present the case of these two separately.

To illustrate the effect of A , Figure 7 shows the effect of A for fixed $\rho = 1/2$.

To illustrate the effect of ρ , Figure 8 shows the effect of ρ for fixed $A = 0$.

6.4 Overall Intuition

Qi Note: This is just my take from discussion with Dick and observation of the figures.

There are three parameters that are of interests to our paper: The average aggregate shock A , and two heterogeneity parameter Δ_a and ρ .

A has effect on the pricing kernel, the optimal demand (thus trading) changes accordingly. However, the heterogeneity is the key here, without which there won't be a trade. The change of the prices for different X cause the different demand of the agent. Overall, the agent who receive a negative background shock $a_m < 0$ tends to buy more at the low X state and sell at the high X state. If overall A is high, the prices of high X will be low, thus agent M sells less of high X .

Different Δ_a cause the difference in trade because the larger the Δ_a , the poor M is. She

will desire more at the low X state. To do so requires selling high X .

Different ρ cause the difference in trade because it changes the difference between ψ_{M1} and average ψ_1 . In the extreme case, for example when $\rho \rightarrow 0$, namely there are infinite amount N agents, then ψ_1 is quite close to ψ_{N1} . The economy is dominated by agent N . However M will take on the trades from each N agent. This is definitely different from the case in which M takes on the trade from one agent N .

7 Conclusion

There is an extensive literature on background risk, which arises from stochastic cash flows generating non-marketable wealth. Since this risk cannot be directly hedged, it affects the derived risk aversion of the individual agent. Generally speaking, as documented by several researchers and synthesized by Gollier (2001), in the presence of background risk, agents generally become more risk-averse in their derived utility functions, and thus, behave like a more risk-averse agent would, in the absence of such a risk. This, in turn, influences the demand for insurance.

There has been rather less attention devoted to the pricing of securities and sharing rules in equilibrium, when agents in the economy face background risk. A notable early paper is by FSS, who analyze the equilibrium in such an economy, and derive the portfolio demand of individual agents in this equilibrium. The agents take into account their non-marketable background risk in optimally determining their demand for the marketable assets. Specifically, FSS show that agents with background risk depart from the linear sharing rule that characterizes behavior in complete markets, and may buy or sell non-linear contingent claims such as options.

In this paper, we take the presence of background risk and its influence on risk taking in a different direction. We explore how the prices of assets are determined in equilibrium by the interplay of portfolio demands across agents in the economy, which take into account the background risks they face. If the agents face different background risks, it is reasonable to expect that their portfolio demands will differ: this is the argument first made by FSS. We extend this argument to the multi-period setting and derive the changes in the portfolio demand of different agents as the background risk is revealed over time. To the extent that these changes differ across agents, it establishes a motive for trading, even in the presence of symmetric (full) information across agents.

The equilibrium we obtain turns out to be fairly complex, since portfolio demands depend on the changed derived risk aversion of agents in the presence of background risk, which

in turn, depends on the portfolio holdings. We break this circularity by considering special cases of the evolution of background risk, as well as by using some approximations. We confirm these results by numerical computations.

We have thus been able to derive a theory of trading in the presence of full information, without running afoul of the powerful no-trade results of Grossman and Stiglitz (1980) and milgrom and Stokey (1982) in the context of asymmetric information models. We believe our theory can be extended in several directions to separate the trading in linear (stocks and bonds) versus non-linear (options) claims. Potentially, our theory is testable, if one can quantify the influences of background risks such as human and housing wealth. This could be of interest to researchers in asset pricing, where the focus is mainly on returns, but could also be related to the aspects of trading analyzed in this paper.

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8 Appendix: Derivation of Demand Equations: The General Case

In this appendix, we derive the demand equations in the general case.

First, we look at the terms in the pricing kernel using the above definition of p and q :

$$(X + A - \psi_1)^{-\gamma} = \left[X + A - \frac{(1 + \gamma)\sigma_\eta^2 X + A}{2pq} \right]^{-\gamma} \quad (62)$$

$$= (X + A)^{-\gamma} \left[1 - \frac{(1 + \gamma)\sigma_\eta^2}{2pq} \right]^{-\gamma} \quad (63)$$

$$\approx (X + A)^{-\gamma} \left[1 + \frac{\gamma(\gamma + 1)\sigma_\eta^2}{2pq} \right], \quad (64)$$

where the last step we use the approximation that $\sigma_\eta^2/(pq)$ is small.

Similarly we obtain the approximation:

$$(X + A - \psi_1)^{1-\gamma} \approx (X + A)^{1-\gamma} \left[1 - \frac{(1 - \gamma^2)\sigma_\eta^2}{2pq} \right]. \quad (65)$$

Thus:

$$\frac{1}{E_1 \left\{ (X + A)^{1-\gamma} \left[1 - \frac{(1 - \gamma^2)\sigma_\eta^2}{2pq} \right] \right\}} \approx \frac{1}{E_1[(X + A)^{1-\gamma}]} \left\{ 1 + \frac{E_1 \left[\frac{(1 - \gamma^2)\sigma_\eta^2(X + A)^{1-\gamma}}{2pq} \right]}{E_1[(X + A)^{1-\gamma}]} \right\} \quad (66)$$

Substituting these into the optimal demand function, it follows:

$$\begin{aligned} p \approx & E_1 \left[(X + A)^{-\gamma} \left(1 + \frac{\gamma(\gamma + 1)\sigma_\eta^2}{2pq} \right) \left(X + a_m - \frac{(1 + \gamma)\sigma_\eta^2}{2p} \right) \right] \\ & \frac{1}{E_1[(X + A)^{1-\gamma}]} \left[1 + \frac{E_1 \left(\frac{(1 - \gamma^2)\sigma_\eta^2(X + A)^{1-\gamma}}{2pq} \right)}{E_1[(X + A)^{1-\gamma}]} \right] \left(X + A - \frac{(1 + \gamma)\sigma_\eta^2(X + A)}{2pq} \right) \\ & + \frac{(1 + \gamma)\sigma_\eta^2}{2p}. \end{aligned} \quad (67)$$

