Bäcklund transforms of chiral fields

D. Kramer and G. Neugebauer
Friedrich-Schiller-Universität Jena, Theoretisch-Physikalisches Institut, Max-Wien-Platz 1, O-6900 Jena, Germany

T. Matos
Centro de Investigacion y de Estudios Avanzados, del I.P.N., Apartado Postal 14-740, Mexico 07000, D.F.

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The action of the Bäcklund transformations on the axisymmetric chiral fields is investigated. In particular, it is calculated how the superpotential Γ associated with any chiral field is affected by the Bäcklund transformations.

I. INTRODUCTION

The axisymmetric chiral field equations are derivable from a two-dimensional variational principle with a Lagrangian L of the particular structure

$$L = \nabla g \nabla g^{-1},$$

the chiral field g being an $N \times N$ matrix. Examples are the stationary axisymmetric vacuum and electrovacuum fields in General Relativity, where g is an element of SU(1,1) and SU(2,1), respectively.

The linear problem associated with the nonlinear chiral field equations[1] leads to the possibility to apply the polynomial method[2] in order to generate new solutions from known (seed) solutions of the chiral field equations. This generation method corresponds to the Bäcklund transformations used in the theory of nonlinear evolution equations.[3,4]

There is a superpotential Γ defined in Eq. (6) below. For stationary axisymmetric vacuum gravitational fields, in the Lewis–Papapetrou form of the metric,

$$ds^2 = (1/f)[e^{2\Gamma}(d\rho^2 + dz^2) + \rho^2\, d\varphi^2)] - f(dt + \omega\, d\varphi)^2 \tag{1}$$

the superpotential Γ is one of the three gravitational potentials $f = f(\rho,z)$, $\omega = \omega(\rho,z)$, and $\Gamma = \Gamma(\rho,z)$. Whereas the matrix g can be calculated from f and ω, the superpotential Γ is related to the conformal factor of the two-metric orthogonal to the group orbits in space-time.

In this paper we will derive explicit expressions for the change of Γ and g under Bäcklund transformations.

All our calculations are valid for *principal* chiral fields, i.e., the chiral fields are not *a priori* subject to special constraints. In physical applications, such constraints have to be imposed on g.

II. THE CHIRAL FIELD EQUATIONS AND THE ASSOCIATED LINEAR PROBLEM

By definition, a cylindrically symmetric or stationary axisymmetric chiral field is represented by an $N \times N$ matrix g satisfying the nonlinear partial differential equation

$$(1/\rho)(\rho g_{,u} g^{-1})_{,v} + (1/\rho)(\rho g_{,v} g^{-1})_{,u} = 0. \tag{2}$$

The chiral field g depends on the two variables u and v which can be either real ($\bar{u} = u, \bar{v} = v$) for cylindrically symmetric fields or complex conjugate ($v = \bar{u}$) for stationary axisymmetric fields. The radial coordinate ρ is defined by $\rho := (u + v)/2$.

The nonlinear equation (2) is implied by the linear equations

$$\Psi_{,u} = \tfrac{1}{2}(1+\lambda)A\Psi, \quad A := g_{,u} g^{-1},$$
$$\Psi_{,v} = \tfrac{1}{2}(1+\lambda^{-1})B\Psi, \quad B := g_{,v} g^{-1},$$
$$\lambda := (K - iv)^{1/2}(K + iu)^{-1/2}, \tag{3}$$

K being a complex constant. For any given chiral field g one can calculate a corresponding $N \times N$ matrix function $\Psi(\lambda,u,v)$ with the normalization

$$\Psi(1) = g. \tag{4}$$

In this article, the dependence of Ψ on u and v is not explicitly indicated, $\Psi(\lambda) \equiv \Psi(\lambda,u,v)$.

Note that g and, consequently, the matrices A and B defined in (3) do not depend on λ.

To show that the linear problem (3) implies the chiral field equation (2), one has to consider the integrability condition $\Psi_{,u,v} = \Psi_{,v,u}$ and to use the formulas

$$\lambda_{,u} = (1/4\rho)(\lambda^2 - 1)\lambda, \quad \lambda_{,v} = (1/4\rho)(\lambda^2 - 1)/\lambda, \tag{5}$$

which follow from the definition of λ as given in (3). The integrability condition, which must be true separately for the terms with different powers of λ, leads to (2), without any further restrictions.

