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BACKWARD–FORWARD STOCHASTIC DIFFERENTIAL EQUATIONS

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This paper shows the existence and uniqueness of the solution of a backward stochastic differential equation inspired from a model for stochastic differential utility in finance theory. We show our results assuming, when possible, no more than the integrability of the terms involved in the equation. We also show the existence and uniqueness of the solution of a backward–forward stochastic differential equation, where the solution depends explicitly on both the past and the future of its own trajectory, under a more restrictive hypothesis on the Lipschitz constant.

1. Introduction. The goal of this work is to show the existence and uniqueness of an *adapted* solution of backward–forward stochastic differential equations of the type

$$\begin{aligned} U_t &= J_t + \int_0^t f_s(U_s, V_s)\, dX_s, \\ V_t &= E\left(\int_t^T g_s(U_s, V_s)\, dZ_s + Y \middle| \mathscr{F}_t\right), \qquad 0 \le t \le T, \\ V_T &= Y, \end{aligned} \tag{1.1}$$

where $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{0 \le t \le T}, P)$ is a complete filtered space satisfying the "usual hypotheses," Y is $\mathscr{F}_T$-measurable, f_s, g_s are uniformly Lipschitz [see (2.2)], X_s and Z_s are semimartingales in $\mathbf{H}^\infty$ and J_t is a cadlag progressively measurable process. This model is called backward–forward because the two components in the system are solutions, respectively, of a forward and a backward equation. If the system is reduced to the forward case only (for instance when U is known), it is possible to find examples of this class of equations mostly in control theory and in economics, when one models phenomena involving knowledge of the terminal value of the solution process. The example that in fact inspired our work is given in Duffie and Epstein (1992) by the construction of a recursively defined stochastic utility function, solving the problem of finding the optimal portfolio for an investor in a market model. For a clear overview of the setting motivating the problem, the reader may also refer to Picqué and Pontier (1990). Duffie and Epstein (1992) show the existence and uniqueness of the adapted solution for a backward equation when $Z_s = s$, $V \in \mathbf{S}^p$ and $Y \in L^p$ for $p > 1$. The case $p = 1$ is left

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open. However, we resolve it here, as well as considering more general equations. The key point in their proof is the use of the strong version of Doob's maximal inequality to show that the operator $G\colon \mathbf{S}^p \to \mathbf{S}^p$ induced by the equation [see (2.1)] is a contraction. This technique is not useful for the case $p = 1$.

Under the further assumption that $\{\mathscr{F}_t\}_{0 \le t \le T}$ is the natural filtration of a k-dimensional Brownian motion, Pardoux and Peng (1990) treat instead the similar problem of finding an *adapted* pair of processes $x(t), y(t)$, $t \in [0,1]$, with values in $\mathbb{R}^d$ and $\mathbb{R}^{d\times k}$ solving

$$(1.2)\quad x(t) + \int_t^1 f(s, x(s), y(s))\, ds + \int_t^1 g(s, x(s), y(s))\, dW_s = X,$$

where W_s is a standard k-dimensional Wiener process generating the underlying filtration, X is an F_1-measurable random variable and various L^2 integrability hypotheses are made. Here the assumption that the filtration is Brownian is important. Indeed the authors first solve a simplified version of (1.2) via the martingale representation theorem.

In this paper we relax all these conditions. We will assume that the filtration $\{\mathscr{F}_t\}_{0 \le t \le T}$ satisfies only the "usual hypotheses" and, when possible, we assume only enough integrability on each term to have the conditional expectation in (1.1) defined. In Section 2 we consider only the backward case and we prove the existence and uniqueness of the solution. By including the particular case when the integrator is just time and $p = 1$ we affirmatively resolve a conjecture of Duffie and Epstein (1992).

Finally in Section 3 we are concerned with the model (1.1) in its generality. Again we show existence and uniqueness of the solution of (1.1) whenever the Lipschitz constant k is small enough. We first consider only finite variation integrators, which is the case when the weakest integrability hypotheses are required, and then we generalize to semimartingales in $\mathbf{H}^\infty$, when strengthening the integrability assumptions. Last, we also provide two examples in a linear setting when k is large, thus explaining our restriction.

Throughout this paper we refer to Protter (1990) for notation and definitions.

2. Finite horizon backward equations. In this section we study only the backward case of (1.1), so U_s will not occur in the model. We will start with the finite variation case, which requires the weakest integrability hypotheses.

Let us consider the equation

$$(2.1)\qquad G(V)_t = E\left(\int_t^T g_s(V_s)\, dA_s + Y \middle| \mathscr{F}_t\right),$$

where A is an adapted process of bounded total variation with $A_0 = 0$ and the function $g\colon [0,T] \times \Omega \times \mathbb{R} \to \mathbb{R}$ is $\mathscr{B}([0,T]) \times \mathscr{F} \times \mathscr{B}(\mathbb{R})/\mathscr{B}(\mathbb{R})$-

measurable and such that:

1. g is uniformly Lipschitz: there exists a $k > 0$ such that

$$|g_s(u) - g_s(v)| \le k|u - v| \qquad \forall\, u, v \in \mathbb{R}, \forall\, s \in [0,T]; \tag{2.2}$$

2. $g_s(\cdot, u)\colon \Omega \to \mathbb{R}$ is $\mathscr{F}_s$-measurable for each s and u;
3. g is 1-integrable; that is, $E(\int_0^T |g_s(0)||dA_s|) < +\infty$.

Moreover,

4. $Y \in L^1(P)$ and it is $\mathscr{F}_T$-measurable.

The total variation process of A_t, $|A|_t$, will be increasing, adapted and bounded, let us say $|A|_T \le \beta$, for some positive constant β and $|A|_0 = 0$. The process $|A|$ therefore induces a finite measure on $[0,T] \times \Omega$ [see Dellacherie and Meyer (1982)] given by

$$\begin{aligned} \mu([0,t] \times B) &= \int_B \int_0^t |dA_s|\, dP \\ &= E(1_B(|A|_t - |A|_0)) = E(1_B|A|_t) \end{aligned} \tag{2.3}$$

for every $B \in \mathscr{F}$, $0 \le t \le T$, which is called the Doléans–Dade measure. By using the monotone class theorem, for any nonnegative progressively measurable process X, the integral

$$\int_{\Omega \times [0,T]} X_s(\omega)\, d\mu(s,\omega) = E\left(\int_0^T X_s(\omega)|dA_s|\right) \tag{2.4}$$

makes sense and so we can define the Banach space: $L^1(\mu) = \{V_t(\omega)$ stochastic processes on $[0,T] \times \Omega$ such that $E(\int_0^T |V_s(\omega)||dA_s|) < +\infty\}$.