Then, under our assumption the terms $\sigma^4/p^4, \sigma^4/p^3q, \sigma^4/p^2q^2 \rightarrow 0$. Thus we have:

$$\begin{aligned}
p &\approx E_1 \left[(X+A)^{-\gamma} \left(X+a_m + \frac{\gamma(1+\gamma)\sigma_\eta^2(X+a_m)}{2pq} - \frac{(\gamma+1)\sigma_\eta^2}{2p} \right) \right] \frac{(X+A)}{E_1[(X+A)^{1-\gamma}]} \\
&\quad \left[1 - \frac{(1+\gamma)\sigma_\eta^2}{2pq} + \frac{E_1 \left(\frac{(1-\gamma^2)\sigma_\eta^2(X+A)^{1-\gamma}}{2pq} \right)}{E_1[(X+A)^{1-\gamma}]} \right] + \frac{(1+\gamma)\sigma_\eta^2}{2p} \\
&= \left\{ \frac{E_1[(X+A)^{-\gamma}(X+a_m)]}{E_1[(X+A)^{1-\gamma}]}(X+A) \right. \\
&\quad \left. + \frac{(1+\gamma)\sigma_\eta^2(X+A)}{2E_1[(X+A)^{1-\gamma}]} \left[E_1 \left(\frac{\gamma(X+A)^{-\gamma}(X+a_m)}{pq} \right) - E_1 \left(\frac{(X+A)^{-\gamma}}{p} \right) \right] \right\} \\
&\quad \left[1 - \frac{(1+\gamma)\sigma_\eta^2}{2pq} + \frac{(1+\gamma)\sigma_\eta^2}{2} \frac{E_1 \left(\frac{(1-\gamma)(X+A)^{1-\gamma}}{pq} \right)}{E_1[(X+A)^{1-\gamma}]} \right] + \frac{(1+\gamma)\sigma_\eta^2}{2p}
\end{aligned}$$

Further multiplying out the above expression, it follows:

$$\begin{aligned}
p &\approx \frac{E_1[(X+A)^{-\gamma}(X+a_m)]}{E_1[(X+A)^{1-\gamma}]}(X+A) \\
&\quad + \frac{(1+\gamma)\sigma_\eta^2}{2E_1[(X+A)^{1-\gamma}]}(X+A) \left[E_1 \left(\frac{\gamma(X+A)^{-\gamma}(X+a_m)}{pq} \right) - E_1 \left(\frac{(X+A)^{-\gamma}}{p} \right) \right] \\
&\quad - \frac{(1+\gamma)\sigma_\eta^2}{2pq} \frac{E_1[(X+A)^{-\gamma}(X+a_m)]}{E_1[(X+A)^{1-\gamma}]}(X+A) \\
&\quad + \frac{(1+\gamma)\sigma_\eta^2}{2} \frac{E_1 \left(\frac{(1-\gamma)(X+A)^{1-\gamma}}{pq} \right)}{E_1[(X+A)^{1-\gamma}]} \frac{E_1[(X+A)^{-\gamma}(X+a_m)]}{E_1[(X+A)^{1-\gamma}]}(X+A) + \frac{(1+\gamma)\sigma_\eta^2}{2p} \quad (68) \\
&= \frac{E_1[(X+A)^{-\gamma}(X+a_m)]}{E_1[(X+A)^{1-\gamma}]}(X+A) \\
&\quad + \frac{(1+\gamma)\sigma_\eta^2}{2E_1[(X+A)^{1-\gamma}]} \left\{ \left[E_1 \left(\frac{\gamma(X+A)^{-\gamma}(X+a_m)}{pq} \right) \right. \right. \\
&\quad \left. \left. - E_1 \left(\frac{(X+A)^{-\gamma}}{p} \right) + E_1 [(X+A)^{-\gamma}(X+a_m)] \frac{E_1 \left(\frac{(1-\gamma)(X+A)^{1-\gamma}}{pq} \right)}{E_1[(X+A)^{1-\gamma}]} \right] \right\} (X+A) \\
&\quad - E_1[(X+A)^{-\gamma}(X+a_m)] \frac{(X+A)}{pq} + \frac{E_1((X+A)^{1-\gamma})}{p} \quad (69)
\end{aligned}$$

Finally, the approximate explicit solution is found by substituting $p = p^*$, $q = q^*$ to obtain

$$\begin{aligned}
p &\approx p^* + \frac{(1+\gamma)\sigma_\eta^2}{2E_1[(X+A)^{1-\gamma}]} \left\{ \left[E_1 \left(\frac{\gamma(X+A)^{-\gamma}(X+a_m)}{p^*q^*} \right) - E_1 \left(\frac{(X+A)^{-\gamma}}{p^*} \right) \right. \right. \\
&+ \left. E_1[(X+A)^{-\gamma}(X+a_m)] \frac{E_1 \left(\frac{(1-\gamma)(X+A)^{1-\gamma}}{p^*q^*} \right)}{E_1[(X+A)^{1-\gamma}]} \right] (X+A) \\
&- \left. E_1[(X+A)^{-\gamma}(X+a_m)] \frac{(X+A)}{p^*q^*} + \frac{E_1((X+A)^{1-\gamma})}{p^*} \right\} \\
&= p^* + \frac{(1+\gamma)\sigma_\eta^2}{2} \left\{ B_{1p}(X+A) - \frac{E_1[(X+A)^{-\gamma}(X+a_m)](X+A)}{E_1((X+A)^{1-\gamma})p^*q^*} + \frac{1}{p^*} \right\} \\
&= p^* + \frac{(1+\gamma)\sigma_\eta^2}{2} \left[B_{1p}(X+A) + B_{2p} \frac{1}{(X+A)} \right],
\end{aligned}$$

where

$$\begin{aligned}
B_{1p} &= \frac{1}{E_1[(X+A)^{1-\gamma}]} \left[E_1 \left(\frac{\gamma(X+A)^{-\gamma}(X+a_m)}{p^*q^*} \right) - E_1 \left(\frac{(X+A)^{-\gamma}}{p^*} \right) \right. \\
&+ \left. E_1[(X+A)^{-\gamma}(X+a_m)] \frac{E_1 \left(\frac{(1-\gamma)(X+A)^{1-\gamma}}{p^*q^*} \right)}{E_1[(X+A)^{1-\gamma}]} \right], \\
B_{2p} &= \frac{E_1[(X+A)^{1-\gamma}]}{E_1[(X+A)^{-\gamma}(X+a_m)]} - \frac{E_1[(X+A)^{1-\gamma}]}{E_1[(X+A)^{-\gamma}(X+a_n)]}.
\end{aligned}$$

