BAER AND QUASI-BAER MODULES

DISSERTATION

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By

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ABSTRACT

We introduce the notions of the Baer and the quasi-Baer properties in a general module theoretic setting. A module M is called (quasi-) Baer if the right annihilator of a (two-sided) left ideal of $End(M)$ is a direct summand of M. We show that a direct summand of a (quasi-) Baer module inherits the property. Every finitely generated abelian group is Baer exactly if it is semisimple or torsion-free. Close connections to the extending property and the FI-extending property are exhibited and it is shown that a module M is (quasi-) Baer and (FI-) $\mathcal{K}$-cononsingular if and only if it is (FI-) extending and (FI-) $\mathcal{K}$-nonsingular. While we show that direct sums of (quasi-) Baer modules are not (quasi-) Baer, we prove that an arbitrary direct sum of mutually subisomorphic quasi-Baer modules is quasi-Baer and that every free (projective) module over a quasi-Baer ring is always a quasi-Baer module. Some results, related to direct sums of Baer modules and direct sums of quasi-Baer modules, are also included. A ring over which every module is Baer is shown to be precisely a semisimple Artinian ring. Among other results, we also show that the endomorphism ring of a (quasi-) Baer module is a (quasi-) Baer ring, while the converse is not true in general. A characterization for this to hold in the Baer modules case is obtained. We provide a type theory of Baer modules and decomposition of a Baer module into into five types, similar to the one provided by Kaplansky for the Baer rings case.

This type theory and type decomposition is applied, in particular, to all nonsingular extending modules. Applications of the results obtained are included.

This work is dedicated to the memory of my father.

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CHAPTER 1

INTRODUCTION

1.1 Background and Motivation

The notions of Baer and quasi-Baer rings have their roots in functional analysis. For example, von Neumann algebras, such as the *-algebra of bounded operators on a Hilbert space containing the identity operator which are closed under the weak operator topology (and are also called W^*-algebras), possess a plethora of structures - algebraic, geometric, topological. For an algebraist, a boon is the rich supply of idempotents which these algebras have. In order to obtain an insight into the theory of von Neumann algebras, several authors started to axiomatize this theory, including S.W.P. Steen, I.M. Gel'fand, M.A. Naĭmark, C.E. Rickart and von Neumann. Algebraically, in any von Neumann algebra (i.e. W^*-algebra) the right annihilator of any subset is generated as a right ideal by a projection (i.e. a self-adjoint idempotent with respect to the involution *). Kaplanksy [20], in 1951, defined the concept of abstract W^*-algebras, or AW^*-algebras, which took into account mainly the algebraic structure of von Neumann algebras (AW^*-algebras are Banach algebras with an involution such that $\|xx^*\| = \|x\|^2$ and which have the property that the right annihilator of any subset is generated by a projection). He also made the connection with von

Neumann's study of continuous geometries, by noticing that the projection lattice of a 'directly finite' AW^*-algebra is a continuous geometry [21]. Kaplansky in 1955 [22] defined the larger class of Baer *-rings by focusing on annihilators and projections of AW^*-algebras. A Baer *-ring is defined as a ring with involution in which the right annihilator of every subset (or left ideal) is a principal right ideal generated by a projection. The name honors Reinhold Baer, who studied this condition earlier in his book "Linear Algebra and Projective Geometry". Dropping the assumption of an involution in this definition, led Kaplansky to the concept of a Baer ring.

A *Baer ring* is defined as a ring in which the right annihilator of any left ideal (or subset) is a right ideal, generated by an idempotent. A number of very interesting properties of Baer rings were shown by Kaplansky and further investigated by several other mathematicians. Examples of Baer rings include right self-injective von Neumann regular rings, von Neumann algebras, any domain (with a unit element) and the endomorphisms rings of semisimple modules (thus, endomorphisms rings of all vector spaces). The concept of Baer rings was generalized to that of *quasi-Baer rings* by W. E. Clark [13] in 1967 by replacing the 'left ideal' by a 'two-sided ideal' in the above definition. Examples of quasi-Baer rings include all prime rings, and rings of matrices over Baer rings. It is easy to see that the Baer and quasi-Baer properties are left-right symmetric for any ring. Large classes of rings satisfy the Baer and the quasi-Baer properties, respectively. On the other hand, these two concepts are distinct: for example, a prime ring with a nonzero singular ideal is quasi-Baer but not Baer, and the n-by-n upper triangular matrix ring over a domain which is not a division ring, is a quasi-Baer ring which is not Baer. An important fact that makes the quasi-Baer rings useful is that the quasi-Baer property is a Morita invariant property, while the

Baer property is not. The theory of Baer and quasi-Baer rings has come to play an important role and major contributions to this theory have been made in recent years, providing a number of interesting results on Baer and quasi-Baer properties in the ring-theoretical setting. Some of the contributors include S.K. Berberian, G. F. Birkenmeier, A. W. Chatters, S. M. Khuri, J. Y. Kim, Y. Hirano, J. K. Park, A. Pollingher, K.G. Wolfson and A. Zaks, among others (see, for example, [38], [33], [31], [12], [9], [10], [3], [4], [6], [5], [7]).

Not much is known about these properties in a general module-theoretic setting. For example, a natural question that can be asked is: if $e^2 = e$ is an idempotent in a (quasi-) Baer ring R, then does the right R-module eR possess any 'kind' of Baer or quasi-Baer properties?