As mentioned in the Introduction, for any chiral field g, there is an associated superpotential Γ defined by

$$\Gamma_{,u} = \tfrac{1}{4}\rho\, \mathrm{Tr}\, A^2, \quad \Gamma_{,v} = \tfrac{1}{4}\rho\, \mathrm{Tr}\, B^2. \tag{6}$$

One can easily verify that the integrability condition $\Gamma_{,u,v} = \Gamma_{,v,u}$ holds as a consequence of the chiral field equation (2).

Starting with the linear equations (3) and using (5) one derives the expressions[5]

$$A = (1/2\rho) \lim_{\lambda \to \infty} (\lambda^2 \Psi' \Psi^{-1}),$$
$$B = -\frac{1}{2\rho} \lim_{\lambda \to 0} (\Psi' \Psi^{-1}), \quad \Psi' := \frac{\partial \Psi}{\partial \lambda}. \tag{7}$$

It is the aim of this paper to calculate Γ, for a special class of solutions to (2), from (6) and (7).

III. THE POLYNOMIAL METHOD

Suppose we have at hand a seed solution g_0 to (2) and the associated solution Ψ_0 to the linear problem (2). Then we try to generate a new solution Ψ by means of the ansatz[2]

$$\Psi = T\Psi_0, \quad T = q_n P_n(\lambda), \tag{8}$$

where $P_n(\lambda)$ is a matrix polynomial in λ of degree n,

$$P_n(\lambda) = \sum_{s=0}^{n} a_s \lambda^s, \tag{9}$$

with λ-independent $N \times N$ matrix coefficients a_s. The scalar term q_n can be appropriately chosen as

$$q_n = \kappa(K)(K + iu)^{n/2}. \tag{10}$$

The matrix Ψ constructed in this way must again satisfy linear equations of the form (3). This behavior is guaranteed if the coefficients a_s in (9) are determined from the following set of algebraic equations:

$$P_n(\lambda_i)\Psi_0(\lambda_i)C_i = 0, \tag{11a}$$

$$P_n(-1) = I, \tag{11b}$$

C_i being constant N-dimensional column eigenvectors. The operation (8)–(11) is called the n-fold Bäcklund transformation (BT) because it corresponds to the BT in the theory of nonlinear evolution equations.[4]

Equation (11a) means that the λ_i's are the Nn zeros of $\det P_n(\lambda)$ provided that $\Psi_0(\lambda_i)$ is regular and $\det \Psi_0(\lambda_i) \neq 0$. (The zeros λ_i have to be prescribed.) Under these conditions one infers that the new Ψ generated from Ψ_0 by the BT (8)–(11) satisfies (3). The proof runs as follows.[6]

The expression

$$Y(\lambda) := \lambda \Psi_{,v}\Psi^{-1} \det P_n$$
$$= \lambda\left[P_{n,v} + \tfrac{1}{2}(1 + \lambda^{-1})P_n B_0\right](P_n^{-1}\det P_n) \tag{12}$$

is a matrix polynomial in λ of degree $Nn + 1$. From (11a) and (8) one gets

$$\kappa_i^{-1}\Psi(\lambda_i)C_i = 0, \quad i = 1 \cdots Nn,$$
$$\lambda_i := (K_i - iv)^{1/2}(K_i + iu)^{-1/2}, \quad \kappa_i := \kappa_i(K_i). \tag{13}$$

The comparison of (13) with the identity

$$\kappa_i^{-1}\Psi(\lambda_i)\kappa_i\Psi^{-1}(\lambda_i)\det P_n(\lambda_i) = 0 \tag{14}$$

shows that the columns of $\kappa_i\Psi^{-1}(\lambda_i)\det P_n(\lambda_i)$ are proportional to C_i. Using this fact one obtains from (13) after differentiation with respect to v (the C_i's are constants)

$$[\kappa_i^{-1}\Psi(\lambda_i)]_{,v}\kappa_i\Psi^{-1}(\lambda_i)\det P_n(\lambda_i) = 0, \tag{15}$$

which yields $Y(\lambda_i) = 0$, i.e., the polynomials $Y(\lambda)$ and $\det P_n(\lambda)$ have the same zeros $\lambda_i, i = 1 \cdots Nn$. Hence, the polynomial $Y(\lambda)/\det P_n(\lambda)$ is linear in λ, and $\Psi_{,v}\Psi^{-1}$ is linear in λ^{-1}. Similarly, it turns out that $\Psi_{,u}\Psi^{-1}$ is linear in λ:

$$\Psi_{,u}\Psi^{-1} = \alpha + \beta\lambda,$$
$$\Psi_{,v}\Psi^{-1} = \gamma + \delta\lambda^{-1}. \tag{16}$$

Finally, from (11b) one finds $\alpha = \beta \equiv \frac{1}{2}A$ and $\gamma = \delta \equiv \frac{1}{2}B$, and we conclude that Ψ satisfies the linear problem (3). According to (4) one can assign to Ψ the new chiral field g generated from g_0 by the BT (8)–(11).