This space will be the natural environment in which to look for a solution of (2.1). Before proving our main result we need a lemma that will enable us to handle the conditional expectation in the equation more easily.

LEMMA 2.1. *Let X be a positive measurable process and let Y be its optional projection. Let C be an adapted increasing cadlag process. Then*

$$E\left(\int_0^\infty X_s\, dC_s\right) = \left(\int_0^\infty Y_s\, dC_s\right). \tag{2.5}$$

PROOF. See Dellacherie and Meyer (1982), Theorem VI.57. □

As a consequence of this lemma we have the following remark, which is what we will actually use in the proof of the theorem.

REMARK 2.2. If we consider the process $X_t(\omega) = a(t)H(\omega)$, where a is a positive Borel function on $\mathbb{R}_+$ and H is a positive or L^1 random variable and we call H_t the cadlag version of $E(H|\mathscr{F}_t)$, then Lemma 2.1 says that

$$E\left(H\int_{[0,\infty)} a(s)\,dC_s\right) = E\left(\int_{[0,\infty)} H_s a(s)\,dC_s\right) \tag{2.6}$$

when C is an adapted, increasing, cadlag process (and hence optional).

LEMMA 2.3. *Let G be the operator defined by* (2.1). *Then* G: $L^1(\mu) \to L^1(\mu)$.

PROOF. By the Lipschitz property (2.2) we note that

$$|G(V)_t| \le E\left(\int_t^T (k|V_s| + |g_s(0)|)|dA_s| + |Y|\,|\mathscr{F}_t\right).$$

Let us call $Z_t = \int_0^t (k|V_s| + |g_s(0)|)|dA_s|$ and $M_t = E(|Y|\,|\mathscr{F}_t)$. So M is a martingale and by Lemma 2.1 we obtain

$$\begin{aligned} E\left(\int_0^T |G(V)_t||dA_t|\right) &\le E\left(\int_0^T E(Z_T - Z_t|\mathscr{F}_t)|dA_t|\right) + E\left(\int_0^T M_t|dA_t|\right) \\ &= E\left(Z_T\int_0^T |dA_t| - \int_0^T Z_t|dA_t|\right) \\ &\quad + E\left(|Y|\int_0^T |dA|_t\right) \\ &\le E(Z_T|A|_T) + E(|Y||A|_T) \\ &\le \beta E(Z_T + |Y|) < +\infty. \end{aligned}$$

□

Using again Lemma 2.1, we are now able to show the main result of this section.

THEOREM 2.4. *Under hypotheses* 1–4 *stated previously, there exists a unique adapted cadlag solution of* (2.1) *in* $L^1(\mu)$ *sense. That is, there is one and only one semimartingale* $V \in L^1(\mu)$ *such that*

$$E\left(\int_0^T |G(V)_t - V_t||dA_t|\right) = 0.$$

PROOF. Let us first introduce some notation. Without loss of generality we may assume that A is increasing so $A = |A|$ and we will indicate by $A_t^{(n)}$ and $A_t^{(n-)}$, respectively, the integrals $((\cdots((A\cdot A)\cdot A)\cdots A)\cdot A)_t$ and $((\cdots((A_-\cdot A)_-\cdot A)_-\ \cdots A)_-\cdot A)_t$ with the convention that $A^{(0)} = A^{(0-)} = 1$ and $A_t^{(1)} = (1\cdot A)_t = A_t = (1_-\cdot A)_t = A_t^{(1-)}$. By using the integration by parts formula for Lebesgue–Stieltjes integrals $A_t^2 = (A\cdot A)_t + (A_-\cdot A)_t$, it is easy to verify that $A_t^{(n)}$ and $A_t^{(n-)}$ are linked by the relation $\sum_{i=0}^n (-1)^i A_t^{((n-i)-)} A_t^{(i)} = 0$, for all $t \in [0,T]$. Finally we set $(A_u - A_t)^{[n]} = \sum_{i=0}^n (-1)^i A_u^{((n-i)-)} A_t^{(i)}$ for $u > t$, the same formula holding also for u_-. Note that when A is continuous there is of course no distinction between $A_t^{(n)}$ and $A_t^{(n-)}$, and $(A_{u-} - A_t)^{[n]} = (A_u - A_t)^{[n]}$ reduces to $(A_u - A_t)^n/n!$. Finally $G^{(n)}$ will indicate the nth iterate of G.

For any $U, V \in L^1(\mu)$ we will establish, by induction, the inequalities

$$|G^{(n)}(U)_t - G^{(n)}(V)_t| \leq k^n E\left(\int_t^T (A_{s-} - A_t)^{[n-1]} |U_s - V_s| \, dA_s | \mathscr{F}_t\right), \tag{2.7}$$

$$E\left(\int_0^T |G^{(n)}(U)_t - G^{(n)}(V)_t| \, dA_t\right) \leq k^n E\left(\int_0^T A_{t-}^{(n-)} |U_t - V_t| \, dA_t\right). \tag{2.8}$$

Because of the Lipschitz property, (2.7) is clear when $n = 1$. As for (2.8) when $n = 1$, let us indicate with $Z_t^{(1)}$ the integral $\int_0^t |U_s - V_s| \, dA_s$; then applying Remark 2.2 to the martingale $E(Z_T^{(1)} | \mathscr{F}_t)$, we obtain

$$\begin{aligned}
E\left(\int_0^T |G(U)_t - G(V)_t| \, dA_t\right) &\leq kE\left(\int_0^T E\left(\int_t^T |U_s - V_s| \, dA_s | \mathscr{F}_t\right) dA_t\right) \\
&= kE\left(\int_0^T E\left(Z_T^{(1)} - Z_t^{(1)} | \mathscr{F}_t\right) dA_t\right) \\
&= kE\left(Z_T^{(1)} \int_0^T dA_t - \int_0^T Z_t^{(1)} \, dA_t\right) \\
&= kE\left(+Z_T^{(1)} A_T - Z_T^{(1)} A_T + \int_0^T A_{t-} \, dZ_t^{(1)}\right) \\
&\leq kE\left(\int_0^T A_{t-} |U_t - V_t| \, dA_t\right).
\end{aligned}$$

Thus also (2.8) is established when $n = 1$.