Recall that a module is *extending* (or CS), if every submodule is essential in a direct summand. This simple property is satisfied by every (quasi-) injective module. Since late 1980s, the development of the extending module theory has been a major area of research interest in Ring Theory. Contributors include M. Harada and his school in Japan, B. Müller and his collaborators in Canada, B. Osofsky, P.F. Smith, D.V. Huynh, N.V. Dung, R. Wisbauer and many others in various parts of the world. Even with numerous papers published in the last two decades related to this theory, a number of open problems remain. Although this generalization of injectivity is very useful, it does not satisfy some important algebraic properties. For example, it is not known when direct sums of extending modules are extending, or when full or upper triangular matrix rings over right extending rings are right extending (we know that these properties in general do not hold true even for finite direct sums or matrix rings). Much work has been done on finding necessary and sufficient conditions to ensure that the extending property is preserved under various extensions, but with only limited success and a full general characterization has not yet been found.

In 1980, A.W. Chatters and S.M. Khuri [12], showed that there are close connections between the Baer rings and the right extending rings. In particular, they proved that every right nonsingular right extending ring can be characterized as a Baer ring which is right cononsingular (Theorem 2.2.1). Another question that can be asked is: can we provide a similar or analogous characterization for nonsingular extending *modules* instead of *rings*?

The study of Baer property for rings (and now proposed for modules) provides a possible new approach to investigations on the extending property, given the close connections between these two properties. Until recently, there were no satisfactory connections made in the general module-theoretical setting, which we now propose to do.

Recently, the concept of FI-extending property for modules was discussed for abelian groups in [8] and introduced, in general module-theoretical setting in [9]. An *FI-extending module* is defined to be the one for which every fully invariant submodule is essential in a direct summand. Obviously, the FI-extending property generalizes the concept of extending property. One advantage of this generalization of the extending property over various other generalizations is that the underpinnings (i.e., the fully invariant submodules) form a complete modular sublattice of the lattice of submodules and are well behaved with respect to endomorphisms. Moreover, the lattice connection naturally follows the lattice theoretic view that was originally indicated in von Neumann's formulation of continuous geometries [32] and Utumi's formulation of continuous (regular) rings [36]. The class of fully invariant submodules includes many of the most significant submodules of a module (e.g., the Jacobson radical, the socle, the singular submodule, etc.). For a ring R, the fully invariant submodules of R_R are precisely the two-sided ideals of R.

Another interesting connection was established by G.F. Birkenmeier, B. Müller and S.T. Rizvi in 2002 [9], this time between the quasi-Baer rings and the FI-extending rings. It was shown that nonsingular, FI-extending rings satisfy the quasi-Baer property, and can also be studied from this point of view. If the ring is semiprime then the two properties coincide (Theorem 4.7 in [9]). However, for an arbitrary ring, the characterization of FI-extending rings in terms of quasi-Baer rings remains open. In addition, no module-theoretical analogue of such a result has been proposed until recently. A question that can be asked is: can one provide a similar characterization for nonsingular FI-extending modules?

In our work, we introduce the notions of the Baer and the quasi-Baer properties for arbitrary modules. Our definitions and techniques allow us to develop a theory which not only helps provide answers to some of the preceding questions, but also enables us to work in a general module theoretic setting in which a number of results and their extensions can be proved efficiently. We believe that these concepts may prove useful in providing answers to some of the open problems in the theory of extending modules or help prove results in a more general setting.

Let M be an R-module and $S = End_R(M)$. We call M a *Baer module* if the right annihilator in M of any left ideal of S is generated by an idempotent of S. M is called a *quasi-Baer* module if the right annihilator in M of any ideal of S is generated by an idempotent of S. It is easy to see that, when $M = R_R$, the two notions coincide with the existing definitions of Baer and quasi-Baer rings, respectively. Any (quasi-) Baer ring R is (quasi-) Baer as an R-module. To exhibit examples of Baer modules we show that this property is satisfied by any nonsingular extending module, any semisimple $\mathbb{Z}$-module, any finitely generated torsion-free $\mathbb{Z}$-module, and any right ideal direct summand of a Baer ring. We can also show that for any extending module M, $M/Z_2(M)$ is always a Baer module, where $Z_2(M)$ is the second singular submodule of M. Examples of quasi-Baer modules include any projective R-module (in particular, any right ideal summand of R) over a quasi-Baer ring R, any nonsingular FI-extending module and any torsion-free abelian group.

1.2 Summary

We begin Chapter 1 with motivation and background. After providing a summary of the dissertation we include preliminary definitions and known results to be used later.

In Chapter 2 we define the concept of a Baer module, and provide examples. We prove a characterization of Baer modules based on strong summand intersection property. We introduce the concepts of $\mathcal{K}$-nonsingularity and $\mathcal{K}$-cononsingularity, which are closely linked to Baer modules and extending modules. Characterizations of the two nonsingularity concepts introduced are proved, and we present examples illustrating these properties. The main result of this chapter shows that a Baer, $\mathcal{K}$-cononsingular module is precisely an extending module that is $\mathcal{K}$-nonsingular. This result is a module-theoretic analogue of the Chatters and Khuri result in [12] (Theorem 2.1). Applications of this result are included. The question whether the Baer property for modules transfers to direct summands and direct sums is investigated. We show that direct summands inherit the Baer property, and give counterexamples for the case of direct sums. How the two nonsingularities introduced behave with respect to direct summands and direct sums is another question studied. We provide conditions for a direct sum of Baer modules to be Baer. Finitely generated abelian groups that are Baer modules as $\mathbb{Z}$-modules are fully characterized as semisimple or torsion-free $\mathbb{Z}$-modules. We study indecomposable Baer modules, and indecomposable decompositions of Baer modules. Chapter 2 concludes with a result that shows that a ring R, for which all right R-modules are Baer, is semisimple Artinian ring.

