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BACKWARD STOCHASTIC DIFFERENTIAL EQUATIONS WITH REFLECTION AND DYNKIN GAMES[1]

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We establish existence and uniqueness results for adapted solutions of backward stochastic differential equations (BSDE's) with two reflecting barriers, generalizing the work of El Karoui, Kapoudjian, Pardoux, Peng and Quenez. Existence is proved first by solving a related pair of coupled optimal stopping problems, and then, under different conditions, via a penalization method. It is also shown that the solution coincides with the value of a certain Dynkin game, a stochastic game of optimal stopping. Moreover, the connection with the backward SDE enables us to provide a pathwise (deterministic) approach to the game.

1. Introduction. The notion of backward stochastic differential equation (BSDE) was introduced by Pardoux and Peng (1990), who proved existence and uniqueness of adapted solutions, under suitable square-integrability assumptions on the coefficients and on the terminal condition. Independently, Duffie and Epstein (1992) introduced stochastic differential utilities in economics models, as solutions to certain BSDE's. More recently, El Karoui, Kapoudjian, Pardoux, Peng and Quenez (1995) generalized these results to BSDE's with reflection, that is, to a setting with an additional continuous, increasing process added in the equation; the function of this additional process is to keep the solution above a certain prescribed lower-boundary process and to do so in a minimal fashion. Moreover, these authors make the crucial observation that the solution is the value function of an optimal stopping problem; their paper provided much of the inspiration and motivation for our work.

We generalize these results to the case of two reflecting barrier processes, that is, to a setting where, in addition to agreeing with a target random variable ξ at the terminal time $t = T$, the solution process of our BSDE has to remain between two prescribed upper- and lower-boundary processes, U and L, respectively, almost surely. This is accomplished by the cumulative action of two continuous, increasing reflection processes, which keep the solution within the prescribed bounds when it attempts to cross either of them. We also establish the connection between this problem and certain stochastic games of stopping (Dynkin games), as well as with a pair of coupled optimal stopping problems.

The BSDE problem with two reflecting barriers is described in Section 2 of the paper. Preliminary results on a related pair of coupled optimal stopping problems are obtained in Section 3. In Section 4 it is shown that any solu-

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tion of the BSDE with two reflecting barriers is also the value of a stochastic game of optimal stopping, usually called Dynkin game, therefore establishing the uniqueness of such a solution. Games of stopping have been studied in Dynkin and Yushkevich (1968), Neveu (1975), Bensoussan and Friedman (1974), Bismut (1977), Stettner (1982), Morimoto (1984), Alario-Nazaret, Lepeltier and Marchal (1982), Lepeltier and Maingueneau (1984) and others. The game involves two players, each of whom can decide to stop it at a random time of his choice; upon termination a certain amount, which is random and depends on the time of termination, is paid by one of the players to the other. In Section 5 we show that the pair of coupled optimal stopping problems from Section 3 has a solution. This, in turn, implies directly that the BSDE with two reflecting barriers has a solution (and that the Dynkin game has a value), in the special case in which the drift does not depend on the solution. The general case is treated using this special case and a fixed point argument.

An alternative method for proving the existence of a solution to the BSDE with two reflecting barriers is presented in Section 6: a standard penalization method is applied, under a condition which roughly says that the barriers can be approximated by semimartingales with absolutely continuous finite variation parts. In this case, the reflection processes which keep the solution between the barriers are also absolutely continuous.

In Section 7 we present a pathwise (deterministic) approach to the Dynkin game, much in the spirit of the pathwise treatment of the optimal stopping problem in Davis and Karatzas (1994). It turns out that there is a game with payoff equal to that of the Dynkin game plus an extra nonadapted process Λ; this game can be solved path-by-path and has a (path-dependent) value whose conditional expectation coincides with the value of the Dynkin game and with the solution process of our BSDE. The nonadapted process Λ is also obtained directly from the solution of the BSDE. Moreover, the optimal stopping times for the players are the same in both games and are also optimal in yet another pair of pathwise, decoupled, optimal stopping problems.

Finally, we collect in the Appendix some useful results on supermartingales, in particular, on potentials and on Snell envelopes.

In future work, we plan to treat the Markovian aspects of this theory, including the associated partial differential equations and variational inequalities.

2. Backward SDE with two reflecting barriers. On a given, complete probability space $(\Omega, \mathscr{F}, P)$, let $B = (B_1, \dots, B_d)'$ be a standard d-dimensional Brownian motion on the finite interval $[0, T]$, and denote by $\mathbf{F} = \{\mathscr{F}(t)\}_{0 \le t \le T}$ the augmentation of the natural filtration $\mathbf{F}^B$, namely $\mathscr{F}^B(t) = \sigma(B(s), 0 \le s \le t)$, $0 \le t \le T$, generated by B. We shall need the following notation. For any given $n \in \mathbb{N}$, let us introduce the following spaces:

$\mathbf{L}_n^2$ of $\mathscr{F}(T)$-measurable random variables ξ: $\Omega \mapsto \mathbb{R}^n$ with $E(\|\xi\|^2) < \infty$;

$\mathbf{H}_n^2$ of $\mathbf{F}$-predictable processes φ: $[0, T] \times \Omega \mapsto \mathbb{R}^n$ with $\int_0^T E\|\varphi(t)\|^2\, dt < \infty$;

$\mathbf{S}_n^k$ of $\mathbf{F}$-progressively measurable processes φ: $[0, T] \times \Omega \mapsto \mathbb{R}^n$ with $E(\sup_{0 \le t \le T} \|\varphi(t)\|^k) < \infty$, $k \in \mathbb{N}$;

$\mathbf{S}^2_{ci}$ of continuous, increasing, $\mathbf{F}$-adapted processes A: $[0, T] \times \Omega \mapsto [0, \infty)$ with $A(0) = 0$, $E(A^2(T)) < \infty$.

Finally, we shall denote by $\mathscr{P}$ the σ-algebra of predictable sets in $[0, T] \times \Omega$.

PROBLEM 2.1 (*Backward stochastic differential equation* (BSDE) *with upper and lower reflecting barriers*). Let ξ be a given random variable in $\mathbf{L}^2_1$, and f: $[0, T] \times \Omega \times \mathbb{R} \times \mathbb{R}^d \mapsto \mathbb{R}$ a given $\mathscr{P} \otimes \mathscr{B}(\mathbb{R}) \otimes \mathscr{B}(\mathbb{R}^d)$-measurable function that satisfies

$$E \int_0^T f^2(t, \omega, 0, 0)\, dt < \infty \tag{2.1}$$

$$|f(t, \omega, x, y) - f(t, \omega, x', y')| \le k(|x - x'| + \|y - y'\|), \quad \forall\, (t, \omega) \in [0, T] \times \Omega;\ x, x' \text{ in } \mathbb{R};\ y,\ y' \text{ in } \mathbb{R}^d \tag{2.2}$$

for some $0 < k < \infty$. Consider also two continuous processes L, U in $\mathbf{S}^2_1$ that satisfy

$$L(t) \le U(t), \qquad \forall\ 0 \le t \le T \quad \text{and} \quad L(T) \le \xi \le U(T) \quad \text{a.s.} \tag{2.3}$$

We say that a triple (X, Y, K) of $\mathbf{F}$-progressively measurable processes X: $[0, T] \times \Omega \mapsto \mathbb{R}$, Y: $[0, T] \times \Omega \mapsto \mathbb{R}^d$ and K: $[0, T] \times \Omega \mapsto \mathbb{R}$ is a solution of the backward stochastic differential equation (BSDE) with reflecting barriers $U(\cdot)$, $L(\cdot)$ (upper and lower, respectively), terminal condition ξ and coefficient f, if the following hold:

(i) $K = K^+ - K^-$, with $K^\pm \in \mathbf{S}^2_{ci}$;
(ii) $Y \in \mathbf{H}^2_d$, and

$$\begin{aligned} X(t) = \xi + \int_t^T f(s, X(s), Y(s))\, ds + K^+(T) - K^+(t) \\ -(K^-(T) - K^-(t)) - \int_t^T Y'(s)\, dB(s), \qquad 0 \le t \le T, \end{aligned} \tag{2.4}$$

$$L(t) \le X(t) \le U(t), \qquad \forall\ 0 \le t \le T, \tag{2.5}$$

$$\int_0^T (X(t) - L(t))\, dK^+(t) = \int_0^T (U(t) - X(t))\, dK^-(t) = 0, \tag{2.6}$$

almost surely.

DISCUSSION. In the setup of Problem 2.1 the processes $L(\cdot)$, $U(\cdot)$ play the role of reflecting barriers; these are allowed to be random and time-varying, and the state-process $X(\cdot)$ is not allowed to cross them [cf. (2.5)] on its way to the prescribed terminal target condition $X(T) = \xi$ [cf. (2.4)].

The state-process $X(\cdot)$ is forced to stay within the region enveloped by the lower and upper barriers $L(\cdot)$, $U(\cdot)$, thanks to the cumulative action of the two increasing reflection processes $K^+(\cdot)$, $K^-(\cdot)$, respectively [cf. (2.4)]; these act only when necessary to prevent $X(\cdot)$ from crossing the respective boundary [cf. (2.6)] and, in this sense, their action can be considered minimal. Finally, it is the freedom to choose the intensity of noise process $Y = (Y_1, \dots, Y_d)'$, $Y_i(t) = (d/dt)\langle X, B_i\rangle(t)$, that allows one to have an **F**-*adapted* solution to (2.4), just as in the case of unconstrained BSDE's [Pardoux and Peng (1990)].

When viewed *backwards* in time, that is, with $\tilde{X}(\theta) := X(T-\theta)$, $\tilde{Y}(\theta) := Y(T-\theta)$, $\tilde{K}^{\pm}(\theta) := K^{\pm}(T) - K^{\pm}(T-\theta)$ and $\tilde{B}(\theta) := B(T) - B(T-\theta)$ in (2.4) written as

$$\begin{aligned}\tilde{X}(\theta) &= \xi + \int_0^\theta f(T-u, \tilde{X}(u), \tilde{Y}(u))\,du + \tilde{K}^+(\theta) - \tilde{K}^-(\theta)\\ &\quad + \int_0^\theta \tilde{Y}'(u)\,d\tilde{B}(u), \qquad 0 \le \theta \le T,\end{aligned} \tag{2.4$'$}$$

the effect of the increasing process $\tilde{K}^+(\cdot)$ [resp., $\tilde{K}^-(\cdot)$] is to push the state-process $\tilde{X}(\cdot)$ upward (resp., downward), in order to prevent it from crossing the lower boundary $\tilde{L}(\cdot) := L(T-\cdot)$ [resp., the upper boundary $\tilde{U}(\cdot) := U(T-\cdot)$], and to do this with minimal effort; that is,

$$\int_0^T (\tilde{X}(\theta) - \tilde{L}(\theta))\,d\tilde{K}^+(\theta) = \int_0^T (\tilde{U}(\theta) - \tilde{X}(\theta))\,d\tilde{K}^-(\theta) = 0. \tag{2.6$'$}$$

When (2.4) is viewed *forwards* in time, in other words as

$$dX(t) = f(t, X(t), Y(t))\,dt - dK^+(t) + dK^-(t) + Y'(t)\,dB(t),$$

$$X(0) = E\left[\xi + \int_0^T f(s, X(s), Y(s))\,ds + K^+(T) - K^-(T)\right], \qquad X(T) = \xi \text{ a.s.},$$

the effect of the increasing process $K^+(\cdot)$ [resp., $K^-(\cdot)$] is to push the state-process $X(\cdot)$ downward (resp., upward), in order to prevent overshooting, that is, $X(T) > \xi$ [resp., undershooting, i.e., $X(T) < \xi$], the prescribed terminal target ξ.

3. Analysis: a pair of coupled optimal stopping problems. In this section and the next, we shall assume that *there exists a solution* (X, Y, K) *to the BSDE of Problem* 2.1 and will try to derive some consequences and representations. We shall set

$$g(t, \omega) := f(t, \omega, X(t, \omega), Y(t, \omega)), \qquad (t, \omega) \in [0, T] \times \Omega; \tag{3.1}$$

from the assumptions on $f(\cdot)$ and (2.1), (2.2), this defines a process $g \in \mathbf{H}_1^2$. For such g, we shall also introduce the processes

$$\begin{aligned}N(t) &:= E\left[\xi + \int_0^T g(s)\,ds \,\middle|\, \mathscr{F}(t)\right] - \int_0^t g(s)\,ds\\ &= E\left[\xi + \int_t^T g(s)\,ds \,\middle|\, \mathscr{F}(t)\right], \qquad 0 \le t \le T,\end{aligned} \tag{3.2}$$

(3.3) $L^{\xi}(t) := L(t)\mathbf{1}_{\{t<T\}} + \xi\mathbf{1}_{\{t=T\}}, \qquad \tilde{L}(t) := L^{\xi}(t) - N(t), \qquad 0 \le t \le T,$

(3.4) $U^{\xi}(t) := U(t)\mathbf{1}_{\{t<T\}} + \xi\mathbf{1}_{\{t=T\}}, \qquad \tilde{U}(t) := U^{\xi}(t) - N(t), \qquad 0 \le t \le T.$

Note that

$$N \text{ is continuous on } [0, T] \text{ a.s., and } N \in \mathbf{S}_1^2 \tag{3.5}$$

(use the stochastic integral representation property of Brownian martingales and Doob's maximal inequality), that

$$\tilde{L},\ \tilde{U} \text{ belong to } \mathbf{S}_1^2 \text{ and are continuous on } [0, T) \text{ a.s.} \tag{3.6}$$

and that we have, almost surely,

$$\begin{gathered} \tilde{L}(t) \le \tilde{U}(t), \qquad \forall\ 0 \le t \le T \quad \text{and} \\ \tilde{L}(T-) \le \tilde{L}(T) = 0 = \tilde{U}(T) \le \tilde{U}(T-). \end{gathered} \tag{3.7}$$

With all this notation, (2.4) and (2.5) give

$$\begin{aligned} L^{\xi}(t) \le X(t) = \xi + \int_t^T g(s)\,ds + K^+(T) - K^+(t) - (K^-(T) - K^-(t)) \\ - \int_t^T Y'(s)\,dB(s) \le U^{\xi}(t) \end{aligned} \tag{3.8}$$

on $[0, T]$, almost surely, and taking conditional expectations with respect to $\mathscr{F}(t)$ in (3.8), we have

$$L^{\xi}(t) \le X(t) = N(t) + \pi_t(K^+) - \pi_t(K^-) \le U^{\xi}(t), \qquad 0 \le t \le T \tag{3.9}$$

almost surely. We have used the notation of (A.2), (3.2) and the fact that the stochastic integral in (3.8) satisfies $E[\int_t^T Y'(s)\,dB(s)|\mathscr{F}(t)] = 0$ a.s., thanks to the assumption $Y \in \mathbf{H}_d^2$.