Similarly,

$$q = q^* + \frac{(1+\gamma)\sigma_\eta^2}{2} \left[B_{1q}(X+A) + B_{2q} \frac{1}{(X+A)} \right],$$

where

$$\begin{aligned}
B_{1q} &= \frac{1}{E_1[(X+A)^{1-\gamma}]} \left[E_1 \left(\frac{\gamma(X+A)^{-\gamma}(X+a_n)}{p^*q^*} \right) - E_1 \left(\frac{(X+A)^{-\gamma}}{q^*} \right) \right. \\
&+ \left. E_1[(X+A)^{-\gamma}(X+a_n)] \frac{E_1 \left(\frac{(1-\gamma)(X+A)^{1-\gamma}}{p^*q^*} \right)}{E_1[(X+A)^{1-\gamma}]} \right] \\
B_{2q} &= \frac{E_1[(X+A)^{1-\gamma}]}{E_1[(X+A)^{-\gamma}(X+a_n)]} - \frac{E_1[(X+A)^{1-\gamma}]}{E_1[(X+A)^{-\gamma}(X+a_m)]}.
\end{aligned}$$

$$p = p^* + \frac{(1+\gamma)\sigma_\eta^2}{2} \left(B_{1p}(X+A) + B_{2p} \frac{1}{(X+A)} \right) \quad (70)$$

$$q = q^* + \frac{(1+\gamma)\sigma_\eta^2}{2} \left(B_{1q}(X + A) + B_{2q} \frac{1}{(X+A)} \right), \quad (71)$$

Figure 1. $\rho = 1, A = 0$

This figure shows the effect of different Δ_a while fixing $A = 0$. The solid line corresponds to $\Delta_a = -1$, the dashed line corresponds to $\Delta_a = -2$, and the dotted line corresponds to $\Delta_a = -3$.

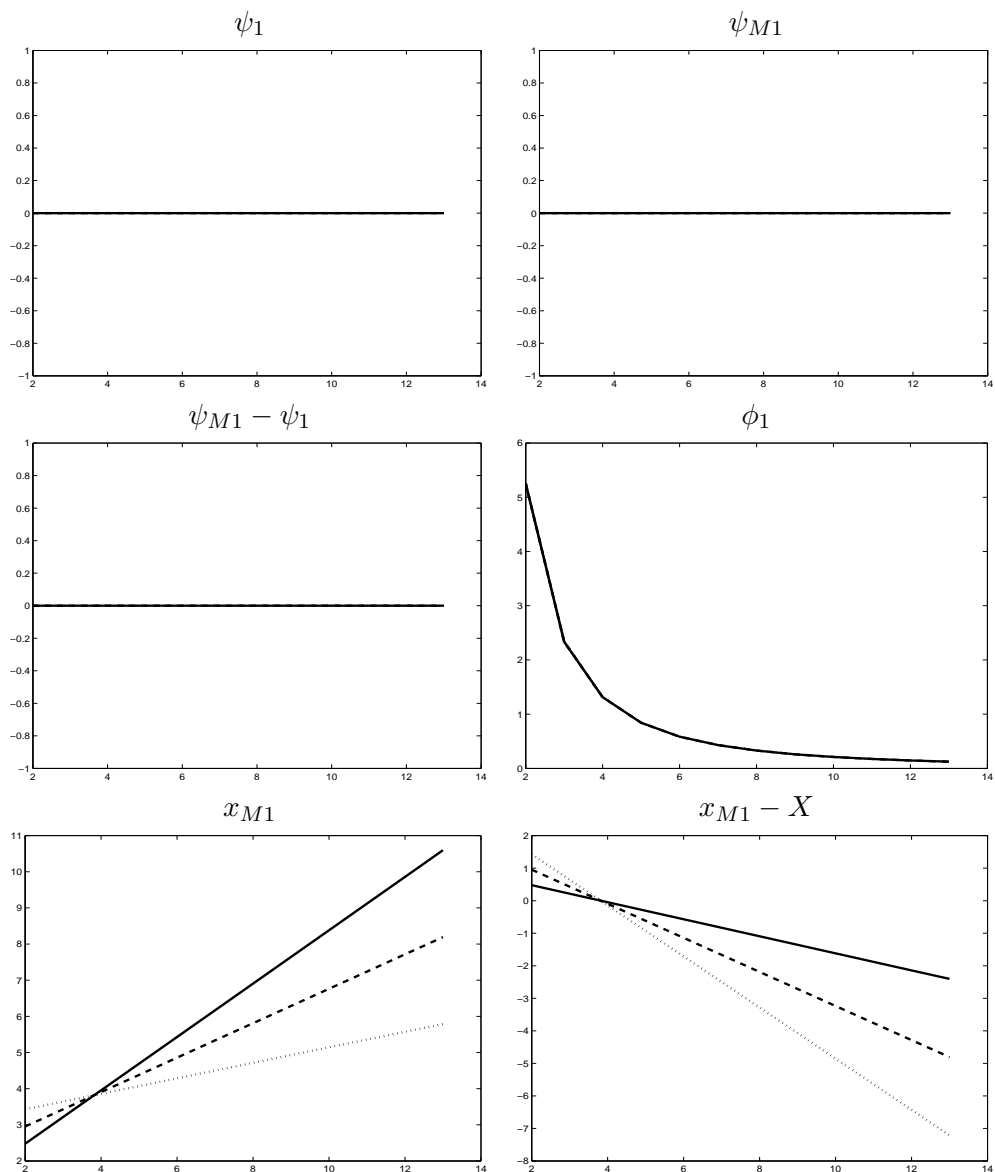


Figure 2. $\rho = 1, A = 0.1$

This figure shows the effect of different Δ_a while fixing $A = 0.1$. The solid line corresponds to $\Delta_a = -1$, the dashed line corresponds to $\Delta_a = -2$, and the dotted line corresponds to $\Delta_a = -3$.