From now on, for convenience, we will indicate $|U_t - V_t|$ by H_t. By the inductive hypothesis, if we assume (2.7) is true for $n - 1$, then, using the Lipschitz property of g, we obtain for n,

$$\begin{aligned}
&|G^{(n)}(U)_t - G^{(n)}(V)_t| \\
&\quad \leq kE\left(\int_t^T |G^{(n-1)}(U)_s - G^{(n-1)}(V)_s| \, dA_s | \mathscr{F}_t\right) \\
&\quad \leq k^n E\left(\int_t^T E\left(\int_s^T (A_{r-} - A_s)^{[n-2]} H_r \, dA_r | \mathscr{F}_s\right) dA_s | \mathscr{F}_t\right) \\
&\quad = k^n E\left(\int_t^T E\left(\int_s^T \sum_{i=0}^{n-2} (-1)^i A_{r-}^{((n-2-i)-)} A_s^{(i)} H_r \, dA_r | \mathscr{F}_s\right) dA_s | \mathscr{F}_t\right) \\
&\quad = k^n \sum_{i=0}^{n-2} (-1)^i E\left(\int_t^T A_s^{(i)} E\left(\int_s^T A_{r-}^{((n-2-i)-)} H_r \, dA_r | \mathscr{F}_s\right) dA_s | \mathscr{F}_t\right) \\
&\quad = k^n \sum_{i=0}^{n-2} (-1)^i E\left(\int_t^T E\left(\int_s^T H_r \, dA_r^{((n-1-i)-)} | \mathscr{F}_s\right) dA_s^{(i+1)} | \mathscr{F}_t\right).
\end{aligned}$$

Let us now set $Z_t^{(n-1-i)} = \int_0^t H_r\, dA_r^{((n-1-i)_-)} = \int_0^t A_{r-}^{((n-2-i)_-)}\, dZ_r^{(1)}$. It is important to notice that both $Z_t^{(n-1-i)}$ and $A_t^{(i+1)}$ are increasing processes, and we can therefore apply Lemma 2.1 to them in its conditional version [see Dellacherie and Meyer (1982)]. This and the integration by parts formula will lead to the following series of equalities for the right side of the preceding inequality:

$$
\begin{aligned}
&= k^n \sum_{i=0}^{n-2} (-1)^i E\left(\int_t^T E\left(Z_T^{(n-1-i)} - Z_s^{(n-1-i)} | \mathscr{F}_s \right) dA_s^{(i+1)} | \mathscr{F}_t \right) \\
&= k^n \sum_{i=0}^{n-2} (-1)^i E\left(Z_T^{(n-1-i)}\left(A_T^{(i+1)} - A_t^{(i+1)} \right) - \int_t^T Z_s^{(n-1-i)}\, dA_s^{(i+1)} | \mathscr{F}_t \right) \\
&= k^n \sum_{i=0}^{n-2} (-1)^i E\left(A_t^{(i+1)}\left(Z_t^{(n-1-i)} - Z_T^{(n-1-i)} \right) + \int_t^T A_{s-}^{(i+1)}\, dZ_s^{(n-1-i)} | \mathscr{F}_t \right) \\
&= k^n \sum_{i=0}^{n-2} (-1)^i E\left(\int_t^T \left(A_{s-}^{(i+1)} - A_t^{(i+1)} \right) dZ_s^{(n-1-i)} | \mathscr{F}_t \right) \\
&= k^n \sum_{i=0}^{n-2} (-1)^i E\left(\int_t^T \left(A_{s-}^{(i+1)} - A_t^{(i+1)} \right) A_{s-}^{((n-2-i)_-)}\, dZ_s^{(1)} | \mathscr{F}_t \right) \\
&= k^n E\left(\int_t^T \left(\sum_{i=0}^{n-2} (-1)^i A_{s-}^{(i+1)} A_{s-}^{(n-2-i)} \right.\right. \\
&\qquad \left.\left. - \sum_{i=0}^{n-2} (-1)^i A_t^{(i+1)} A_{s-}^{((n-2-i)_-)} \right) dZ_s^{(1)} | \mathscr{F}_t \right) \\
&= k^n E\left(\int_t^T \left(\sum_{i=1}^{n-1} (-1)^{i-1} A_{s-}^{(i)} A_{s-}^{(n-1-i)} \right.\right. \\
&\qquad \left.\left. - \sum_{i=1}^{n-1} (-1)^{i-1} A_t^{(i)} A_{s-}^{((n-1-i)_-)} \right) dZ_s^{(1)} | \mathscr{F}_t \right) \\
&= k^n E\left(\int_t^T \left(A_{s-}^{(n-1)} + \sum_{i=1}^{n-1} (-1)^i A_t^{(i)} A_{s-}^{((n-1-i)_-)} \right) dZ_s^{(1)} | \mathscr{F}_t \right) \\
&= k^n E\left(\int_t^T (A_{s-} - A_t)^{[n-1]} |U_s - V_s|\, dA_s | \mathscr{F}_t \right),
\end{aligned}
$$

or summarizing,

$$
|vG^{(n)}(U)_t - G^{(n)}(V)_t| \le k^n E\left(\int_t^T (A_{s-} - A_t)^{[n-1]} |U_s - V_s|\, dA_s | \mathscr{F}_t \right),
$$

which is exactly the formula (2.7) we wanted to establish.