The BT method described in this section generates from any seed g_0 a new chiral field g which contains the additional parameters K_i corresponding to the zeros λ_i of $\det P_n(\lambda)$, and C_i.

IV. THE CALCULATION OF THE SUPERPOTENTIAL Γ

Now we will derive an expression for the superpotential Γ defined in (6) in terms of the seed potential Γ_0, and the constants K_i and C_i arising from the BT (8)–(11). For the class of chiral fields generated in this way, one can calculate the corresponding superpotential Γ algebraically whereas otherwise the calculation of Γ leads to line integrals, see (6).

First we determine A and B according to (7) from the seed quantities A_0 and B_0,

$$A_0 = (1/2\rho)\lim_{\lambda\to\infty}(\lambda^2\Psi_0'\Psi_0^{-1}),$$
$$B_0 = -(1/2\rho)\lim_{\lambda\to 0}(\Psi_0'\Psi_0^{-1}). \tag{17}$$

The relations (8)–(10) link the new Ψ with the original Ψ_0 and the term q_n in (10) can be chosen in the form

$$q_n = \prod_{i=1}^{n}\left(\frac{1-\lambda_i^2}{\lambda^2 - \lambda_i^2}\right)^{1/2}, \tag{18}$$

where the product is formed with n of the Nn zeros λ_i. From (8) it follows that the quantity $\Psi'\Psi^{-1}$ in (7) reads

$$\Psi'\Psi^{-1} = P_n'P_n^{-1} + P_n\Psi_0'\Psi_0^{-1}P_n^{-1} + q_n'q_n^{-1}. \tag{19}$$

To calculate A from (7) it is convenient to introduce $\tilde{q}_n := \lambda^n q_n$ and the polynomial (in λ^{-1}) $\tilde{P}_n := \lambda^{-n}P_n$ with the limits

$$\lim_{\lambda\to\infty}\tilde{P}_n = a_n, \quad \lim_{\lambda\to\infty}\lambda^2\tilde{P}_n'\tilde{P}_n^{-1} = -a_{n-1}a_n^{-1},$$
$$\lim_{\lambda\to\infty}\lambda^2\tilde{q}_n'\tilde{q}_n^{-1} = 0. \tag{20}$$

On the other hand, the calculation of B from (7) requires the limit $\lambda \to 0$ of (19); one obtains

$$\lim_{\lambda\to 0}P_n = a_0, \quad \lim_{\lambda\to 0}P_n'P_n^{-1} = a_1a_0^{-1}, \quad \lim_{\lambda\to 0}q_n'q_n^{-1} = 0. \tag{21}$$

Using the limits (20), (21) one derives from (7) and (19) the relations

$$A = a_nA_0a_n^{-1} - (1/2\rho)a_{n-1}a_n^{-1},$$
$$B = a_0B_0a_0^{-1} - (1/2\rho)a_1a_0^{-1}, \tag{22}$$

and the terms $\rho \operatorname{Tr} A^2$ and $\rho \operatorname{Tr} B^2$, which occur in the definition (6) of the superpotential Γ, are given by

$$\rho \operatorname{Tr} A^2 = \rho \operatorname{Tr} A_0^2 - \operatorname{Tr}(A_0a_n^{-1}a_{n-1}) + (1/4\rho)\operatorname{Tr}(a_n^{-1}a_{n-1})^2,$$
$$\rho \operatorname{Tr} B^2 = \rho \operatorname{Tr} B_0^2 - \operatorname{Tr}(B_0a_0^{-1}a_1) + (1/4\rho)\operatorname{Tr}(a_0^{-1}a_1)^2. \tag{23}$$