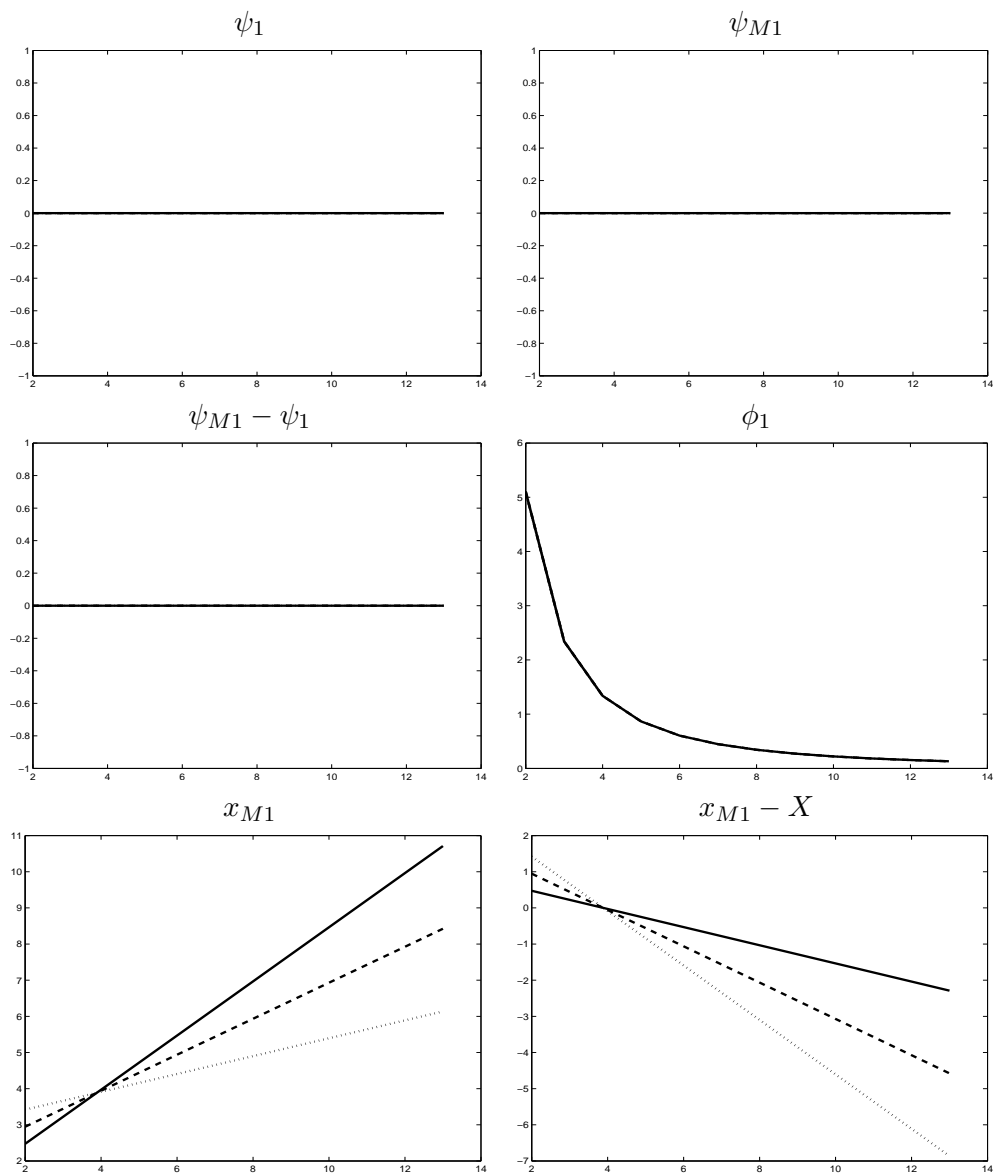


Figure 3. $\rho = 1, A = -0.1$

This figure shows the effect of different Δ_a while fixing $A = -0.1$. The solid line corresponds to $\Delta_a = -1$, the dashed line corresponds to $\Delta_a = -2$, and the dotted line corresponds to $\Delta_a = -3$.