Chapter 3 focuses on the study of quasi-Baer modules. The notions of FI-$\mathcal{K}$-nonsingularity and FI-$\mathcal{K}$-cononsingularity are introduced to obtain the connections of quasi-Baer modules to the FI-extending modules, analogous to the result of Chatters-Khuri for Baer rings ([12]). We show that direct summands of quasi-Baer modules inherit the property, and that a direct sum of relatively subisomorphic quasi-Baer modules (in particular, a direct sum of copies of a quasi-Baer module) is always quasi-Baer. Thus, any projective module over a quasi-Baer ring is a quasi-Baer module. Some conditions necessary for direct sums of arbitrary quasi-Baer modules to be quasi-Baer are also provided.

The aim of our investigation in Chapter 4 is on the endomorphism rings of modules with Baer property. Since we defined the Baer and quasi-Baer modules in terms of their idempotent endomorphisms, it is of interest to investigate connections of the properties of endomorphisms rings with those of the underlying module. In particular, we investigate the transfer of some properties between a module and its endomorphism ring. A characterization for the module to be Baer is provided in terms of its endomorphism ring. Kaplansky ([22]) introduced a type theory for Baer rings. The type theory was further developed and extended by Goodearl ([16], [17]) for self-injective regular rings and for nonsingular injective modules. In this chapter, we also use the properties of idempotents of the endomorphism ring of a Baer module to provide a type theory for Baer modules, similar to the type theory for Baer rings.

The dissertation concludes with Appendix A. A topology is shown to exist on a Baer module M with the property that every endomorphisms of M is continuous in the topology.

1.3 Preliminaries

All the rings are assumed to be with unit, and not necessarily commutative. The modules are unital right modules. We usually denote the base ring by R, the module by M and its endomorphism ring by $S = End_R(M)$. The notation $End(M)$ will be used instead of $End_R(M)$ when there is no danger of confusion.

The right annihilator of $X \subseteq M$ in R (i.e. all elements $r \in R$ so that $Xr = 0$) is denoted by $r_R(X)$, the left annihilator of $X \subseteq M$ in S (i.e. all elements $\varphi \in S$ so that $\varphi X = 0$) is denoted by $l_S(X)$; the right annihilator of $T \subseteq S$ in M (i.e. all elements $m \in M$ so that $Tm = 0$) is denoted by $r_M(T)$ and the left annihilator of $P \subseteq R$ in M (i.e. all elements $m \in M$ so that $mP = 0$) is denoted by $l_M(P)$.

Notation: $N \leq^e M$ means N is *essential* in M, i.e. $N \cap P \neq 0$, $\forall\ 0 \neq P \leq M$; $N \leq^c M$ means N is *essentially closed* in M, i.e. if $\nexists\ N \neq P \leq M$ with $N \leq^e P$; $N \leq^{\oplus} M$ means N is *direct summand* of M; $N \trianglelefteq M$ means N is *fully invariant* in M (i.e. $\forall\ \varphi \in End(M)$, $\varphi(N) \subseteq N$); $N \trianglelefteq^{\#} M$, where $\#$ stands for e, c and, respectively, $\oplus$, means N is fully invariant and essential, closed and respectively direct summand submodule of M.

Definition 1.3.1. A module M is called an *extending module* if, for any $N \leq M$, there exists a direct summand $N' \leq^{\oplus} M$ such that $N \leq^{e} N'$.

Definition 1.3.2. A module M is called an *FI-extending module* if, for any $N \trianglelefteq M$, there exists a direct summand $N' \leq^{\oplus} M$ such that $N \leq^{e} N'$.

Definition 1.3.3. A ring R is called a *Baer ring* if the right annihilator in R of any left ideal is generated, as a right ideal, by an idempotent element of R (in other words, for all $I \leq {}_RR$, $r_R(I) = eR$ where $e^2 = e \in R$).

Definition 1.3.4. A ring R is called a *quasi-Baer ring* if the right annihilator in R of any two-sided ideal is generated, as a right ideal, by an idempotent element of R (for all $I \trianglelefteq R$, $r_R(I) = eR$, where $e^2 = e \in R$).

Remark 1.3.5. The Baer and quasi-Baer properties for rings are left-right symmetric: a ring R is a (quasi-) Baer ring if and only if the left annihilator in R of any (two-sided) right ideal is generated, as a left ideal, by an idempotent element of R.

Definition 1.3.6. A module M_R is called nonsingular if the *singular submodule of* M, $Z(M) = \{m \in M | mI = 0, \text{ where } I \leq^{e} R_R\} = 0$. A ring R is right nonsingular if R_R is nonsingular.

Definition 1.3.7. The *second singular submodule of* M, denoted by $Z_2(M)$, is the submodule of M containing $Z(M)$, so that $Z_2(M)/Z(M) = Z(M/Z(M))$.

Definition 1.3.8. A ring R is called *right cononsingular* if any right ideal, with zero left annihilator, is essential in R_R.

Definition 1.3.9. A ring R is called *right Utumi* if R is right nonsingular and right cononsingular. Equivalently, R is right Utumi ring if, for some $I \leq R$, $I \leq^{e} R_R$ if and only if $rI \neq 0$, $\forall\, r \in R$.

Definition 1.3.10. An idempotent $e^2 = e \in R$ is called a *left* (respectively, *right*) *semicentral idempotent* if eR (respectively, Re) is a two-sided ideal of R.

Definition 1.3.11. An idempotent $e^2 = e \in R$ is called a *central idempotent* if e commutes with every element of R. Equivalently, e is central if eR and $(1-e)R$ are both two-sided ideals of R.

Definition 1.3.12. Let M and N be two R-modules. We say that M is N*-injective* if, $\forall\ N' \leq N$ and $\forall\ \varphi : N' \to M$, $\exists\ \overline{\varphi} : N \to M$ such that $\overline{\varphi}|_{N'} = \varphi$.