The linear algebraic equations (11a) for the matrices $a_0,\ldots,a_n$ can be written in the form

$$a_n^{-1}(a_0,a_1,\ldots,a_{n-1})\mathbb{D} = -(\Psi_1,\Psi_2,\ldots,\Psi_n)\Lambda^n, \tag{24}$$

where the following definitions are used. The elements of the

$nN \times nN$ block matrices $\mathbb{D}$ and Λ,

$$\mathbb{D}:=\begin{pmatrix} \Psi_1 & \cdots & \Psi_n \\ \Psi_1\Lambda_1 & \cdots & \Psi_n\Lambda_n \\ \vdots & \ddots & \vdots \\ \Psi_1\Lambda_1^{n-1} & \cdots & \Psi_n\Lambda_n^{n-1} \end{pmatrix}, \qquad (25)$$

$$\Lambda:=\begin{pmatrix} \Lambda_1 & 0 & \cdots & 0 \\ 0 & \Lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \Lambda_n \end{pmatrix} \qquad (26)$$

are the $N \times N$ matrices Λ_k and Ψ_k $(k = 1 \cdots n)$ defined by

$$\Lambda_k:=\begin{pmatrix} \lambda_{(k-1)N+1} & 0 & \cdots & 0 \\ 0 & \lambda_{(k-1)N+2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_{kN} \end{pmatrix}, \qquad (27)$$

$$\Psi_k:=(\Psi_0(\lambda_{(k-1)N+1})C_{(k-1)N+1},\ldots,\Psi_0(\lambda_{kN})C_{kN}). \qquad (28)$$

Note that the C_i's are column vectors.

The nN zeros λ_i, and the corresponding eigenvectors C_i, are arbitrarily ordered into n sets of N elements each. The brackets in (24) denote block row vectors.

Because of (5) and

$$\Psi_{0,u} = \tfrac{1}{2}(1+\lambda)A_0\Psi_0,$$
$$\Psi_{0,v} = \tfrac{1}{2}(1+1/\lambda)B_0\Psi_0, \qquad (29)$$

the block matrix $\mathbb{D}$ satisfies the relations

$$\mathbb{D}_{,u} = \tfrac{1}{2}\mathbb{A}_0\mathbb{D}(\Lambda+\mathbb{I}) + (1/4\rho)\mathbb{N}\mathbb{D}(\Lambda^2-\mathbb{I}),$$
$$\mathbb{D}_{,v} = \tfrac{1}{2}\mathbb{B}_0\mathbb{D}(\Lambda^{-1}+\mathbb{I}) - (1/4\rho)\mathbb{N}\mathbb{D}(\Lambda^{-2}-\mathbb{I}), \qquad (30)$$

where the block diagonal matrices $\mathbb{A}_0, \mathbb{B}_0, \mathbb{N}, \mathbb{I}$ are defined by

$$\mathbb{A}_0:=\begin{pmatrix} A_0 & 0 & \cdots & 0 \\ 0 & A_0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_0 \end{pmatrix},$$

$$\mathbb{B}_0:=\begin{pmatrix} B_0 & 0 & \cdots & 0 \\ 0 & B_0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & B_0 \end{pmatrix}$$

$$\mathbb{N}:=\begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & I & 0 & \cdots & 0 \\ 0 & 0 & 2I & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & (n-1)I \end{pmatrix},$$

$$\mathbb{I}:=\begin{pmatrix} I & 0 & \cdots & 0 \\ 0 & I & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & I \end{pmatrix}, \qquad (31)$$

I being the $N \times N$ unit matrix.

From the first part of (30) we conclude

$$(\ln\det\mathbb{D})_{,u} = \mathrm{Tr}(\mathbb{D}_{,u}\mathbb{D}^{-1})$$
$$= \tfrac{1}{2}\mathrm{Tr}(\mathbb{A}_0\mathbb{D}\Lambda\mathbb{D}^{-1} + \mathbb{A}_0)$$
$$+ (1/4\rho)\,\mathrm{Tr}(\mathbb{N}\mathbb{D}\Lambda^2\mathbb{D}^{-1} - \mathbb{N}). \qquad (32)$$

The definitions (31) and $A_0 = g_{0,u}g_0^{-1}$ imply

$$\mathrm{Tr}\,\mathbb{A}_0 = n\,\mathrm{Tr}\,A_0 = n\,(\ln\det g_0)_{,u}$$
$$\mathrm{Tr}\,\mathbb{N} = (Nn/2)(n-1). \qquad (33)$$