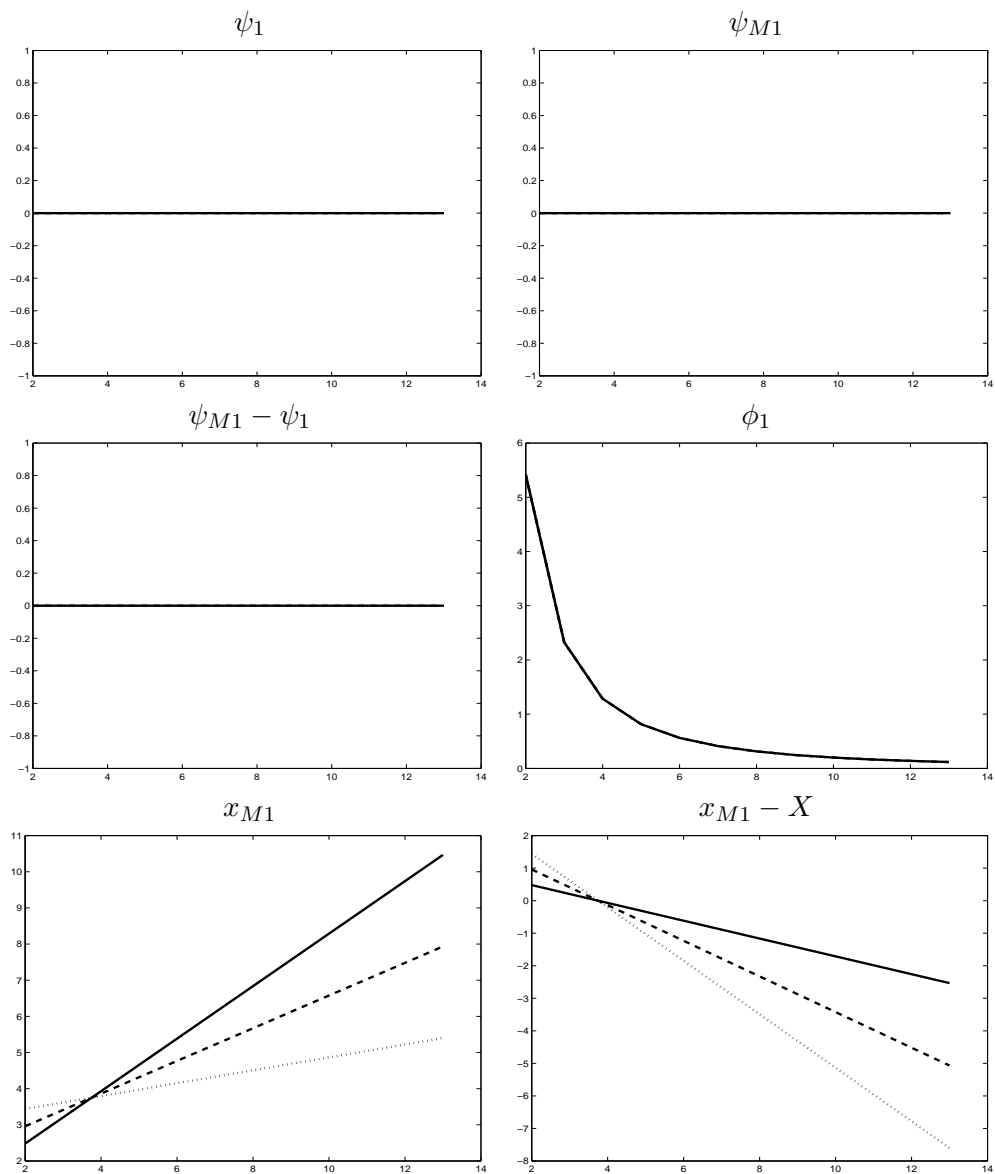


Figure 4. $\rho = 1, \Delta_a = -1$

This figure shows the effect of different A while fixing $\Delta_a = -1$. The solid line corresponds to $A = 1$, the dashed line corresponds to $A = 0$, and the dotted line corresponds to $A = -1$.

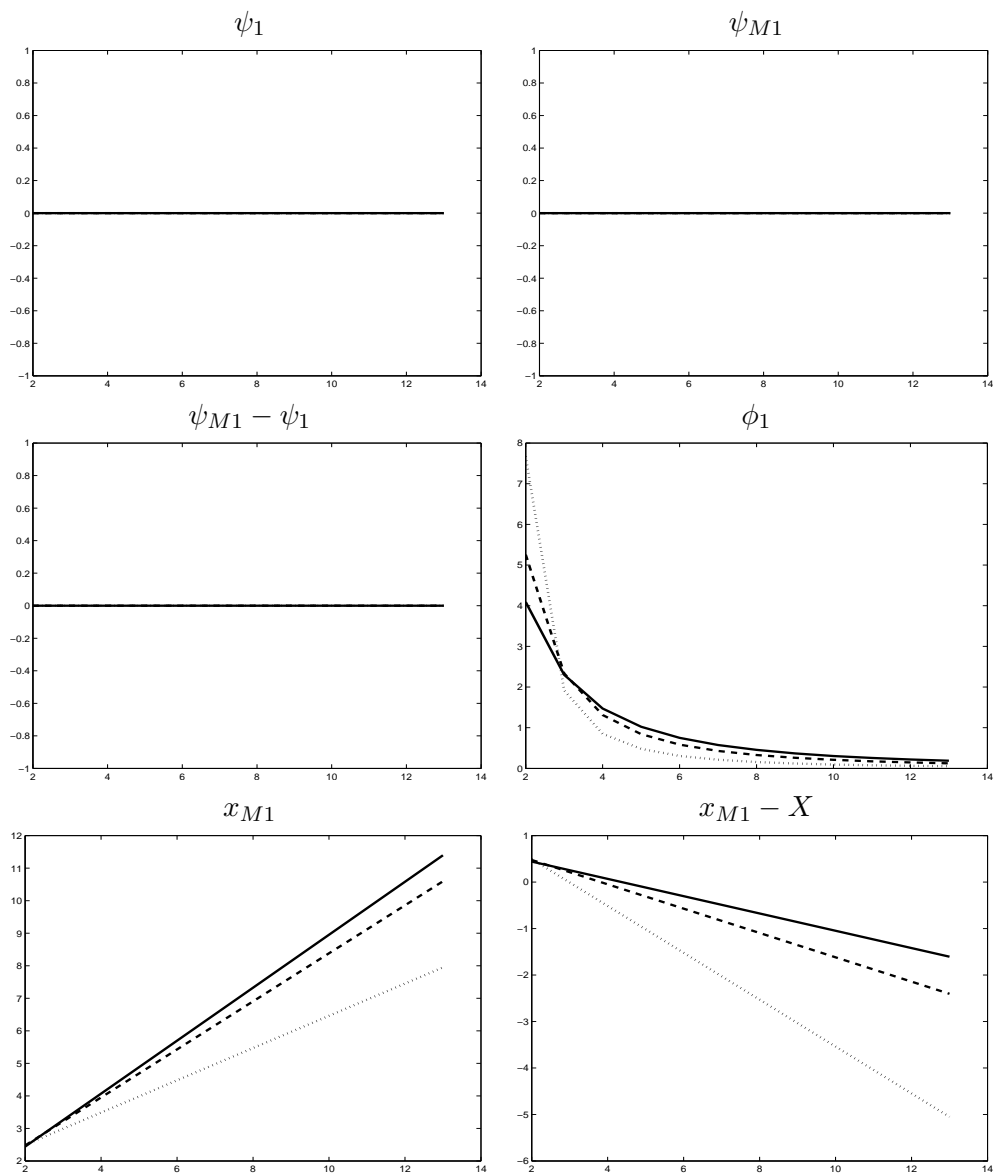


Figure 4A. Full resolution, $A = 0.5$

This figure shows the effect of different ρ while fixing $\rho = 5$. The solid line corresponds to $A = 1$, the dashed line corresponds to $\rho = 1$, and the dotted line corresponds to $\rho = 1/5$.