In particular, (3.9) gives

$$\begin{aligned} \pi(K^+) &\ge \tilde{L} + \pi(K^-) =: \eta^+ \\ \pi(K^-) &\ge -\tilde{U} + \pi(K^+) =: \eta^-. \end{aligned} \tag{3.10}$$

Now $K^{\pm} \in \mathbf{S}_{\text{ci}}^2$, so $\pi(K^{\pm})$ belong to the space $\mathbf{\Pi}_c^2$ of Definition A.2, by Corollary A.3 (Appendix). From this observation and (3.6), the processes $\eta^{\pm}$ of (3.10) are seen to be in $\mathbf{S}_1^2$, and to have paths which are continuous on $[0, T)$ and quasi-left-continuous on $[0, T]$, almost surely [since, if they have a jump at $t = T$, this jump is upwards, by (3.7)]. Therefore, using the notation of (A.8), the Snell envelopes

$$S_t(\eta^{\pm}) := \operatorname*{ess\,sup}_{\tau \in \mathscr{M}_{t,T}} E[\eta^{\pm}(\tau)|\mathscr{F}(t)], \qquad 0 \le t \le T \tag{3.11}$$

are potentials in the space $\mathbf{\Pi}_c^2$ of Definition A.2. Hence

$$S(\eta^{\pm}) = \pi(A^{\pm}) \quad \text{for suitable, uniquely determined } A^{\pm} \in \mathbf{S}_{\text{ci}}^2, \tag{3.12}$$

thanks to Lemma A.4 and Corollary A.3. We have denoted by $\mathscr{M}_{t,T}$ the class of $\mathbf{F}$-stopping times $\tau\colon \Omega \mapsto [t, T]$. Furthermore,

$$\pi(K^{\pm}) \geq \pi(A^{\pm}) \tag{3.13}$$

from (3.10), and

$$\int_0^T (\pi_t(A^{\pm}) - \eta^{\pm}(t))\, dA^{\pm}(t) = 0 \quad \text{a.s.} \tag{3.14}$$

from (A.13). In particular, from (3.10), (3.3), (3.4) and the continuity of $A^{\pm}$, (3.14) reads

$$\begin{aligned} &\int_0^T [(\pi_t(A^+) - \pi_t(K^-) + N(t)) - L(t)]\, dA^+(t) = 0 \quad \text{a.s.},\\ &\int_0^T [U(t) - (\pi_t(K^+) - \pi_t(A^-) + N(t))]\, dA^-(t) = 0 \quad \text{a.s.} \end{aligned} \tag{3.15}$$

We have established the following result.

PROPOSITION 3.1. *For a given, fixed* $g \in \mathbf{H}_1^2$, *the mapping* $(K^+, K^-) \to (A^+, A^-)$ *of* (3.12) *and* (3.10), *namely*

$$\begin{aligned} \pi(A^+) &= S(\tilde{L} + \pi(K^-)),\\ \pi(A^-) &= S(-\tilde{U} + \pi(K^+)), \end{aligned} \tag{3.16}$$

maps $\mathbf{S}^2_{\mathrm{ci}} \times \mathbf{S}^2_{\mathrm{ci}}$ *into itself and satisfies* (3.13) *and* (3.15).

In Section 5, we shall show that this mapping has a *fixed point* $(K^+, K^-) \in \mathbf{S}^2_{\mathrm{ci}} \times \mathbf{S}^2_{\mathrm{ci}}$; for this fixed point, (3.13) holds as *equality*, and (3.15) is then equivalent to (2.6), in light of the equality $X = N + \pi(K^+) - \pi(K^-)$ in (3.9).

Open question. Can we deduce $\pi(K^{\pm}) = \pi(A^{\pm})$, thus also $A^{\pm} = K^{\pm}$, from (3.15), (3.13) and (2.6)? This would show that every solution to the BSDE induces a fixed point of the mapping of (3.16).

4. Analysis: a stochastic game of E. B. Dynkin. Our purpose in this section is to show that the existence of a solution (X, Y, K) to the BSDE of Problem 2.1 implies that X is the value of a certain stochastic game of stopping. First introduced by Dynkin and Yushkevich (1968) and later studied, in different contexts, by several authors, including Neveu (1975), Bensoussan and Friedman (1974), Bismut (1977), Stettner (1982), Morimoto (1984), Alario-Nazaret, Lepeltier and Marchal (1983), Lepeltier and Maingueneau (1984) and others, such stochastic games are known as Dynkin games. As a corollary to our results, we shall present in Section 7 a very simple, pathwise approach to this game, in the spirit of a similar treatment in Davis and Karatzas (1994) for the optimal stopping problem.

THEOREM 4.1. *Let (X, Y, K) be a solution to the BSDE of Problem* 2.1 *and retain the notation of* (3.1). *For any $0 \le t \le T$ and any two stopping times σ, τ in the class $\mathscr{M}_{t,T}$, consider the payoff*

$$R_t(\sigma, \tau) := \int_t^{\sigma\wedge\tau} g(u)\,du + \xi \mathbf{1}_{\{\sigma\wedge\tau = T\}} + L(\tau)\mathbf{1}_{\{\tau<T,\, \tau\le\sigma\}} + U(\sigma)\mathbf{1}_{\{\sigma<\tau\}}, \tag{4.1}$$

as well as the upper and lower values, respectively,

$$\begin{aligned}\overline{V}(t) &:= \operatorname*{ess\,inf}_{\sigma\in\mathscr{M}_{t,T}} \operatorname*{ess\,sup}_{\tau\in\mathscr{M}_{t,T}} E[R_t(\sigma,\tau)|\mathscr{F}(t)],\\ \underline{V}(t) &:= \operatorname*{ess\,sup}_{\tau\in\mathscr{M}_{t,T}} \operatorname*{ess\,inf}_{\sigma\in\mathscr{M}_{t,T}} E[R_t(\sigma,\tau)|\mathscr{F}(t)]\end{aligned} \tag{4.2}$$

of a corresponding stochastic game. This game has value $V(t)$, given by the state-process X of the solution to the BSDE, that is,

$$V(t) = \overline{V}(t) = \underline{V}(t) = X(t) \quad a.s. \qquad \forall\ 0 \le t \le T, \tag{4.3}$$

as well as a saddlepoint $(\hat\sigma_t, \hat\tau_t) \in \mathscr{M}_{t,T} \times \mathscr{M}_{t,T}$ given by

$$\begin{aligned}\hat\sigma_t &:= \inf\{s \in [t,T)/X(s) = U(s)\} \wedge T,\\ \hat\tau_t &:= \inf\{s \in [t,T)/X(s) = L(s)\} \wedge T,\end{aligned} \tag{4.4}$$

namely

$$\begin{aligned}E[R_t(\hat\sigma_t, \tau)|\mathscr{F}(t)] &\le E[R_t(\hat\sigma_t, \hat\tau_t)|\mathscr{F}(t)]\\ &= X(t) \le E[R_t(\sigma, \hat\tau_t)|\mathscr{F}(t)] \quad a.s.\end{aligned} \tag{4.5}$$

for every $(\sigma, \tau) \in \mathscr{M}_{t,T} \times \mathscr{M}_{t,T}$.

In the game of (4.1) and (4.2) with $t = 0$, player 1 chooses the stopping time σ, player 2 chooses the stopping time τ, and $R_0(\sigma, \tau)$ represents the amount paid by player 1 to player 2. It is the expectation $ER_0(\sigma, \tau)$ of this random payoff that player 1 tries to minimize and player 2 tries to maximize. The game stops when one player decides to stop, that is, at the stopping time $\sigma \wedge \tau$, or at T if $\sigma = \tau = T$; the payoff $R_0(\sigma, \tau)$ then equals

$$\int_0^{\sigma\wedge\tau} g(u)\,du + \begin{cases} U(\sigma), & \text{if player 1 stops the game first,}\\ L(\tau), & \text{if player 2 stops the game first or if both stop}\\ & \quad\text{the game simultaneously,}\\ \xi, & \text{if neither player stops the game before } T.\end{cases}$$

According to Theorem 4.1, the pair $(\hat\sigma_0, \hat\tau_0)$ of (4.4) and (4.5) provides a saddlepoint of optimal stopping rules for the players, in the following sense: if player 1 chooses the rule $\hat\sigma_0$, then $\hat\tau_0$ is an optimal stopping time for player 2, and if player 2 chooses the rule $\hat\tau_0$, then $\hat\sigma_0$ is an optimal stopping time for player 1.

PROOF. It suffices to show (4.5), since (4.3) follows directly from this.

(i) First, let us take $\sigma = \hat{\sigma}_t$ and arbitrary $\tau \in \mathscr{M}_{t,T}$. On the event $\{\hat{\sigma}_t < \tau\}$, we have $U(\hat{\sigma}_t) = X(\hat{\sigma}_t)$, $K^-(\hat{\sigma}_t) = K^-(t)$ from (4.4) and (2.6). Thus

$$
\begin{aligned}
R_t(\hat{\sigma}_t, \tau) &= \int_t^{\hat{\sigma}_t} g(u)\,du + X(\hat{\sigma}_t) - (K^-(\hat{\sigma}_t) - K^-(t)) \\
&\le \int_t^{\hat{\sigma}_t} g(u)\,du + X(\hat{\sigma}_t) + (K^+(\hat{\sigma}_t) - K^+(t)) \\
&\quad -(K^-(\hat{\sigma}_t) - K^-(t)) \\
&= \int_t^{\hat{\sigma}_t} g(u)\,du + X(\hat{\sigma}_t) + K(\hat{\sigma}_t) - K(t) \\
&= X(t) + \int_t^{\hat{\sigma}_t} Y'(u)\,dB(u)
\end{aligned}
\tag{4.6}
$$

almost surely, with equality if $\tau = \hat{\tau}_t$, and on the event $\{\tau \le \hat{\sigma}_t\}$ we have,

$$
\begin{aligned}
R_t(\hat{\sigma}_t, \tau) &= \int_t^{\tau} g(u)\,du + \xi \mathbf{1}_{\{\tau=T\}} + L(\tau)\mathbf{1}_{\{\tau<T\}} \\
&= -(K^-(\tau) - K^-(t)) \\
&\le \int_t^{\tau} g(u)\,du + X(\tau) + (K^+(\tau) - K^+(t)) \\
&\quad -(K^-(\tau) - K^-(t)) \\
&= \int_t^{\tau} g(u)\,du + X(\tau) + K(\tau) - K(t) \\
&= X(t) + \int_t^{\tau} Y'(u)\,dB(u),
\end{aligned}
\tag{4.7}
$$

almost surely, with equality if $\tau = \hat{\tau}_t$. Putting (4.6) and (4.7) together, we obtain

$$
R_t(\hat{\sigma}_t, \tau) \le X(t) + \int_t^{\tau \wedge \hat{\sigma}_t} Y'(u)\,dB(u) \quad \text{a.s.,} \tag{4.8}
$$

with equality if $\tau = \hat{\tau}_t$, and taking conditional expectations with respect to $\mathscr{F}(t)$,

$$
E[R_t(\hat{\sigma}_t, \tau) | \mathscr{F}(t)] \le X(t) = E[R_t(\hat{\sigma}_t, \hat{\tau}_t) | \mathscr{F}(t)] \quad \text{a.s.} \qquad \forall\, \tau \in \mathscr{M}_{t,T} \tag{4.9}
$$

because $Y \in \mathbf{H}_d^2$.

