BACKWARD STOCHASTIC DIFFERENTIAL EQUATIONS WITH CONSTRAINTS ON THE GAINS-PROCESS

By Jakša Cvitanić,[1] Ioannis Karatzas[1] and H. Mete Soner[2]

Columbia University, Columbia University and Bogazici University

We consider backward stochastic differential equations with convex constraints on the gains (or intensity-of-noise) process. Existence and uniqueness of a minimal solution are established in the case of a drift coefficient which is Lipschitz continuous in the state and gains processes and convex in the gains process. It is also shown that the minimal solution can be characterized as the unique solution of a functional stochastic control-type equation. This representation is related to the penalization method for constructing solutions of stochastic differential equations, involves change of measure techniques, and employs notions and results from convex analysis, such as the support function of the convex set of constraints and its various properties.

1. Introduction and summary. The standard theory for stochastic differential equations (SDE) of the type

$$(1.1) \qquad dX(t) = -f(t, X(t))\,dt + \sigma'(t, X(t))\,dB(t), \qquad 0 \le t \le T$$

with initial condition $X(0) = x \in \mathbb{R}$, driven by the d-dimensional Brownian motion $B(\cdot)$, was developed by Itô (1942, 1946, 1951). It asserts that (1.1) has a pathwise-unique solution $X(\cdot)$, a measurable process on the given probability space $(\Omega, \mathscr{F}, P)$ that satisfies

$$(1.2) \qquad E\Big[\sup_{0 \le t \le T} |X(t)|^2\Big] < \infty$$

and is *adapted to the filtration* F *generated by the driving Brownian motion* $B(\cdot)$, provided that the drift $f\colon [0, T] \times \mathbb{R} \to \mathbb{R}$ and dispersion $\sigma\colon [0, T] \times \mathbb{R} \to \mathbb{R}^d$ coefficients satisfy appropriate Lipschitz and growth conditions; see, for instance, Karatzas and Shreve (1991), Section 5.2.

In a very interesting paper, Pardoux and Peng (1990) recently developed a similar theory for equations analogous to (1.1), but in which one specifies a terminal rather than initial condition. More precisely, with $f(\cdot, \cdot)$ and $\sigma(\cdot, \cdot)$ as above and with ξ a square-integrable and $\mathscr{F}(T)$-measurable random variable, they showed that there exists a unique pair of F-adapted processes $(X(\cdot), Y(\cdot))$ that satisfy (1.2),

$$(1.3) \qquad E\int_0^T \|Y(t)\|^2\,dt < \infty,$$

as well as the *backward stochastic differential equation* (BSDE),

(1.4)
$$X(t) = \xi + \int_t^T f(s, X(s)) \, ds$$
$$- \int_t^T [\sigma(s, X(s)) + Y(s)]' \, dB(s), \qquad 0 \le t \le T.$$

In other words, one tries to "steer" the state process $X: [0, T] \times \Omega \to \mathbb{R}$ to the specified terminal condition $X(T) = \xi$ at time $t = T$, while keeping it adapted to the filtration F generated by the driving Brownian motion $B(\cdot)$. The abilility to accomplish this depends crucially on the freedom to choose the "gains," or intensity-of-noise, process $Y: [0, T] \times \Omega \to \mathbb{R}^d$, again in a nonanticipative manner. Indeed, one could try to solve the SDE (1.1) using a time reversal, that is, for the process $\tilde{X}(s) := X(T-s), 0 \le s \le T$, starting with the condition $\tilde{X}(0) = X(T) = \xi$, but the resulting state process $X(\cdot)$ would then be adapted to the "reversed-time" filtration $\tilde{\mathscr{F}}(s) := \sigma(W(u) - W(s), s \le u \le T), 0 \le s \le T$, not to F.

The freedom to choose the "gains" process $Y(\cdot)$ as an element of control, is the crucial difference between the theory for BSDEs and the more classical Itô theory for SDEs. Suppose, however, that the controller's ability to choose this gains process $Y(\cdot)$ is limited, say, by the requirement that $Y(\cdot)$ take values in a given nonempty, closed convex set K of $\mathbb{R}^d$. Then it is, generally, no longer possible to find a pair of F-adapted processes $(X(\cdot), Y(\cdot))$ that satisfy this requirement, in addition to (1.2)–(1.4). One needs to give the controller freedom to take more swift "corrective action," captured by an F-adapted process $C: [0, T] \times \Omega \to [0, \infty)$ with increasing, right-continuous paths and

(1.5)
$$E(C(T))^2 < \infty;$$

here $C(t)$ represents the cumulative effect of the corrective actions taken by time $t \in [0, T]$. More precisely, one seeks a triple of F-adapted processes $(X(\cdot), Y(\cdot), C(\cdot))$ as above, that satisfies almost surely the analogue

(1.6)
$$X(t) = \xi + \int_t^T f(s, X(s)) \, ds$$
$$- \int_t^T [\sigma(s, X(s)) + Y(s)]' \, dB(s) + C(T) - C(t), \qquad 0 \le t \le T$$

of the BSDE (1.4), the conditions (1.2), (1.3), (1.5), as well as the constraint

(1.7)
$$Y(t) \in K, \qquad 0 \le t \le T,$$

and is the minimal solution of (1.6) with these properties [meaning that for any other such triple $(\tilde{X}(\cdot), \tilde{Y}(\cdot), \tilde{C}(\cdot))$ that satisfies the system (1.2), (1.3), (1.5)–(1.7) we have $X(\cdot) \le \tilde{X}(\cdot)$, a.s.].

The constrained backward stochastic differential equation (CBSDE) of (1.6) and (1.7) is the focus of this paper. In order to simplify things and help focus attention on the constraint (1.7), we have taken $\sigma \equiv 0$ throughout. Using notions, tools and results from convex analysis and ideas from our earlier papers

[Cvitanić and Karatzas (1992, 1993)] that dealt with constrained optimization and hedging problems in the special context of mathematical finance, we discuss first the case of constrained backward stochastic equations (CBSE), that is, with $\sigma \equiv 0$ and $f(\cdot, \cdot)$ replaced by an F-adapted process $g(\cdot)$ in (1.6) (Section 2). Next, we develop in Section 3 the solvability and properties of the "penalized" version

$$
\begin{aligned}
X_n(t) = \xi + \int_t^T f(s, X(s))\,ds \\
- \int_t^T Y_n(s)'\,dB(s) + C_n(T) - C_n(t), \qquad 0 \le t \le T
\end{aligned}
$$

(1.8)

of (1.6) with $\sigma \equiv 0$ and

$$
C_n(t) := n \int_0^t \rho(Y_n(s))\,ds, \qquad \rho(y) := \operatorname{dist}(y, K),
$$

again with the help of tools from convex analysis. We then put together the theory of Section 2 and the properties of the penalization scheme (1.8) to study the CBSDE (1.6) in the case of a general Lipschitz-continuous drift function $f(t, \omega, \cdot)$ via martingale and stochastic control methods.

A crucial element of our approach, developed in Section 4, is the *functional stochastic control-type equation*

$$
X^*(t) = \operatorname*{ess\,sup}_{\nu \in \mathscr{D}} E^{\nu}\left[\xi + \int_t^T [f(u, X^*(u)) - \delta(\nu(u))]\,du \ \Big| \ \mathscr{F}(t) \right],
$$

(1.9)

$$
0 \le t \le T,
$$

which seems to be encountered and studied in this paper for the first time. Here $\delta(z) = \sup_{y \in K}(z'y)$ is the support function of the set K of (1.7), $\mathscr{D}$ is the class of bounded, F-adapted processes $\nu(\cdot)$ with values in the effective domain $\tilde{K} := \{x \in \mathbb{R}^d \ / \ \delta(x) < \infty\}$ of $\delta(\cdot)$ and E^{ν} denotes expectation with respect to the auxiliary probability measure $P^{\nu}(A) := E[\exp\{\int_0^T \nu'(s)\,dB(s) - \frac{1}{2}\int_0^T \|\nu(s)\|^2\,ds\}1_A]$, $A \in \mathscr{F}(T)$ for every "adjoint variable" process $\nu(\cdot)$ in $\mathscr{D}$. We show in Section 4 that (1.9) admits a unique solution $X^*(\cdot)$ with the property (1.2); this process is dominated by the state process of any solution to the constrained BSDE of (1.6), (1.7) leading, as we demonstrate, to the minimal solution of this equation. In Sections 5 and 6 we show how to extend those results to the case of a drift coefficient $f(t, x, y)$, which depends also on the current value $Y(t) = y$ of the gains process, but in a convex fashion, and to the case of a reflecting lower barrier for the state process $X(\cdot)$; each of these cases necessitates the introduction of an additional "adjoint variable" [a process $\mu(\cdot)$ or a stopping time τ, respectively]. In subsequent work we expect to be able to extend the methodology of this paper to cover the case of general dispersion $\sigma(t, x)$ and drift $f(t, x, y)$ coefficients.

Related existence results are obtained by Buckdahn and Hu (1997) for the special, one-dimensional case ($d = 1$), in the context of BSDEs with a lower barrier process (as in our Section 6), driven by both a Brownian motion and

a Poisson random measure. These authors do not use a stochastic control approach or representations of the type (1.9).

Backward stochastic differential equations were apparently first studied in the context of the stochastic version of Pontryagin's "maximum principle" for the *optimal control of diffusions* [see Saksonov (1989), Arkin and Saksonov (1979), Peng (1990, 1993), Elliott (1990), as well as Haussmann (1986), Bensoussan (1981), Bismut (1978), and the references therein, for earlier work]. They also arose in the context of "recursive utility" for *mathematical economics* in the work of Duffie and Epstein (1992). Since their formal and systematic study by Pardoux and Peng (1990) in a general framework, they have found an enormous range of applications in such diverse fields as *partial differential equations* [cf. Peng (1991), Barles, Buckdahn and Pardoux (1997), Darling and Pardoux (1997)], *variational inequalities* and *obstacle problems* [cf. Pardoux and Tang (1996), El Karoui, Kapoudjian, Pardoux, Peng and Quenez (1997) (abbreviated [EKPPQ]), Ma and Cvitanić (1997)], *stochastic PDEs* [Pardoux and Peng (1994)], *stochastic control* [cf. Peng (1990, 1993), Hamadène and Lepeltier (1995a)], *stochastic games* [cf. Hamadène and Lepeltier (1995b), Cvitanić and Karatzas (1996)] and *mathematical finance* [cf. Cvitanić and Karatzas (1993), El Karoui, Peng and Quenez (1997), Buckdahn and Hu (1996, 1997)].

2. **Backward stochastic equations with constraints.** On a given, complete probability space $(\Omega, \mathscr{F}, P)$, let $B(\cdot) = (B_1(\cdot), \ldots, B_d(\cdot))'$ be a standard d-dimensional Brownian motion over the finite interval $[0, T]$, and denote by $\mathsf{F} = \{\mathscr{F}(t)\}_{0 \leq t \leq T}$ the augmentation of the natural filtration F^B generated by $B(\cdot)$, namely $\mathscr{F}^B(t) = \sigma(B(s), \ 0 \leq s \leq t)$, $0 \leq t \leq T$. We shall need the following notation: for any given $n \in \mathbb{N}$, let us introduce the spaces:

L_n^2 of $\mathscr{F}(T)$-measurable random variables $\xi \colon \Omega \to \mathbb{R}^n$ with $E(\|\xi\|^2) < \infty$;

H_n^2 of F-progressively measurable processes $\varphi \colon [0, T] \times \Omega \to \mathbb{R}^n$ with $\int_0^T E\|\varphi(t)\|^2 \, dt < \infty$;

S_n^k of F-progressively measurable processes $\varphi \colon [0, T] \times \Omega \to \mathbb{R}^n$ with the property $E(\sup_{0 \leq t \leq T} \|\varphi(t)\|^k) < \infty$, $k \in \mathbb{N}$;

A_i^2 of RCLL, F-adapted, predictable increasing processes $A \colon [0, T] \times \Omega \to [0, \infty)$ with $A(0) = 0$, $E(A^2(T)) < \infty$.

Finally, we shall denote by $\mathscr{P}$ the σ-algebra of predictable sets in $[0, T] \times \Omega$.

Suppose now that we are given a random variable $\xi \colon \Omega \to \mathbb{R}$ in the space L_1^2, as well as a process $g \colon [0, T] \times \Omega \to \mathbb{R}$ in the space H_1^2. Suppose also that we are given a closed, convex set $K \subset \mathbb{R}^d$ which contains the origin, with support function

$$(2.1) \qquad \delta(z) := \sup_{y \in K}(y'z), \qquad z \in \mathbb{R}^d,$$

which is bounded on compact subsets of its effective domain

$$(2.2) \qquad \tilde{K} := \{x \in \mathbb{R}^d \ / \ \delta(x) < \infty\},$$

the "barrier cone" of the set K [cf. Rockafellar (1970), page 114]. Here and in the sequel, $y'x$ denotes the inner product of the vectors y and x.