At this point (2.8) can be proved for n by virtue of (2.7), following the same procedure. In fact we have

$$\begin{aligned}
&\|G^{(n)}(U) - G^{(n)}(V)\|_{L^1} \\
&= E\left(\int_0^T |G^{(n)}(U)_t - G^{(n)}(V)_t|\, dA_t\right) \\
&\le k^n E\left(\int_0^T E\left(\int_t^T \left(\sum_{i=0}^{n-1} (-1)^i A_t^{(i)} A_{s-}^{((n-1-i)_-)}\right) dZ_s^{(1)} \Big| \mathcal{F}_t\right) dA_t\right) \\
&= k^n \sum_{i=0}^{n-1} (-1)^i E\left(\int_0^T E\left(\int_t^T A_{s-}^{((n-1-i)_-)}\, dZ_s^{(1)} \Big| \mathcal{F}_t\right) dA_t^{(i+1)}\right) \\
&= k^n \sum_{i=0}^{n-1} (-1)^i E\left(\int_0^T A_{s-}^{((n-1-i)_-)} A_{s-}^{(i+1)}\, dZ_s^{(1)}\right) \\
&= k^n E\left(\int_0^T \sum_{i=1}^{n} (-1)^{i-1} A_{s-}^{((n-i)_-)} A_{s-}^{(i)}\, dZ_s^{(1)}\right) \\
&= k^n E\left(\int_0^T A_{s-}^{(n_-)} H_s\, dA_s\right) = k^n E\left(\int_0^T A_{s-}^{(n_-)} |U_s - V_s|\, dA_s\right).
\end{aligned}$$

It is now important to notice that $A_t^{(n_-)} \le A_t^n/n!$. As a matter of fact, when A is continuous we have the equality; otherwise, we have to add the contribution of the quadratic variation process in the left side to obtain the right one. But under our hypotheses A is bounded by β and so

$$\|G(U) - G(V)\|_{L^1(\mu)} \le k^n(\beta^n/n!)\|U - V\|_{L^1(\mu)}.$$

This means that for n sufficiently large, $G^{(n)}$ is a contraction on the Banach space $L^1(\mu)$ and by the Banach fixed point theorem there exists one and only one $V \in L^1(\mu)$ such that $E(\int_0^T |G^{(n)}(V)_t - V_t|\, dA_t) = 0$. Applying G again, the Lipschitz property (2.2) implies that $E(\int_0^T |G^{(n)}(G(V))_t - G(V)_t|\, dA_t) = 0$ and by the uniqueness of fixed points, $G(V) = V$ in the $L^1(\mu)$ sense.

Finally let us note that the solution V will be automatically adapted because of the conditional expectation. Moreover, it will be a semimartingale by virtue of the decomposition

$$V_t = E\left(\int_0^T g_s(V_s)\, dA_s + Y \Big| \mathcal{F}_t\right) - \int_0^t g_s(V_s)\, dA_s, \tag{2.9}$$

which is possible because A is an adapted process and $V \in L^1(\mu)$. □

In the special case $A_t = t$ our proof simplifies. In fact, the random measure μ reduces to $dt \times dP$ and Lemma 2.1 becomes the conditional version of Fubini's theorem. This case answers affirmatively the conjecture stated in Duffie and Epstein (1992) for $p = 1$.

Theorem 2.4 also implies that (2.1) has a solution in $\mathbb{D}$ under the uniform convergence on compacts in probability topology. Indeed assuming without loss of generality that G is already a contraction on $L^1(\mu)$, the sequence of iterates $V_t^0 = 0, V_t^1 = G(V^0)_t, \ldots, V_t^n = G(V^{n-1})_t, \ldots$ converges to the solution V in $L^1(\mu)$ and by Doob's weak inequality for martingales, we obtain for any $\lambda > 0$,

$$
\begin{aligned}
P\Big(\sup_{0<t\le T} |V_t^{n+1} - V_t| \ge \lambda\Big) &\le P\bigg(\sup_{0<t\le T} kE\bigg(\int_t^T |V_s^n - V_s||dA_s||\mathscr{F}_t\bigg) \ge \lambda\bigg)\\
&\le P\bigg(\sup_{0<t\le T} kE\bigg(\int_0^T |V_s^n - V_s||dA_s||\mathscr{F}_t\bigg) \ge \lambda\bigg)\\
&\le \frac{E(\int_0^T |V_s^n - V_s||dA_s|)}{\lambda} \to 0 \quad \text{as } n \to \infty.
\end{aligned}
$$

So $V^n \to V$ in $\mathbb{D}$. Moreover, by the Lipschitz property of g, V^n converges in $\mathbb{D}$ also to $F(V)$. Thus we must have $F(V) = V$ because of the uniqueness of the limit in $\mathbb{D}$.

REMARK 2.5. It is important to note that if we require more integrability on the terms in (2.1), then we can obtain better integrability also for the solution. Indeed, if we assume that $Y \in L^p$ and $(E(\int_0^T |g_s(0)||dA_s|)^p)^{1/p} < +\infty$ for some $p > 1$, we have that $G\colon \mathbf{S}^p \to \mathbf{S}^p$ and, using the strong version of Doob's martingale inequality and the fact that $|A|_T$ is bounded, it is possible to show that the iterate $G^{(n)}$ will be a contraction on $\mathbf{S}^p$ for n large enough. This implies as before the existence and uniqueness of a $V \in \mathbf{S}^p$ verifying (2.1) [see Duffie and Epstein (1992) for the case $A_s = s$].

This allows us to extend our model even further. In fact, if we take any semimartingale X in $\mathbf{H}^\infty$ with canonical decomposition $X = M + A$, where M is a martingale and A is a bounded total variation process, and consider the equation

$$G(V)_t = E\bigg(\int_t^T g_{s-}(V_{s-})\, dX_s + Y|\mathscr{F}_t\bigg) \tag{2.10}$$

under the additional hypotheses that $g_s(0) \in \mathbf{S}^p$, we have that (2.10) reduces to

$$G(V)_t = E\bigg(\int_t^T g_{s-}(V_{s-})\, dA_s + Y|\mathscr{F}_t\bigg) \quad \text{for any } V \in \mathbf{S}^p, \tag{2.11}$$

because the martingale term gets cancelled by the action of the conditional expectation. Again we can find a unique solution in $\mathbf{S}^p$, which will coincide with the solution of (2.10).

3. Backward–forward stochastic differential equations. We are now going to consider the system (1.1) in its more general form when a forward and a backward stochastic differential equation intervene at the same time. It is clear that in this case the solution may depend simultaneously upon its own past and future; that is, at each time it may depend on its whole trajectory. This peculiarity is exactly what led us to impose stricter assumptions on our model. For simplicity here we will consider only the two-dimensional case. Again we start our study by restricting to integrator processes of bounded total variation, because this is the case when the least integrability is required. We will proceed later on to generalize our model to semimartingales in $\mathbf{H}^\infty$.