To calculate the remaining terms in (32) one needs the diagonal elements of the block matrices $\mathbb{R}:=\mathbb{D}\Lambda\mathbb{D}^{-1}$ and $\mathbb{R}^2$. In the block matrix formulation, Eq. (24) can be rewritten as

$$\begin{pmatrix} 0 & I & 0 & \cdots & 0 \\ 0 & 0 & I & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & I \\ x_1 & x_2 & x_3 & \cdots & x_n \end{pmatrix}\begin{pmatrix} \Psi_1 & \cdots & \Psi_n \\ \Psi_1\Lambda_1 & \cdots & \Psi_n\Lambda_n \\ \vdots & \ddots & \vdots \\ \Psi_1\Lambda_1^{n-2} & \cdots & \Psi_n\Lambda_n^{n-2} \\ \Psi_1\Lambda_1^{n-1} & \cdots & \Psi_n\Lambda_n^{n-1} \end{pmatrix}$$
$$=\begin{pmatrix} \Psi_1\Lambda_1 & \cdots & \Psi_n\Lambda_n \\ \Psi_1\Lambda_1^2 & \cdots & \Psi_n\Lambda_n^2 \\ \vdots & \ddots & \vdots \\ \Psi_1\Lambda_1^{n-1} & \cdots & \Psi_n\Lambda_n^{n-1} \\ \Psi_1\Lambda_1^{n} & \cdots & \Psi_n\Lambda_n^{n} \end{pmatrix}, \qquad (34)$$

with

$$(x_1,x_2,\ldots,x_n):= -a_n^{-1}(a_0,a_1,\ldots,a_{n-1}). \qquad (35)$$

The second matrix in (34) is $\mathbb{D}$ as defined in (25) and the matrix on the right-hand side of (34) is $\mathbb{D}\Lambda$. Hence, one infers from (34)

$$\mathbb{R}=\begin{pmatrix} 0 & I & 0 & \cdots & 0 \\ 0 & 0 & I & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & I \\ x_1 & x_2 & x_3 & \cdots & x_n \end{pmatrix} \qquad (36)$$

and the diagonal elements of $\mathbb{R}^2$ are found to be

$$\mathbb{R}^2=\begin{pmatrix} 0 & & & & & \\ & 0 & & & & \\ & & \ddots & & & \\ & & & 0 & & \\ & & & & x_{n-1} & \\ & & & & & y_n \end{pmatrix},$$

$$x_{n-1} = -a_n^{-1}a_{n-2},$$
$$x_n = -a_n^{-1}a_{n-1}, \qquad (37)$$
$$y_n = x_n^2 + x_{n-1}.$$

(The irrelevant off-diagonal elements have been omitted.) From (37) one gets

$$\mathrm{Tr}\,\mathbb{R}^2 = \mathrm{Tr}\,\Lambda^2 = \sum_{i=1}^{Nn}\lambda_i^2$$
$$= \mathrm{Tr}(a_n^{-1}a_{n-1})^2 - 2\,\mathrm{Tr}(a_n^{-1}a_{n-2}). \qquad (38)$$

With (36), (37) the terms $\mathrm{Tr}(\mathbb{A}_0\mathbb{R})$ and $\mathrm{Tr}(\mathbb{N}\mathbb{R}^2)$ in (32) can be written as

$$\mathrm{Tr}(\mathbb{A}_0\mathbb{R}) = -\mathrm{Tr}(A_0 a_n^{-1} a_{n-1}),$$
$$\mathrm{Tr}(\mathbb{N}\mathbb{R}^2) = -(2n-3)\,\mathrm{Tr}(a_n^{-1}a_{n-2}) + (n-1)\,\mathrm{Tr}(a_n^{-1}a_{n-1})^2, \tag{39}$$

and one obtains from (32), (33), and (39)

$$(\ln\det\mathbb{D})_{,u} = -\frac{1}{2}\mathrm{Tr}(A_0 a_n^{-1}a_{n-1}) + \frac{n}{2}(\ln\det g_0)_{,u} - \frac{(2n-3)}{4\rho}\mathrm{Tr}(a_n^{-1}a_{n-2}) + \frac{n-1}{4\rho} \times \mathrm{Tr}(a_n^{-1}a_{n-1})^2 - \frac{Nn}{8\rho}(n-1). \tag{40}$$