We shall denote by $\mathscr{H}$ the class of F-progressively measurable processes $\nu: [0, T] \times \Omega \to \tilde{K}$ with $E \int_0^T \|\nu(t)\|^2 \, dt < \infty$; for every $\nu(\cdot) \in \mathscr{H}$, the exponential process

$$(2.3) \qquad Z_\nu(t) := \exp\left\{ \int_0^t \nu'(s) \, dB(s) - \tfrac{1}{2} \int_0^t \|\nu(s)\|^2 \, ds \right\}, \qquad 0 \le t \le T$$

is a local martingale and a supermartingale; it is a martingale if and only if $EZ_\nu(T) = 1$, in which case

$$(2.4) \qquad\qquad P^\nu(A) := E[Z_\nu(T)1_A], \qquad A \in \mathscr{F}(T)$$

is a probability measure. In particular, this is the case for every process $\nu(\cdot)$ in the space

$$(2.5) \qquad \begin{aligned} \mathscr{D} &= \bigcup_{n=1}^\infty \mathscr{D}_n, \\ \mathscr{D}_n &:= \left\{ \nu \in \mathscr{H} \ / \ \|\nu(t, \omega)\| \le n \text{ for a.e. } (t, \omega) \in [0, T] \times \Omega \right\} \end{aligned}$$

of bounded processes in $\mathscr{H}$. [For the unconstrained case $K = \mathbb{R}^d$ we have trivially $\tilde{K} = \{0\}$; then $\mathscr{D}$ contains only the evanescent processes $\nu(\cdot) \equiv 0$, a.e. on $[0, T] \times \Omega$ and $P^0 = P$.]

We first consider the problem of a backward stochastic equation (BSE) with constraints on the "gains" or "intensity-of-noise" process; the solution for this problem was provided in a slightly different context by Cvitanić and Karatzas (1993), hereafter abbreviated [CK93].

PROBLEM 2.1. Find a triple of F-progressively measurable processes $(X(\cdot), Y(\cdot), C(\cdot))$ with $X(\cdot) \in S_1^2$, $Y(\cdot) \in H_d^2$, $C(\cdot) \in A_i^2$, such that the *backward stochastic equation* (BSE)

$$(2.6) \quad X(t) = \xi + \int_t^T g(u) \, du - \int_t^T Y'(u) \, dB(u) + C(T) - C(t), \qquad 0 \le t \le T$$

and the constraint

$$(2.7) \qquad\qquad Y(t) \in K \quad \text{for } \lambda\text{-a.e. } t \in [0, T]$$

hold almost surely, and such that for any other triple $(\tilde{X}(\cdot), \tilde{Y}(\cdot), \tilde{C}(\cdot)) \in S_1^2 \times H_d^2 \times A_i^2$ that satisfies (2.6) and (2.7) we have

$$X(t) \le \tilde{X}(t), \qquad 0 \le t \le T$$

almost surely. Here and in the sequel, λ denotes Lebesgue measure on $[0, T]$.

In the interest of readability and completeness, we recall here the main results from [CK93] related to this problem, modified and adapted to our framework. First, we notice that for any solution to the BSE of (2.6), we have

$$X(t) = E\left[\xi + \int_t^T g(u)\,du + C(T) - C(t) \mid \mathscr{F}(t)\right]$$

(2.8)
$$- E\left[\int_t^T Y'(u)\,dB(u) \mid \mathscr{F}(t)\right]$$

$$\geq E\left[\xi + \int_t^T g(u)\,du \mid \mathscr{F}(t)\right] =: X_0(t), \qquad 0 \leq t \leq T.$$

This process $X_0(\cdot)$ is the solution of the unconstrained version

$$X_0(t) = \xi + \int_t^T g(u)\,du - \int_t^T Y_0'(u)\,dB(u), \qquad 0 \leq t \leq T$$

of (2.6), with $C_0(\cdot) \equiv 0$ and with a suitable process $Y_0(\cdot) \in \mathsf{H}_d^2$ that takes values in $\mathbb{R}^d$ (unconstrained); the existence and uniqueness of such a process $Y_0(\cdot)$ follows from the integral representation property for square-integrable martingales of the Brownian filtration [cf. Karatzas and Shreve (1991), pages 182–184]. Furthermore, let us notice that the process $X(\cdot) + \int_0^\bullet g(s)\,ds$ dominates the square-integrable, P-martingale

$$X_0(t) + \int_0^t g(u)\,du = E\left[\xi + \int_0^T g(u)\,du \,\Big|\, \mathscr{F}(t)\right]$$

(2.9)
$$= E\left[\xi + \int_0^T g(u)\,du\right]$$

$$+ \int_0^t Y_0'(u)\,dB(u), \qquad 0 \leq t \leq T.$$

Moreover, for every $\nu(\cdot) \in \mathscr{D}$ we know from Girsanov's theorem [e.g., Karatzas and Shreve (1991), Section 3.5] that the process

(2.10)
$$B_\nu(t) := B(t) - \int_0^t \nu(s)\,ds, \qquad 0 \leq t \leq T$$

is Brownian motion under the probability measure P^ν of (2.4).

PROPOSITION 2.1. *For any triple* $(X(\cdot), Y(\cdot), C(\cdot))$ *that solves the constrained BSE of Problem 2.1, the process*

(2.11)
$$X(t) + \int_0^t [g(u) - \delta(\nu(u))]\,du, \qquad 0 \leq t \leq T$$

is a P^ν*-supermartingale with RCLL paths.*

PROOF.   It is easily seen from (2.6) and (2.10) that

$$
(2.12) \quad
\begin{aligned}
&X(t) + \int_0^t [g(u) - \delta(\nu(u)] \, du + \left[C(t) + \int_0^t (\delta(\nu(u)) - \nu'(u)Y(u)) \, du \right] \\
&= X(0) + \int_0^t Y'(u) \, dB_\nu(u),
\end{aligned}
$$

for all $0 \le t \le T$. The stochastic integral on the right-hand side is a P^ν-martingale by virtue of the Burkholder–Davis–Gundy inequalities [cf. Karatzas and Shreve (1991), Theorem 3.3.28], since we have

$$
E^\nu \left(\int_0^T \|Y(u)\|^2 \, du \right)^{1/2} \le \left(EZ_\nu^2(T) \cdot E \int_0^T \|Y(u)\|^2 \, du \right)^{1/2} < \infty;
$$

we are using here the boundedness of the process $\nu(\cdot)$, the assumption $Y(\cdot) \in H_d^2$ and the Cauchy–Schwarz inequality. Here and in the sequel, E^ν denotes the expectation operator under the probability measure P^ν of (2.4). The statement of the proposition follows then from (2.12), after noting that $C(\cdot) + \int_0^\bullet [\delta(\nu(u)) - \nu'(u)Y(u)] \, du$ is an increasing process. □

PROPOSITION 2.2.   *For any triple $(X(\cdot), Y(\cdot), C(\cdot))$ that solves the constrained BSE of Problem 2.1, we have*

$$
(2.13) \quad X(t) \ge \hat{X}(t) := \operatorname*{ess\,sup}_{\nu \in \mathscr{D}} E^\nu \left[\xi + \int_t^T [g(u) - \delta(\nu(u)] \, du \,\Big|\, \mathscr{F}(t) \right] \quad a.s.
$$

for every $t \in [0, T]$.

PROOF.   From Proposition 2.1 we have

$$
X(t) \ge E^\nu \left[X(T) + \int_t^T [g(u) - \delta(\nu(u))] \, du \,\Big|\, \mathscr{F}(t) \right] \quad \text{a.s.}
$$

for every $\nu(\cdot) \in \mathscr{D}$, and we are done, because $X(T) = \xi$. □

It is clear now that, in order to find the minimal solution to the constrained BSE of Problem 2.1, *it suffices to show that there exist processes $\hat{Y}(\cdot) \in H_d^2$ and $\hat{C}(\cdot) \in A_i^2$ such that $(\hat{X}(\cdot), \hat{Y}(\cdot), \hat{C}(\cdot))$ is a solution.* Then this triple has to be the minimal solution, and the processes $\nu(\cdot) \in \mathscr{D}$ are seen [by comparing (2.13) with (2.9)] to play the role of "adjoint variables" that enforce the constraint $\hat{Y}(\cdot) \in K$. We shall do this by imposing the following, very mild assumption.

ASSUMPTION 2.1.   There exists at least one solution $(\tilde{X}(\cdot), \tilde{Y}(\cdot), \tilde{C}(\cdot)) \in S_1^2 \times H_d^2 \times A_i^2$ to the constrained BSE of Problem 2.1; or equivalently, we have

$$
(2.14) \quad\quad\quad\quad \xi + \int_0^T g(u) \, du \le \eta \quad \text{a.s.}
$$

for some random variable $\eta \in L_1^2(\Omega)$ that can be represented in the form $\eta = c + \int_0^T Y'_\eta(u) \, dB(u)$ for suitable $c \in \mathbb{R}$ and $Y_\eta(\cdot) \in H_d^2$ (thus $c = E\eta$) such that $P[Y_\eta(t) \in K, \ \lambda\text{-a.e. } t \in [0, T]] = 1$.

Let us show that the two assumptions are indeed equivalent. If $(\tilde{X}(\cdot), \tilde{Y}(\cdot),$ $\tilde{C}(\cdot))$ is a solution of Problem 2.1, then we can take $\eta := \tilde{X}(0) + \int_0^T \tilde{Y}'(u) \, dB(u)$ and obtain the inequality (2.14) from (2.6) with $t = 0$. Conversely, given η as in the inequality (2.14), we can define $\tilde{X}(t) := E\eta + \int_0^t Y'_\eta(u) \, dB(u) - \int_0^t g(u) \, du$ and $\tilde{C}(t) := 0$ for $0 \leq t < T$, as well as $\tilde{X}(T) := \xi$ and $\tilde{C}(T) := \tilde{X}(T-) - \xi \geq 0$; it is easily seen that $(\tilde{X}(\cdot), Y_\eta(\cdot), \tilde{C}(\cdot))$ is then a solution of Problem 2.1.

Assumption 2.1 is satisfied, in its form (2.14), for example if both ξ and $g(\cdot)$ are bounded. Many more examples can be found in [CK93] and in Broadie, Cvitanić and Soner (1996).

We state now a result which is analogous to Proposition 6.3 of [CK93] and has a similar proof (sketched in the Appendix).

PROPOSITION 2.3.   *The process* $\hat{X}(\cdot)$ *of* (2.13) *can be considered in its RCLL modification; then the process* $\hat{X}(t) + \int_0^t [g(u) - \delta(\nu(u))] \, du$, $0 \leq t \leq T$ *is a* P^ν-*supermartingale with RCLL paths for every* $\nu(\cdot) \in \mathcal{D}$, *and we obtain the stronger version*

(2.15) $P[X(t) \geq \hat{X}(t) \qquad \forall \, 0 \leq t \leq T] = 1$

of the result in Proposition 2.2.

Next, we have the following result.

PROPOSITION 2.4.   *The process*

(2.16) $\hat{Q}(t) := \hat{X}(t) + \int_0^t g(u) \, du, \qquad 0 \leq t \leq T$

belongs to the space S_1^2, *that is,* $E[\sup_{0 \leq t \leq T} (\hat{Q}(t))^2] < \infty$.

PROOF.   From (2.13) we have

$$\hat{Q}(t) \geq E\left[\xi + \int_0^T g(u) \, du \,\Big|\, \mathcal{F}(t)\right], \qquad 0 \leq t \leq T.$$

The process on the right-hand side is a martingale in the space S_1^2, by Doob's maximal inequality. On the other hand, (2.13) and Assumption 2.1 imply

$$\hat{Q}(t) \leq \tilde{X}(t) + \int_0^t g(u) \, du, \qquad 0 \leq t \leq T.$$

The process on the right-hand side is also in S_1^2, and we are done. □

COROLLARY 2.1.   *For every given* $\nu(\cdot) \in \mathcal{D}$, *the process*

$$\hat{X}(t) + \int_0^t [g(u) - \delta(\nu(u))] \, du = \hat{Q}(t) - \int_0^t \delta(\nu(u)) \, du =: \hat{Q}_\nu(t), \qquad 0 \leq t \leq T$$

is a supermartingale of class $\mathcal{D}([0, T])$ *under* P^ν; *in other words, the family* $\{\hat{Q}_\nu(\tau)\}_{\tau \in \mathcal{S}_{0,T}}$ *is* P^ν-*uniformly integrable, where* $\mathcal{S}_{0,T}$ *is the set of all* F-*stopping times* $\tau \colon \Omega \to [0, T]$.