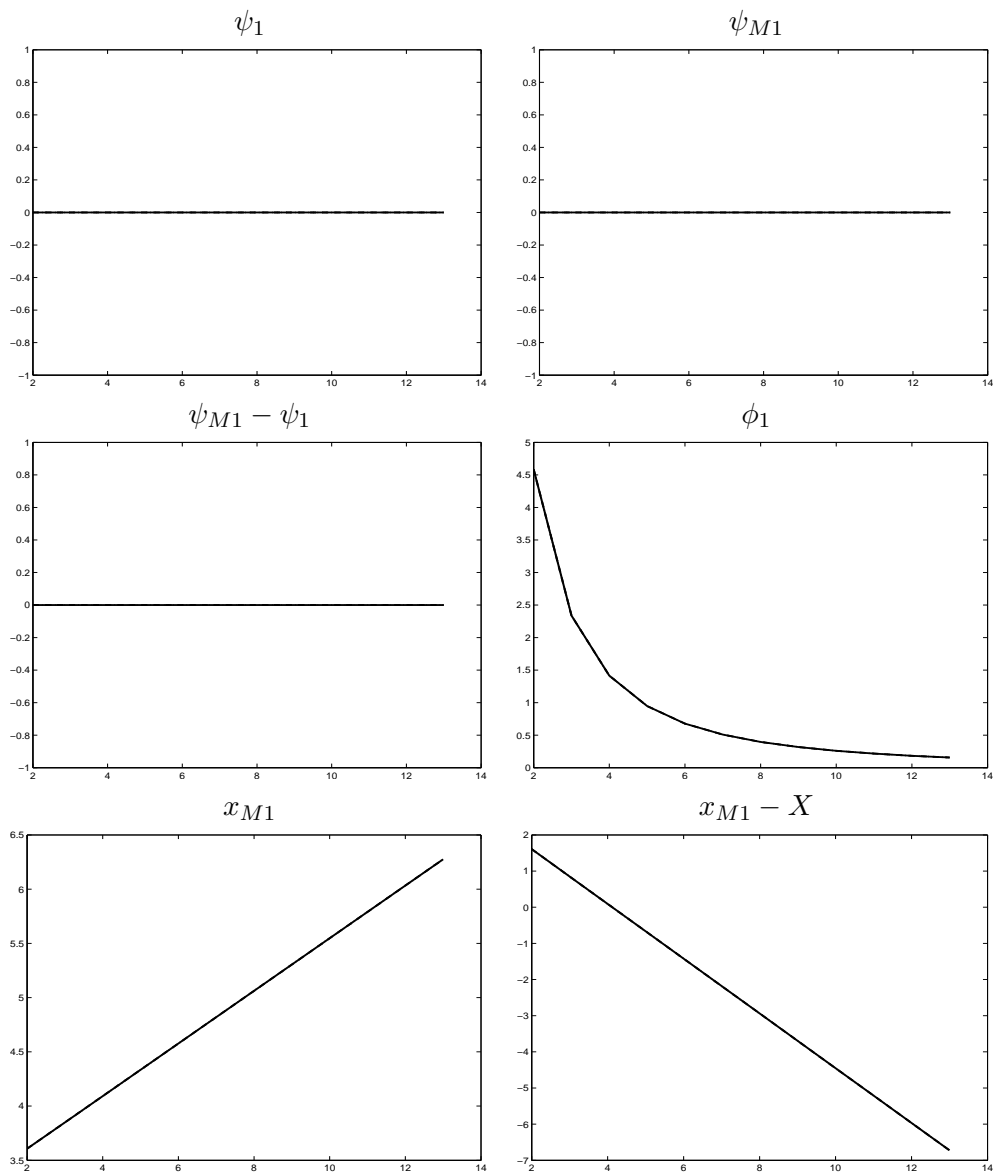


Figure 5. General Case: $\rho = 1$

This figure shows the effect of different A while fixing $a_m = -3$. The solid line corresponds to $a_n = 4, A = 0.5$, the dashed line corresponds to $a_n = 3, A = 0$, and the dotted line corresponds to $a_n = 2, A = -0.5$.

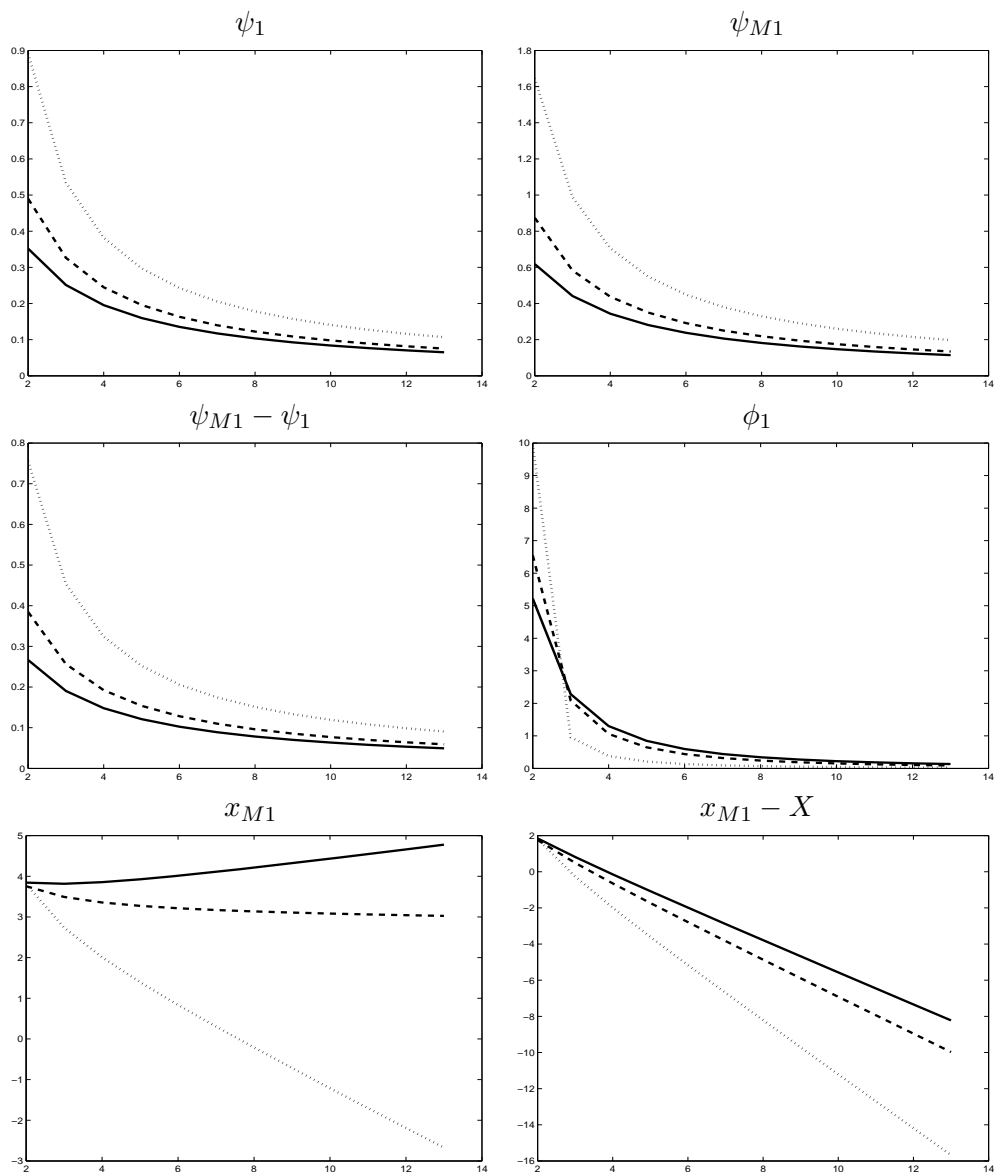


Figure 6. General Case: $a_m = 2, a_n = -2$

This figure shows the effect of holding individual shocks a_m, a_n fixed while the number of agents are changing. The solid line corresponds to $\rho = 3$, the dashed line corresponds to $\rho = 1$, and the dotted line corresponds to $\rho = 1/3$.