Definition 1.3.13. A module Q that is M-injective, $\forall\ M$ R-module, is called *injective.*

Notation. For a module M, we denote by $E(M)$ its injective hull, i.e., $E(M)$ is an injective module such that $M \leq^e E(M)$. The existence and uniqueness up to isomorphism of injective hulls are known.

An interesting consequence of relative injectivity is the way it characterizes direct summands of a direct sum of two modules, when one is relative injective to the other (for example see [14]).

Lemma 1.3.14. *Let M and N be R-modules, so that M is N-injective. Then, if $P \leq M \oplus N$ so that $P \cap M = 0$, $\exists\ \overline{P} \leq^{\oplus} M \oplus N$ so that $P \leq \overline{P}$ and $M \oplus \overline{P} = M \oplus N$.*

Proof. If $M \cap P = 0 \Rightarrow \pi_N(p) = 0$ with $p \in P$ implies $p \in P \cap M = 0 \Rightarrow p = 0$ (where π_N is the canonical projection of $M \oplus N$ onto N). Hence we can define the following morphism, $\varphi : \pi_N(P) \to \pi_M(P)$ (where π_M is the canonical projection of $M \oplus N$ onto M) by $\varphi(\pi_N(p)) = \pi_M(p)$. This is a well-defined function as 0 can only be mapped in 0; it is easy to check that it is also a morphism. By N-injectivity of M, this morphism can be extended to a morphism $\overline{\varphi} : N \to M$, and we can construct the following submodule of $M \oplus N$: $\overline{P} = \{n + \overline{\varphi}(n) | n \in N\}$ (it is easy to check that it is, in fact, a submodule). $P \subseteq \overline{P}$ since $\overline{\varphi}$ extends φ. Also note that $\overline{P} \cap M = 0$, since $n + \overline{\varphi}(n) \in M \Rightarrow n = 0 \Rightarrow \overline{\varphi}(n) = 0$. Since $N \subseteq M + \overline{P} \Rightarrow M \oplus N = M + \overline{P} \Rightarrow M \oplus N = M \oplus \overline{P}$. □

Lemma 1.3.15. *If M_i is N injective, for $i = 1, \ldots, n$ ($n \in \mathbb{N}$), then $\bigoplus_{i \leq n} M_i$ is N-injective. If N is M_i injective, for $i = 1, \ldots, n$. then N is $\bigoplus_{i \leq n} M_i$-injective.*

The next lemma will be useful.

Lemma 1.3.16. *For $N \leq M$, $I \leq R_R$, $K \leq {}_S S$, $P \trianglelefteq M$, $J \trianglelefteq R$, $L \trianglelefteq S$, the following hold:*

1. $l_M(r_R(l_M(I))) = l_M(I)$
2. $r_R(l_M(r_R(N))) = r_R(N)$
3. $l_S(r_M(l_S(N))) = l_S(N)$
4. $r_M(l_S(r_M(K))) = r_M(K)$
5. $l_M(J) \trianglelefteq M$
6. $r_R(P) \trianglelefteq R$
7. $l_S(P) \trianglelefteq S$
8. $r_M(L) \trianglelefteq M$.

Proof. It is well-known that the pairs $r_R(\cdot)$-$l_M(\cdot)$, respectively $l_S(\cdot)$-$r_M(\cdot)$ are Galois pairs, hence equalities 1 through 4 hold true (for example, see [1]).

For assertion 5 we observe that, in general, $l_M(J) \leq {}_S M$. On the other hand, if $J \trianglelefteq R \Rightarrow rJ \subseteq J$, and so, if $m \in l_M(J)$, $mr \in l_M(J)$: $mrJ \subseteq mJ = 0$. Hence $l_M(J) \trianglelefteq M$. The last three statements follow similarly. $\square$

Next, we provide below some results concerning fully invariant submodules, to be used later.

Lemma 1.3.17. *Let M be a module, and let $M = M_1 \oplus M_2$ be a direct sum decomposition. If $N \trianglelefteq M$ then $N = N_1 \oplus N_2$, where $N_i = N \cap M_i \trianglelefteq M_i$, for $i = 1, 2$.*

Proof. Let π_i be the canonical projection of M onto M_i, for $i = 1, 2$. Since $N \trianglelefteq M$, $\pi_i(N) \subseteq N$, and so $\pi_i(N) = N \cap M_i = N_i$, for $i = 1, 2$. Hence $N \subseteq \pi_1(N) + \pi_2(N) = N_1 + N_2$. But since $N_i \subseteq N$, for $i = 1, 2$, $N_1 + N_2 \subseteq N$. As $N_1 \cap N_2 = N \cap M_1 \cap M_2 = 0$ we get that $N = N_1 \oplus N_2$. $\square$

Lemma 1.3.18. *Let M be a module, and let $M = M_1 \oplus M_2$ be a direct sum decomposition so that $M_1, M_2 \trianglelefteq M$. If $N \leq^{\oplus} M$ then $N = N_1 \oplus N_2$, where $N_i = N \cap M_i$, for $i = 1, 2$.*

Proof. Let $M = N \oplus N'$. Then, by Lemma 1.3.17, $M_i = M_i' \oplus M_i''$, where $M_i' = M_i \cap N$ and $M_i'' = M_i \cap N'$, for $i = 1, 2$. Clearly, $M_1' \oplus M_2' \subseteq N$. Then, $\forall\, n \in N$, $n = n_1 \oplus n_2$, $n_1 = n_1' + n_1''$ and $n_2 = n_2' + n_2''$, where $n_i \in M_i$, $n_i' \in M_i'$ and $n_i'' \in M_i''$, $i = 1, 2$. By uniqueness of writing $n \in N$ we have that $n = (n_1' + n_2') + (n_1'' + n_2'') \Rightarrow n_1'' + n_2'' = 0 \Rightarrow n_1'' = n_2'' = 0 \Rightarrow n \in M_1' \oplus M_2'$. $\square$