PROOF.   Since the support function $\delta(\cdot)$ is bounded on compact subsets of its effective domain $\tilde{K}$, and the process $\nu(\cdot)$ is bounded, it suffices to show $E^\nu[\sup_{0 \le t \le T} |\hat{Q}(t)|] < \infty$. But this follows from Proposition 2.4, the Cauchy–Schwarz inequality and the boundedness of the process $\nu(\cdot)$.   □

From Corollary 2.1, we have for every $\nu(\cdot) \in \mathscr{D}$ the *Doob–Meyer decomposition*

$$
(2.17) \quad \hat{X}(t) + \int_0^t \big(g(u) - \delta(\nu(u))\big)\, du = \hat{Q}(t) - \int_0^t \delta(\nu(u))\, du
$$
$$
= \hat{X}(0) + M^{(\nu)}(t) - A^{(\nu)}(t), \qquad 0 \le t \le T.
$$

Here $A^{(\nu)}(\cdot)$ is an F-predictable process with increasing, right-continuous paths and $A^{(\nu)}(0) = 0$, $E^\nu A^{(\nu)}(T) < \infty$. On the other hand, $M^{(\nu)}(\cdot)$ is a uniformly integrable P^ν-martingale of the Brownian filtration F; as such, it can be represented in the form

$$
(2.18) \qquad M^{(\nu)}(t) = \int_0^t \big(Y^{(\nu)}(u)\big)'\, dB_\nu(u), \qquad 0 \le t \le T
$$

for some process $Y^{(\nu)} \colon [0, T] \times \Omega \to \mathbb{R}^d$ which is F-progressively measurable and satisfies $\int_0^T \|Y^{(\nu)}(t)\|^2\, dt < \infty$ a.s. [cf. Karatzas and Shreve (1991), page 375].

The proof of the following proposition proceeds along lines similar to those in the proof of Theorem 6.4 in [CK93]; we sketch its main arguments in the Appendix.

PROPOSITION 2.5.   *The integrand*

$$
(2.19) \qquad\qquad \hat{Y}(\cdot) := Y^{(0)}(\cdot) \equiv Y^{(\nu)}(\cdot)
$$

of (2.18) *does not depend on the process* $\nu(\cdot) \in \mathscr{D}$, *and neither does the predictable increasing, right-continuous process*

$$
(2.20) \qquad \hat{C}(\cdot) := A^{(0)}(\cdot) \equiv A^{(\nu)}(\cdot) - \int_0^\bullet [\delta(\nu(u)) - \nu'(u)\hat{Y}(u)]\, du.
$$

Furthermore, we have

$$
(2.21) \quad \hat{X}(t) = \xi + \int_t^T g(u)\, du - \int_t^T \hat{Y}'(u)\, dB(u) + \hat{C}(T) - \hat{C}(t), \qquad 0 \le t \le T
$$

and

$$
(2.22) \qquad\qquad \hat{Y}(t) \in K \quad \text{for } \lambda\text{-a.e. } t \in [0, T]
$$

almost surely.

Finally, we obtain the following identification of the minimal solution.

THEOREM 2.1.   *Under Assumption 2.1, the triple $(\hat{X}(\cdot), \hat{Y}(\cdot), \hat{C}(\cdot))$, as defined in* (2.13), (2.19) *and* (2.20) *provides the minimal solution of the constrained BSE of Problem 2.1.*

PROOF.   It remains to prove

$$(2.23) \qquad\qquad E\Big[\sup_{0 \le t \le T} (\hat{X}(t))^2\Big] < \infty,$$

$$(2.24) \qquad\qquad E[\hat{C}(T)]^2 < \infty$$

and

$$(2.25) \qquad\qquad E \int_0^T \|\hat{Y}(t)\|^2 \, dt < \infty.$$

The inequality (2.23) follows from Proposition 2.4. The inequality (2.25) will follow from (2.24), because (2.17) with $\nu(\cdot) \equiv 0$ implies then that $M^{(0)}(\cdot)$ is a square-integrable martingale. Thus, it remains to show (2.24).

Let $Q_* := \sup_{0 \le t \le T} |\hat{Q}(t)|$, $q(t) := E[Q_* | \mathscr{F}(t)]$. Moreover, for every $k \in \mathbb{N}$, let $\rho_k := \inf\{t \in [0, T)/\hat{C}(t) \ge k\} \wedge T$. These are F-stopping times, and we have $\rho_k \uparrow T$ as $k \to \infty$, a.s. Clearly,

$$
\begin{aligned}
E[\hat{C}(\rho_k)]^2 &= 2E \int_0^{\rho_k} [\hat{C}(\rho_k) - \hat{C}(t)] \, d\hat{C}(t) \\
&= 2E \int_0^{\rho_k} E[\hat{C}(\rho_k) - \hat{C}(t) \mid \mathscr{F}(t)] \, d\hat{C}(t) \\
&= 2E \int_0^{\rho_k} E[\hat{Q}(t) - \hat{Q}(\rho_k) + M^{(0)}(\rho_k) - M^{(0)}(t) \mid \mathscr{F}(t)] \, d\hat{C}(t) \\
&= 2E \int_0^{\rho_k} E[\hat{Q}(t) - \hat{Q}(\rho_k) \mid \mathscr{F}(t)] \, d\hat{C}(t) \\
&\le 4E \int_0^{\rho_k} q(t) \, d\hat{C}(t) \le 4E\Big[\sup_{0 \le t \le T} (q(t)) \cdot \hat{C}(\rho_k)\Big] \\
&\le 4\Big(E\Big[\sup_{0 \le t \le T} q^2(t)\Big] \cdot E[\hat{C}(\rho_k)]^2\Big)^{1/2}.
\end{aligned}
$$

Therefore, we have

$$E[\hat{C}(\rho_k)]^2 \le 16E\Big[\sup_{0 \le t \le T} q^2(t)\Big]$$

for all $k \in \mathbb{N}$. Furthermore, by Doob's maximal inequality and Proposition 2.4,

$$E\Big[\sup_{0 \le t \le T} q^2(t)\Big] \le 4Eq^2(T) = 4E[Q_*^2] = 4E\Big[\sup_{0 \le t \le T} (\hat{Q}(t))^2\Big] < \infty.$$

Thus, letting $k \uparrow \infty$, we obtain

$$E[\hat{C}(T-)]^2 \le 64E\Big[\sup_{0 \le t \le T} (\hat{Q}(t))^2\Big] < \infty.$$

On the other hand, since $\int_0^{\cdot} \hat{Y}'(s)\,dB(s)$ is continuous, (2.21) implies

$$\hat{C}(T) - \hat{C}(T-) = \hat{Q}(T-) - \hat{Q}(T) \in L_1^2(\Omega);$$

thus $\hat{C}(T) \in L_1^2(\Omega)$ as well, and we are done. □

3. Penalization and BSDEs with constraints. Suppose now that the process $g(\cdot) \in H_1^2$ is replaced by the random field $f: [0, T] \times \Omega \times \mathbb{R} \to \mathbb{R}$, a given $\mathscr{P} \otimes \mathscr{B}(\mathbb{R})$-measurable mapping that satisfies

$$(3.1) \qquad E \int_0^T f^2(t, \omega, 0)\,dt < \infty,$$

as well as

$$(3.2) \qquad |f(t, \omega, x) - f(t, \omega, x')| \leq \kappa |x - x'|$$

for all $(t, \omega) \in [0, T] \times \Omega$ and $(x, x') \in \mathbb{R}^2$, for some $0 < \kappa < \infty$. Thus, instead of the constrained BSE of Problem 2.1, our focus now is the following constrained backward stochastic differential equation (CBSDE) problem.

PROBLEM 3.1. Find a triple of F-progressively measurable processes $(X(\cdot), Y(\cdot), C(\cdot))$ with $X(\cdot) \in S_1^2$, $Y(\cdot) \in H_d^2$, $C(\cdot) \in A_i^2$, such that the *backward stochastic differential equation* (BSDE)

$$(3.3) \qquad \begin{aligned} X(t) &= \xi + \int_t^T f(u, X(u))\,du \\ &\quad - \int_t^T Y'(u)\,dB(u) + C(T) - C(t), \qquad 0 \leq t \leq T \end{aligned}$$

and the constraint

$$(3.4) \qquad Y(t) \in K \quad \text{for } \lambda\text{-a.e. } t \in [0, T]$$

hold almost surely, and such that for any other solution $(\tilde{X}(\cdot), \tilde{Y}(\cdot), \tilde{C}(\cdot)) \in S_1^2 \times H_d^2 \times A_i^2$ satisfying (3.3) and (3.4), we have

$$X(t) \leq \tilde{X}(t), \qquad 0 \leq t \leq T$$

almost surely.

In order to solve Problem 3.1, we introduce the penalized BSDE

$$(3.5) \qquad \begin{aligned} X_n(t) &= \xi + \int_t^T [f(u, X_n(u)) + n\rho(Y_n(u))]\,du \\ &\quad - \int_t^T Y_n'(u)\,dB(u), \qquad 0 \leq t \leq T \end{aligned}$$

for every $n \in \mathbb{N}$, where $\rho(y) := \inf_{\xi \in K} \|y - \xi\|$ denotes the distance of the vector $y \in \mathbb{R}^d$ to the set K. Since the function $y \mapsto \rho(y)$ satisfies the Lipschitz condition

$$|\rho(y) - \rho(z)| \leq |y - z| \qquad \forall (y, z) \in (\mathbb{R}^d)^2,$$

(3.5) has a unique solution $(X_n(\cdot), Y_n(\cdot)) \in S_1^2 \times H_d^2$, by the standard theory of Pardoux and Peng (1990). We have the following characterization of this solution.

PROPOSITION 3.1. *The solution $X_n(\cdot)$ of the penalized BSDE* (3.5) *satisfies the following stochastic equation*

$$(3.6) \quad X_n(t) = \operatorname*{ess\,sup}_{\nu \in \mathscr{D}_n} E^\nu \left[\xi + \int_t^T [f(u, X_n(u)) - \delta(\nu(u))]\,du \;\Big|\; \mathscr{F}(t) \right],$$
$$0 \le t \le T$$

almost surely.

In order to prove this result, we need a property of the support function $\delta(\cdot)$ in (2.1).

LEMMA 3.1.

$$(3.7) \qquad \sup_{y \in \mathbb{R}^d}[\nu'y - n\rho(y)] = \begin{cases} \delta(\nu), & \nu \in \tilde{K} \cap B_n, \\ \infty, & \nu \notin \tilde{K} \cap B_n, \end{cases}$$

where $B_n := \{\nu \in \mathbb{R}^d;\ \|\nu\| \le n\}$.

PROOF. For every $\nu \in \tilde{K}$, we have

$$\delta(\nu) = \sup_{y \in K}(\nu'y) = \sup_{y \in K}(\nu'y - n\rho(y)) \le \sup_{y \in \mathbb{R}^d}(\nu'y - n\rho(y)).$$

If, moreover, $\|\nu\| \le n$, and we denote by y_K the projection of y on K [i.e., $\rho(y) = \|y - y_K\|$], we get

$$\nu'y - n\rho(y) = \nu'y_K + \nu'(y - y_K) - n\|y - y_K\|$$
$$\le \delta(\nu) + \|y - y_K\|(\|\nu\| - n) \le \delta(\nu)$$

for all $y \notin K$. For $y \in K$, we have clearly $\nu'y - n\rho(y) \le \delta(\nu)$ again, and thus

$$\delta(\nu) = \sup_{y \in \mathbb{R}^d}[\nu'y - n\rho(y)] \quad \text{for } \nu \in \tilde{K} \cap B_n.$$

Next, for any $\nu \in \mathbb{R}^d$ and $k \in \mathbb{N}$ with $\|\nu\| > n + \varepsilon$ for some $\varepsilon > 0$, there exists $y \in \mathbb{R}^d$, such that $\nu'(y/\|y\|) \ge n + \varepsilon$ and $\|y\| \ge k$. Thus,

$$\nu'y - n\rho(y) = \|y\| \left[\nu'\frac{y}{\|y\|} - n\frac{\rho(y)}{\|y\|} \right]$$
$$\ge \|y\| \left[\varepsilon + n\left(1 - \frac{\rho(y)}{\|y\|}\right) \right] \ge \varepsilon\|y\| \ge \varepsilon k,$$

and letting $k \uparrow \infty$ we obtain $\sup_{y \in \mathbb{R}^d}[\nu'y - n\rho(y)] = \infty$, for all $\nu \notin B_n$.