We want to consider the system

$$U_t = J_t + \int_0^t f_s(U_s, V_s)\, dA_s, \tag{3.1}$$

$$V_t = E\left(\int_t^T g_s(U_s, V_s)\, dC_s + Y|\mathscr{F}_t\right), \tag{3.2}$$

where A and C are two finite variation processes in $\mathbf{H}^\infty$, say $|A|_T, |C|_T < \beta$ for some constant β and with $A_0 = C_0 = 0$. The functions $f, g\colon [0,T] \times \Omega \times \mathbb{R}^2 \to \mathbb{R}$ are $\mathscr{B}([0,T]) \times \mathscr{F} \times \mathscr{B}(\mathbb{R}^2)/\mathscr{B}(\mathbb{R})$-measurable and such that:

1. f, g are k-Lipschitz in (x, y) uniformly in (ω, s); that is,

$$|\zeta_s(\omega, x, y) - \zeta_s(\omega, \bar{x}, \bar{y})| \le k\big(|x - \bar{x}| + |y - \bar{y}|\big),$$

   where $\zeta = f, g$ for any $(x, y), (\bar{x}, \bar{y}) \in \mathbb{R}^2$, $s \in [0,T]$ and $\omega \in \Omega$.
2. $f_s(\cdot, x, y), g_s(\cdot, x, y)\colon \Omega \to \mathbb{R}$ are $\mathscr{F}_s$-measurable for any $(x, y) \in \mathbb{R}^2$.
3. If $D_t = \max(|A|_t, |C|_t)$, then

$$E\left(\int_0^T |f_s(0,0)|\, dD_s\right) < +\infty, \qquad E\left(\int_0^T |g_s(0,0)|\, dD_s\right) < +\infty.$$

Finally Y is a random variable $\mathscr{F}_T$-measurable in $L^1(P)$ and J_t is an adapted process such that $E(\int_0^T |J_t|\, dD_t) < k_1 < +\infty$.

As before, D_t induces a finite measure μ. Let us consider the space $L^1(\mu) \otimes L^1(\mu)$ with the norm $\|(U,V)\|_{L^1 \otimes L^1} = \|U\|_{L^1} + \|V\|_{L^1}$. Again this is a Banach space and we can view (3.1)–(3.2) as the operator

$$\Gamma(U,V)_t = \begin{pmatrix} F(U,V)_t \\ G(U,V)_t \end{pmatrix} = \begin{pmatrix} J_t + \int_0^t f_s(U_s, V_s)\, dA_s \\ E\left(\int_t^T g_s(U_s, V_s)\, dC_s + Y|\mathscr{F}_t\right) \end{pmatrix}. \tag{3.3}$$

Under the previous hypotheses, $\Gamma\colon L^1(\mu) \otimes L^1(\mu) \to L^1(\mu) \otimes L^1(\mu)$. Indeed, for any $(U,V) \in L^1(\mu) \otimes L^1(\mu)$, we obtain

$$\begin{aligned}
\|\Gamma(U,V)\|_{L^1 \otimes L^1} &= \left\| J_t + \int_0^t f_s(U_s, V_s)\, dA_s \right\|_{L^1} + \left\| E\left(\int_t^T g_s(U_s, V_s)\, dC_s + Y \middle| \mathscr{F}_t \right) \right\|_{L^1} \\
&= E\left(\int_0^T \left(\left| J_t + \int_0^t f_s(U_s, V_s)\, dA_s \right| \right.\right. \\
&\qquad \left.\left. + \left| E\left(\int_t^T g_s(U_s, V_s)\, dC_s + Y \middle| \mathscr{F}_t \right) \right| \right) dD_t \right) \\
&\le E\left(\int_0^T |J_t|\, dD_t \right) + E\left(\int_0^T \int_0^t |f_s(U_s, V_s)| |dA_s|\, dD_t \right) \\
&\qquad + E\left(\int_0^T E\left(\int_t^T |g_s(U_s, V_s)| |dC_s| + |Y| \middle| \mathscr{F}_t \right) dD_t \right) \\
&\le k_1 + \beta E(|Y|) + E\left(\int_0^T \int_0^t |f_s(U_s, V_s)| |dA_s|\, dD_t \right) \\
&\qquad + E\left(\int_0^T E\left(\int_t^T |g_s(U_s, V_s)| |dC_s| \middle| \mathscr{F}_t \right) dD_t \right).
\end{aligned}$$

But we have to note that, by the choice of D_t, $|A|_t \ll D_t$ and $|C|_t \ll D_t$ with respective Radon–Nikodym derivatives H_t, K_t, which are positive and less than or equal 1. So we can write the last side of the inequality as

$$\begin{aligned}
\|\Gamma(U,V)\|_{L^1 \otimes L^1} &\le k_1 + \beta E(|Y|) + E\left(\int_0^T \int_0^t |f_s(U_s, V_s)| H_s\, dD_s\, dD_t \right) \\
&\qquad + E\left(\int_0^T E\left(\int_t^T |g_s(U_s, V_s)| K_s\, dD_s \middle| \mathscr{F}_t \right) dD_t \right) \\
&\le k_1 + \beta E(|Y|) + E\left(\int_0^T \int_0^t |f_s(U_s, V_s)|\, dD_s\, dD_t \right) \\
&\qquad + E\left(\int_0^T E\left(\int_t^T |g_s(U_s, V_s)|\, dD_s \middle| \mathscr{F}_t \right) dD_t \right).
\end{aligned}$$

Looking more closely at the two integral terms, for the first one we have

$$\begin{aligned}
&E\left(\int_0^T \int_0^t z |f_s(U_s, V_s)|\, dD_s\, dD_t \right) \\
&\qquad = E\left(D_T \int_0^T |f_s(U_s, V_s)|\, dD_s - \int_0^T |f_s(U_s, V_s)| D_{s-}\, dD_s \right) \\
&\qquad \le E\left(D_T \int_0^T |f_s(U_s, V_s)|\, dD_s \right) \\
&\qquad \le \beta E\left(\int_0^T (|f_s(0,0)| + k|U_s| + k|V_s|)\, dD_s \right) < +\infty.
\end{aligned}$$