Lemma 1.3.19. *Let M be a module, with $M = N_1 \oplus N_2$ and let $F_1 \trianglelefteq N_1$. Then there exists $F_2 \trianglelefteq N_2$ so that $F_1 \oplus F_2 \trianglelefteq M$.*

Proof. Let

$$F_2 = \sum_{\varphi \in Hom(N_1,N_2)} \varphi(F_1) \leq N_2.$$

Take any $\psi \in End(N_2)$. Since $\psi\varphi \in Hom(N_1, N_2)$ $\forall$ $\varphi \in Hom(N_1, N_2)$, we obtain $\psi(F_2) = \psi(\Sigma\varphi(F_1)) = \Sigma\psi\varphi(F_1) \subseteq F_2$. Hence $F_2 \trianglelefteq N_2$. Consider $\chi \in End(M)$; then $\chi = (\chi_{ij})_{i,j=1,2}$, $\chi_{ij} : N_j \to N_i$, with $i, j = 1, 2$. Note that $\chi_{ii}(F_i) \subseteq F_i$, since $F_i \trianglelefteq N_i$, $i = 1, 2$, and $\chi_{21}(F_1) \subseteq F_2$, from the definition of F_2. For $\varphi \in Hom(N_1, N_2)$, $\chi_{12}\varphi \in End(N_1)$; it follows that $\chi_{12}(F_2) = \chi_{12}(\Sigma\varphi(F_1)) = \Sigma\chi_{12}\varphi(F_1) \subseteq F_1$. Since each component of χ maps $F_1 \oplus F_2$ back into $F_1 \oplus F_2$, $F_1 \oplus F_2 \trianglelefteq M$. $\square$

We present a simple known result.

Lemma 1.3.20. *A ring R is right nonsingular if and only if the right annihilator of any subset X of R is always a closed right ideal.*

Proof. The sufficiency is clear: assume that there exists $r \in R$ so that $r_R(r) \leq^e R$; since $r_R(r) \leq^c R$, it implies that $r_R(r) = R \Rightarrow r = 0$.

To prove necessity, assume $X \subseteq R$ so that $r_R(X) \leq^e Y$ for some $Y \leq R_R$. Let $y \in Y$; the fact that the set $E = \{r \in R \,|\, yr \in r_R(X)\}$ is essential in R_R follows easily. Then $X(yE) = 0 \Rightarrow (Xy)E = 0 \Rightarrow Xy = 0$, by the nonsingularity of R. Hence $y \in r_R(X)$. Thus $r_R(X) \leq^c R$. $\square$

Close connections between the extending and the Baer property in the presence of some form of nonsingularity are evident by Lemma 1.3.20, since when R is right extending all closed right ideals are generated by an idempotent. Thus, a right annihilator of any subset of R is generated by an idempotent, forcing R to be Baer.

CHAPTER 2

BAER MODULES

In this chapter we introduce the concept of Baer modules. We provide a characterization of Baer modules based on strong summand intersection property. We define the concepts of $\mathcal{K}$-nonsingularity and $\mathcal{K}$-cononsingularity, which are closely linked to Baer modules and extending modules. We provide characterizations of the two nonsingularity concepts introduced, and present examples illustrating their properties. Then we use these nonsingularities to prove one of our main results of this chapter, that a Baer, $\mathcal{K}$-cononsingular module is precisely an extending module that is $\mathcal{K}$-nonsingular. This result is a module-theoretic analogue of the Chatters and Khuri result in [12] (Theorem 2.1). We also show results which are instrumental in producing large classes of Baer modules.

We study afterwards whether the Baer property for modules transfers to direct summands and direct sums. We prove it in the positive for the first, and give counterexamples for the general situation of the latter. We also study how the two nonsingularities introduced behave with respect to direct summands and direct sums. We analyze certain conditions, necessary and, respectively, sufficient for a direct sum of Baer modules to be Baer. We characterize finitely generated abelian groups that are

Baer modules as $\mathbb{Z}$-modules. We study indecomposable Baer modules, and indecomposable decompositions of Baer modules.

We conclude this chapter by proving that a ring R, for which all right R-modules are Baer, is semisimple Artinian ring.

2.1 Definitions and characterizations of Baer modules and nonsingularities

Definition 2.1.1. A right R-module M is called a *Baer module* if $\forall\, N \leq M$, $l_S(N) = Se$, with $e^2 = e \in S$. Equivalently, $\forall\, I \leq {}_S S$, $r_M(I) = eM$ where $e^2 = e \in S$.

Example 2.1.2. All semisimple modules are obviously Baer modules, as are all Baer rings viewed as right modules over themselves. $\mathbb{Z}^n$ is a Baer $\mathbb{Z}$-module, $\forall\, n \in \mathbb{N}$. More examples will be provided later.

Summand intersection property for modules was studied in several papers (e.g. [37], [19]).

Definition 2.1.3. A module M is said to have the *summand intersection property (SIP)* if the intersection of any two direct summands of M is a direct summand. M is said to have the *strong summand intersection property (SSIP)* if the intersection of any family of direct summands of M is a direct summand.

We first provide a useful characterization of Baer modules based on SSIP. In this characterization we show that Baer modules are exactly those that have SSIP and the property that annihilators of single endomorphisms (i.e. kernels) are direct summands.

Theorem 2.1.4. *A module M is Baer if and only if M has the strong summand intersection property and $Ker(\varphi) \leq^{\oplus} M$, $\forall\, \varphi \in S$.*

Proof. The second assertion of the necessary condition is obviously true, as the set of principal left ideals is a subset of the set of all left ideals.