Finally, for $\nu \notin \tilde{K}$, we have $\sup_{y \in \mathbb{R}^d}[\nu'y - n\rho(y)] \ge \sup_{y \in K}(\nu'y) = \delta(\nu) = \infty$. □

PROOF OF PROPOSITION 3.1.   Let $\nu(\cdot) \in \mathscr{D}_n$ and $t \in [0, T]$. From the BSDE (3.5) and Lemma 3.1, we have

$$X_n(t) + \int_t^T \delta(\nu(s))\, ds = \xi + \int_t^T f(s, X_n(s))\, ds - \int_t^T Y_n'(s)\, dB_\nu(s)$$

(3.8)
$$+ \int_t^T [n\rho(Y_n(s)) - Y_n'(s)\nu(s) + \delta(\nu(s))]\, ds$$

$$\geq \xi + \int_t^T f(s, X_n(s))\, ds - \int_t^T Y_n'(s)\, dB_\nu(s).$$

By analogy with the proof of Proposition 2.1, the stochastic integral $I_n(\cdot) = \int_0^\cdot Y_n'(s)\, dB_\nu(s)$ is a P^ν-martingale. Hence $E^\nu[I_n(T) - I_n(t)|\mathscr{F}(t)] = 0$ and, after taking conditional expectations in (3.8), we obtain

$$X_n(t) \geq E^\nu\left[\xi + \int_t^T [f(s, X_n(s)) - \delta(\nu(s))]\, ds \,\Big|\, \mathscr{F}(t)\right]$$

almost surely. On the other hand, because the function $n\rho(\cdot)$ is Lipschitz-continuous and convex, we conclude as in page 36 of El Karoui, Peng and Quenez (1997) (hereafter abbreviated [EPQ]) that there exists a process $\hat{\nu}_n(\cdot) \in \mathscr{D}_n$ with

(3.9) $n\rho(Y_n) - Y_n'\hat{\nu}_n + \delta(\hat{\nu}_n) \equiv 0$ a.e. on $[0, T] \times \Omega$.

Setting $\nu(\cdot) = \hat{\nu}_n(\cdot)$ in (3.8) we get equality there, and therefore also

$$X_n(t) = E^{\hat{\nu}_n}\left[\xi + \int_t^T [f(s, X_n(s)) - \delta(\hat{\nu}_n(s))]\, ds \,\Big|\, \mathscr{F}(t)\right],$$

almost surely. Thus we obtain the a.s. equality of (3.6), first for fixed $t \in [0, T]$ and then for all $0 \leq t \leq T$ simultaneously, from the continuity of its left-hand side $X_n(\cdot)$ and the right continuity of its right-hand side [recall (3.5) and Proposition 2.3, respectively].   □

We now embark on the problem of finding and characterizing the limit of the sequence $\{X_n(\cdot)\}_{n\in\mathbb{N}}$. The standard comparison theorem for BSDEs (see [EPQ], page 23) implies that

(3.10) $X_n(t) \leq X_{n+1}(t),$ $0 \leq t \leq T$

holds almost surely for all $n \in \mathbb{N}$, since $n\rho(\cdot) \leq (n+1)\rho(\cdot)$. We also impose the following analogue of Assumption 2.1.

ASSUMPTION 3.1.   There exists at least one solution $(\tilde{X}(\cdot), \tilde{Y}(\cdot), \tilde{C}(\cdot))$ to the constrained BSDE of Problem 3.1.

Unlike the situation with Assumption 2.1, we do not have conditions on the data ξ, f, K of Problem 3.1, which are both necessary and sufficient for the validity of Assumption 3.1. See, however, Section 7 for discussion and sufficient conditions.

LEMMA 3.2. *Let Assumption 3.1 hold and* $(\tilde{X}(\cdot), \tilde{Y}(\cdot), \tilde{C}(\cdot))$ *be any solution to the constrained BSDE of Problem 3.1. Then, we have*

$$X_n(t) \le \tilde{X}(t), \qquad 0 \le t \le T$$

almost surely, for every $n \in \mathbb{N}$.

PROOF. Choose $\hat{\nu}_n(\cdot)$ as in the proof of Proposition 3.1 so that, by (3.8), the process $X_n(\cdot)$ satisfies the BSDE

$$X_n(t) = \xi + \int_t^T [f(s, X_n(s)) - \delta(\hat{\nu}_n(s))]\,ds - \int_t^T Y_n'(s)\,dB_{\hat{\nu}_n}(s), \qquad 0 \le t \le T.$$

We also observe from (3.3) and (2.10) that $\tilde{X}(\cdot)$ satisfies the BSDE

$$\tilde{X}(t) = \xi + \int_t^T [f(s, \tilde{X}(s)) - \tilde{Y}'(s)\hat{\nu}_n(s)]\,ds$$

$$+ \tilde{C}(T) - \tilde{C}(t) - \int_t^T \tilde{Y}'(s)\,dB_{\hat{\nu}_n}(s), \qquad 0 \le t \le T.$$

However, $0 \le \tilde{C}(T) - \tilde{C}(\cdot)$ and $-\delta(\hat{\nu}_n(\cdot)) \le -\tilde{Y}'(\cdot)\hat{\nu}_n(\cdot)$, so that the comparison theorem for BSDEs ([EPQ], page 23) applies again to give $X_n(\cdot) \le \tilde{X}(\cdot)$. [Note that, even though these BSDEs are driven by $B_{\hat{\nu}_n}(\cdot)$ rather than by $B(\cdot)$, the comparison theorem cited earlier is still valid because the stochastic integrals $\int_0^t Y_n'(s)\,dB_{\hat{\nu}_n}(s)$, $\int_0^t \tilde{Y}'(s)\,dB_{\hat{\nu}_n}(s)$, $0 \le t \le T$ are $P^{\hat{\nu}_n}$-martingales.] □

We conclude from (3.10) and Lemma 3.2 that the limit

(3.11) $$X^*(t) := \lim_{n\to\infty} X_n(t), \qquad 0 \le t \le T$$

exists almost surely. In the next section we prove that the limit process $X^*(\cdot)$ leads to the minimal solution of the constrained BSDE of Problem 3.1.

4. Constrained BSDE and a stochastic equation. We shall impose Assumption 3.1 throughout this section and establish with its help the following main result.

THEOREM 4.1. *The process* $X^*(\cdot)$ *of (3.11) is the unique solution, in the space* S_1^2, *of the stochastic equation*

(4.1) $$X^*(t) = \operatorname*{ess\,sup}_{\nu \in \mathscr{D}} E^\nu\!\left[\xi + \int_t^T [f(u, X^*(u)) - \delta(\nu(u))]\,du \,\Big|\, \mathscr{F}(t)\right],$$
$$0 \le t \le T.$$

COROLLARY 4.1 (Existence and uniqueness for Problem 3.1). *There exist processes* $Y^*(\cdot) \in H_d^2$ *and* $C^*(\cdot) \in A_i^2$ *such that the triple* $(X^*(\cdot), Y^*(\cdot), C^*(\cdot))$ *is the minimal solution to the constrained BSDE of Problem* 3.1.

PROOF OF COROLLARY 4.1.   Since $X_n(\cdot) \le X^*(\cdot) \le \tilde{X}(\cdot)$, we have $X^*(\cdot) \in$ S_1^2. From this, and from Theorem 4.1, it is easily checked that the analogue of Proposition 2.4 holds, with $\hat{X}(\cdot)$ replaced by $X^*(\cdot)$ and $g(\cdot)$ replaced by $f(\cdot, X^*(\cdot))$. Then, using the theory developed in Section 2, one constructs processes $Y^*(\cdot) \in H_d^2$ and $C^*(\cdot) \in A_i^2$ such that the triple $(X^*(\cdot), Y^*(\cdot), C^*(\cdot))$ is a solution to the constrained BSDE of Problem 3.1. From Lemma 3.2 we also conclude that this solution is minimal. □

The following "change of variable" result will be needed in the proof of Theorem 4.1.

PROPOSITION 4.1.   *For a given process* $g(\cdot) \in H_1^2$ *and random variable* $\xi \in$ L_1^2, *let*

$$\hat{X}(t) := \operatorname*{ess\,sup}_{\nu \in \mathscr{D}} E^\nu \left[\xi + \int_t^T [g(u) - \delta(\nu(u))]\, du \,\Big|\, \mathscr{F}(t) \right], \qquad 0 \le t \le T$$

as in (2.13). *Then, for any* $\lambda \in \mathbb{R}$, *we have*

$$(4.2) \quad e^{\lambda t} \hat{X}(t) = \operatorname*{ess\,sup}_{\nu \in \mathscr{D}} E^\nu \left[\xi e^{\lambda T} + \int_t^T e^{\lambda u} [G(u) - \delta(\nu(u))]\, du \,\Big|\, \mathscr{F}(t) \right],$$
$$0 \le t \le T$$

almost surely, where $G(u) := g(u) - \lambda \hat{X}(u)$.

PROOF.   We recall from (2.21) that the equation

$$\hat{X}(t) = \xi + \int_t^T g(u)\, du - \int_t^T \hat{Y}'(u)\, dB(u) + \hat{C}(T) - \hat{C}(t)$$

$$= \xi + \int_t^T [g(u) - \delta(\nu(u))]\, du - \int_t^T \hat{Y}'(u)\, dB_\nu(u) + m(t, T; \nu)$$

holds almost surely for every process $\nu(\cdot)$ in $\mathscr{D}$, where we have set

$$m(t, r; \nu) := \hat{C}(r) - \hat{C}(t) + \int_t^r [\delta(\nu(u)) - \hat{Y}'(u)\nu(u)]\, du, \qquad 0 \le t \le r \le T.$$

Since $\hat{Y}(\cdot) \in K$, the nonnegative random field $(t, r) \mapsto m(t, r; \nu)$ is nonincreasing in the first variable (t) and nondecreasing in the second variable (r). As in [CK93], page 677, there exists a sequence of processes $\{\nu_n(\cdot)\}_{n \in \mathbb{N}} \subseteq \mathscr{D}$ such that

$$\hat{X}(t) = \lim_{n \to \infty} E^{\nu_n} \left[\xi + \int_t^T [g(u) - \delta(\nu_n(u))]\, du \,\Big|\, \mathscr{F}(t) \right], \qquad 0 \le t \le T$$

holds almost surely [in fact, one can take $\nu_n(\cdot) \equiv \hat{\nu}_n(\cdot)$ as selected in (3.9), proof of Proposition 3.1]. Recalling that $\int_0^\bullet \hat{Y}'(u)\, dB_\nu(u)$ is a P^ν-martingale, we have then

$$(4.3) \quad E^{\nu_n} \left[\xi + \int_t^T [g(u) - \delta(\nu_n(u))]\, du \,\Big|\, \mathscr{F}(t) \right] = \hat{X}(t) - M_n(t) \quad \text{a.s.}$$

and

(4.4)
$$\lim_{n\to\infty} M_n(t) = 0 \quad \text{a.s.}$$

for every fixed $t \in [0, T]$. Here

(4.5)
$$M_n(t) := E^{\nu_n}[m(t, T; \nu_n) \,|\, \mathscr{F}(t)]$$
$$= E^{\nu_n}[m(0, T; \nu_n) \,|\, \mathscr{F}(t)] - m(0, t; \nu_n), \qquad 0 \le t \le T$$

is a nonnegative P^{ν_n}-supermartingale with RCLL paths [recall Theorem 1.1.13 in Karatzas and Shreve (1991)].

We deduce from (4.3) that the process $\hat{X}(t) - M_n(t) + \int_0^t [g(u) - \delta(\nu_n(u))] \, du$ is a P^{ν_n}-martingale. Therefore, by Itô's rule on $e^{\lambda t} \hat{X}(t)$, the process

$$e^{\lambda t} \hat{X}(t) - \int_0^t e^{\lambda u} \, dM_n(u) + \int_0^t e^{\lambda u} [g(u) - \lambda \hat{X}(u) - \delta(\nu_n(u))] \, du, \qquad 0 \le t \le T$$

is also a P^{ν_n}-martingale. This implies the equation

(4.6)
$$E^{\nu_n}\left[e^{\lambda T}\xi + \int_t^T e^{\lambda u}[G(u) - \delta(\nu_n(u))] \, du \Big| \mathscr{F}(t) \right]$$
$$= e^{\lambda t} \hat{X}(t) + E^{\nu_n}\left[\int_t^T e^{\lambda u} \, dM_n(u) \Big| \mathscr{F}(t) \right].$$

We want to show that the last term on the right-hand side of (4.6) tends to zero, as $n \to \infty$. First, recall that $M_n(\cdot)$ of (4.5) is an (F, P^{ν_n})-supermartingale and integrate by parts to obtain

(4.7)
$$0 \le -E^{\nu_n}\left[\int_t^T e^{\lambda u} dM_n(u) \Big| \mathscr{F}(t) \right]$$
$$= e^{\lambda t} M_n(t) + \lambda E^{\nu_n}\left[\int_t^T e^{\lambda u} M_n(u) \, du \Big| \mathscr{F}(t) \right].$$

Suppose first that $\lambda \le 0$; since $M_n(\cdot)$ is nonnegative, the right-hand side of (4.7) is bounded from above by $e^{\lambda t} M_n(t)$, which converges to zero as $n \to \infty$. On the other hand, if $\lambda > 0$, the supermartingale property of $M_n(\cdot)$ gives

$$\lambda E^{\nu_n}\left[\int_t^T e^{\lambda u} M_n(u) \, du \Big| \mathscr{F}(t) \right] \le M_n(t) \int_t^T \lambda e^{\lambda u} \, du \le (e^{\lambda T} - 1)M_n(t).$$

Recalling (4.7) and (4.4), and letting n tend to infinity, we conclude that

(4.8)
$$\lim_{n\to\infty} E^{\nu_n}\left[\int_t^T e^{\lambda u} \, dM_n(u) \Big| \mathscr{F}(t) \right] = 0$$

holds almost surely, for every $t \in [0, T]$ fixed, and for any $\lambda \in \mathbb{R}$.