(ii) Second, we take $\tau = \hat{\tau}_t$ and arbitrary $\sigma \in \mathscr{M}_{t,T}$. Arguments similar to those of case (2), now lead to

$$
R_t(\sigma, \hat{\tau}_t) \ge X(t) + \int_t^{\sigma \wedge \hat{\tau}_t} Y'(u)\,dB(u) \quad \text{a.s.,} \tag{4.10}
$$

which holds with equality if $\sigma = \hat{\sigma}_t$, and to

$$
E[R_t(\sigma, \hat{\tau}_t) | \mathscr{F}(t)] \ge X(t) = E[R_t(\hat{\sigma}_t, \hat{\tau}_t) | \mathscr{F}(t)] \quad \text{a.s.} \qquad \forall\, \sigma \in \mathscr{M}_{t,T} \tag{4.11}
$$

by taking conditional expectations as before. Now (4.5) follows from (4.9) and (4.11). □

COROLLARY 4.2 (Uniqueness for Problem 2.1 in a special case.). *Suppose that, for some given* $g \in \mathbf{H}_d^2$, *we have*

$$f(t, \omega, x, y) = g(t, \omega) \qquad \forall\ (t, \omega, x, y) \in [0, T] \times \Omega \times \mathbb{R} \times \mathbb{R}^d \tag{4.12}$$

in the BSDE of Problem 2.1. *Then this problem can have at most one solution.*

PROOF. Suppose that the triple (X, Y, K) solves the BSDE of Problem 2.1; then the state-process X of this triple is uniquely determined, from (4.3) of Theorem 4.1, as the value of the Dynkin game (4.1) and (4.2). But X is also a continuous semimartingale of the Brownian filtration $\mathbf{F} = \mathbf{F}^B$, with

$$X(t) = X(0) - \left(K(t) + \int_0^t g(u)\, du\right) + \int_0^t Y'(u)\, dB(u), \qquad 0 \le t \le T \tag{2.4$'$}$$

as its decomposition; in it, both K and Y are uniquely determined as well. □

5. Synthesis: existence and uniqueness for the BSDE. We show in this section how to construct, for a given, fixed $g \in \mathbf{H}_1^2$, a (unique) solution (X, Y, K) for Problem 2.1 with $f \equiv g$ as in (4.12), starting from a fixed point (K^+, K^-) of the mapping (3.16) in Proposition 3.1. We address then the questions of existence of such a fixed point and of constructing a (unique) solution to the BSDE of Problem 2.1 for general coefficient functions f.

PROBLEM 5.1. Let $\xi \in \mathbf{L}_1^2$, $g \in \mathbf{H}_1^2$ be given, as well as two continuous processes L, U in $\mathbf{S}_1^2$ as in Problem 2.1. We say that a triple (X, Y, K) of $\mathbf{F}$-progressively measurable processes is a solution of the backward stochastic equation (BSE) with reflecting barriers $U(\cdot)$, $L(\cdot)$, terminal condition ξ and coefficient g, if $K = K^+ - K^-$ with $K^\pm \in \mathbf{S}_{\mathrm{ci}}^2$, $Y \in \mathbf{H}_d^2$ and if (2.5), (2.6) as well as

$$X(t) = \xi + \int_t^T g(u)\, du + K(T) - K(t) - \int_t^T Y'(u)\, dB(u), \qquad 0 \le t \le T \tag{5.1}$$

are satisfied almost surely.

THEOREM 5.2. *For a given* $g \in \mathbf{H}_1^2$, *suppose that the mapping* (3.16) *in Proposition* 3.1 *has a fixed point* (K^+, K^-), *namely,*

$$\begin{aligned} \pi(K^+) &= S(\tilde{L} + \pi(K^-)) \\ \pi(K^-) &= S(-\tilde{U} + \pi(K^+)) \end{aligned} \tag{5.2}$$

for some $(K^+, K^-) \in \mathbf{S}_{\mathrm{ci}}^2 \times \mathbf{S}_{\mathrm{ci}}^2$. *Then the triple* (X, Y, K), *with*

$$K := K^+ - K^-, \qquad X := N + \pi(K^+) - \pi(K^-) \tag{5.3}$$

and with $Y \in \mathbf{H}_d^2$ uniquely determined via

$$E\left[\xi + \int_0^T g(s)\,ds + K(T)\middle|\mathscr{F}(t)\right] = N(0) + E(K(T)) \qquad (5.4)$$
$$+ \int_0^t Y'(u)\,dB(u), \qquad 0 \le t \le T,$$

is the unique solution to the BSE of Problem 5.1.

PROOF. We have, from (5.3), (5.4) and (3.2),

$$X(t) + \int_0^t g(u)\,du + K(t) = E\left[\xi + \int_0^T g(u)\,du + K(T)\middle|\mathscr{F}(t)\right]$$
$$= X(0) + \int_0^t Y'(u)\,dB(u),$$

for $0 \le t \le T$, where $X(0) = N(0) + E(K(T))$; in particular, $X(T) = \xi$ and thus

$$\xi + \int_0^T g(u)\,du + K(T) = X(0) + \int_0^T Y'(u)\,dB(u).$$

Subtracting memberwise we obtain (5.1). On the other hand, (5.2) gives

$$\pi(K^+) \ge \tilde{L} + \pi(K^-),$$
$$\pi(K^-) \ge -\tilde{U} + \pi(K^+)$$

and thus, in conjunction with (5.3) (3.3) and (3.4), we obtain

$$L \le N + \tilde{L} \le X = N + \pi(K^+) - \pi(K^-) \le \tilde{U} + N \le U;$$

in other words, (2.5) is satisfied. Finally, the fact that the pair (K^+, K^-) solves (5.2) implies that the equalities of (3.15) hold with $A^\pm = K^\pm$; in conjunction with $X = N + \pi(K^+) - \pi(K^-)$, these equalities read

$$\int_0^T (X(t) - L(t))dK^+(t) = \int_0^T (U(t) - X(t))\,dK^-(t) = 0$$

almost surely; that is, (2.6) holds as well. Uniqueness follows from Corollary 4.2. □

It develops from Theorem 5.2 that, in order to establish the existence and uniqueness of solution to the BSE of Problem 5.1, it suffices to show the existence of a pair $(K^+, K^-) \in \mathbf{S}_{\mathrm{ci}}^2 \times \mathbf{S}_{\mathrm{ci}}^2$ that solves the system (5.2), or equivalently, from Corollary A.3, the existence of a pair $(Z^+, Z^-) \in \mathbf{\Pi}_c^2 \times \mathbf{\Pi}_c^2$ that solves

$$Z^+ = S(\tilde{L} + Z^-), \qquad (5.5)$$
$$Z^- = S(-\tilde{U} + Z^+)$$

in the notation of (A.8). This system was introduced by Bismut (1977) and was studied by him and by Alario-Nazaret (1982), among others, as a crucial step towards solving Dynkin games of the type (4.1), (4.2) by reducing them to a pair of coupled optimal stopping problems. For completeness, we include a proof of the basic existence result in this direction.

THEOREM 5.3. *For the existence of a solution* $(Z^+, Z^-) \in \mathbf{\Pi}_c^2 \times \mathbf{\Pi}_c^2$ *to the system* (5.5) [*equivalently, of a solution* $(K^+, K^-) \in \mathbf{S}_{\mathrm{ci}}^2 \times \mathbf{S}_{\mathrm{ci}}^2$ *to the system* (5.2)], *it is necessary that*

$$L^\xi \le h - \theta + E[\xi|\mathscr{F}(\cdot)] \le U^\xi \tag{5.6}$$

for some $h \in \mathbf{\Pi}_c^2$, $\theta \in \mathbf{\Pi}_c^2$; *condition* (5.6) *is also sufficient, provided that*

$$L(t) < U(t), \qquad 0 \le t < T \tag{5.7}$$

holds almost surely.

REMARK. For any given $g \in \mathbf{H}_1^2$, the condition (5.6) is equivalent to

$$\tilde{L} \le H - \Theta \le \tilde{U} \quad \text{for some } H \in \mathbf{\Pi}_c^2 \text{ and } \Theta \in \mathbf{\Pi}_c^2. \tag{5.6$'$}$$

This can be seen easily from (3.2)–(3.4). Indeed, if (5.6′) holds, then we can take

$$h(t) = H(t) + E\left[\int_0^T g_+(u)\,du \,\middle|\, \mathscr{F}(t)\right] - \int_0^t g_+(u)\,du,$$

$$\theta(t) = \Theta(t) + E\left[\int_0^T g_-(u)\,du \,\middle|\, \mathscr{F}(t)\right] - \int_0^t g_-(u)\,du, \qquad 0 \le t \le T$$

in (5.6), with $g_+ = g \vee 0$, $g_- = (-g) \vee 0$; on the other hand, if (5.6) holds, then (5.6′) holds as well, with

$$H(t) = h(t) + E\left[\int_0^T g_-(u)\,du \,\middle|\, \mathscr{F}(t)\right] - \int_0^t g_-(u)\,du,$$

$$\Theta(t) = \theta(t) + E\left[\int_0^T g_+(u)\,du \,\middle|\, \mathscr{F}(t)\right] - \int_0^t g_+(u)\,du, \qquad 0 \le t \le T.$$

PROOF OF THEOREM 5.3. Suppose that $Z^\pm \in \mathbf{\Pi}_c^2$ satisfy (5.5); then we have $Z^+ \ge \tilde{L} + Z^-$, $Z^- \ge -\tilde{U} + Z^+$, whence

$$\tilde{L} \le Z^+ - Z^- \le \tilde{U},$$

and (5.6′) is satisfied.

For the remainder of this proof, let us assume that (5.6′) holds and try to establish the existence of a solution to (5.5) by considering the iterative scheme

$$\begin{aligned} Z_{n+1}^+ &:= S(\tilde{L} + Z_n^-), \\ Z_{n+1}^- &:= S(-\tilde{U} + Z_n^+), \qquad n \in \mathbb{N}_0 \text{ and } Z_0^\pm \equiv 0. \end{aligned} \tag{5.8}$$

We claim that

$$Z_n^{\pm} \in \mathbf{\Pi}_c^2, \qquad \forall\, n \in \mathbb{N} \tag{5.9}$$

and

$$\begin{aligned} \tilde{L} \vee 0 \le Z_n^+ \le Z_{n+1}^+ \le H \\ (-\tilde{U}) \vee 0 \le Z_n^- \le Z_{n+1}^- \le \Theta, \qquad \forall\, n \in \mathbb{N} \end{aligned} \tag{5.10}$$

hold a.s.

PROOF OF (5.9). Clearly $Z_0^{\pm} \in \mathbf{\Pi}_c^2$. Suppose $Z_n^{\pm} \in \mathbf{\Pi}_c^2$ for some $n \in \mathbb{N}$; then both $\tilde{L} + Z_n^-$, $-\tilde{U} + Z_n^+$ are in $\mathbf{S}_1^2$ and have paths which are a.s. continuous on $[0, T)$ and quasi-left-continuous on $[0, T]$, because of the continuity of $Z_n^{\pm}$ on $[0, T]$ and (3.7). From Lemma A.4, (A.10) and Corollary A.3, we deduce $Z_{n+1}^{\pm} \in \mathbf{\Pi}_c^2$, and (5.9) follows. □

PROOF OF (5.10). The inequalities $Z_1^+ = S(\tilde{L}) \ge \tilde{L}$ and $Z_1^- = S(-\tilde{U}) \ge -\tilde{U}$, $Z_1^{\pm} \ge 0$ are obvious, by the definition (5.8). Clearly also, $Z_2^+ = S(\tilde{L}+Z_1^-) \ge S(\tilde{L}) = Z_1^+$, $Z_2^- = S(-\tilde{U} + Z_1^+) \ge S(-\tilde{U}) = Z_1^-$; assuming $Z_n^{\pm} \ge Z_{n-1}^{\pm}$, we obtain similarly $Z_{n+1}^+ = S(\tilde{L} + Z_n^-) \ge S(\tilde{L} + Z_{n-1}^-) = Z_n^+$, $Z_{n+1}^- = S(-\tilde{U} + Z_n^+) \ge S(-\tilde{U} + Z_{n-1}^+) = Z_n^-$. This establishes, by induction, the monotonicity of $\{Z_n^{\pm}\}_{n\in\mathbb{N}}$.

For the last inequalities in (5.10), observe from (5.6′) that $H \ge \Theta + \tilde{L} \ge \tilde{L}$, $\Theta \ge H - \tilde{U} \ge -\tilde{U}$; therefore, $H \ge S(\tilde{L}) = Z_1^+$ and $\Theta \ge S(-\tilde{U}) = Z_1^-$. Suppose that $H \ge Z_n^+$, $\Theta \ge Z_n^-$ for some $n \in \mathbb{N}$; then $\Theta + \tilde{U} \ge H \ge Z_n^+$, $H - \tilde{L} \ge \Theta \ge Z_n^-$ and thus $H \ge S(\tilde{L} + Z_n^-) = Z_{n+1}^+$, $\Theta \ge S(-\tilde{U} + Z_n^+) = Z_{n+1}^-$, establishing (by induction) the last inequalities in (5.10). □

It follows from (5.10) and from Exercise 3.30, page 21 in Karatzas and Shreve (1991) that the pointwise, increasing limits

$$Z^{\pm} := \lim_n Z_n^{\pm} \tag{5.11}$$

are potentials, that is, nonnegative $\mathbf{F}$-supermartingales, with RCLL (Right Continuous with Left Limits) paths and $Z^{\pm}(T) = 0$ a.s., and satisfy $E[\sup_{0\le t\le T}(Z^{\pm}(t))^2] < \infty$. From (5.8) and Lemma A.5 in the Appendix, it develops then that $Z^{\pm}$ solve the system (5.5). In the remainder of the proof we show that we can assume, without loss of generality, that $Z^{\pm}$ have continuous paths on $[0, T]$.