For the second term, by the optional projection theorem, we obtain

$$E\left(\int_0^T E\left(\int_t^T |g_s(U_s, V_s)|\, dD_s | \mathscr{F}_t\right) dD_t\right)$$
$$= E\left(\int_0^T |g_s(U_s, V_s)| D_{s-}\, dD_s\right)$$
$$\le \beta E\left(\int_0^T (|g_s(0,0)| + k|U_s| + k|V_s|)\, dD_s\right) < +\infty.$$

In conclusion $\|\Gamma(U,V)\|_{L^1 \otimes L^1} < +\infty$, $\forall\, (U,V) \in L^1(\mu) \otimes L^1(\mu)$. We can therefore hope that Γ is a contraction on $L^1(\mu) \otimes L^1(\mu)$. This is unfortunately not true for any pair of Lipschitz functions f, g as we will see by the following theorem and examples.

THEOREM 3.1. *Under the previously stated hypotheses and under the additional assumption that $k\|D\|_{\mathbf{H}^\infty} < 1$, then Γ is a contraction on $L^1(\mu) \otimes L^1(\mu)$. Consequently there exists a unique adapted solution (U, V) that satisfies the system* (3.1) *and* (3.2) *in the $L^1(\mu) \otimes L^1(\mu)$ sense.*

PROOF. Let $(U,V), (\tilde{U}, \tilde{V}) \in L^1(\mu) \times L^1(\mu)$. Let us recall that by our definition of norm, $\|\Gamma(U,V)\|_{L^1 \otimes L^1} = \|F(U,V)\|_{L^1} + \|G(U,V)\|_{L^1}$. Also, by absolute continuity, we have that

$$|F(U,V)_t - F(\tilde{U},\tilde{V})_t| = \left|\int_0^t \left(f(U_s, V_s) - f(\tilde{U}_s, \tilde{V}_s)\right) dA_s\right|$$
$$\le k\int_0^t \left(|U_s - \tilde{U}_s| + |V_s - \tilde{V}_s|\right)|dA_s|$$
$$\le k\int_0^t \left(|U_s - \tilde{U}_s| + |V_s - \tilde{V}_s|\right) dD_s$$

and

$$|G(U,V)_t - G(\tilde{U},\tilde{V})_t| = \left|E\left(\int_t^T \left(g(U_s, V_s) - g(\tilde{U}_s, \tilde{V}_s)\right) dC_s | \mathscr{F}_t\right)\right|$$
$$\le kE\left(\int_t^T \left(|U_s - \tilde{U}_s| + |V_s - \tilde{V}_s|\right)|dC_s| | \mathscr{F}_t\right)$$
$$\le kE\left(\int_t^T \left(|U_s - \tilde{U}_s| + |V_s - \tilde{V}_s|\right) dD_s | \mathscr{F}_t\right).$$

Hence by the adaptedness of the processes we obtain

$$|F(U,V)_t - F(\tilde{U},\tilde{V})_t| + |G(U,V)_t - G(\tilde{U},\tilde{V})_t|$$
$$\le kE\left(\int_0^T \left(|U_s - \tilde{U}_s| + |V_s - \tilde{V}_s|\right) dD_s | \mathscr{F}_t\right)$$

and finally

$$E\left(\int_0^T |F(U,V)_t - F(\tilde{U},\tilde{V})_t|\,dD_t\right) + E\left(\int_0^T |G(U,V)_t - G(\tilde{U},\tilde{V})_t|\,dD_t\right)$$
$$\le kE\left(\int_0^T E\left(\int_0^T \left(|U_s - \tilde{U}_s| + |V_s - \tilde{V}_s|\right) dD_s \,|\mathscr{F}_t\right) dD_t\right).$$

Once again by the optional projection theorem we can conclude

$$\|F(U,V) - F(\tilde{U},\tilde{V})\|_{L^1} + \|G(U,V) - G(\tilde{U},\tilde{V})\|_{L^1}$$
$$\le kE\left(D_T \int_0^T \left(|U_s - \tilde{U}_s| + |V_s - \tilde{V}_s|\right) dD_s\right)$$
$$\le k\|D\|_{\mathbf{H}^\infty} E\left(\int_0^T \left(|U_s - \tilde{U}_s| + |V_s - \tilde{V}_s|\right) dD_s\right),$$

or

$$\|\Gamma(U,V) - \Gamma(\tilde{U},\tilde{V})\|_{L^1 \otimes L^1} \le k\|D\|_{\mathbf{H}^\infty} \|(U,V) - (\tilde{U},\tilde{V})\|_{L^1 \otimes L^1}.$$

Because of our assumption $k\|D\|_{\mathbf{H}^\infty} < 1$, the operator Γ is a contraction on $L^1(\mu) \times L^1(\mu)$, which is a Banach space. Again by the Banach fixed point theorem there is a unique pair of processes $(U,V) \in L^1(\mu) \otimes L^1(\mu)$ solving the system (3.1) and (3.2). The adaptedness of each process is clear from the structure of the equations. □

As before, if we increase the integrability requirements in our system we are able to generalize the model to integrating semimartingales. We now want to consider the system

$$F(U,V)_t = J_t + \int_0^t f_{s-}(U_{s-}, V_{s-})\,dX_s, \tag{3.4}$$

$$G(U,V)_t = E\left(\int_t^T g_{s-}(U_{s-}, V_{s-})\,dZ_s + Y|\mathscr{F}_t\right), \tag{3.5}$$

where X and Z are two semimartingales in $\mathbf{H}^\infty$ with respective canonical decompositions $X = M + A$ and $Z = N + C$, with M and N martingales and A and C bounded total variation processes, and such that $X_0 = Z_0 = 0$. Here f and g satisfy the foregoing hypotheses 1 and 2 and instead of 3, we will have the substitute hypothesis

3′. $f_s(0,0) \in \mathbf{S}^p$ and $g_s(0,0) \in \mathbf{S}^p$, for $p > 1$.