Returning to (4.6), we obtain in conjunction with (4.8) the representation

$$\lim_{n\to\infty} E^{\nu_n}\left[\xi e^{\lambda T} + \int_t^T e^{\lambda u}[G(u) - \delta(\nu_n(u))] \, du \Big| \mathscr{F}(t) \right] = e^{\lambda t} \hat{X}(t)$$

and thus also

$$e^{\lambda t}\hat{X}(t) \le \operatorname*{ess\,sup}_{\nu\in\mathscr{D}} E^\nu\left[\xi e^{\lambda T} + \int_t^T e^{\lambda u}[G(u) - \delta(\nu(u))]\,du \,\Big|\, \mathscr{F}(t)\right],$$

almost surely. The reverse inequality follows as in the previous section (first part in the proof of Proposition 3.1), after noting that the triple $(e^{\lambda t}\hat{X}(t), \int_0^t e^{\lambda u}\,d\hat{C}(u), e^{\lambda t}\hat{Y}(t))$ solves the BSDE (3.3), with the terminal condition ξ replaced by $\xi e^{\lambda T}$, with $f(t, \hat{X}(t))$ replaced by $e^{\lambda t}G(t)$, and with the constraint $\hat{Y}(t) \in K$ replaced by $e^{\lambda t}\hat{Y}(t) \in e^{\lambda t}K$, for λ-a.e. $t \in [0, T]$.

We conclude that the representation (4.2) holds almost surely, first for $t \in [0, T]$ fixed, and then for all $0 \le t \le T$ simultaneously, thanks to the RCLL regularity of both sides in (4.2) (recall Proposition 2.3). □

PROOF OF THEOREM 4.1.

Existence. We have to show that the process $X^*(\cdot)$ of (3.11) solves the stochastic equation (4.1). Fix a process $\nu(\cdot)$ in $\mathscr{D}$ and select an integer n sufficiently large, so that $\nu(\cdot)$ belongs to $\mathscr{D}_n$. From Proposition 3.1 we get

$$X^*(t) \ge X_n(t) \ge E^\nu\left[\xi + \int_t^T [f(u, X_n(u)) - \delta(\nu(u))]\,du \,\Big|\, \mathscr{F}(t)\right], \qquad 0 \le t \le T.$$

The comparison theorem ([EPQ], page 23) implies $X^{(0)}(\cdot) \le X_n(\cdot)$, for all $n \in \mathbb{N}$, where $X^{(0)}(\cdot) \in S_1^2$ is the state process in the solution $(X^{(0)}(\cdot), Y^{(0)}(\cdot), 0)$ to the unconstrained version

$$X^{(0)}(t) = \xi + \int_t^T f(u, X^{(0)}(u))\,du - \int_t^T (Y^{(0)}(u))'\,dB(u), \qquad 0 \le t \le T$$

of the BSDE (3.3). Since we also have $X_n(\cdot) \le X^*(\cdot) \le \tilde{X}(\cdot) \in S_1^2$, by the Lipschitz property of f, we can use the dominated convergence theorem for conditional expectations to conclude that

$$X^*(t) \ge E^\nu\left[\xi + \int_t^T [f(u, X^*(u)) - \delta(\nu(u))]\,du \,\Big|\, \mathscr{F}(t)\right]$$

holds almost surely for all $\nu(\cdot) \in \mathscr{D}$; thus

$$X^*(t) \ge \operatorname*{ess\,sup}_{\nu\in\mathscr{D}} E^\nu\left[\xi + \int_t^T [f(u, X^*(u)) - \delta(\nu(u))]\,du \,\Big|\, \mathscr{F}(t)\right], \qquad 0 \le t \le T.$$

In order to prove the reverse inequality, let us observe that the function

(4.9) $$F(s, x) := -\lambda x + e^{\lambda s}f(s, e^{-\lambda s}x), \qquad 0 \le s \le T, \ x \in \mathbb{R}$$

is nondecreasing in the variable x, provided we select $\lambda = -\kappa$, where κ is the Lipschitz constant of the function f as in (3.2). Then, using Proposition 3.1

and the analogue of Proposition 4.1, we get

$$e^{\lambda t} X_n(t)$$

$$= \operatorname*{ess\,sup}_{\nu \in \mathscr{D}_n} E^\nu \left[\xi e^{\lambda T} + \int_t^T [F(u, e^{\lambda u} X_n(u)) - e^{\lambda u} \delta(\nu(u))]\, du \;\Big|\; \mathscr{F}(t) \right]$$

$$\leq \operatorname*{ess\,sup}_{\nu \in \mathscr{D}} E^\nu \left[\xi e^{\lambda T} + \int_t^T [F(u, e^{\lambda u} X^*(u)) - e^{\lambda u} \delta(\nu(u))]\, du \;\Big|\; \mathscr{F}(t) \right]$$

$$= \operatorname*{ess\,sup}_{\nu \in \mathscr{D}} E^\nu \left[\xi e^{\lambda T} + \int_t^T e^{\lambda u}[f(u, X^*(u)) - \lambda X^*(u) - \delta(\nu(u))]\, du \;\Big|\; \mathscr{F}(t) \right]$$

$$=: X^{(\lambda)}(t).$$

Therefore, letting $n \to \infty$ leads to $X^*(t) \leq e^{-\lambda t} X^{(\lambda)}(t)$, $0 \leq t \leq T$; yet another application of Proposition 4.1, this time to the process $e^{-\lambda t} X^{(\lambda)}(t)$, $0 \leq t \leq T$, implies

$$X^*(t) \leq \operatorname*{ess\,sup}_{\nu \in \mathscr{D}} E^\nu \left[\xi + \int_t^T [f(u, X^*(u)) - \delta(\nu(u))]\, du \right.$$

$$\left. + \int_t^T \lambda [e^{-\lambda u} X^{(\lambda)}(u) - X^*(u)]\, du \;\Big|\; \mathscr{F}(t) \right]$$

$$\leq \operatorname*{ess\,sup}_{\nu \in \mathscr{D}} E^\nu \left[\xi + \int_t^T [f(u, X^*(u)) - \delta(\nu(u))]\, du \;\Big|\; \mathscr{F}(t) \right],$$

$$0 \leq t \leq T. \quad \square$$

Uniqueness. Let $\tilde{X}(\cdot) \in \mathbf{S}_1^2$ be another solution to the stochastic equation (4.1). As in Corollary 4.1, there exist processes $\tilde{C}(\cdot)$ and $\tilde{Y}(\cdot)$, such that $(\tilde{X}(\cdot), \tilde{Y}(\cdot), \tilde{C}(\cdot))$ is a solution to the BSDE (3.3). In particular then, $\tilde{X}(\cdot)$ has RCLL paths, and Lemma 3.2 implies $X^*(\cdot) \leq \tilde{X}(\cdot)$ a.s. In order to prove the reverse inequality, let $\lambda = \kappa$, where again κ is the Lipschitz constant of f as in (3.2), and observe that the function $x \mapsto F(s, x) = -\lambda x + e^{\lambda s} f(s, e^{-\lambda s} x)$ of (4.9) is now nonincreasing. Using Proposition 4.1, we obtain

$$e^{\lambda t} \tilde{X}(t) \geq e^{\lambda t} X^*(t)$$

$$= \operatorname*{ess\,sup}_{\nu \in \mathscr{D}} E^\nu \left[\xi e^{\lambda T} + \int_t^T [F(u, e^{\lambda u} X^*(u)) - e^{\lambda u} \delta(\nu(u))]\, du \;\Big|\; \mathscr{F}(t) \right]$$

$$\geq \operatorname*{ess\,sup}_{\nu \in \mathscr{D}} E^\nu \left[\xi e^{\lambda T} + \int_t^T [F(u, e^{\lambda u} \tilde{X}(u)) - e^{\lambda u} \delta(\nu(u))]\, du \;\Big|\; \mathscr{F}(t) \right]$$

$$= e^{\lambda t} \tilde{X}(t), \qquad 0 \leq t \leq T$$

almost surely, and uniqueness follows. $\square$

5. **The case of convex drift $f(t, \omega, x, \bullet)$.** In this section we study the case of a drift random field f which is also a function of the gains process $Y(\cdot)$.

More precisely, we consider a random field $f: [0, T] \times \Omega \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ which is $\mathscr{P} \otimes \mathscr{B}(\mathbb{R}) \otimes \mathscr{B}(\mathbb{R}^d)/\mathscr{B}(\mathbb{R})$-measurable and satisfies

$$(5.1) \qquad E \int_0^T f^2(t, \omega, 0, 0) \, dt < \infty,$$

as well as

$$(5.2) \qquad |f(t, \omega, x, y) - f(t, \omega, x', y')| \leq \kappa(|x - x'| + |y - y'|)$$

for all $(t, \omega) \in [0, T] \times \Omega$, $(x, x') \in \mathbb{R}^2$ and $(y, y') \in \mathbb{R}^{2d}$, for some $0 < \kappa < \infty$. Our aim is to study the analogue of Problem 3.1, in which (3.3) is replaced by

$$(5.3) \qquad \begin{aligned} X(t) = \xi + \int_t^T f(u, X(u), Y(u)) \, du \\ - \int_t^T Y'(u) \, dB(u) + C(T) - C(t), \qquad 0 \leq t \leq T. \end{aligned}$$

We shall refer to this modified problem as Problem 3.1′. We shall be able to study the modified problem with minimal extra effort, but under the following assumption.

ASSUMPTION 5.1. The function $y \mapsto f(t, \omega, x, y)$ is convex on $\mathbb{R}^d$, for every $(t, \omega, x) \in [0, T] \times \Omega \times \mathbb{R}$.

Following [EPQ], we introduce for every fixed $(t, \omega, x) \in [0, T] \times \Omega \times \mathbb{R}$ the dual

$$(5.4) \qquad \tilde{f}(t, \omega, x, \mu) := \sup_{y \in \mathbb{R}^d} [\mu' y - f(t, \omega, x, y)], \qquad \mu \in \mathbb{R}^d$$

of the convex function $f(t, \omega, x, \cdot)$, as well as its *effective domain*

$$(5.5) \qquad \tilde{O} := \{(t, \omega, x, \mu) \in [0, T] \times \Omega \times \mathbb{R} \times \mathbb{R}^d \, / \, \tilde{f}(t, \omega, x, \mu) < \infty\}.$$

As in [EPQ], one can show that each (t, ω, x)-section of $\tilde{O}$, denoted as $\tilde{O}^{t, \omega, x}$, is included in a bounded set $\tilde{R}$ in $\mathbb{R}^d$, independent of (t, ω, x). Moreover, we have the following result.

LEMMA 5.1. *For any given* $(t, \omega) \in [0, T] \times \Omega$, *the set* $\tilde{O}^{t, \omega, x}$ *does not depend on* x.

PROOF. Let $\mu \in \tilde{O}^{t, \omega, x}$ for some $(t, \omega, x) \in [0, T] \times \Omega \times \mathbb{R}$. Let $x' \in \mathbb{R}$ be arbitrary. There exists a sequence $\{y_n\}_{n \in \mathbb{N}} \in \mathbb{R}^d$ attaining the (possibly infinite) supremum in the definition of $\tilde{f}(t, \omega, x', \mu)$. We have

$$\tilde{f}(t, \omega, x', \mu) - \tilde{f}(t, \omega, x, \mu) \leq \lim_n [\mu' y_n - f(t, \omega, x', y_n)]$$

$$+ \liminf_n [f(t, \omega, x, y_n) - \mu' y_n]$$

$$\leq \kappa |x - x'|$$

and thus $\tilde{f}(t, \omega, x', \mu) < \infty$. $\square$

Consequently, we may omit x in the notation $\tilde{O}^{t,\,\omega,\,x}$, and write $\tilde{O}^{t,\,\omega}$ instead. Let us also introduce the class $\mathscr{A}$ of F−progressively measurable processes $\mu(\cdot)$: $[0, T] \times \Omega \to \tilde{R}$ which satisfy $E \int_0^T \tilde{f}^2(t, 0, \mu(t)) \, dt < \infty$.