Let us consider the events on which the processes $Z^{\pm}$ of (5.11) undergo a (left) jump at $t = T$, namely,

$$B^{\pm} := \{Z^{\pm}(T-) > 0\} = \{Z^{\pm}(T-) \ne Z^{\pm}(T)\};$$

by El Karoui (1981), we have

$$\begin{aligned} B^+ \subset \{Z^+(T-) = Z^-(T-) + \tilde{L}(T-)\}, \\ B^- \subset \{Z^-(T-) = Z^+(T-) - \tilde{U}(T-)\}, \end{aligned}$$

and thus from (3.7),

$$B := B^+ \cap B^- \subset \{\tilde{L}(T-) = \tilde{U}(T-) = 0\} \cap \{Z^+(T-) = Z^-(T-)\}, \text{ mod } P.$$

CASE 1. Suppose that, in addition to (5.7), we have as well

$$P[L(T) < U(T)] = 1 \quad \text{or equivalently } P[\tilde{L}(T-) = \tilde{U}(T-) = 0] = 0.$$

Then Alario-Nazaret, Lepeltier and Marchal [(1982), page 30] prove that the (left) jumps of Z^+ and Z^- occur on disjoint events; this implies, as they also show, the regularity property (A.4) for the potentials $Z^\pm$, and hence the Doob–Meyer decomposition

$$Z^\pm = \pi(K^\pm), \qquad K^\pm \in \mathbf{S}^2_{\text{ci}}$$

[cf. the Appendix, condition (A.4), Lemma A.1(ii) and Definition A.2]. Corollary A.3 gives then $Z^\pm \in \mathbf{\Pi}^2_c$, and this completes the proof.

CASE 2. Suppose now that $P[\tilde{L}(T-) = \tilde{U}(T-) = 0] > 0$. Then the above argument guarantees the continuity of $Z^\pm$, $K^\pm$ only on $[0, T)$. To overcome this, we introduce the random variable

$$\zeta := Z^+(T-)\mathbf{1}_B = Z^-(T-)\mathbf{1}_B \geq 0 \quad \text{a.s.}$$

and the nonnegative supermartingales

$$\tilde{Z}^\pm(t) := \begin{cases} Z^\pm(t) - E[\zeta|\mathscr{F}(t)], & 0 \leq t < T, \\ 0, & t = T. \end{cases} \tag{5.12}$$

The nonnegativity follows easily from Fatou's lemma and the supermartingale property of $Z^\pm$ since, for $0 \leq t < T$, we have

$$\begin{aligned}\tilde{Z}^\pm(t) &= Z^\pm(t) - E\Big[Z^\pm(T-)\mathbf{1}_B\Big|\mathscr{F}(t)\Big] \geq Z^\pm(t) - E[Z^\pm(T-)|\mathscr{F}(t)] \\ &= Z^\pm(t) - E\left[\lim_{n\to\infty} Z^\pm\left(T - \frac{1}{n}\right)\middle|\mathscr{F}(t)\right] \\ &\geq Z^\pm(t) - \liminf_{n\to\infty} E\left[Z^\pm\left(T - \frac{1}{n}\right)\middle|\mathscr{F}(t)\right] \geq 0 \quad \text{a.s.}\end{aligned}$$

On the other hand, the processes of (5.12) are easily seen to be supermartingales; their paths are continuous on $[0, T)$ and we have $\tilde{Z}^\pm(T-) = Z^\pm(T-)\mathbf{1}_{B^c}$, so that $\{\tilde{Z}^+(T-) > 0, \tilde{Z}^-(T-) > 0\} = B^c \cap (B^+ \cap B^-) = \varnothing$. In other words, the (possible, left) jumps of $\tilde{Z}^\pm$ at $t = T$ occur on disjoint events. If we manage to show that $\tilde{Z}^\pm$ of (5.12) also solve the system (5.5), then we can just repeat the arguments of Case 1, this time for the new pair $(\tilde{Z}^+, \tilde{Z}^-)$, to obtain $\tilde{Z}^\pm \in \mathbf{\Pi}^2_c$ and thereby complete the proof of the theorem.

Now, let us observe from (5.12) that $\tilde{Z}^+ \geq \tilde{Z}^- + \tilde{L}$; this is obvious for $t = T$ since $\tilde{Z}^+(T) = 0 = \tilde{Z}^-(T) + \tilde{L}(T)$, whereas on $[0, T)$ we have

$$\tilde{Z}^+ = Z^+ - E[\zeta|\mathscr{F}(\cdot)] \geq Z^- + \tilde{L} - E[\zeta|\mathscr{F}(\cdot)] = \tilde{Z}^- + \tilde{L}.$$

Let R be another RCLL supermartingale with $R \geq \tilde{Z}^- + \tilde{L}$; then

$$\tilde{R}(t) := \begin{cases} R(t) + E[\zeta|\mathscr{F}(t)], & 0 \leq t < T, \\ R(T), & t = T \end{cases}$$

is a supermartingale and satisfies $\tilde{R}(T) \geq \tilde{L}(T) = Z^-(T) + \tilde{L}(T)$, as well as

$$\tilde{R} \geq \tilde{Z}^- + \tilde{L} + E[\zeta|\mathscr{F}(\cdot)] = Z^- + \tilde{L} \quad \text{on } [0, T),$$

almost surely. Therefore, $\tilde{R}$ dominates $Z^- + \tilde{L}$, and thus $\tilde{R} \geq S(Z^- + \tilde{L}) = Z^+$; in other words, $R \geq \tilde{Z}^+$, almost surely.

We conclude that $\tilde{Z}^+ = S(\tilde{Z}^- + \tilde{L})$. A completely similar argument then leads to $\tilde{Z}^- = S(\tilde{Z}^+ - \tilde{U})$, and this leads to the fact that the pair $(\tilde{Z}^+, \tilde{Z}^-)$ solves the system (5.5). □

Putting Theorems 5.2 and 5.3 together, we have the following conclusion.

(5.13) Under conditions (5.6) and (5.7), the BSE of Problem 5.1 has a unique solution.

Let us discuss now the solvability of our original problem, the BSDE of Problem 2.1. We shall do that by adapting, to our situation at hand, a fixed point method due to Pardoux and Peng (1990) and modified by El Karoui, Kapoudjian, Pardoux, Peng and Quenez (1995).

THEOREM 5.4. *For fixed, given $\xi \in \mathbf{L}_1^2$ and continuous L, U in $\mathbf{S}_1^2$ satisfying* (2.3), *suppose that Problem* 5.1 *has a unique solution for every $g \in \mathbf{H}_1^2$. Then there is also a unique solution (X, Y, K) to the BSDE of Problem* 2.1, *and the state process X admits the stochastic game representation* (4.3) *of Theorem* 4.1.

COROLLARY 5.5. *Under conditions* (5.6) *and* (5.7), *the BSDE of Problem* 2.1 *has a unique solution (X, Y, K), and the representation* (4.3) *of Theorem* 4.1 *holds.*

This follows directly from (5.13) and Theorem 5.4.

PROOF OF THEOREM 5.4. Let us start with a pair (χ, Ψ) in the set

$$\mathscr{L} := \{(X, Y) \in \mathbf{S}_1^2 \times \mathbf{H}_d^2 \big| X \text{ has continuous paths with } L(t) \leq X(t) \leq U(t)\ \forall\ 0 \leq t \leq T \text{ and } X(T) = \xi \text{ a.s.}\} \tag{5.14}$$

and define $g \in \mathbf{H}_1^2$ by setting $g(t, \omega) := f(t, \omega, \chi(t, \omega), \Psi(t, \omega))$, where f: $[0, T] \times \Omega \times \mathbb{R} \times \mathbb{R}^d \mapsto \mathbb{R}$ is the coefficient function of Problem 2.1. For this $g \in \mathbf{H}_1^2$, the BSE of Problem 5.1 has, by assumption, a unique solution (X, Y, K); in particular, the pair (X, Y) belongs to the set $\mathscr{L}$ of (5.14). This way, we have constructed a mapping

$$\varphi\colon \mathscr{L} \mapsto \mathscr{L} \quad \text{via } (X, Y) = \varphi(\chi, \Psi). \tag{5.15}$$

In order to establish the unique solvability of the BSDE of Problem 2.1, it is clearly sufficient to show that the mapping φ of (5.15) is a contraction with respect to an appropriate norm in $\mathbf{S}_1^2 \times \mathbf{H}_d^2$; as such, we shall take

$$\|(X, Y)\|_\beta := \left(E\int_0^T e^{\beta t}(X^2(t) + \|Y(t)\|^2)\,dt\right)^{1/2}, \tag{5.16}$$

for an appropriate $\beta \in (0, \infty)$ to be determined in (5.18) below.

Let (χ_0, Ψ_0) be another pair in the set $\mathscr{L}$ of (5.14), denote by (X_0, Y_0, K_0), $(X_0, Y_0) = \varphi(\chi_0, \Psi_0)$, $K_0 = K_0^+ - K_0^-$, the unique solution of the BSE of Problem 5.1 with $g(t, \omega) = f(t, \omega, \chi_0(t, \omega), \Psi_0(t, \omega))$, and define

$$\bar\chi = \chi - \chi_0, \qquad \bar\Psi = \Psi - \Psi_0, \qquad \bar X = X - X_0, \qquad \bar Y = Y - Y_0, \qquad \bar K = K - K_0.$$

Clearly $d\bar X(t) = [f(t, \chi_0(t), \Psi_0(t)) - f(t, \chi(t), \Psi(t))]\,dt - d\bar K(t) + \bar Y'(t)\,dB(t)$, and from Itô's rule,

$$\begin{aligned} d(e^{\beta t}\bar X^2(t)) = e^{\beta t}[&(\beta \bar X^2(t) + \|\bar Y(t)\|^2)\,dt - 2\bar X(t)\,d\bar K(t) + 2\bar X(t)\bar Y'(t)\,dB(t)\\ &+2\bar X(t)\{f(t, \chi_0(t), \Psi_0(t)) - f(t, \chi(t), \Psi(t))\}\,dt], \end{aligned}$$

we obtain the bounds (as argued below):

$$\begin{aligned} &e^{\beta t}E[\bar X(t)]^2 + E\int_t^T e^{\beta u}[\beta(\bar X(u))^2 + \|\bar Y(u)\|^2]\,du\\ &\quad= 2E\int_t^T e^{\beta u}\bar X(u)\,d\bar K(u) - 2E\int_t^T e^{\beta u}\bar X(u)\bar Y'(u)\,dB(u)\\ &\qquad+2E\int_t^T e^{\beta u}\bar X(u)[f(u, \chi(u), \Psi(u)) - f(u, \chi_0(u), \Psi_0(u))]\,du\\ &\quad\le 2kE\int_0^T e^{\beta u}|\bar X(u)|(|\bar\chi(u)| + \|\bar\Psi(u)\|)\,du\\ &\quad\le 4k^2E\int_t^T e^{\beta u}(\bar X(u))^2\,du + \tfrac12 E\int_t^T e^{\beta u}((\bar\chi(u))^2 + \|\bar\Psi(u)\|^2)\,du, \end{aligned} \tag{5.17}$$

where k is the Lipschitz constant of (2.2). We have used in (5.17) the elementary inequality

$$2k|y|(|a| + |b|) = 2(2k|y|)\frac{|a| + |b|}{2} \le 4k^2y^2 + \frac{a^2 + b^2}{2},$$

the bounds

$$\begin{aligned} &E\left(\int_0^T e^{2\beta u}(\bar X(u))^2\|\bar Y(u)\|^2\,du\right)^{1/2}\\ &\qquad\le e^{\beta T}E\left(\sup_{0\le t\le T}|\bar X(t)|\left(\int_0^T \|\bar Y(u)\|^2\,du\right)^{1/2}\right)\\ &\qquad\le \tfrac12 e^{\beta T}E\left[\sup_{0\le t\le T}(\bar X(t))^2 + \int_0^T \|\bar Y(u)\|^2\,du\right] < \infty, \end{aligned}$$

which, together with the Burkholder–Davis–Gundy inequalities [e.g., Karatzas and Shreve (1991), page 166] imply that the stochastic integral in (5.17) is a martingale and thus has zero expectation, and the inequality

$$\begin{aligned}\bar{X}\,d\bar{K} &= (X - X_0)\,dK - (X - X_0)\,dK_0 \\ &= (X - X_0)\,dK^+ + (X_0 - X)\,dK^- + (X_0 - X)\,dK_0^+ + (X - X_0)\,dK_0^- \\ &= (X - L)\,dK^+ + (L - X_0)\,dK^+ + (X_0 - U)\,dK^- + (U - X)\,dK^- \\ &\quad +(X_0 - L)\,dK_0^+ + (L - X)\,dK_0^+ + (X - U)\,dK_0^- + (U - X_0)\,dK_0^- \\ &\le 0,\end{aligned}$$

which is a consequence of (2.5) and (2.6). Now choose

$$\beta = 1 + 4k^2 \tag{5.18}$$

in (5.17) to obtain

$$E\int_0^T e^{\beta u}[(\bar{X}(u))^2 + \|\bar{Y}(u)\|^2]\,du \le \tfrac{1}{2}E\int_0^T e^{\beta u}[(\bar{\chi}(u))^2 + \|\bar{\Psi}(u)\|^2]\,du, \tag{5.19}$$

the contraction property that we sought for the norm of (5.16). The proof of Theorem 5.4 is complete. □

6. Existence by penalization. We shall present in this section a second approach to the question of existence of solutions to the BSE of Problem 5.1, which complements the result of (5.13) as it establishes existence under slightly different conditions than (5.6) and (5.7); see (6.3) and (6.4). Under these conditions one shows, in fact, that the reflection processes $K^\pm$ are absolutely continuous with respect to Lebesgue measure, almost surely.