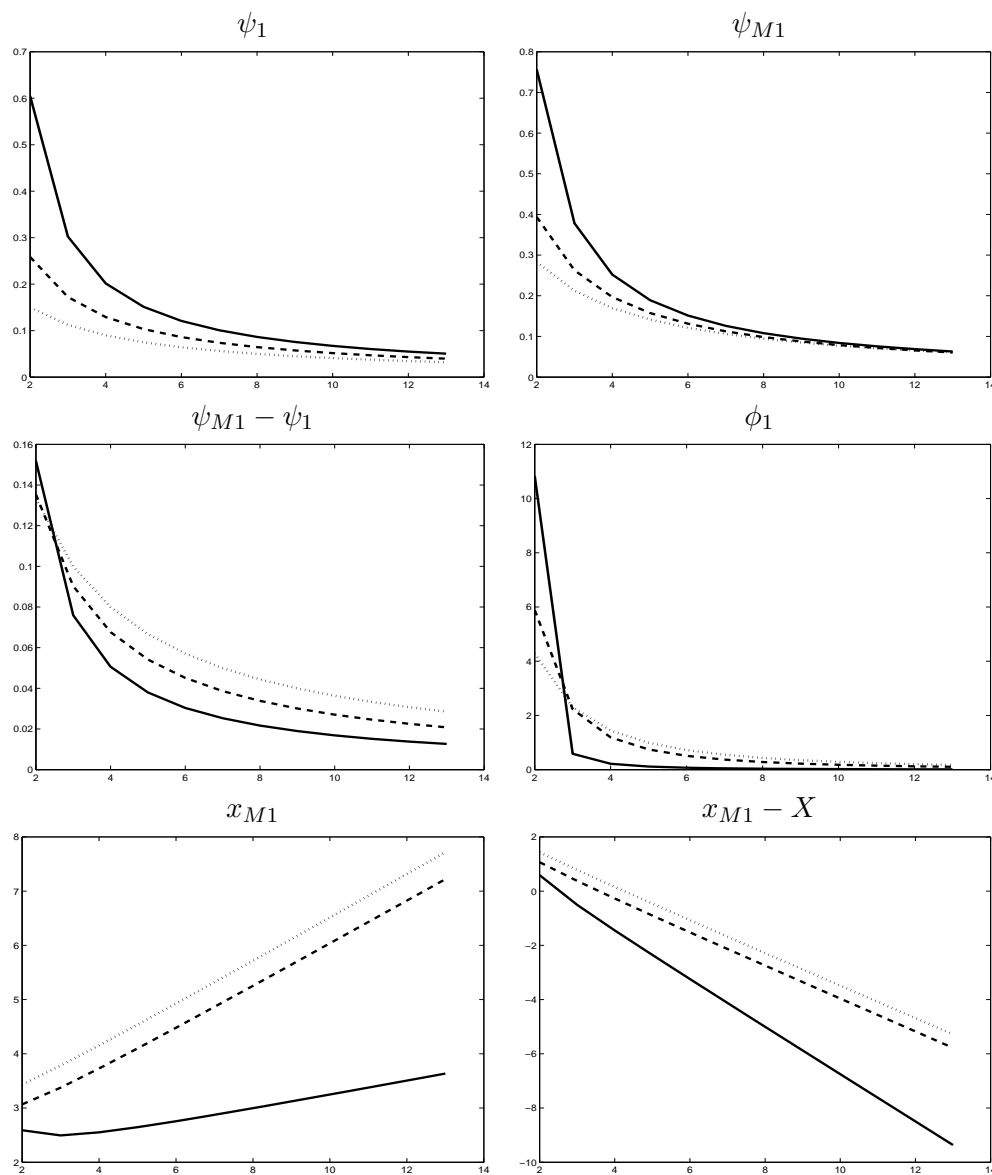


Figure 7. General Case: $\rho = 1/2$ changing a_n , thus changing A

This figure shows the effect of holding $\rho = 1/2$ fixed while the number of agents are changing. The solid line corresponds to $A = .5$, the dashed line corresponds to $A = 0$, and the dotted line corresponds to $A = -.5$.