LEMMA 5.2.   *For any pair of processes $(X(\cdot), Y(\cdot)) \in \mathsf{S}_1^2 \times \mathsf{H}_d^2$, there exists a process $\mu(\cdot) \in \mathscr{A}$ such that*

$$(5.6) \qquad f(t, X(t), Y(t)) = \mu'(t)Y(t) - \tilde{f}(t, X(t), \mu(t)), \qquad 0 \le t \le T$$

holds almost surely.

This result is proved in [EPQ]. The proof of Lemma 5.1 also implies the following.

LEMMA 5.3.   *The function $\tilde{f}(t, \omega, \cdot, \mu)$ is uniformly Lipschitz in x; more precisely, there exists a constant $C > 0$ such that, for any given $(t, \omega) \in [0, T] \times \Omega$, $(x, x') \in \mathbb{R}^2$, and $\mu \in \tilde{O}^{t,\omega}$, we have*

$$\left| \tilde{f}(t, \omega, x, \mu) - \tilde{f}(t, \omega, x', \mu) \right| \le C|x - x'|.$$

For any given pair of processes $(\nu(\cdot), \mu(\cdot)) \in \mathscr{D} \times \mathscr{A}$, let us introduce now the exponential martingale

$$(5.7) \qquad Z_{\nu,\,\mu}(t) = \exp\left\{ \int_0^t (\nu(s) + \mu(s))' \, dB(s) - \frac{1}{2} \int_0^t \|\nu(s) + \mu(s)\|^2 \, ds \right\},$$
$$0 \le t \le T,$$

as well as the probability measure

$$(5.8) \qquad P^{\nu,\,\mu}(A) := E[Z_{\nu,\,\mu}(T)1_A], \qquad A \in \mathscr{F}(T),$$

under which the process

$$(5.9) \qquad B_{\nu,\,\mu}(t) := B(t) - \int_0^t [\nu(s) + \mu(s)] \, ds, \qquad 0 \le t \le T$$

is Brownian motion. We also denote by $E^{\nu,\,\mu}$ the expectation with respect to the probability measure of (5.8). Moreover, we introduce the penalized BSDEs

$$(5.10) \qquad \begin{aligned} X_n(t) = \xi + \int_t^T [f(u, X_n(u), Y_n(u)) + n\rho(Y_n(u))] \, du \\ - \int_t^T Y_n'(u) \, dB(u), \qquad 0 \le t \le T \end{aligned}$$

for every $n \in \mathbb{N}$, by analogy with (3.5).

PROPOSITION 5.1. *The solution $X_n(\cdot)$ of the penalized BSDE* (5.10) *satisfies the stochastic equation*

(5.11)
$$X_n(t) = \operatorname*{ess\,sup}_{(\nu,\mu)\in\mathscr{D}_n\times\mathscr{A}} E^{\nu,\mu}\left[\xi - \int_t^T \left[\tilde{f}(u, X_n(u), \mu(u)) + \delta(\nu(u))\right] du \,\Big|\, \mathscr{F}(t)\right],$$

$$0 \le t \le T$$

almost surely.

The proof is completely analogous to that of Proposition 3.1 and uses Lemma 5.2. In particular, to show that the supremum of (5.11) is attained, we choose $(\nu_n(\cdot), \mu_n(\cdot)) \in \mathscr{D}_n \times \mathscr{A}$ so as to have $n\rho(Y_n(\cdot)) - Y_n'(\cdot)\nu_n(\cdot) + \delta(\nu_n(\cdot)) \equiv 0$ and $\tilde{f}(\cdot, X_n(\cdot), \mu_n(\cdot)) \equiv -f(\cdot, X_n(\cdot), Y_n(\cdot)) + \mu_n'(\cdot)Y_n(\cdot)$, a.e. on $[0, T] \times \Omega$.

ASSUMPTION 5.2. *There exists at least one solution $(\tilde{X}(\cdot), \tilde{Y}(\cdot), \tilde{C}(\cdot))$ to the constrained BSDE* (5.3) *of Problem 3.1′.*

Under this assumption, one shows as before that the limit

(5.12)
$$X^*(t) := \lim_{n\to\infty} X_n(t), \qquad 0 \le t \le T$$

exists almost surely and establishes the following analogues of Theorem 4.1 and Corollary 4.1.

THEOREM 5.1. *Under Assumption 5.2, the process $X^*(\cdot)$ of* (5.12) *is the unique solution, in the space S_1^2, of the stochastic equation*

(5.13)
$$X^*(t) = \operatorname*{ess\,sup}_{(\nu,\mu)\in\mathscr{D}\times\mathscr{A}} E^{\nu,\mu}\left[\xi - \int_t^T [\tilde{f}(u, X^*(u), \mu(u)) + \delta(\nu(u))] du \,\Big|\, \mathscr{F}(t)\right],$$

$$0 \le t \le T.$$

COROLLARY 5.1. *There exist processes $Y^*(\cdot) \in H_d^2$ and $C^*(\cdot) \in A_i^2$ such that the triple $(X^*(\cdot), Y^*(\cdot), C^*(\cdot))$ is the minimal solution to the constrained BSDE* (5.3) *of Problem 3.1′.*

The proofs of these results are parallel to those of Theorem 4.1 and Corollary 4.1, with the help of Lemma 5.2. In particular, the proof of Theorem 5.1 uses the following analogue of Proposition 4.1.

PROPOSITION 5.2. *For a given process $W(\cdot) \in S_1^2$ and a random variable variable $\xi \in L_1^2$, let*

$$\hat{X}(t) := \operatorname*{ess\,sup}_{(\nu,\mu)\in\mathscr{D}\times\mathscr{A}} E^{\nu,\mu}\left[\xi - \int_t^T [\tilde{f}(u, W(u), \mu(u)) + \delta(\nu(u))] du \,\Big|\, \mathscr{F}(t)\right],$$

$$0 \le t \le T.$$

Then, for any $\lambda \in \mathbb{R}$, we have

$$
\begin{aligned}
(5.14) \quad e^{\lambda t}\hat{X}(t) = \operatorname*{ess\,sup}_{(\nu,\,\mu)\in\mathscr{D}\times\mathscr{A}} E^{\nu,\,\mu}\bigg[& \xi e^{\lambda T} - \int_t^T e^{\lambda u}\Big[\tilde{f}(u, W(u), \mu(u)) \\
& + \lambda\hat{X}(u) + \delta(\nu(u))\Big]\,du \,\Big|\, \mathscr{F}(t)\bigg]
\end{aligned}
$$

for all $0 \le t \le T$, almost surely.

We only sketch the beginning of the proof of this result, since the rest is similar to that of Proposition 4.1. By analogy with the proof of Proposition 2.5 in the Appendix (and using Lemma 5.2), one shows that the following analogue of (2.21),

$$
\begin{aligned}
\hat{X}(t) &= \xi + \int_t^T f(u, W(u), \hat{Y}'(u))\,du - \int_t^T \hat{Y}'(u)\,dB(u) + \hat{C}(T) - \hat{C}(t) \\
&= \xi - \int_t^T [\tilde{f}(u, W(u), \mu(u)) + \delta(\nu(u))]\,du \\
&\quad - \int_t^T \hat{Y}'(u)\,dB_{\nu,\,\mu}(u) + m(t, T; \nu, \mu), \qquad 0 \le t \le T,
\end{aligned}
$$

holds almost surely, for some process $\hat{Y}(\cdot) \in \mathsf{H}_d^2$ taking values in K, some $\hat{C}(\cdot) \in \mathsf{A}_i^2$ and for every pair of processes $(\nu(\cdot), \mu(\cdot))$ in $\mathscr{D} \times \mathscr{A}$. Here we have set

$$
\begin{aligned}
m(t, r; \nu, \mu) := \; &\hat{C}(r) - \hat{C}(t) \\
&+ \int_t^r \Big[\delta(\nu(u)) + \tilde{f}(u, W(u), \mu(u)) \\
&\qquad + f(u, W(u), \hat{Y}(u)) - \hat{Y}'(u)(\nu(u) + \mu(u))\Big]\,du, \\
&\hspace{8cm} 0 \le t \le r \le T.
\end{aligned}
$$

By the definitions of the functions δ in (2.1) and $\tilde{f}$ in (5.4), the nonnegative random field $(t, r) \mapsto m(t, r; \nu, \mu)$ is nonincreasing in the first variable (t), and nondecreasing in the second variable (r). Moreover, there is a sequence $\{\nu_n(\cdot), \mu_n(\cdot)\}_{n\in\mathbb{N}} \subseteq \mathscr{D} \times \mathscr{A}$ such that

$$
\hat{X}(t) = \lim_{n\to\infty} E^{\nu_n,\,\mu_n}\bigg[\xi - \int_t^T [\tilde{f}(u, W(u), \mu_n(u)) + \delta(\nu_n(u))]\,du \,\Big|\, \mathscr{F}(t)\bigg],
$$
$$
0 \le t \le T
$$

holds almost surely. One can take $\nu_n(\cdot) \equiv \hat{\nu}_n(\cdot)$, as in (3.9) in the proof of Proposition 3.1, while $\mu_n(\cdot)$ is selected as in Lemma 5.2, so that $\tilde{f}(\cdot, X_n(\cdot), \mu_n(\cdot)) \equiv \mu_n'(\cdot)Y_n(\cdot) - f(\cdot, X_n(\cdot), Y_n(\cdot))$ a.e. on $[0, T] \times \Omega$. The rest of the proof is similar to that of Proposition 4.1. $\square$

6. **The case of a lower barrier.** Let us suppose now that we are given a process $L(\cdot) \in S_1^2$ with continuous paths and $L(T) \leq \xi$ almost surely, and consider Problem 2.1 with the a.s. requirement

$$(6.1) \qquad\qquad X(t) \geq L(t), \qquad 0 \leq t \leq T$$

on its state process, in addition to (2.6) and (2.7). Similarly, consider the analogue of Problem 3.1 where, along with (3.3) and (3.4), we impose the lower bound (6.1) on the state process.

In both these so-modified problems, denoted henceforth as Problem 2.1″ and Problem 3.1″, respectively, we treat $L(\cdot)$ as a lower barrier that the state process $X(\cdot)$ is not allowed to cross on its way to the terminal condition $X(T) = \xi \geq L(T)$. As before, we seek a minimal solution to each of these problems (assuming, of course, that at least one solution exists).

For the unconstrained case $K = \mathbb{R}^d$, these problems were discussed thoroughly in [EKPPQ]. In our setting, it is not hard to modify the theory developed in Sections 2–4 in order to take into account the imposition of the lower bound (6.1). For instance, the minimal solution to Problem 2.1″ is given as

$$(2.13)'' \qquad \hat{X}(t) = \operatorname*{ess\,sup}_{\substack{\nu \in \mathscr{D} \\ \tau \in \mathscr{S}_{t,T}}} E^{\nu}\Bigg[\xi 1_{\{\tau=T\}} + L(\tau)1_{\{\tau<T\}} \\ + \int_t^{\tau} [g(u) - \delta(\nu(u))]\,du \;\Big|\; \mathscr{F}(t) \Bigg]$$

for $0 \leq t \leq T$, by analogy with Theorem 4.1, where $\mathscr{S}_{t,T}$ denotes the class of F-*stopping times* τ with values in the interval $[t, T]$.

Notice here the need to introduce a double optimization problem, of mixed stochastic control–stopping type, in order to represent this minimal solution. The maximization over control processes $\nu(\cdot)$ ensures that the constraint (2.7) on the gains process is observed; whereas the optimization over stopping times τ guarantees that the state process $X(\cdot)$ satisfies the constraint (6.1). In other words, $\nu(\cdot)$ and τ play the roles of "dual (adjoint) variables" that enforce the constraints (2.7) and (6.1), respectively.