This new approach considers a sequence of penalized versions

$$\begin{aligned} X_n(t) = \xi_n + \int_t^T g(s)\,ds + n\int_t^T ((L_n(s) - X_n(s)) \vee 0)\,ds \\ - n\int_t^T ((X_n(s) - U_n(s)) \vee 0)\,ds - \int_t^T Y_n'(s)\,dB(s), \\ 0 \le t \le T,\end{aligned} \tag{6.1}$$

for $n \in \mathbb{N}$, of the backward stochastic equation (5.1), with suitable random functions ξ_n, L_n, U_n and $n \in \mathbb{N}$. From the standard theory of unconstrained BSDE's [Pardoux and Peng (1990)], equation (6.1) has, for every $n \in \mathbb{N}$, a unique $\mathbf{F}$-adapted solution $(X_n, Y_n) \in \mathbf{H}_1^2 \times \mathbf{H}_d^2$. Then, with

$$\begin{aligned} K_n^\pm(t) = \int_0^t k_n^\pm(u)\,du; \qquad k_n^+ := n(L_n(t) - X_n(t)) \vee 0), \\ k_n^-(t) := n((X_n(t) - U_n(t)) \vee 0) \quad \text{and} \quad K_n := K_n^+ - K_n^-,\end{aligned} \tag{6.2}$$

the idea is to show that $\{(X_n, Y_n, K_n)\}_{n\in\mathbb{N}}$ converges to a triple (X, Y, K) of processes with $X \in \mathbf{S}_1^2$, $Y \in \mathbf{H}_d^2$ and $K = K^+ - K^-$, $K^\pm \in \mathbf{S}_{\mathrm{ci}}^2$ (in fact, with $K^\pm$

absolutely continuous with respect to Lebesgue measure, as we shall show), which solves the BSE of Problem 5.1.

We shall assume in this section that

$$g \in \mathbf{S}_1^2 \tag{6.3}$$

and that

(6.4) there exist sequences $\{U_n\}_{n\in\mathbb{N}}$, $\{L_n\}_{n\in\mathbb{N}}$ of Itô processes $dU_n(t) = u_n(t)\,dt + v_n'(t)\,dB(t)$, $dL_n(t) = l_n(t)\,dt + m_n'(t)\,dB(t)$ with $\{u_n\}_{n\in\mathbb{N}}$, $\{l_n\}_{n\in\mathbb{N}}$ bounded in $\mathbf{S}_1^2$, $\{v_n\}_{n\in\mathbb{N}} \subset \mathbf{H}_d^2$, $\{m_n\}_{n\in\mathbb{N}} \subset \mathbf{H}_d^2$ and $\{\xi_n\}_{n\in\mathbb{N}} \subset \mathbf{L}_1^2$, such that $L_n(t) \le U_n(t)$, $\forall\ 0 \le t \le T$ and $L_n(T) \le \xi_n \le U_n(T)$ hold almost surely for every $n \in \mathbb{N}$, and, as $n \to \infty$, $\xi_n \to \xi$, $\sup_{0\le t\le T} |U_n(t) - U(t)| \to 0$, $\sup_{0\le t\le T} |L_n(t) - L(t)| \to 0$ both almost surely and in $\mathbf{L}_1^2$.

Here, of course, ξ and U, L are the data (terminal condition and barriers, respectively) for Problem 5.1. Condition (6.4) imposes some regularity on the boundary processes U and L in the form of uniform approximation by Itô processes; it is satisfied trivially, if U and L are themselves Itô processes in $\mathbf{S}_1^2$.

THEOREM 6.1. *Under conditions* (6.3) *and* (6.4), *the BSE of Problem* 5.1 *has a unique solution* (X, Y, K); *in particular*, K *is absolutely continuous with respect to Lebesgue measure*, *and* X *admits the stochastic game representation* (4.3).

The uniqueness is again a consequence of Theorem 4.1 (arguing just as in Corollary 4.2), so we shall devote the rest of this section to proving existence. From Theorem 3.1 in Pardoux and Peng (1990), the unconstrained BSDE of (6.1) has a unique, $\mathbf{F}$-adapted solution $(X_n, Y_n) \in \mathbf{H}_1^2 \times \mathbf{H}_d^2$, for every $n \in \mathbb{N}$. By analogy with (4.2), there is an interpretation of this solution in terms of a suitable stochastic game, which, again, provides as a corollary the uniqueness of the solution to (6.1).

PROPOSITION 6.2. *For every* $n \in \mathbb{N}$, *let* $\mathscr{D}_n$ *denote the class of* $\mathbf{F}$-*progressively measurable processes* μ: $[0, T] \times \Omega \mapsto [0, n]$, *and introduce the payoff*:

$$\begin{aligned} R_t^{(n)}(\mu, \nu) := \xi_n \exp\left(-\int_t^T (\mu(u) + \nu(u))\,du\right) \\ + \int_t^T \exp\left(-\int_t^s (\mu(u) + \nu(u))\,du\right) \\ \times [g(s) + \mu(s)L_n(s) + \nu(s)U_n(s)]\,ds \end{aligned} \tag{6.5}$$

for every $\mu \in \mathscr{D}_n$, $\nu \in \mathscr{D}_n$. *We have then*

$$\begin{aligned} E[R_t^{(n)}(\mu, \hat{\nu}_n)|\mathscr{F}(t)] &\leq E[R_t^{(n)}(\hat{\mu}_n, \hat{\nu}_n)|\mathscr{F}(t)] \\ &= X_n(t) \leq E[R_t^{(n)}(\hat{\mu}_n, \nu)|\mathscr{F}(t)] \quad a.s. \end{aligned} \tag{6.6}$$

for every $t \in [0, T]$, $(\mu, \nu) \in \mathscr{D}_n \times \mathscr{D}_n$, *where*

$$\hat{\mu}_n := n\mathbf{1}_{\{X_n < L_n\}}, \qquad \hat{\nu}_n := n\mathbf{1}_{\{X_n > U_n\}}. \tag{6.7}$$

COROLLARY 6.3. *The pair* $(\hat{\mu}_n, \hat{\nu}_n) \in \mathscr{D}_n \times \mathscr{D}_n$ *of* (6.7) *is a saddlepoint for the stochastic game with upper- and lower-values*:

$$\begin{aligned} \overline{V}_n(t) &:= \operatorname*{ess\,inf}_{\nu \in \mathscr{D}_n} \operatorname*{ess\,sup}_{\mu \in \mathscr{D}_n} E[R_t^{(n)}(\mu, \nu)|\mathscr{F}(t)], \\ \underline{V}_n(t) &:= \operatorname*{ess\,sup}_{\mu \in \mathscr{D}_n} \operatorname*{ess\,inf}_{\nu \in \mathscr{D}_n} E[R_t^{(n)}(\mu, \nu)|\mathscr{F}(t)], \end{aligned} \tag{6.8}$$

respectively. This game has value, namely $V_n(t)$, *given as*

$$V_n(t) = \overline{V}_n(t) = \underline{V}_n(t) = X_n(t) = E[R_t^{(n)}(\hat{\mu}_n, \hat{\nu}_n)|\mathscr{F}(t)], \quad a.s. \tag{6.9}$$

PROOF. From (6.1), and by applying Itô's rule to the product of the proceses $X_n(\cdot)$ and $\exp\{-\int_0^{\cdot}(\mu(u) + \nu(u))\,du\}$, we obtain

$$\begin{aligned} X_n(t) = E\bigg[&\xi_n \exp\left(-\int_t^T (\mu(u) + \nu(u))\,du\right) \\ &+ \int_t^T \exp\left(-\int_t^s (\mu(u) + \nu(u))\,du\right) \\ &\qquad \times [g(s) + \mu(s)L_n(s) + \nu(s)U_n(s)]\,ds \\ &+ \int_t^T \exp\left(-\int_t^s (\mu(u) + \nu(u))\,du\right) \\ &\qquad \times [\mu(s)(X_n(s) - L_n(s)) + n((L_n(s) - X_n(s)) \vee 0)]\,ds \\ &+ \int_t^T \exp\left(-\int_t^s (\mu(u) + \nu(u))\,du\right) \\ &\qquad \times [\nu(s)(X_n(s) - U_n(s)) - n((X_n(s) - U_n(s)) \vee 0)]\,ds \Big| \mathscr{F}(t)\bigg] \end{aligned} \tag{6.10}$$

almost surely, for every $t \in [0, T]$, $\mu \in \mathscr{D}_n$, $\nu \in \mathscr{D}_n$, after taking conditional expectations with respect to $\mathscr{F}(t)$ and noting that the conditional expectation of the stochastic integral vanishes, since $Y_n \in \mathbf{H}_d^2$. In the last two terms of (6.10), the integrands satisfy

$$\begin{aligned} &\mu(X_n - L_n) + n((L_n - X_n) \vee 0) \geq 0 \quad \text{with equality for } \mu = \hat{\mu}_n, \\ &\nu(X_n - U_n) - n((X_n - U_n) \vee 0) \leq 0 \quad \text{with equality for } \nu = \hat{\nu}_n. \end{aligned}$$

Now (6.6) follows from these observations and from (6.5), and leads directly to (6.9). □

Let us define now

$$\bar{X}_n := X_n - U_n, \qquad \bar{g}_n := g - u_n \tag{6.11}$$

and observe that

$$\begin{aligned}\bar{X}_n(t) &= \xi_n - U_n(T) + \int_t^T \bar{g}_n(s)\,ds \\ &\quad + n\int_t^T ((L_n(s) - U_n(s) - \bar{X}_n(s)) \vee 0)\,ds \\ &\quad - n\int_t^T (\bar{X}_n(s) \vee 0)\,ds + \int_t^T (v_n(s) - Y_n(s))'\,dB(s), \\ &\qquad\qquad 0 \le t \le T,\end{aligned} \tag{6.12}$$

from (6.1) and (6.4), as well as

$$\begin{aligned}\bar{X}_n(t) = \operatorname*{ess\,sup}_{\mu\in\mathscr{D}_n}\operatorname*{ess\,inf}_{\nu\in\mathscr{D}_n} E\bigg[&(\xi_n - U_n(T))\exp\bigg(-\int_t^T(\mu(u)+\nu(u))\,du\bigg) \\ &+ \int_t^T \exp\bigg(-\int_t^s (\mu(u)+\nu(u))\,du\bigg) \\ &\times [\bar{g}_n(s) + \mu(s)(L_n(s) - U_n(s))]\,ds\bigg|\mathscr{F}(t)\bigg] \quad \text{a.s.}\end{aligned} \tag{6.13}$$

by analogy with the stochastic game representation (6.9). From (6.13), and with $g^* := \sup_{t,n}|g(t) - u_n(t)| \in \mathbf{L}_1^2$ by (6.3) and (6.4), we have

$$\begin{aligned}\bar{X}_n(t) \vee 0 &\le \operatorname*{ess\,sup}_{\mu\in\mathscr{D}_n}\operatorname*{ess\,inf}_{\nu\in\mathscr{D}_n} E\bigg[\int_t^T \exp\bigg(-\int_t^s(\mu(u)+\nu(u))\,du\bigg)|\bar{g}_n(s)|\,ds\bigg|\mathscr{F}(t)\bigg] \\ &\le \operatorname*{ess\,sup}_{\mu\in\mathscr{D}_n} E\bigg[\int_t^T \exp\bigg(-\int_t^s(\mu(u)+n)\,du\bigg)|\bar{g}_n(s)|\,ds\bigg|\mathscr{F}(t)\bigg] \\ &\le E\bigg[\int_t^T \exp(-n(s-t))|\bar{g}_n(s)|\,ds\bigg|\mathscr{F}(t)\bigg] \le \frac{1}{n}E[g^*|\mathscr{F}(t)],\end{aligned}$$

and from Doob's maximal inequality,

$$E\bigg[\sup_{0\le t\le T}((X_n(t) - U_n(t)) \vee 0)^2\bigg] \le \frac{c}{n^2}. \tag{6.14}$$

A similar analysis yields

$$E\bigg[\sup_{0\le t\le T}((L_n(t) - X_n(t)) \vee 0)^2\bigg] \le \frac{c}{n^2}, \qquad \forall\, n \in \mathbb{N}, \tag{6.15}$$

whence

$$E\Big[\sup_{0\le t\le T} X_n^2(t)\Big] \le c, \qquad E[(K_n^+(T))^2 + (K_n^-(T))^2] \le c,$$
$$E\Big[\sup_{0\le t\le T} (K_n(t))^2\Big] \le c \qquad \forall\, n \in \mathbb{N} \tag{6.16}$$

follow as well, from (6.2). Here and in the sequel, $c > 0$ denotes a real constant, whose value may vary from line to line.