To show the SSIP, take $e_i^2 = e_i \in S$, $i \in \mathcal{I}$ (for an index set $\mathcal{I}$) and let $I = \Sigma_{i\in\mathcal{I}} S(1-e_i)$. Then $Ker((1-e_i)) \supseteq r_M(I)\ \forall\ i \in \mathcal{I}$ (for any $m \in M$ with $\varphi m = 0$, $\forall\ \varphi \in I$, we have that $(1-e_i)m = 0$, as $(1-e_i) \in I$). Let $N = \bigcap_{i\in\mathcal{I}} e_iM = \bigcap_{i\in\mathcal{I}} Ker((1-e_i))$; then $r_M(I) \subseteq N$. For the reverse inclusion, for $m \in M \setminus N$, there exists i_0 so that $(1-e_{i_0})m \neq 0$, and hence $m \notin r_M(I)$; thus, $r_M(I) = N$. This yields $\bigcap_{i\in\mathcal{I}} e_iM = N = r_M(I) \leq^{\oplus} M$, since M is Baer. Therefore M satisfies the SSIP.

For sufficiency, take an arbitrary $I \leq_S S$. For each $\varphi \in I$ we have, by hypothesis, $Ker(\varphi) \leq^{\oplus} M$. Then $r_M(I) = \bigcap_{\varphi\in I} Ker(\varphi) \leq^{\oplus} M$, by the SSIP. Hence we get that M is Baer. $\square$

Since in [12], Theorem 2.1, it was shown that a right extending right nonsingular ring coincides with a Baer right cononsingular ring, we need to introduce a module-theoretical analogue of cononsingularity, and a weaker form of nonsingularity, to obtain a similar characterization for Baer modules.

Definition 2.1.5. We say a module M is $\mathcal{K}$-*nonsingular* if, for all $\varphi \in S$, $r_M(\varphi) = Ker\varphi \leq^e M$ implies $\varphi = 0$.

Definition 2.1.6. A module M is called $\mathcal{K}$-*cononsingular* if, for all $N \leq M$, $l_S(N) = 0$ implies $N \leq^e M$ (equivalently, $\varphi(N) \neq 0$ for all $0 \neq \varphi \in S$ implies $N \leq^e M$).

Example 2.1.7. Semisimple modules are $\mathcal{K}$-nonsingular. Uniform modules are $\mathcal{K}$-cononsingular.

Next, we show that the concept of $\mathcal{K}$-nonsingularity of modules is strictly weaker than the 'usual' concept of nonsingularity for modules.

Proposition 2.1.8. *Every nonsingular module M is $\mathcal{K}$-nonsingular.*

Proof. Assume M is not $\mathcal{K}$-nonsingular; hence $\exists\ 0 \neq \varphi \in S$ so that $Ker(\varphi) \leq^e M$. Since $\varphi \neq 0$, $\exists\ 0 \neq m \in M \setminus Ker(\varphi)$. The set $I = \{r \in R | mr \in Ker(\varphi)\}$ is a nonzero, essential right ideal in R: $r \notin I \Rightarrow mr \notin Ker(\varphi) \Rightarrow \exists r'$ so that $0 \neq mrr' \in Ker(\varphi) \Rightarrow 0 \neq rr' \in I$. But for $0 \neq \varphi(m)$, $\varphi(m)I = 0$, contradiction to the nonsingularity of M. $\square$

Example 2.1.9. The $\mathbb{Z}$-module $\mathbb{Z}_p$, where p is prime, is $\mathcal{K}$-nonsingular (it is a simple module, hence all non-zero endomorphisms are automorphisms); however, it is easy to check that the module $\mathbb{Z}_p$ is not nonsingular (for all $\hat{x} \in \mathbb{Z}_p$, $\hat{x} \cdot p\mathbb{Z} = 0$, and $p\mathbb{Z} \leq^e \mathbb{Z}$).

The following definition was included in [14].

Definition 2.1.10. A module M is called *polyform* (or *non-M-singular*) if, for any $K \leq M$ and $f : K \to M$, $Kerf \leq^c M$.

It is known that every nonsingular module M is polyform. We show that we can improve Proposition 2.1.8.

Proposition 2.1.11. *Every non-M-singular module (or polyform) is $\mathcal{K}$-nonsingular.*

Proof. Using the definition, we have that $\forall\, K \leq M$ and $\forall\, \varphi : K \to M$, $Ker\varphi \leq^e K \Rightarrow \varphi = 0$. If we choose $K = M$, we obtain that $\forall\ \varphi : M \to M$, $Ker\varphi \leq^e M \Rightarrow \varphi = 0$, thus M is $\mathcal{K}$-nonsingular. $\square$

We provide characterizations of $\mathcal{K}$-nonsingularity and $\mathcal{K}$-cononsingularity in the following.

Proposition 2.1.12. *Let M be an R-module.*

(i) *M is $\mathcal{K}$-nonsingular if and only if, for all $I \leq {}_S S$, $r_M(I) \leq^e eM$ for $e^2 = e \in S$, implies $I \cap Se = 0$;*

(ii) *M is $\mathcal{K}$-cononsingular if and only if, for all $N \leq M$, $r_M(l_S(N)) \leq^{\oplus} M$ implies $N \leq^e r_M(l_S(N))$.*

Proof. (i) Let $I \leq S$ so that $r_M(I) \leq^e eM$. Then $r_M(I \cap Se) = r_M(I) \oplus (1-e)M \leq^e M \Rightarrow I \cap Se = 0$ (by $\mathcal{K}$-nonsingularity of M).

Conversely, to show $\mathcal{K}$-nonsingularity of M, let $I \leq {}_S S$ such that $r_M(I) \leq^e M = 1_M \cdot M$, where 1_M is the identity map. Then, by hypothesis, we have that $I \cap S \cdot 1_M = 0$, thus $I = 0$.