By analogy with Theorem 4.1 and Corollary 4.1, there is now a unique process $X^*(\cdot)$ in the space S_1^2 that solves the *stochastic functional equation*

$$(4.1)'' \qquad X^*(t) = \operatorname*{ess\,sup}_{\substack{\nu \in \mathscr{D} \\ \tau \in \mathscr{S}_{t,T}}} E^{\nu}\Bigg[\xi 1_{\{\tau=T\}} + L(\tau)1_{\{\tau<T\}} \\ + \int_t^{\tau} [f(u, X^*(u)) - \delta(\nu(u))]\,du \;\Big|\; \mathscr{F}(t) \Bigg]$$

for $0 \leq t \leq T$, and this $X^*(\cdot)$ is the state process of the minimal solution to Problem 3.1″. As in Section 3, the state process $X^*(\cdot)$ is constructed through a penalization scheme which now takes a more complicated form due to the

presence of the "reflecting lower barrier," namely,

$$X_n(t) = \xi + \int_t^T [f(u, X_n(u)) + n\rho(Y_n(u))]\,du$$

(3.5)''
$$\qquad\qquad - \int_t^T Y_n'(u)\,dB(u) + C_n(T) - C_n(t),$$

$$X_n(t) \geq L(t), \qquad 0 \leq t \leq T,$$

$$C_n(\cdot) \text{ continuous, increasing and } \int_0^T [X_n(t) - L(t)]\,dC_n(t) = 0$$

almost surely, for a suitable triple $(X_n(\cdot), Y_n(\cdot), C_n(\cdot)) \in S_1^2 \times H_d^2 \times A_i^2,\ n \in \mathbb{N}$.

The solvability of the system (3.5)'' and the a.s. comparison $X_n(\cdot) \leq X_{n+1}(\cdot),\ n \in \mathbb{N}$, are consequences of Theorems 4.1, 5.2 in [EKPPQ]. The state process of the (unique) solution to (3.5)'' satisfies the equation

$$X_n(t) = \operatorname*{ess\,sup}_{\substack{\nu \in \mathscr{D}_n \\ \tau \in \mathscr{S}_{t,T}}} E^\nu \Bigg[\xi 1_{\{\tau=T\}} + L(\tau)1_{\{\tau<T\}}$$

(3.6)''
$$\qquad\qquad + \int_t^\tau [f(u, X_n(u)) - \delta(\nu(u))]\,du \ \Big|\ \mathscr{F}(t) \Bigg]$$

for $0 \leq t \leq T$. This supremum is attained by the pair $(\nu(\cdot), \tau) = (\nu_n(\cdot), \tau_n)$, where $\nu_n(\cdot)$ satisfies $\rho(Y_n(\cdot)) - Y_n'(\cdot)\nu_n(\cdot) + \delta(\nu_n(\cdot)) = 0$ a.e. on $[0, T]$ as in (3.9) of the proof of Proposition 3.1 and

(6.2)
$$\tau_n(t) := \inf\{u \in [t, T) \,/\, X_n(u) = L(u)\} \wedge T,$$

namely

$$X_n(t) = E^{\nu_n} \Bigg[\xi 1_{\{\tau_n(t)=T\}} + L(\tau_n(t))1_{\{\tau_n(t)<T\}}$$
$$\qquad\qquad + \int_t^{\tau_n(t)} [f(u, X_n(u)) - \delta(\nu_n(u))]\,du \ \Big|\ \mathscr{F}(t) \Bigg].$$

One can also show that the limit process $X^*(t) := \lim_{n\to\infty} \uparrow X_n(t),\ 0 \leq t \leq T$ is the minimal solution of Problem 3.1''.

The details of these derivations are more or less straightforward, with the possible exception of the proof of the change-of-variable formula

$$e^{\lambda t}\hat{X}(t) = \operatorname*{ess\,sup}_{\substack{\nu \in \mathscr{D} \\ \tau \in \mathscr{S}_{t,T}}} E^\nu \Bigg[\xi e^{\lambda T} 1_{\{\tau=T\}} + L(\tau)e^{\lambda\tau} 1_{\{\tau<T\}}$$

(4.2)''
$$\qquad\qquad + \int_t^\tau e^{\lambda u}[g(u) - \lambda\hat{X}(u) - \delta(\nu(u))]\,du \ \Big|\ \mathscr{F}(t) \Bigg],$$

valid for every $\lambda \in \mathbb{R}$, for the process $\hat{X}(\cdot)$ of (2.13)'' [analogue of Proposition (4.1)]. This formula plays again a crucial role in establishing the existence and uniqueness of the solution to the stochastic functional equation (4.1)''. We shall leave these details to the care of the diligent reader. $\square$

7. On Assumption 3.1.   In Section 4 we identified the process $X^*(\cdot)$ of (3.11) as the state process in the minimal solution of the constrained BSDE Problem 3.1. This identification used Assumption 3.1 only inasmuch as to guarantee that the sequence $\{X_n(\cdot)\}$ of "penalized solutions" in (3.5) is bounded from above by some process in S_1^2:

(7.1) $X_n(\cdot) \le \tilde{X}(\cdot)$ a.s. for all $n \in \mathbb{N}$, some $\tilde{X}(\cdot) \in S_1^2$.

We provide here some sufficient conditions for (7.1)—and thus also for Assumption 3.1, as well, since the process $X^*(\cdot)$ of (3.11) leads then to a solution of Problem 3.1 (as in Corollary 4.1). The first set of sufficient conditions is as follows.

ASSUMPTION 7.1.   (i) The drift random field $f(t, \omega, x)$ satisfies the conditions of Section 3, including (3.1) and (3.2), as: well as:

$$f(t, \omega, x) \le C \qquad \forall\, (t, \omega, x) \in [0, T] \times \Omega \times \mathbb{R}$$

for some real constant C.
   (ii) The terminal random variable ξ satisfies

(7.2) $\xi \le C + \displaystyle\int_0^T \varphi'(u)\, dB(u)$ a.s.

for some real constant C, and some process $\varphi(\cdot) \in H_d^2$ with $P[\varphi(t) \in K,\ \lambda\text{-a.e.}\ t \in [0, T]] = 1$.

Indeed, the expression on the right-hand side of (3.6) admits the a.s. upper bounds

$$E^\nu\left[\xi + \int_t^T [f(u, X_n(u)) - \delta(\nu(u))]\, du \,\Big|\, \mathscr{F}(t)\right]$$

$$\le E^\nu\left[C + \int_0^t \varphi'(u)\, dB(u) + \int_t^T \varphi'(u)(dB_\nu(u) + \nu(u)\, du)\right.$$

$$\left. + \int_t^T [C - \delta(\nu(u))]\, du \,\Big|\, \mathscr{F}(t)\right]$$

$$\le C' + \int_0^t \varphi'(u)\, dB(u) + E^\nu\left[\int_t^T \varphi'(u)\, dB_\nu(u) \,\Big|\, \mathscr{F}(t)\right]$$

$$= C' + \int_0^t \varphi'(u)\, dB(u), \qquad 0 \le t \le T$$

for every $\nu(\cdot) \in \mathscr{D}_n$, $n \in \mathbb{N}$. We have used (2.1), as well as the fact that $\int_0^\cdot \varphi'(u)\, dB_\nu(u)$ is a P^ν-martingale (same reasoning as in the proof of Proposition 2.1). Thus (3.6) leads to the a.s. upper bound

(7.3) $X_n(\cdot) \le C' + \displaystyle\int_0^\cdot \varphi'(u)\, dB(u)$

for all $n \in \mathbb{N}$. The process on the right-hand side of (7.3) is in S_1^2, and (7.1) follows.

The drawback of the conditions of Assumption 7.1 is that they exclude the case

$$(7.4) \qquad f(t, \omega, x) = \alpha(t, \omega) + k(t, \omega)x$$

of linear drift. This can be remedied by imposing, instead of conditions (i) and (ii) of Assumption 7.1, the following assumption.

ASSUMPTION 7.2. (i) The drift random field $f(t, \omega, x)$ satisfies the conditions of Section 3, including (3.1) and (3.2), as well as:

- $x \mapsto f(t, \omega, x)$ is convex in $\mathbb{R}$, for every $(t, \omega) \in [0, T] \times \Omega$,

and

- the dual function $\tilde{f}(t, \omega, \lambda) := \sup_{x \in \mathbb{R}}[\lambda x - f(t, \omega, x)]$ is bounded from below on its effective domain

$$(7.5) \qquad \tilde{O} := \big\{(t, \omega, \lambda) \in [0, T] \times \Omega \times \mathbb{R} \, / \, \tilde{f}(t, \omega, \lambda) < \infty\big\}.$$

(ii) The terminal random variable ξ satisfies

$$(7.6) \qquad \xi \exp\bigg(\int_0^T \lambda(s)\, ds\bigg) \le C + \int_0^T \exp\bigg(\int_0^u \lambda(s)\, ds\bigg) \varphi'(u)\, dB(u) \quad \text{a.s.}$$

for some C, $\varphi(\cdot)$ as in (7.2), and for every F-progressively measurable process $\lambda(\cdot)$: $[0, T] \times \Omega \to \mathbb{R}$ that satisfies $E\int_0^T (\tilde{f}(t, \lambda(t)))^2\, dt < \infty$.

We denote by Λ the space of all such processes $\lambda(\cdot)$. Every (t, ω)-section of the set $\tilde{O}$ in (7.5) is included in a bounded set $\tilde{R}$ of $\mathbb{R}$ (recall Section 5), so that the elements of Λ are uniformly bounded. By taking $\varphi(\cdot) \equiv 0$, this implies that (7.6) is satisfied by every ξ which is a.s. bounded from above by a real constant. Note also that the conditions of Assumption 7.2(i) are satisfied by linear drifts of the type (7.4).

The conditions of Assumption 7.2 lead to those of Assumption 3.1. Indeed, it can be shown then (as in Section 5) that the sequence of "penalized state processes" $\{X_n(\cdot)\}$ of (3.5), satisfies the analogue

$$X_n(t) = \operatorname*{ess\,sup}_{(\nu, \lambda) \in \mathscr{D}_n \times \Lambda} E^\nu\bigg[\xi \exp\bigg(\int_t^T \lambda(u)\, du\bigg) - \int_t^T \exp\bigg(\int_t^u \lambda(s)\, ds\bigg)$$

$$(7.7)$$

$$\times [\tilde{f}(u, \lambda(u)) + \delta(\nu(u))]\, du \,\Big|\, \mathscr{F}(t)\bigg],$$

$$0 \le t \le T$$

of the a.s. representations (3.6), (5.11). Then an argument similar to that described above leads to (7.1) as before.

APPENDIX

In this section, we sketch the proofs of Propositions 2.3 and 2.5 by adapting to our current situation the techniques developed in [CK93] and El Karoui and Quenez (1995).

PROOF OF PROPOSITION 2.3.   With the notation $g_\nu(\cdot) := g(\cdot) - \delta(\nu(\cdot))$, let us start by establishing the *equation of dynamic programming*,

$$(A.1) \qquad \hat{X}(t) = \operatorname*{ess\,sup}_{\nu \in \mathscr{D}_{t,\theta}} E^\nu\left[\hat{X}(\theta) + \int_t^\theta g_\nu(u)\,du \,\Big|\, \mathscr{F}(t)\right] \quad \text{a.s.,}$$

valid for every $0 \le t \le \theta \le T$. We have denoted by $\mathscr{D}_{t,\theta}$ the restriction of $\mathscr{D}$ to the set $[t, \theta] \times \Omega$; note that (A.1) with $\theta = T$ becomes just the definition of $\hat{X}(t)$ in (2.13), since $\hat{X}(T) = \xi$. Let us observe also that, for any $\nu(\cdot) \in \mathscr{D}$ and with the notation $Z_\nu(t, \theta) := Z_\nu(\theta)/Z_\nu(t)$ as in (2.3), the random variable

$$(A.2) \qquad \begin{aligned} J_\nu(\theta) &:= E^\nu\left[\xi + \int_\theta^T g_\nu(u)\,du \,\Big|\, \mathscr{F}(\theta)\right] \\ &= E\left[Z_\nu(\theta, T)\left\{\xi + \int_\theta^T g_\nu(u)\,du\right\} \,\Big|\, \mathscr{F}(\theta)\right] \end{aligned}$$

depends only on the restriction of the process $\nu(\cdot)$ to $[\theta, T] \times \Omega$. In particular, from (2.13) written in the form

$$(A.3) \qquad \hat{X}(\theta) = \operatorname*{ess\,sup}_{\nu \in \mathscr{D}} J_\nu(\theta),$$

we obtain that

$$\begin{aligned} \hat{X}(t) &= \operatorname*{ess\,sup}_{\nu \in \mathscr{D}} E^\nu\left[J_\nu(\theta) + \int_t^\theta g_\nu(u)\,du \,\Big|\, \mathscr{F}(t)\right] \\ &\le \operatorname*{ess\,sup}_{\nu \in \mathscr{D}_{t,\theta}} E^\nu\left[\hat{X}(\theta) + \int_t^\theta g_\nu(u)\,du \,\Big|\, \mathscr{F}(t)\right], \end{aligned}$$