LEMMA 6.4. *We have*

$$E\Big[\sup_{0\le t\le T} (X_n(t) - X_m(t))^2 + \sup_{0\le t\le T} (K_n(t) - K_m(t))^2 + \int_0^T \|Y_n(t) - Y_m(t)\|^2\, dt\Big] \to 0, \tag{6.17}$$

as $m, n \to \infty$.

PROOF. From (6.1), (6.2) and Itô's rule

$$\begin{aligned} d(X_n(t) - X_m(t))^2 &= 2(X_n(t) - X_m(t)) \\ &\quad\times [(dK_n^-(t) - dK_m^-(t)) - (dK_n^+(t) - dK_m^+(t)) \\ &\qquad + (Y_n(t) - Y_m(t))'\, dB(t)] + \|Y_n(t) - Y_m(t)\|^2\, dt, \end{aligned}$$

we have

$$\begin{aligned} E\Big[(X_n(t) - X_m(t))^2 + \int_t^T \|Y_n(s) - Y_m(s)\|^2\, ds\Big] &= 2E\int_t^T [(X_n(s) - L_n(s)) - (X_m(s) - L_m(s)) \\ &\qquad + (L_n(s) - L_m(s))](dK_n^+(s) - dK_m^+(s)) \\ &\quad + 2E\int_t^T [(X_m(s) - U_m(s)) - (X_n(s) - U_n(s)) \\ &\qquad + (U_m(s) - U_n(s))](dK_n^-(s) - dK_m^-(s)). \end{aligned} \tag{6.18}$$

Now, from (6.2), $(X_n - L_n)(dK_n^+/dt) \le 0$ and $(X_n - U_n)(dK_n^-/dt) \ge 0$, so we obtain from (6.18)

$$E\Big[(X_n(t) - X_m(t))^2 + \int_t^T \|Y_n(s) - Y_m(s)\|^2\, ds\Big] \le 2c_{n,m} \to 0 \quad \text{as } m, n \to \infty,\ \forall\, 0 \le t \le T \tag{6.19}$$

because

$$\begin{aligned}
c_{n,m} &:= E\Big[\sup_{0\le t\le T}\int_t^T (X_n(s)-X_m(s))(dK_n(s)-dK_m(s))\Big]\\
&\le E\Big[\sup_{0\le t\le T}\int_t^T [((L_n(s)-X_n(s))\vee 0)\,dK_m^+(s)\\
&\qquad\qquad + ((L_m(s)-X_m(s))\vee 0)\,dK_n^+(s)\\
&\qquad\qquad + |L_n(s)-L_m(s)|(dK_n^+(s)+dK_m^+(s))\\
&\qquad\qquad + |U_n(s)-U_m(s)|(dK_n^-(s)+dK_m^-(s))\\
&\qquad\qquad + ((X_m(s)-U_m(s))\vee 0)\,dK_n^-(s)\\
&\qquad\qquad\quad + ((X_n(s)-U_n(s))\vee 0)\,dK_m^-(s)]\,ds\Big]
\end{aligned}$$

goes to zero as $n\to\infty$, $m\to\infty$, by virtue of (6.14)–(6.16) and (6.4). In particular, (6.19) gives

$$\sup_{0\le t\le T} E(X_n(t)-X_m(t))^2 + E\int_0^T \|Y_n(t)-Y_m(t)\|^2\,dt \to 0 \qquad \text{as } m,n\to\infty. \tag{6.20}$$

On the other hand, again by Itô's rule we have

$$\begin{aligned}
&E\Big[\sup_{0\le t\le T}(X_n(t)-X_m(t))^2\Big]\\
&\quad\le E\Big[\sup_{0\le t\le T}\Big((X_n(t)-X_m(t))^2+\int_t^T\|Y_n(s)-Y_m(s)\|^2\,ds\Big)\Big]\\
&\quad= 2E\Big[\sup_{0\le t\le T}\Big(\int_t^T (X_n(s)-X_m(s))(dK_n(s)-dK_m(s))\\
&\qquad\qquad + \int_t^T (X_n(s)-X_m(s))(Y_n(s)-Y_m(s))'\,dB(s)\Big)\Big]\\
&\quad\le 2c_{n,m}+2E\Big(\sup_{0\le t\le T}\Big|\int_t^T (X_n(s)-X_m(s))\\
&\qquad\qquad\times (Y_n(s)-Y_m(s))'\,dB(s)\Big|\Big).
\end{aligned}\tag{6.21}$$

By the Burkholder–Davis–Gundy inequalities, this last term is bounded by

$$\begin{aligned}
&2cE\Big(\int_0^T (X_n(t)-X_m(t))^2\|Y_n(t)-Y_m(t)\|^2\,dt\Big)^{1/2}\\
&\quad\le E\Big[\sup_{0\le t\le T}|X_n(t)-X_m(t)|\Big(4c^2\int_0^T\|Y_n(t)-Y_m(t)\|^2\,dt\Big)^{1/2}\Big]\\
&\quad\le \tfrac12 E\Big[\sup_{0\le t\le T}(X_n(t)-X_m(t))^2\Big]+2c^2E\int_0^T\|Y_n(t)-Y_m(t)\|^2\,dt,
\end{aligned}\tag{6.22}$$

for a suitable constant $0 < c < \infty$ and it follows from (6.19)–(6.22) that

$$E\Big[\sup_{0\le t\le T}(X_n(t)-X_m(t))^2\Big]\to 0 \quad \text{as } m,n\to\infty. \tag{6.23}$$

Finally, from basic properties of the Itô integral and Doob's maximal inequality,

$$E\bigg(\sup_{0\le t\le T}\bigg|\int_0^t Y_n'(s)\,dB(s)-\int_0^t Y_m'(s)\,dB(s)\bigg|^2\bigg) \le cE\int_0^T \|Y_n(s)-Y_m(s)\|^2\,ds \to 0,$$

as $m,n\to\infty$, in conjunction with (6.20), which leads to

$$E\Big[\sup_{0\le t\le T}|K_n(t)-K_m(t)|^2\Big]\to 0 \quad \text{as } m,n\to\infty, \tag{6.24}$$

along with (6.23) and $K_n(t)=X_n(0)-X_n(t)+\int_0^t g(u)\,du+\int_0^t Y_n'(s)\,dB(s)$, $0\le t\le T$. $\square$

We conclude from (6.17) that *there exist continuous adapted processes* X, K *in* $\mathbf{S}_1^2$, *as well as a process* Y *in* $\mathbf{H}_d^2$, such that

$$E\Big[\sup_{0\le t\le T}(X_n(t)-X(t))^2+\sup_{0\le t\le T}(K_n(t)-K(t))^2 + \int_0^T \|Y_n(t)-Y(t)\|^2\,dt\Big]\to 0 \quad \text{as } n\to\infty. \tag{6.25}$$

Thus, passing to the limit as $n\to\infty$ in (6.14) and (6.15), we obtain, thanks to (6.4) and (6.25),

$$E\Big[\sup_{0\le t\le T}((X(t)-U(t))\vee 0)^2\Big]=E\Big[\sup_{0\le t\le T}((L(t)-X(t))\vee 0)^2\Big]=0,$$

so that the triple (X,Y,K) satisfies (2.5), almost surely.

On the other hand, passing to the limit as $n\to\infty$ in

$$X_n(t)=\xi_n+\int_t^T g(u)\,du+K_n(T)-K_n(t)-\int_t^T Y_n'(u)\,dB(u), \quad 0\le t\le T, \tag{6.1$'$}$$

we obtain from (6.25) that the BSE (5.1) is satisfied as well. Now (6.16) shows that the sequences $\{k_n^\pm\}_{n\in\mathbb{N}}$ of nonnegative, $\mathbf{F}$-progressively measurable processes of (6.2) are bounded in $\mathbf{L}^2([0,T]\times\Omega)$; consequently, there exist $\mathbf{F}$-progressively measurable processes $k^\pm\colon [0,T]\times\Omega\mapsto[0,\infty)$ such that (along a relabelled subsequence),

$$k_n^\pm\to k^\pm \quad \text{as } n\to\infty, \text{ weakly in } \mathbf{L}^2([0,T]\times\Omega). \tag{6.26}$$

For these processes, we have

$$
\begin{aligned}
E\bigg|\int_0^T &(X_n(t)-L_n(t))k_n^+(t)\,dt - \int_0^T (X(t)-L(t))k^+(t)\,dt\bigg| \\
&= E\bigg|\int_0^T (X_n(t)-X(t))k_n^+(t)\,dt + \int_0^T (X(t)-L(t))(k_n^+(t)-k^+(t))\,dt \\
&\qquad\qquad + \int_0^T (L(t)-L_n(t))k_n^+(t)\,dt\bigg| \\
&\le E\Big[\sup_{0\le t\le T}|X_n(t)-X(t)|K_n^+(T)\Big] + E\Big[\sup_{0\le t\le T}|L_n(t)-L(t)|K_n^+(T)\Big] \\
&\quad + E\bigg|\int_0^T (X(t)-L(t))(k_n^+(t)-k^+(t))\,dt\bigg| \to 0, \quad \text{as } n\to\infty,
\end{aligned}
$$

thanks to (6.16), $E(\sup_{0\le t\le T}|X_n(t)-X(t)|^2)\to 0$ in (6.25), $E(\sup_{0\le t\le T}|L_n(t)-L(t)|^2)\to 0$ in (6.4) and $X-L\in\mathbf{L}^2([0,T]\times\Omega)$ in conjunction with (6.26). In particular,

$$
E\int_0^T (X_n(t)-L_n(t))k_n^+(t)\,dt \to E\int_0^T (X(t)-L(t))k^+(t)\,dt, \quad \text{as } n\to\infty, \tag{6.27}
$$

but

$$
\begin{aligned}
\int_0^T (X(t)-L(t))k^+(t)\,dt \ge 0 &\ge \int_0^T (X_n(t)-L_n(t))k_n^+(t)\,dt \\
&= \int_0^T (X_n(t)-L_n(t))((L_n(t)-X_n(t))\vee 0)\,dt
\end{aligned}
$$

holds almost surely, for every $n\in\mathbb{N}$. Therefore, the processes $k^\pm$ of (6.26) satisfy

$$
\int_0^T (X(t)-L(t))k^+(t)\,dt = 0 \quad \text{a.s.} \tag{6.28}
$$

thanks to (6.27), and a similar argument gives

$$
\int_0^T (U(t)-X(t))k^-(t)\,dt = 0 \quad \text{a.s.} \tag{6.29}
$$

In order to finish the proof of existence in Theorem 6.1, it thus remains to show that

$$
K(t) = \int_0^t k^+(u)\,du - \int_0^t k^-(u)\,du =: \tilde K(t) \tag{6.30}
$$

holds almost surely, for every fixed $t\in[0,T]$, because, since both $K,\tilde K$ have continuous paths, this implies $K=\tilde K$, a.s.

PROOF OF (6.30). The key idea here is to turn the weak convergence of (6.26) into strong by considering convex combinations; namely, the Banach–Mazur lemma [Dunford and Schwartz (1963), page 422, Corollary 14; Ekeland

and Temam (1976), page 6] shows that there exist, for every $n \in \mathbb{N}$, an integer $N(n) \geq n$ and weights $\lambda_j^{(n)} \geq 0$, $j = n, \dots, N(n)$ with $\sum_{j=n}^{N(n)} \lambda_j^{(n)} = 1$, such that

$$\tilde{k}_n^{\pm} := \sum_{j=n}^{N(n)} \lambda_j^{(n)} k_j^{\pm} \to k^{\pm} \quad \text{as } n \to \infty, \text{ in } \mathbf{L}^2([0,T] \times \Omega). \tag{6.31}$$

For fixed $t \in [0, T]$, (6.31) and Jensen's inequality,

$$\left(\int_0^t (\tilde{k}_n^{\pm}(s) - k^{\pm}(s))\, ds \right)^2 \leq t \int_0^t (\tilde{k}_n^{\pm}(s) - k^{\pm}(s))^2\, ds,$$

give

$$E\left(\int_0^t (\tilde{k}_n^{\pm}(s) - k^{\pm}(s))\, ds \right)^2 \leq T \cdot E \int_0^t (\tilde{k}_n^{\pm}(s) - k^{\pm}(s))^2\, ds \to 0 \quad \text{as } n \to \infty.$$

Therefore,

$$E(\tilde{K}_n(t) - \tilde{K}(t))^2 \to 0 \quad \text{as } n \to \infty, \qquad \text{where } \tilde{K}_n(\cdot) = \int_0^{\cdot} (\tilde{k}_n^{+}(s) - \tilde{k}_n^{-}(s))\, ds. \tag{6.32}$$

On the other hand, denoting by $\|\eta\|_2 = \sqrt{E(\eta^2)}$ the norm of $\mathbf{L}_1^2$, we have from (6.25) that $\|K_n(t) - K(t)\|_2 < \varepsilon$ holds for arbitrary $\varepsilon > 0$ and all sufficiently large $n \in \mathbb{N}$; therefore, we have as well

$$\|\tilde{K}_n(t) - K(t)\|_2 = \left\| \sum_{j=n}^{N(n)} \lambda_j^{(n)} (K_j(t) - K(t)) \right\|_2 \leq \sum_{j=n}^{N(n)} \lambda_j^{(n)} \|K_j(t) - K(t)\|_2 < \varepsilon,$$

so that

$$E[\tilde{K}_n(t) - K(t)]^2 \to 0 \quad \text{as } n \to \infty, \tag{6.33}$$

and (6.30) follows from (6.32) and (6.33). □

The proof of Theorem 6.1 is complete.