Inserting (38) into (41) one finds the relation

$$(\ln\det\mathbb{D})_{,u} = -\frac{1}{2}\mathrm{Tr}(A_0 a_n^{-1}a_{n-1}) + \frac{n}{2}(\ln\det g_0)_{,u} + \frac{1}{8\rho}\mathrm{Tr}(a_n^{-1}a_{n-1})^2 + \frac{2n-3}{8\rho}\sum_{i=1}^{Nn}\lambda_i^2 - \frac{Nn}{8\rho}(n-1). \tag{41}$$

Taking into account Eq. (23) and the definitions (6) together with the corresponding relations

$$\Gamma_{0,u} = \tfrac{1}{4}\rho\,\mathrm{Tr}\,A_0^2, \quad \Gamma_{0,v} = \tfrac{1}{4}\rho\,\mathrm{Tr}\,B_0^2 \tag{42}$$

for the seed solution, Eq. (41) reads

$$(\ln\det\mathbb{D})_{,u} = 2(\Gamma-\Gamma_0)_{,u} + \frac{n}{2}(\ln\det g_0)_{,u} + \frac{2n-3}{8\rho}\sum_{i=1}^{Nn}\lambda_i^2 - \frac{Nn}{8\rho}(n-1). \tag{43}$$

A similar consideration yields

$$(\ln\det\mathbb{D})_{,v} = 2(\Gamma-\Gamma_0)_{,v} + \frac{n}{2}(\ln\det g_0)_{,v} - \frac{1}{8\rho}\sum_{i=1}^{Nn}\lambda_i^{-2} + \frac{Nn}{8\rho}(n-1). \tag{44}$$

Finally, the integration of (43) and (44) leads to the expression

$$e^{2\Gamma} = Me^{2\Gamma_0}\det\mathbb{D}\,\frac{\rho^{Nn(n-2)/4}}{(\det g_0)^{n/2}}\prod_{i=1}^{Nn}\frac{(\lambda_i^2-1)^{1-n/2}}{\lambda_i^{1/2}} \tag{45}$$

for the Bäcklund transform of the superpotential Γ. (M is an arbitrary constant of integration.) This formula contains as a particular case (for $N=2$) the results derived in Refs. 1 and 7 for stationary axisymmetric vacuum fields.

V. THE CALCULATION OF THE CHIRAL FIELD g

The new chiral field g generated from g_0 by means of a BT is given by

$$g = P_n(1)g_0, \quad P_n(1) = \sum_{s=0}^{n} a_s. \tag{46}$$

The matrices $a_0,\ldots,a_n$ are determined from the algebraic equations (11) which are equivalent to

$$P_n(-1) = \sum_{s=0}^{n}(-1)^s a_s = I \tag{47}$$

and (24),

$$a_n^{-1}(a_0,a_1,\ldots,a_{n-1}) = -(\Psi_1\Lambda_1^n,\ldots,\Psi_n\Lambda_n^n)\mathbb{D}^{-1}, \tag{48}$$

where the definitions (25)–(28) have been used.

From (46) one gets

$$a_n^{-1}(P_n(1)-a_n) = a_n^{-1}P_n(1) - I = a_n^{-1}(a_0,a_1,\ldots,a_{n-1})\begin{pmatrix} I \\ I \\ \vdots \\ I \\ I \end{pmatrix}, \tag{49}$$

whereas (47) implies

$$a_n^{-1}(P_n(-1)-a_n) = a_n^{-1} - I = a_n^{-1}(a_0,a_1,\ldots,a_{n-1})\begin{pmatrix} I \\ -I \\ \vdots \\ I \\ -I \end{pmatrix}. \tag{50}$$

From (48), and the last two equations, one obtains the final result:

$$g = P_n(1)g_0,$$
$$P_n(1) = \left[I - (\Psi_1\Lambda_1^n,\ldots,\Psi_n\Lambda_n^n)\mathbb{D}^{-1}\begin{pmatrix} I \\ -I \\ \vdots \\ I \\ -I \end{pmatrix}\right]^{-1} \times \left[I - (\Psi_1\Lambda_1^n,\ldots,\Psi_n\Lambda_n^n)\mathbb{D}^{-1}\begin{pmatrix} I \\ I \\ \vdots \\ I \\ I \end{pmatrix}\right]. \tag{51}$$

Summarizing our results, the Bäcklund transforms of Γ and g are given by the formulas (45) and (51), respectively.

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