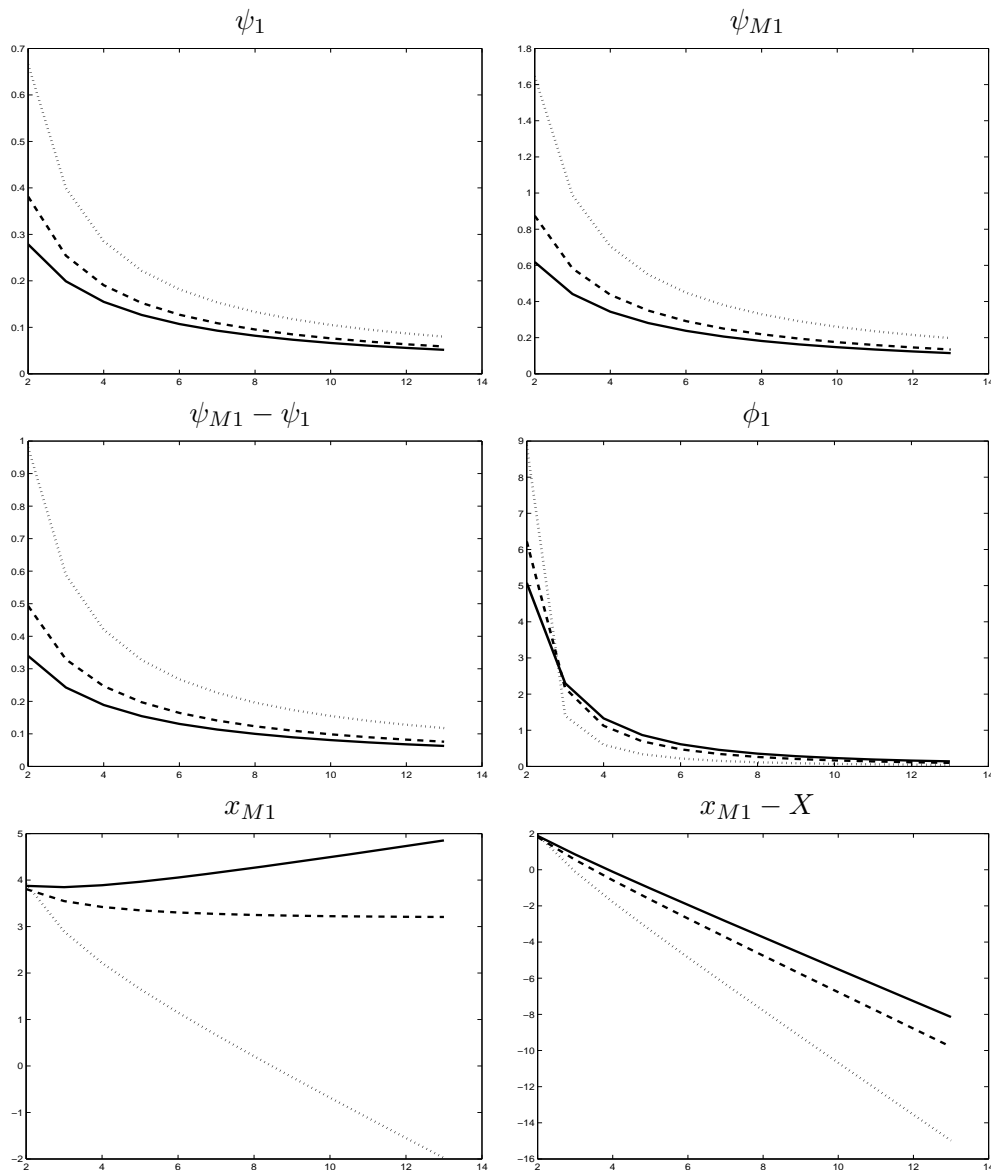


Figure 8. General Case: $A = 0$ changing ρ and a_n

This figure shows the effect of holding $A = 0$ fixed while the number of agents are changing. The solid line corresponds to $\rho = 5$, the dashed line corresponds to $\rho = 1$, and the dotted line corresponds to $\rho = 1/5$.

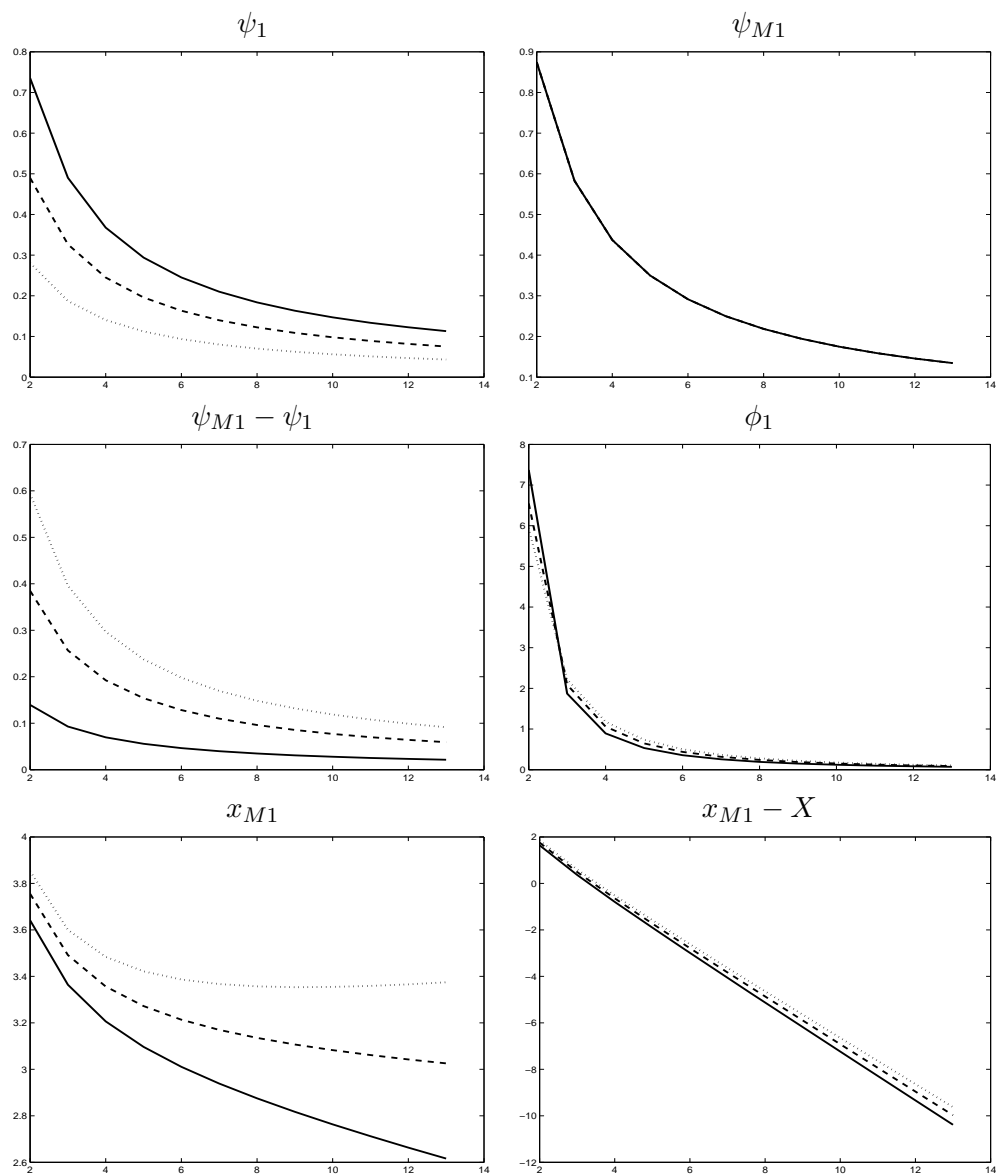


Figure 9. General Case: $A = 0.5$ changing ρ and a_n

This figure shows the effect of holding $A = 0.5$ fixed while the number of agents are changing. The solid line corresponds to $\rho = 5$, the dashed line corresponds to $\rho = 1$, and the dotted line corresponds to $\rho = 1/5$.