Let us end this section by extending the results of Theorem 6.1 for the BSE of Problem 5.1 to the BSDE of Problem 2.1.

THEOREM 6.5. *Under the conditions* (6.4),

$$f(t, \omega, x, y) \equiv f(t, \omega, x) \quad \textit{does not depend on } y \in \mathbb{R}^d \tag{6.34}$$

and

$$E\left[\sup_{0 \leq t \leq T} f^2(t, \omega, 0) \right] < \infty, \tag{2.1$'$}$$

the BSDE of Problem 2.1 *has a unique solution* (X, Y, K) *and the stochastic game representation* (4.3) *of Theorem* 4.1 *holds, with*

$$g(t, \omega) \equiv f(t, \omega, X(t, \omega)). \tag{3.1$'$}$$

PROOF. For given $\xi \in \mathbf{L}_1^2$ and continuous L, U in $\mathbf{S}_1^2$ satisfying (2.3) and (6.4), Problem 5.1 has a unique solution (X, Y, K) in $\mathbf{S}_1^2 \times \mathbf{H}_d^2 \times \mathbf{S}_1^2$, for every given $g \in \mathbf{S}_1^2$; see Theorem 6.1. Arguing now as in Theorem 5.4 and Corollary 5.5, with the additional observation that the process of (3.1′) belongs to the space $\mathbf{S}_1^2$ [thanks to the assumptions (6.34), (2.1′) and (2.2) on f, as well as the fact $X \in \mathbf{S}_1^2$], we conclude that the BSDE of Problem 2.1 also has a unique solution. □

7. A pathwise approach to Dynkin's game. We shall show in this section that the theory we have developed allows for a pathwise (deterministic) approach to the (stochastic) Dynkin game of Theorem 4.1, an approach analogous to the pathwise treatment of the optimal stopping problem in Davis and Karatzas (1994).

Let (X, Y, K) solve the BSE of Problem 5.1 for given $g \in \mathbf{H}_1^2$ [or, let (X, Y, K) solve the BSDE of Problem 2.1, and then define g in $\mathbf{H}_1^2$ by (3.1)]; we shall assume for simplicity,

$$L(T) = \xi = U(T) \quad \text{a.s.} \tag{7.1}$$

We obtain from (3.8), almost surely,

$$\begin{aligned} &X(t) - L(t) \\ &\quad = \left(\xi - L(t) + \int_t^T g(s)\,ds - (K^-(T) - K^-(t)) - \int_t^T Y'(s)\,dB(s)\right) \\ &\qquad + (K^+(T) - K^+(t)) \\ &\quad \geq 0, \\ &U(t) - X(t) \\ &\quad = \left(U(t) - \xi - \int_t^T g(s)\,ds - (K^+(T) - K^+(t)) + \int_t^T Y'(s)\,dB(s)\right) \\ &\qquad + (K^-(T) - K^-(t)) \\ &\quad \geq 0 \end{aligned} \tag{7.2}$$

for all $0 \leq t \leq T$. In the more suggestive backward notation

$$\begin{aligned} x^+(\theta) &:= X(T-\theta) - L(T-\theta), \\ x^-(\theta) &:= U(T-\theta) - X(T-\theta), \\ a^\pm(\theta) &:= K^\pm(T) - K^\pm(T-\theta), \\ y^+(\theta) &:= \xi - L(T-\theta) + \int_{T-\theta}^T g(s)\,ds - a^-(\theta) - \int_{T-\theta}^T Y'(s)\,dB(s), \\ y^-(\theta) &:= U(T-\theta) - \xi - \int_{T-\theta}^T g(s)\,ds - a^+(\theta) + \int_{T-\theta}^T Y'(s)\,dB(s), \\ &\qquad\qquad 0 \leq \theta \leq T, \end{aligned} \tag{7.3}$$

the equations (7.2) read

$$x^{\pm}(\theta) = y^{\pm}(\theta) + a^{\pm}(\theta) \geq 0, \qquad 0 \leq \theta \leq T, \tag{7.4}$$

and (2.6) reads

$$\int_0^T x^{\pm}(\theta)\, da^{\pm}(\theta) = 0, \tag{7.5}$$

almost surely. Now all processes in (7.3) have continuous paths, and those of $a^{\pm}(\cdot)$ are also increasing with $a^{\pm}(0) = 0$; it follows from (7.4) and (7.5) that $a^{\pm}$ solve the Skorohod reflection problem [e.g., Karatzas and Shreve (1991), page 210] associated with $y^{\pm}$, and thus

$$a^{\pm}(\theta) = \max\left[0, \max_{0\leq u\leq\theta}(-y^{\pm}(u)\right] = \max_{0\leq u\leq\theta}(-y^{\pm}(u)), \qquad 0 \leq \theta \leq T,$$

since $y^{\pm}(0) = 0$, or equivalently

$$\begin{aligned} K^+(T) &= K^+(t) + \max_{t\leq\tau\leq T}\biggl(L(\tau) - \xi - \int_\tau^T g(u)\, du \\ &\qquad\qquad + K^-(T) - K^-(\tau) + \int_\tau^T Y'(u) dB(u)\biggr) \\ &= K^-(T) + K(t) - \xi - \int_t^T g(u)\, du \\ &\qquad + \max_{t\leq\tau\leq T}\biggl(L(\tau) - (K^-(\tau) - K^-(t)) + \int_t^\tau g(u)\, du + \lambda(\tau)\biggr), \end{aligned} \tag{7.6}$$

$$\begin{aligned} K^-(T) &= K^-(t) + \max_{t\leq\sigma\leq T}\biggl(\xi - U(\sigma) + \int_\sigma^T g(u)\, du \\ &\qquad\qquad + K^+(T) - K^+(\sigma) - \int_\sigma^T Y'(u)\, dB(u)\biggr) \\ &= K^+(T) - K(t) + \xi + \int_t^T g(u)\, du \\ &\qquad - \min_{t\leq\sigma\leq T}\biggl(U(\sigma) + (K^+(\sigma) - K^+(t)) + \int_t^\sigma g(u)\, du + \lambda(\sigma)\biggr), \end{aligned} \tag{7.7}$$

where $\Lambda = \{\lambda(t),\ 0 \leq t \leq T\}$ is the continuous, nonadapted process

$$\lambda(t) = M(T) - M(t) \quad \text{with } M(t) := \int_0^t Y'(u)\, dB(u), \qquad 0 \leq t \leq T. \tag{7.8}$$

THEOREM 7.1. *For given $\omega \in \Omega$, consider the pathwise (deterministic) game with payoff*

$$\begin{aligned} Q_t(\sigma, \tau; \omega) &:= \int_t^{\sigma\wedge\tau} g(u, \omega)\, du + \xi(\omega)\mathbf{1}_{\{\sigma\wedge\tau=T\}} + L(\tau, \omega)\mathbf{1}_{\{\tau<T,\, \tau\leq\sigma\}} \\ &\qquad + U(\sigma, \omega)\mathbf{1}_{\{\sigma<\tau\}} + \lambda(\sigma\wedge\tau, \omega) \\ &= R_t(\sigma, \tau; \omega) + \lambda(\sigma\wedge\tau, \omega), \qquad 0 \leq \sigma, \tau \leq T, \end{aligned} \tag{7.9}$$

and upper and lower values

$$\begin{aligned}\overline{W}(t,\omega) &:= \min_{t\le\sigma\le T}\max_{t\le\tau\le T} Q_t(\sigma,\tau,\omega),\\ \underline{W}(t,\omega) &:= \max_{t\le\tau\le T}\min_{t\le\sigma\le T} Q_t(\sigma,\tau,\omega),\end{aligned}\tag{7.10}$$

respectively. Then, for a.e. $\omega\in\Omega$, *the following conditions hold*:

(i) *This game has a value. That is,*

$$W(t,\omega) := \overline{W}(t,\omega) = \underline{W}(t,\omega) = X(t,\omega)+\lambda(t,\omega).\tag{7.11}$$

(ii) *The pair* $(\hat{\sigma}_t(\omega),\ \hat{\tau}_t(\omega))\in[t,T]^2$, *as in* (4.4), *is a saddlepoint for this game. That is,*

$$\begin{aligned}Q_t(\hat{\sigma}_t(\omega),\tau;\omega) &\le Q_t(\hat{\sigma}_t(\omega),\hat{\tau}_t(\omega);\omega) = X(t,\omega)+\lambda(t,\omega)\\ &\le Q_t(\sigma,\hat{\tau}_t(\omega);\omega),\ \forall\,(\sigma,\tau)\in[t,T]^2.\end{aligned}\tag{7.12}$$

(iii) $\hat{\sigma}_t(\omega)$ *attains*

$$\begin{aligned}&\min_{t\le\sigma\le T}\left[U(\sigma,\omega)+K^+(\sigma,\omega)-K^+(t,\omega)+\lambda(\sigma,\omega)+\int_t^\sigma g(u,\omega)\,du\right]\\ &\qquad = X(t,\omega)+\lambda(t,\omega) = W(t,\omega).\end{aligned}\tag{7.13}$$

(iv) $\hat{\tau}_t(\omega)$ *attains*

$$\begin{aligned}&\max_{t\le\tau\le T}\left[L(\tau,\omega)-K^-(\tau,\omega)+K^-(t,\omega)+\lambda(\tau,\omega)+\int_t^\tau g(u,\omega)\,du\right]\\ &\qquad = X(t,\omega)+\lambda(t,\omega) = W(t,\omega).\end{aligned}\tag{7.14}$$

Finally, the value of the stochastic (*Dynkin*) *game of Theorem* 4.1 *is given as the optional projection*

$$V(t) = X(t) = E[W(t)\,|\,\mathscr{F}(t)]\quad a.s.,\tag{7.15}$$

of the value of the pathwise game, for every $0\le t\le T$.

In other words, the effect of the nonadapted compensator process Λ of (7.8) is to enforce the nonanticipativity constraint in the passage from the pathwise to the Dynkin game—to ensure that the random times $\omega\mapsto\hat{\sigma}_t(\omega)$, $\omega\mapsto\hat{\tau}_t(\omega)$ in the saddle-point $(\hat{\sigma}_t(\omega),\hat{\tau}_t(\omega))$ of the pathwise game are stopping times. Then the pair $(\hat{\sigma}_t,\hat{\tau}_t)\in(\mathscr{M}_{t,T})^2$ provides a saddlepoint for the Dynkin game [cf. (4.5)], and the value of this game is simply the conditional expectation of the value for the pathwise game, given $\mathscr{F}(t)$ [cf. (7.15)].

On the other hand, the pair $(\hat{\sigma}_t(\omega), \hat{\tau}_t(\omega)) \in [t, T]^2$ is also a saddle-point for yet another pathwise game, this one with payoff

$$\frac{1}{2}\left[U(\sigma, \omega) + K^+(\sigma, \omega) - K^+(t, \omega) + \lambda(\sigma, \omega) + \int_t^\sigma g(u, \omega)\, du\right]$$
$$+ \frac{1}{2}\left[L(\tau, \omega) - (K^-(\tau, \omega) - K^-(t, \omega)) + \lambda(\tau, \omega) + \int_t^\tau g(u, \omega)\, du\right],$$
$$t \le \sigma, \tau \le T,$$

separable in the two variables σ and τ; from (7.13) and (7.14), the value of this game coincides with the value $W(t, \omega) = X(t, \omega) + \lambda(t, \omega)$ of the original, pathwise game of (7.9)–(7.11).