Finally we also want $J \in \mathbf{S}^p$ and $Y \in L^p$ for the same $p > 1$. These assumptions imply that the operator

$$\Gamma(U,V)_t = \begin{pmatrix} F(U,V)_t \\ G(U,V)_t \end{pmatrix}$$

defined by (3.4) and (3.5) acts from $\mathbf{S}^p \otimes \mathbf{S}^p$ to $\mathbf{S}^p \otimes \mathbf{S}^p$ with the norm $\|(U,V)\|_{\mathbf{S}^p \otimes \mathbf{S}^p} = \|U\|_{\mathbf{S}^p} + \|V\|_{\mathbf{S}^p}$. By Emery's inequality and by the relationship between the $\mathbf{S}^p$ norm and the $\mathbf{H}^p$ norm [see Protter (1990), pages 190–191] we have

$$\|F(U,V) - J\|_{\mathbf{S}^p} \le c_p \|F(U,V) - J\|_{\mathbf{H}^p} \le c_p \|f(U,V)\|_{\mathbf{S}^p}\|X\|_{\mathbf{H}^\infty}$$
$$\le c_p\|X\|_{\mathbf{H}^\infty}(k\|U\|_{\mathbf{S}^p} + k\|V\|_{\mathbf{S}^p} + \|f(0,0)\|_{\mathbf{S}^p}) < +\infty$$

because of the Lipschitz property of f. Thus $\|F(U,V)\|_{\mathbf{S}^p} < +\infty$. Moreover, for (3.5) we remark that, if U, V and $g(0,0) \in \mathbf{S}^p$, then $\int_0^t g_{s-}(U_{s-}, V_{s-})\,dN_s$ is a true martingale (and not only a local martingale) because

$$\left\| \int_0^T g_{s-}(U_{s-}, V_{s-})\,dN_s \right\|_{\mathbf{H}^p} \le \|g(U,V)\|_{\mathbf{S}^p}\|N\|_{\mathbf{H}^\infty}$$
$$\le (k(\|U\|_{\mathbf{S}^p} + \|V\|_{\mathbf{S}^p}) + \|g(0,0)\|_{\mathbf{S}^p})\|N\|_{\mathbf{H}^\infty} < +\infty.$$

Thus the martingale part gives no contribution to (3.5), whence

$$|G(U,V)_t| \le E\left(\int_t^T |g_{s-}(U_{s-}, V_{s-})||dC_s| + |Y| \Big| \mathscr{F}_t \right)$$
$$\le E\left(\int_0^T |g_{s-}(U_{s-}, V_{s-})||dC_s| + |Y| \Big| \mathscr{F}_t \right)$$

and by Doob's inequality we can conclude

$$\|G(U,V)\|_{\mathbf{S}^p} \le q \left\| \int_0^T |g_{s-}(U_{s-}, V_{s-})||dC_s| + Y \right\|_{L^p}$$
$$\le q(k(\|U\|_{\mathbf{S}^p} + \|V\|_{\mathbf{S}^p}) + \|g(0,0)\|_{\mathbf{S}^p})\|C\|_{\mathbf{H}^\infty} + q\|Y\|_{L^p} < +\infty,$$

where $1/p + 1/q = 1$. Thus we have the following theorem.

THEOREM 3.2. *Let $\Gamma(U,V)$ be defined by the system* (3.4) *and* (3.5). *If f, g and Y verify the foregoing hypotheses and if f, g are Lipschitz functions with constant k such that $k \max(c_p\|X\|_{\mathbf{H}^\infty}, q\|C\|_{\mathbf{H}^\infty}) < 1$, then there exists a unique pair of processes (U,V) in $\mathbf{S}^p \otimes \mathbf{S}^p$ that is the solution to the system* (3.4) *and* (3.5).

PROOF. For $(U,V), (\tilde{U}, \tilde{V}) \in \mathbf{S}^p \otimes \mathbf{S}^p$, because of Emery's inequality, we obtain from (3.4):

$$\|F(U,V) - F(\tilde{U},\tilde{V})\|_{\mathbf{H}^p} \le \|f(U,V) - f(\tilde{U},\tilde{V})\|_{\mathbf{S}^p}\|X\|_{\mathbf{H}^\infty}$$
$$\le k\big(\|U - \tilde{U}\|_{\mathbf{S}^p} + \|V - \tilde{V}\|_{\mathbf{S}^p}\big)\|X\|_{\mathbf{H}^\infty}.$$

Consequently, because of the relationship between the $\mathbf{H}^p$ and the $\mathbf{S}^p$ norms we can conclude

$$\|F(U,V) - F(\tilde{U},\tilde{V})\|_{\mathbf{S}^p} \le c_p\|F(U,V) - F(\tilde{U},\tilde{V})\|_{\mathbf{H}^p}$$
$$\le kc_p\|X\|_{\mathbf{H}^\infty}\big(\|U - \tilde{U}\|_{\mathbf{S}^p} + \|V - \tilde{V}\|_{\mathbf{S}^p}\big).$$

As for (3.5), we get, by the Lipschitz property of g,

$$\begin{aligned}|G(U,V)_t - G(\tilde{U},\tilde{V})_t| &\le kE\left(\int_t^T\left(|U_s-\tilde{U}_s|+|V_s-\tilde{V}_s|\right)|dC_s||\mathscr{F}_t\right)\\ &\le k\left(E\left((U-\tilde{U})_T^* + (V-\tilde{V})_T^*\right)\right.\\ &\qquad \left.\times(|C|_T-|C|_t)|\mathscr{F}_t\right)\\ &\le k\left(E\left((U-\tilde{U})_T^*+(V-\tilde{V})_T^*\right)|C|_T|\mathscr{F}_t\right)\end{aligned}$$

and by Doob's inequality for $p > 1$ we can finally deduce

$$\|G(U,V) - G(\tilde{U},\tilde{V})\|_{\mathbf{S}^p} \le kq\left(\|U-\tilde{U}\|_{\mathbf{S}^p} + \|V-\tilde{V}\|_{\mathbf{S}^p}\right)\|C\|_{\mathbf{H}^\infty}.$$

In conclusion we obtain

$$\begin{aligned}\|\Gamma(U,V) - \Gamma(\tilde{U},\tilde{V})\|_{\mathbf{S}^p\otimes\mathbf{S}^p} &\le k\max\left(c_p\|X\|_{\mathbf{H}^\infty}, q\|C\|_{\mathbf{H}^\infty}\right)\\ &\quad\times\|(U,V)-(\tilde{U},\tilde{V})\|_{\mathbf{S}^p\otimes\mathbf{S}^p}.\end{aligned}$$

Therefore, under our additional assumption on the Lipschitz constant k, it is clear that Γ is a contraction on the Banach space $\mathbf{S}^p \otimes \mathbf{S}^p$. Hence a unique fixed point for Γ exists and this is the solution of our system. □

It would certainly be appealing to be able to remove the preceding restriction on the Lipschitz constant of the coefficients in the equations. Unfortunately this is not possible as the following two examples show.