(ii) $r_M(l_S(N)) = eM$ for $e^2 = e \in S$ implies $l_S(N) \subseteq S(1-e)$. Since $N \leq r_M(l_S(N)) = eM$ we obtain that $l_S(N \oplus (1-e)M) = 0$. By $\mathcal{K}$-cononsingularity, $N \oplus (1-e)M \leq^e M \Rightarrow N \leq^e eM = r_M(l_S(N))$.

Conversely, let $N \leq M$ with $l_S(N) = 0 \Rightarrow r_M(l_S(N)) = M$. Then $N \leq^e r_M(l_S(N)) = M$. □

Example 2.1.13. If $M_R = R_R$ then $\mathcal{K}$-nonsingularity of M coincides with nonsingularity of M.

Under the condition of nonsingularity, it is known that essential closures are unique. For the more general concept of $\mathcal{K}$-nonsingularity we obtain a similar result, for essential closures which are summands.

Proposition 2.1.14. *Let M be a $\mathcal{K}$-nonsingular module, and let $N \leq M$. If $N \leq^e N_i \leq^{\oplus} M$, for $i = 1, 2$, then $N_1 = N_2$.*

Proof. Consider the endomorphism $(1-\pi_1)\pi_2$, where π_i is the canonical projection of M onto N_i, $i = 1, 2$. Then $((1-\pi_1)\pi_2)N = (1-\pi_1)(\pi_2 N) = (1-\pi_1)(\pi_1 N) =$

$((1-\pi_1)\pi_1)N = 0$, since $N \subseteq N_1 \cap N_2$. Taking N_2' so that $N_2 \oplus N_2' = M$, $((1-\pi_1)\pi_2)N_2' = (1-\pi_1)(\pi_2 N_2') = (1-\pi_1)(0) = 0$. Hence $N \oplus N_2' \subseteq Ker((1-\pi_1)\pi_2)$, but $N \oplus N_2' \leq^e N_2 \oplus N_2' = M \Rightarrow Ker((1-\pi_1)\pi_2) \leq^e M \Rightarrow ((1-\pi_1)\pi_2) = 0 \Rightarrow \pi_2 = \pi_1\pi_2 \Rightarrow N_2 \subseteq N_1$.

Similarly, by taking the endomorphism $(1-\pi_2)\pi_1$ and showing it is zero, we obtain that $N_2 \subseteq N_1$. $\square$

2.2 Connections to extending modules

In [12], Chatters and Khuri established strong connections between the extending property of rings and the Baer property of rings, and provided a useful characterization as follows.

Theorem 2.2.1. *(Theorem 2.1, [12], Theorem 12.2, [14]) Let R be a ring. Then R is a right nonsingular, right extending ring if and only if R is a right cononsingular, Baer ring.*

In one of our main theorems of this section we extend this result to the general setting of modules and provide an analogous characterization for Baer modules. We mention that, even though our result extends that of Chatters and Khuri, our proof follows different arguments.

Theorem 2.2.2. *A module M is extending and $\mathcal{K}$-nonsingular if and only if M is Baer and $\mathcal{K}$-cononsingular.*

The proof of Theorem 2.2.2 is comprised of the following four lemmas, which may be of interest in their own right to the reader. These results also provide us with a good source of examples.

Lemma 2.2.3. *Every extending module M is $\mathcal{K}$-cononsingular.*

Proof. Let $N \leq M$ so that $\varphi(N) \neq 0$, $\forall\, 0 \neq \varphi \in S$. If $N \not\leq^e M$, by extending property we have $N \leq^e eM$, for some idempotent $e \in S$, such that $e \neq 1$. Hence $(1-e) \neq 0$; but $(1-e)N = 0$, thus getting a contradiction. Hence, M is $\mathcal{K}$-cononsingular. $\square$

Lemma 2.2.4. *Every $\mathcal{K}$-nonsingular extending module M is a Baer module.*

Proof. Assume that M is a $\mathcal{K}$-nonsingular extending module. Let $N \leq M$. By the extending property, there exists $e^2 = e \in S$ so that $N \leq^e eM$. Hence $l_S(N) \supseteq l_S(eM) = S(1-e)$. Assume that the inclusion is strict; then there exists $\varphi \in l_S(N) \setminus S(1-e)$. Since $S = Se \oplus S(1-e)$ (as a left S-module) we have that $\varphi = s_1 e + s_2(1-e)$ for some $s_1, s_2 \in S$ with $s_1 e \neq 0$; replacing φ with $\varphi - s_2(1-e) \in l_S(N)$, we can safely assume $0 \neq \varphi$ is in Se. We obtain that $\varphi(N) = 0$ and $\varphi((1-e)M) = 0$ and so $\varphi(N \oplus (1-e)M) = 0$. But $N \oplus (1-e)M \leq^e M$, hence by $\mathcal{K}$-nonsingularity of M we get that $\varphi = 0$ which contradicts our hypothesis. Therefore $l_S(N) = S(1-e)$, and so M is Baer. $\square$

Lemma 2.2.5. *Every Baer module M is $\mathcal{K}$-nonsingular.*

Proof. Let M be Baer. Let $\varphi \in S$ be any endomorphism of M with $Ker\varphi \leq^e M$. Since M is Baer, $Ker\varphi = r_M(S\varphi) = fM$ for some $f^2 = f \in S$. Being a summand and an essential submodule in M implies that $Ker\varphi = M$. Thus $\varphi = 0$. This proves that M is $\mathcal{K}$-nonsingular. $\square$

Lemma 2.2.6. *Every $\mathcal{K}$-cononsingular Baer module M is an extending module.*

Proof. Assume M to be $\mathcal{K}$-cononsingular and Baer. From Lemma 2.2.5 it follows that M is also $\mathcal{K}$-nonsingular. To show that M is extending, let $N \leq M$. Then