PROOF OF THEOREM 7.1. The relations of (7.12) follow directly from (4.8), (4.10) and the definition (7.9) whereas (7.11) is a simple consequence of (7.12). On the other hand, (7.15) follows from (7.11), (4.3) and the martingale property of the process M in (7.8) since

$$E[\lambda(t)\big|\mathscr{F}(t)] = E[M(T) - M(t)\big|\mathscr{F}(t)] = 0 \quad \text{a.s.}$$

It remains to establish (7.13) and (7.14). Let us drop the dependence on $\omega \in \Omega$ in the notation of what follows. Observe that equation (5.1) reads

$$X(t) + \lambda(t) = X(\rho) + \int_t^\rho g(u)\, du + K(\rho) - K(t) + \lambda(\rho) \quad \text{for } t \le \rho \le T \tag{7.16}$$

in our notation of (7.8) and recall

$$\begin{aligned} &X(\hat{\sigma}_t) = U(\hat{\sigma}_t), \qquad K^-(\hat{\sigma}_t) = K^-(t) \quad \text{and} \\ &X(\hat{\tau}_t) = L(\hat{\tau}_t), \qquad K^+(\hat{\tau}_t) = K^+(t) \end{aligned} \tag{7.17}$$

from (4.4), (2.6) and (7.1). Now the first claim in (7.14) is equivalent to

$$\begin{aligned} &L(\tau) - K^-(\tau) + K^-(t) + \lambda(\tau) + \int_t^\tau g(u)\, du \\ &\quad \le L(\hat{\tau}_t) - K^-(\hat{\tau}_t) + K^-(t) + \lambda(\hat{\tau}_t) + \int_t^{\hat{\tau}_t} g(u)\, du \qquad \forall\ \tau \in [t, T] \end{aligned} \tag{7.14$'$}$$

and, from (7.16) and (7.17), the right-hand side of (7.14$'$) equals

$$\begin{aligned} &X(\hat{\tau}_t) + K(\hat{\tau}_t) - K(t) + \lambda(\hat{\tau}_t) + \int_t^{\hat{\tau}_t} g(u)\, du \\ &\quad = X(t) + \lambda(t) = X(\tau) + K(\tau) - K(t) + \lambda(\tau) + \int_t^\tau g(u)\, du; \end{aligned}$$

this clearly dominates $L(\tau) - K^-(\tau) + K^-(t) + \lambda(\tau) + \int_t^\tau g(u)\, du$, establishing (7.14$'$). Similarly, the first claim in (7.13) amounts to

$$\begin{aligned} &U(\sigma) + K^+(\sigma) - K^+(t) + \lambda(\sigma) + \int_t^\sigma g(u)\, du \\ &\quad \ge U(\hat{\sigma}_t) + K^+(\hat{\sigma}_t) - K^+(t) + \lambda(\hat{\sigma}_t) + \int_t^{\hat{\sigma}_t} g(u)\, du \qquad \forall\ \sigma \in [t, T]; \end{aligned} \tag{7.13$'$}$$

again from (7.16) and (7.17), the right-hand side of (7.13′) equals

$$X(\hat\sigma_t) + K(\hat\sigma_t) - K(t) + \lambda(\hat\sigma_t) + \int_t^{\hat\sigma_t} g(u)\,du$$
$$= X(t) + \lambda(t) = X(\sigma) + K(\sigma) - K(t) + \lambda(\sigma) + \int_t^{\sigma} g(u)\,du,$$

which is dominated by $U(\sigma)+K^+(\sigma)-K^+(t)+\lambda(\sigma)+\int_t^\sigma g(u)\,du$, establishing (7.13′). Finally, the equalities in (7.13) and (7.14) are consequences of (7.7) and (7.6), respectively, as well as of (5.1) and (7.11). □

APPENDIX

Consider a complete probability space $(\Omega, \mathscr{F}, P)$ equipped with a filtration $\mathbf{F} = \{\mathscr{F}(t)\}_{0\le t\le T}$ which satisfies $\mathscr{F}_0 = \{\varnothing, \Omega\} \bmod P$, as well as the usual conditions of right-continuity and augmentation by P-null sets. On this space, let $Z = \{Z(t),\ 0 \le t \le T\}$ be a potential, that is, an $\mathbf{F}$-supermartingale with paths which are nonnegative, RCLL [right-continuous on $[0, T)$, with left-limits on $(0, T]$] and satisfy $Z(T) = 0$, almost surely. If this potential is of class $\mathscr{D}([0, T])$, that is, if

(A.1) $\qquad$ the family $\{Z(\tau)\}_{\tau\in\mathscr{M}}$ is uniformly integrable,

where $\mathscr{M}$ is the class of $\mathbf{F}$-stopping times τ: $\Omega \to [0, T]$, then there exists a unique natural increasing process $A = \{A(t),\ 0 \le t \le T\}$, adapted to $\mathbf{F}$, with right-continuous paths and $A(0) = 0$, $EA(T) < \infty$, such that Z is indistinguishable from the potential

$$\pi_t(A) := E[A(T)|\mathscr{F}(t)] - A(t), \qquad 0 \le t \le T \tag{A.2}$$

generated by A:

$$Z(t) = \pi_t(A), \qquad \forall\ 0 \le t \le T \tag{A.3}$$

almost surely. Furthermore, if Z is regular in the sense

$$\lim_{n\to\infty} EZ(\tau_n) = EZ(\tau) \quad \text{for any } \tau \in \mathscr{M},\ \{\tau_n\} \subset \mathscr{M} \text{ with } \tau_n \uparrow \tau, \quad \text{a.s.,} \tag{A.4}$$

then A has continuous paths. This is the classical Doob–Meyer decomposition of supermartingales and is well known [e.g., Section 1.4 in Karatzas and Shreve (1991)].

The following result is also known [cf. Dellacherie and Meyer (1975), (1980), El Karoui, Kapoudjian, Pardoux, Peng and Quenez (1995)], but we include a proof for the reader's convenience.

LEMMA A.1. (i) *Let A be a continuous, adapted and increasing process, with $A(0) = 0$ and $E[A^2(T)] < \infty$; then the potential $Z = \pi(A)$ of* (A.2) *is regular and satisfies*

$$E\Big[\sup_{0\le t\le T} Z^2(t)\Big] < \infty. \tag{A.5}$$

(ii) *Conversely, let Z be a regular potential that satisfies* (A.5); *then the continuous increasing process A of its Doob–Meyer decomposition* (A.3) *satisfies* $E[A^2(T)] < \infty$.

PROOF. (i) From (A.2) and Doob's maximal inequality [e.g., Karatzas and Shreve (1991)] we have

$$E\Big[\sup_{0\le t\le T} \pi_t^2(A)\Big] \le E\Big[\sup_{0\le t\le T} M^2(t)\Big] \le 4E[M^2(T)] = 4E[A^2(T)] < \infty,$$

where $M(t) := E[A(T)|\mathscr{F}(t)]$, $0 \le t \le T$. Regularity follows easily.

(ii) The condition (A.5) implies (A.1). That is, that Z is of class $\mathscr{D}[0, T]$. Together with regularity, this gives a unique Doob–Meyer decomposition (A.3) with A adapted, increasing and continuous; in particular,

$$\begin{aligned} A(T) &= \lim\nolimits_{n\to\infty} \uparrow A(\rho_n), \quad \text{where} \\ \rho_n &:= \inf\{t \in [0, T)/|A(t) \ge n\} \wedge T \in \mathscr{M}. \end{aligned} \tag{A.6}$$

Now

$$\begin{aligned} E[A^2(\rho_n)] &= 2E\int_0^{\rho_n} (A(\rho_n) - A(t))\, dA(t) \\ &= 2E\int_0^{\rho_n} E[A(\rho_n) - A(t)|\mathscr{F}(t)]\, dA(t) \\ &= 2E\int_0^{\rho_n} E[Z(t) - Z(\rho_n)|\mathscr{F}(t)]\, dA(t) \le 2E\int_0^{\rho_n} Z(t)\, dA(t) \\ &\le 2E\Big[\sup_{0\le t\le T} Z(t)\cdot A(\rho_n)\Big] \le 2\sqrt{E\Big[\sup_{0\le t\le T} Z^2(t)\Big] E[A^2(\rho_n)]}, \end{aligned}$$

whence $E[A^2(\rho_n)] \le 4E[\sup_{0\le t\le T} Z^2(t)] := c < \infty$; letting $n \to \infty$, we obtain, from (A.6) and monotone convergence, $E[A^2(T)] \le c < \infty$.

DEFINITION A.2. Denote by $\mathbf{\Pi}_c^2$ the space of potentials Z which have continuous paths and satisfy (A.5). Denote also by $\mathbf{S}_{\mathrm{ci}}^2$ the space of continuous, increasing, adapted processes A with $A(0) = 0$ and $E[A^2(T)] < \infty$.

COROLLARY A.3. *Suppose that $\mathbb{F}$ coincides with the filtration $\mathbf{F}^B = \{\mathscr{F}^B(t)\}_{0\le t\le T}$ generated by some standard* (d-*dimensional*) *Brownian motion process B. Then the spaces $\mathbf{\Pi}_c^2$, $\mathbf{S}_{\mathrm{ci}}^2$ of Definition* A.2 *can be put into a one-to-one correspondence via*

$$Z = \pi(A); \qquad A \in \mathbf{S}_{\mathrm{ci}}^2, \qquad Z \in \mathbf{\Pi}_c^2.$$

The proof follows directly from Lemma A.1 and the fact that $\mathbf{F}^B$- (Brownian) martingales are representable as stochastic integrals with respect to B (and have, thus, continuous paths).

For the remainder of this section, let $\eta\colon [0, T]\times\Omega \mapsto \mathbb{R}$ be a given, $\mathbf{F}$-adapted process with RCLL paths and $\eta(T) = 0$, a.s. and assume

$$\eta^* := \sup_{0\le t\le T} |\eta(t)| \in \mathbf{L}^1(\Omega). \tag{A.7}$$

The Snell envelope of η is the process $S(\eta)$ given as

$$S_t(\eta) := \operatorname*{ess\,sup}_{\tau\in\mathscr{M}_{t,T}} E[\eta(\tau)|\mathscr{F}(t)], \qquad 0\le t\le T, \tag{A.8}$$

where

$$\mathscr{M}_{t,\theta} := \{\tau\in\mathscr{M} \;/\; t\le\tau\le\theta \text{ a.s.}\} \quad \text{for } 0\le t\le\theta\le T. \tag{A.8$'$}$$

Then $S(\eta)$ is a potential, of class $\mathscr{D}[0, T]$ (as it is dominated by the martingale $E[\eta^*|\mathscr{F}(t)]$, $0 \le t \le T$), and is the smallest nonnegative supermartingale dominating η. Thus, it has a Doob–Meyer decomposition of the form

$$S_t(\eta) = E[A(T)|\mathscr{F}(t)] - A(t), \qquad 0\le t\le T, \tag{A.9}$$

for a unique natural increasing and adapted process $A = A^\eta$ with right-continuous paths and $A(0) = 0$, $EA(T) < \infty$. Furthermore,

$$\begin{cases} S(\eta) \text{ is regular, thus } A \text{ has continuous paths, if } \eta \text{ is quasi-} \\ \text{left-continuous: } \limsup_{n\to\infty} \eta(\tau_n) \le \eta(\tau) \text{ a.s., for every} \\ \{\tau_n\}\subset\mathscr{M} \text{ with } \tau_n\uparrow\tau, \text{ a.s.} \end{cases} \tag{A.10}$$

In terms of the Snell envelope of (A.8), we can obtain the solution of the optimal stopping problem

$$u(t) := \sup_{\tau\in\mathscr{M}_{t,T}} E\eta(\tau), \qquad 0\le t\le T, \tag{A.11}$$

associated with η, as follows:

$$u(t) = ES_t(\eta), \qquad 0\le t\le T.$$

$$\begin{cases} \text{The stopping time } \rho_t^* := \inf\{u\in[t, T)/\eta(u) = S_u(\eta)\}\wedge T \\ \text{attains the supremum in (A.11), and is thus optimal for this} \\ \text{problem: } u(t) = E\eta(\rho_t^*). \end{cases} \tag{A.12}$$

If, furthermore, η is quasi-left-continuous, so that A continuous by (A.10), then we also have

$$\int_0^T (S_t(\eta) - \eta(t))\, dA(t) = 0 \quad \text{a.s.} \tag{A.13}$$

(in other words, A increases only on $\{S = \eta\}$) and

$$u(t) = E\eta(\nu_t^*), \quad \text{where } \nu_t^* := \inf\{u\in[t, T)/A(u) > A(t)\}\wedge T \ge \rho_t^*. \tag{A.14}$$

All these results are standard in the general theory of optimal stopping [e.g., Neveu (1975), El Karoui (1981) and Karatzas (1993)]. We shall also need the following properties.

LEMMA A.4. *If η has quasi-left-continuous paths and $E[\sup_{0\le t\le T}\eta^2(t)] < \infty$, then we have $E[A^2(T)] < \infty$ for the continuous, increasing process of* (A.9).

PROOF. In the notation of (A.7), we have then

$$E\left(\sup_{0\le t\le T} S_t^2(\eta)\right) \le E\left[\sup_{0\le t\le T}(E[\eta^*|\mathscr{F}(t)])^2\right] \le 4E(\eta^*)^2 < \infty \tag{A.15}$$

from Doob's maximal inequality; the result follows then from (A.10) and Lemma A.1(ii).

LEMMA A.5. *If η, $\{\eta_n\}_{n\in\mathbb{N}}$ are adapted, RCLL processes with $\eta(T) = \eta_n(T) = 0$ and $\eta_n(t) \uparrow \eta(t)$, $\forall\ 0 \le t \le T$ almost surely, as well as $E[\sup_{0\le t\le T}(|\eta(t)| + |\eta_n(t)|)] < \infty$ for all $n \in \mathbb{N}$, then*

$$S_t(\eta_n) \uparrow S_t(\eta), \quad \forall\ 0 \le t \le T \quad a.s.$$

PROOF. Clearly $S_n := S(\eta_n) \ge \eta_n$ and $S_{n+1} = S(\eta_{n+1}) \ge S(\eta_n) = S_n$ for all $n \in \mathbb{N}$, so $\{S_n\}_{n\in\mathbb{N}}$ is an increasing (pointwise) sequence of nonnegative, RCLL supermartingales, of class $\mathscr{D}[0, T]$; it converges to a nonnegative, RCLL supermartingale $S := \lim_n \uparrow S_n$ [e.g., Karatzas and Shreve (1991), page 21]. Clearly $S \ge \eta$, and thus $S \ge S(\eta)$ as well; on the other hand, $\eta_n \le \eta$ implies $S_n = S(\eta_n) \le S(\eta)$, whence $S \le S(\eta)$.

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