EXAMPLE 1. In this example we consider the linear system

$$U_t = J_0 + \int_0^t (U_s + |V_s|)\,ds, \tag{3.6}$$

$$V_t = E\left(\int_t^T (U_s + V_s)\,ds + Y|\mathscr{F}_t\right), \tag{3.7}$$

where we are assuming that J_0 is $\mathscr{F}_0$-measurable and positive and Y is positive and integrable. Here $g(u,v) = u + v$ and $f(u,v) = u + |v|$ are both uniformly Lipschitz in (u,v) with constant $k = 1$. Let us assume we have a solution (U,V). Then we will obtain a contradiction. First of all let us notice that, if the solution exists, we may write U and V in terms of each other:

$$U_t = e^t\left(J_0 + \int_0^t e^{-s}|V_s|\,ds\right) \quad \text{for any given } V,$$

$$V_t = E\left(e^{(T-t)}Y + \int_t^T e^{(s-t)}U_s\,ds|\mathscr{F}_t\right) \quad \text{for any given } U.$$

It is clear that U_s, V_s are both positive, so we can remove the absolute value from (3.6) and conclude that U_s, V_s are also solutions to

$$\tilde{U}_t = J_0 + \int_0^t \left(\tilde{U}_s + \tilde{V}_s\right) ds, \tag{3.8}$$

$$\tilde{V}_t = E\left(\int_t^T \left(\tilde{U}_s + \tilde{V}_s\right) ds + Y | \mathscr{F}_t\right). \tag{3.9}$$

Adding up (3.8) and (3.9) we obtain that

$$\tilde{U}_t + \tilde{V}_t = J_0 + E\left(\int_0^T \left(\tilde{U}_s + \tilde{V}_s\right) ds + Y | \mathscr{F}_t\right)$$

is a martingale; thus its mean will be a constant, let us say β. This implies

$$E\left(\tilde{U}_t + \tilde{V}_t\right) = E(J_0) + \int_0^T E\left(\tilde{U}_s + \tilde{V}_s\right) ds + E(Y)$$

or better

$$\beta = E(J_0) + \beta T + E(Y),$$

which implies $\beta(1 - T) = E(J_0 + Y) > 0$. Hence if $T > 1$, then

$$0 < E(U_t + V_t) = E\left(\tilde{U}_t + \tilde{V}_t\right) = \beta = \frac{E(Y + J_0)}{1 - T} < 0$$

and we have a contradiction to the existence of the solution of (3.6) and (3.7) whenever $T \geq 1$. We would also like to note that the system (3.8) and (3.9) already provides a counterexample to the case $T = 1$.

A pleasant feature of the preceding example is that the contradiction occurs as soon as the sufficient condition of Theorem 3.1 that $kT < 1$ is not satisfied. What is not fully satisfactory is that we used the singular matrix $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, and one may therefore wonder if the contradiction is really due to the singularity of the operator rather than to the large size of the Lipschitz constant. The next example shows that we also need a small Lipschitz constant for a nonsingular linear operator. We still consider Example 1 important because in the following example kT can be indeed bigger than 1, although still bounded.

EXAMPLE 2. Taking again J_0 positive and $\mathscr{F}_0$-measurable and Y positive and even bounded, let us focus our attention on the system

$$U_t = J_0 + \int_0^t V_s \, ds, \tag{3.10}$$

$$V_t = E\left(\int_t^T U_s \, ds + Y | \mathscr{F}_t\right). \tag{3.11}$$

Clearly, here $f(u, v) = v$ and $g(u, v) = u$ are Lipschitz with constant $k = 1$. If the solution of (3.10) and (3.11) exists, then U_T has to be integrable in order to have $V_T = Y$. Let us treat U_T as if it were known for the time being. From (3.10) we get $U_T - U_t = \int_t^T V_s\,ds$ and taking conditional expectations,

$$U_t = E\left(U_T - \int_t^T V_s\,ds|\mathscr{F}_t\right).$$

Substituting the latter in (3.11) we obtain

$$V_t = E\left(\int_t^T E(U_T|\mathscr{F}_s)\,ds - \int_t^T E\left(\int_s^T V_r\,dr|\mathscr{F}_s\right)ds|\mathscr{F}_t\right) + E(Y|\mathscr{F}_t),$$

and by the conditional version of Fubini's theorem,

$$V_t = E(U_T|\mathscr{F}_t)(T - t) - E\left(\int_t^T (r - t)V_r\,dr + Y|\mathscr{F}_t\right). \tag{3.12}$$

Hence, by Picard iterations we can find a solution of (3.12) in terms of Y and U_T. In particular, if we define $M_t = E(U_T|\mathscr{F}_t)$ and $Y_t = E(Y|\mathscr{F}_t)$ we obtain

$$V_t = M_t \sin(T - t) + Y_t \cos(T - t).$$

Substituting this in (3.10) for U_t we get

$$U_t = J_0 + \int_0^t M_s \sin(T - s)\,ds + \int_0^t Y_s \cos(T - s)\,ds. \tag{3.13}$$

Rewriting (3.13) for U_T and taking expectations on both sides, using Fubini's theorem once again, we obtain

$$\begin{aligned} E(U_T) &= E(J_0) + \int_0^T E(U_T)\sin(T - s)\,ds + \int_0^T E(Y)\cos(T - s)\,ds \\ &= E(J_0) + E(U_T)(1 - \cos T) + E(Y)\sin T; \end{aligned}$$

thus

$$E(U_T) = \frac{E(J_0) + E(Y)\sin T}{\cos T}.$$

It is then clear that if $T = \pi/2$, then U_T is no longer integrable and our system does not make sense.

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