holds almost surely. In order to prove the reverse inequality, it suffices to fix an arbitrary process $\mu(\cdot)$ in $\mathscr{D}$ and show that

$$(A.4) \qquad \hat{X}(t) \ge E^\mu\left[\hat{X}(\theta) + \int_t^\theta g_\mu(u)\,du \,\Big|\, \mathscr{F}(t)\right]$$

holds almost surely, for any $0 \le t \le \theta \le T$. To this end, notice that the family of random variables $\{J_\nu(\theta)\}_{\nu \in \mathscr{D}}$ in (A.2), is *directed upward*: for any two processes $\mu(\cdot)$ and $\nu(\cdot)$ in $\mathscr{D}$, there exists a third process $\lambda(\cdot) \in \mathscr{D}$, such that $J_\lambda(\theta) \ge \max(J_\mu(\theta), J_\nu(\theta))$ holds almost surely. Thus [e.g., Neveu (1975)] we can write the essential supremum of (A.3) in the form

$$(A.5) \qquad \hat{X}(\theta) = \lim_{k \to \infty} \uparrow J_{\nu_k}(\theta) \quad \text{a.s.}$$

of an increasing limit, for some sequence $\{\nu_k(\cdot)\}_{k \in \mathbb{N}}$ of processes in $\mathscr{D}_{\theta,T}$; without loss of generality, this sequence can be selected from the class $\mathscr{M}_{t,\theta} := \{\nu(\cdot) \in \mathscr{D}/\nu(\cdot) \equiv \mu(\cdot) \text{ on } [t, \theta] \times \Omega\}$. Now we have

$$\begin{aligned} \hat{X}(t) &\ge E^\nu\left[J_\nu(\theta) + \int_t^\theta g_\nu(u)\,du \,\Big|\, \mathscr{F}(t)\right] \\ &= E^\mu\left[J_\nu(\theta) + \int_t^\theta g_\mu(u)\,du \,\Big|\, \mathscr{F}(t)\right] \quad \text{a.s.} \end{aligned}$$

for every process $\nu(\cdot)$ in $\mathcal{M}_{t,\theta}$; thus, by (A.5) and the monotone convergence theorem, we obtain

$$\hat{X}(t) \geq \lim_{k \to \infty} \uparrow E^\mu \left[J_{\nu_k}(\theta) + \int_t^\theta g_\mu(u)\,du \,\Big|\, \mathscr{F}(t) \right]$$

$$= E^\mu \left[\lim_{k \to \infty} \uparrow J_{\nu_k}(\theta) + \int_t^\theta g_\mu(u)\,du \,\Big|\, \mathscr{F}(t) \right]$$

$$= E^\mu \left[\hat{X}(\theta) + \int_t^\theta g_\mu(u)\,du \,\Big|\, \mathscr{F}(t) \right] \quad \text{a.s.}$$

This proves (A.4) and thus also the P^μ-supermartingale property of the process $\hat{X}(t) + \int_0^t g_\mu(u)\,du$, $0 \leq t \leq T$. The RCLL regularity of the process $\hat{X}(\cdot)$ is then argued as in [CK93], pages 679 and 680.

PROOF OF PROPOSITION 2.5. For any process $\mu(\cdot)$ in the class $\mathscr{D}$ of (2.5), we have from (2.16)–(2.18) and (2.10),

$$\hat{Q}(t) = \hat{X}(t) + \int_0^t g(u)\,du$$

$$= \hat{X}(0) + \int_0^t \delta(\nu(u))\,du$$

$$+ \int_0^t (Y^{(\nu)}(u))'[dB_\mu(u) + (\mu(u) - \nu(u))\,du] - A^{(\nu)}(t)$$

$$= \hat{X}(0) + \int_0^t (Y^{(\nu)}(u))'\,dB_\mu(u)$$

$$+ \int_0^t [\delta(\nu(u)) + (\mu(u) - \nu(u))'(Y^{(\nu)}(u))']\,du - A^{(\nu)}(t)$$

for $0 \leq t \leq T$. But again from (2.17), now read with $\nu(\cdot)$ replaced by $\mu(\cdot)$, the process $\hat{Q}(\cdot)$ has the P^μ-supermartingale representation

$$\hat{Q}(t) = \hat{X}(0) + \int_0^t (Y^{(\mu)}(u))'\,dB_\mu(u) + \int_0^t \delta(\mu(u))\,du - A^{(\mu)}(t), \qquad 0 \leq t \leq T.$$

The equality of these two decompositions leads to the identities of (2.19) and (2.20), whereas (2.21) follows from the P^0-decomposition of $\hat{Q}(\cdot)$.

It remains to prove (2.22). Consider the process $\hat{\nu}(\cdot)$ of Lemma A.1 below, observe that for any $k > 0$ the process $k\hat{\nu}(\cdot)$ belongs to $\mathscr{D}$ and note that we have

(A.6) $$A^{(k\hat{\nu})}(T) = \hat{C}(T) + k \int_0^T (\delta(\hat{\nu}(t)) - \hat{\nu}'(t)\hat{Y}(t))\,dt \quad \text{a.s.}$$

from (2.20). The integrand on the right-hand side of this expression is non-positive [a consequence of (A.8), (A.9) in Lemma A.1 below]. Furthermore, if the product set

$$F_{\hat{\nu}} := \left\{ (t, \omega) \in [0, T] \times \Omega \,/\, \delta(\hat{\nu}(t, \omega)) < \hat{\nu}'(t, \omega)\hat{Y}(t, \omega) \right\}$$

has positive $\lambda \otimes P$-measure, the right-hand side of (A.6) can be made negative with positive probability. But $P[A^{(k\hat{\nu})}(T) \geq 0] = 1$, which implies $(\lambda \otimes P)(F_{\hat{\nu}}) = 0$, and thus

$$\delta(\hat{\nu}(t)) = \hat{\nu}'(t)\hat{Y}(t) \quad \text{for } \lambda\text{-a.e. } t \in [0, T]$$

holds almost surely. The conclusion (2.22) follows from this, in conjuction with (A.8) and (A.9). $\square$

We have used the following result from Karatzas and Shreve (1998), Lemma 5.4.2.

LEMMA A.1. *For any given* F *-progressively measurable process* $\hat{Y}: [0, T] \times \Omega \to \mathbb{R}^d$, *there exists an* F *-progressively measurable process* $\hat{\nu}: [0, T] \times \Omega \to \tilde{K}$ *with*

(A.7) $$\|\hat{\nu}(t)\| \leq 1, \qquad |\delta(\hat{\nu}(t))| \leq 1 \qquad \forall\, 0 \leq t \leq T$$

valid almost surely and

(A.8) $$\{\hat{Y}(t) \in K\} = \{\hat{\nu}(t) = 0\} \mod P,$$

(A.9) $$\{\hat{Y}(t) \notin K\} = \{\delta(\hat{\nu}(t)) < \hat{\nu}'(t)\hat{Y}(t)\} \mod P$$

for every $t \in [0, T]$.

REFERENCES

ARKIN, V. and SAXONOV, M. (1979). Necessary optimality conditions for stochastic diferrential equations. *Soviet Math. Doklady* 20 1–15.

BARLES, G., BUCKDAHN, R. and PARDOUX, E. (1997). Backward stochastic differential equations and integral-partial differential equations. *Stochastics Stochastics Rep.* 60 57–83.

BENSOUSSAN, A. (1981). Lectures on stochastic control. *Lecture Notes in Math.* 972 1–62. Springer, Berlin.

BISMUT, J. M. (1978). An introductory approach to duality in optimal stochastic control. *SIAM Rev.* 20 62–78.

BROADIE, M., CVITANIĆ, J. and SONER, H. M. (1998). Optimal replication of contingent claims under portfolio constraints. *Rev. Financial Studies* 11 59–79.

BUCKDAHN, R. and HU, Y. (1996). Hedging contingent claims for a large investor in an incomplete market. Preprint.

BUCKDAHN, R. and HU, Y. (1997). Pricing of American contingent claims with jump stock price and constrained portfolios. Preprint.

CVITANIĆ, J. and KARATZAS, I. (1992). Convex duality for constrained porfolio optimization. *Ann. Appl. Probab.* 2 767–818.

CVITANIĆ, J. and KARATZAS, I. (1993). Hedging contingent claims with constrained portfolios. *Ann. Appl. Probab.* 3 652–681.

CVITANIĆ, J. and KARATZAS, I. (1996). Backward stochastic differential equations with reflection and Dynkin games. *Ann. Probab.* 24 2024–2056.

DARLING, R. W. R. and PARDOUX, E. (1997). Backward SDE with random terminal time and applications to semilinear elliptic PDE. *Ann. Probab.* 25 1135–1159.

DUFFIE, D. and EPSTEIN, L. (1992). Stochastic differential utility. *Econometrica* 60 353–394.

ELLIOTT, R. J. (1990). The optimal control of diffusions. *Appl. Math. Optim.* 22 229–240.

EL KAROUI, N., KAPOUDJIAN, C., PARDOUX, E., PENG, S. and QUENEZ, M. C. (1997). Reflected solutions of backward SDE's, and related obstacle problems for PDE's. *Ann. Probab.* 25 702–737.

EL KAROUI, N., PENG, S. and QUENEZ, M. C. (1997). Backward stochastic differential equations in finance. *Math. Finance* 7 1–71.

EL KAROUI, N. and QUENEZ, M. C. (1995). Dynamic programming and pricing of contingent claims in an incomplete market. *SIAM J. Control Optim.* 33 29–66.

HAMADÈNE, S. and LEPELTIER, J. M. (1995a). Backward equations, stochastic control, and zero-sum stochastic differential games. *Stochastics Stochastics Rep.* 54 221–231.

HAMADÈNE, S. and LEPELTIER, J. M. (1995b). Zero-sum stochastic differential games and backward equations. *Systems Control Lett.* 24 259–263.

HAUSSMANN, U. G. (1986). *A Stochastic Maximum Principle for Optimal Control of Diffusions.* Longman, Essex, U.K.

ITÔ, K. (1942). Differential equations determining Markov processes. *Zenkoku Shijō Sūgaku Danwakai* 1077 1352–1400 (in Japanese).

ITÔ, K. (1946). On a stochastic integral equation. *Proc. Imperial Acad. Tokyo* 22 32–35.

ITÔ, K. (1951). On stochastic differential equations. *Mem. Amer. Math. Soc.* 4 1–51.

KARATZAS, I. and SHREVE, S. E. (1991). *Brownian Motion and Stochastic Calculus*, 2nd ed. Springer, New York.

KARATZAS, I. and SHREVE, S. E. (1998). *Methods of Mathematical Finance.* Springer, New York. To appear.

MA, J. and CVITANIĆ, J. (1997). Reflected Forward-Backward SDEs and obstacle problems with boundary conditions. Preprint.

NEVEU, J. (1975). *Discrete-Parameter Martingales.* North-Holland, Amsterdam.

PARDOUX, E. and PENG, S. (1990). Adapted solution of a backward stochastic differential equation. *Systems Control Lett.* 14 55–61.

PARDOUX, E. and PENG, S. (1994). Backward doubly stochastic differential equations and systems of quasilinear SPDEs. *Probab. Theory Related Fields* 98 209–227.

PARDOUX, E. and TANG, S. (1996). The study of forward-backward stochastic differential equation and its application in quasilinear PDEs. Preprint.

PENG, S. (1990). A general stochastic maximum principle for optimal control problems. *SIAM J. Control Optim.* 28 966–979.

PENG, S. (1991). Probabilistic interpretation for systems of quasilinear parabolic partial differential equations. *Stochastics Stochastics Rep.* 37 61–74.

PENG, S. (1993). Backwards stochastic differential equations and applications to optimal control. *Appl. Math. Optim.* 27 125–144.

ROCKAFELLAR, R. T. (1970). *Convex Analysis.* Princeton Univ. Press.

SAKSONOV, M. (1989). *The Variational Principles in Stochastic Control.* Dushanbe State Educational Institute, Preprint.

J. CVITANIĆ
DEPARTMENT OF STATISTICS
COLUMBIA UNIVERSITY
NEW YORK, NEW YORK 10027
E-MAIL: cj@stat.columbia.edu

I. KARATZAS
DEPARTMENTS OF MATHEMATICS
AND STATISTICS
COLUMBIA UNIVERSITY
NEW YORK, NEW YORK 10027
E-MAIL: ik@math.columbia.edu

H. M. SONER
DEPARTMENT OF MATHEMATICS
BOGAZICI UNIVERSITY
BEBEK 80815 ISTANBUL
TURKEY
E-MAIL: sonermet@boun.edu.tr