$l_S(N) = Sf$ for $f^2 = f \in S$. Hence $N \subseteq r_M(l_S(N)) = (1-f)M$. Assume that $N \not\leq^e (1-f)M$ (if it were essential then we would be done). Hence there exists $P \leq (1-f)M$ so that $N \cap P = 0$. Take $\overline{N} \supset N$ a complement of P in M. Note that $l(\overline{N}) \neq 0$ by $\mathcal{K}$-cononsingularity since, clearly, $\overline{N} \not\leq^e M$. Let $0 \neq s \in S$, $s\overline{N} = 0$. Then $sN = 0$ and since $l_S(N) = Sf \Rightarrow s(1-f) = 0 \Rightarrow s((1-f)M) = 0$. It follows that $sP = 0$, and so $s(\overline{N} \oplus P) = 0$. But $P \oplus \overline{N} \leq^e M$, hence, by $\mathcal{K}$-nonsingularity, $s = 0$, a contradiction. Thus M is an extending module. $\square$

The following two results provide a rich source of examples of Baer modules.

Corollary 2.2.7. *Let M be an extending module. Then $M/Z_2(M)$ is a Baer module, where $Z_2(M)$ is the second singular submodule of M.*

Proof. Since M is extending, $M = M' \oplus Z_2(M)$, where M', $Z_2(M)$ are extending. But $M' \cong M/Z_2(M)$ is nonsingular, thus $\mathcal{K}$-nonsingular. By 2.2.4, $M/Z_2(M) \cong M'$ is Baer. $\square$

Corollary 2.2.8. *If R_R is extending, then every nonsingular cyclic module M is extending, hence M is a Baer module.*

Proof. The fact that M is extending is shown in [10]. Since M is extending and nonsingular, hence extending and $\mathcal{K}$-nonsingular, we obtain that M is a Baer module. $\square$

Example 2.2.9. Let R be a domain which is not right Ore domain. Then $M = R_R$ is a Baer module which is not extending. Thus, M is a right $\mathcal{K}$-nonsingular module which is not $\mathcal{K}$-cononsingular. For a specific example, let k be any field and $R = k\langle X\rangle$, where X is a set of cardinality greater than or equal to 2; take $M = R_R$. (Example 1.31, [29]).

Example 2.2.10. Take $M = \mathbb{Z}_{p^n}$, where $p \in \mathbb{Z}$ is a prime number, and $n \in \mathbb{N}$, $n > 1$. Then M is extending (in fact it is uniform) but not Baer, since $\varphi : M \Rightarrow M$, $\varphi(\hat{a}) = p\hat{a}$ has nonzero kernel, which is essential in M. This module is, hence, $\mathcal{K}$-cononsingular, but not $\mathcal{K}$-nonsingular.

2.3 Direct summands of Baer modules

A natural question about any algebraic property is if the property is inherited by direct sums or not, if it is inherited by direct summands or not. Our first result shows that direct summands of a Baer module inherit the property. We will also show that direct sums do not inherit this property in general.

Theorem 2.3.1. *Let M be a Baer module. Then every direct summand N of M is also a Baer module.*

Proof. By Theorem 2.1.4, M has the SSIP and the property that $Ker\varphi \leq^{\oplus} M$, $\forall$ $\varphi \in S$.

Let $M = N \oplus N'$. Then every direct summand of N is also a summand of M; it is known that, in general, every summand of M which is a subset of N is a summand of N. Thus, N will have the SSIP.

For any $\psi \in End(N)$, we can extend ψ to an endomorphism of M, by taking $\overline{\psi} = \psi\pi_N : M \to N \subseteq M$, where π_N is the canonical projection of M onto N. $Ker\overline{\psi} \leq^{\oplus} M$, but $Ker\overline{\psi} = N' \oplus Ker\psi$ as it is easily checked. This implies that $Ker\psi \leq^{\oplus} N$ (by SSIP).

In conclusion, N satisfies the conditions in Theorem 2.1.4, hence N is a Baer module. $\square$

As a consequence of this result we see that right direct summands of any Baer ring are Baer modules as right R-modules, fact which provides another rich source of examples of Baer modules.

Corollary 2.3.2. *Let R be a Baer ring, and let $e^2 = e \in R$ be any idempotent of R. Then $M = eR$ is an R-module which is Baer.*

As an application of the above results, we can now characterize all Baer modules in the class of finitely generated $\mathbb{Z}$-modules.

Proposition 2.3.3. *A finitely generated $\mathbb{Z}$-module M is Baer if and only if M is semisimple or torsion-free.*

Proof. If M is semisimple then M is obviously Baer. If M is finitely generated and torsion-free, $M \cong \mathbb{Z}^n$, where $n \in \mathbb{N}$; $\mathbb{Z}^n$ is extending and nonsingular, hence by Theorem 2.2.2 it is Baer.

Next assume that M is a finitely generated Baer module. We can always decompose $M = t(M) \oplus f(M)$, where $t(M)$ is the torsion submodule of M and $f(M)$ is the torsion free submodule of M. Assume $t(M) \neq 0$ and $f(M) \neq 0$; by the structure Theorem 8.4 in [15], $t(M) \cong \bigoplus_{p\in\mathcal{P}} \mathbb{Z}_{p^{n(p)}}$, where $\mathcal{P} \subseteq \mathbb{Z}$ is a finite collection of primes (with some possible repetitions); $n(p) \in \mathbb{N}$, for all $p \in \mathcal{P}$. Also, $f(M) \cong \mathbb{Z}^n$, $0 \neq n \in \mathbb{N}$. Let p_0 be a prime so that $n(p_0) \neq 0$ (such a prime must exist), and let $\varphi : \mathbb{Z} \to \mathbb{Z}_{p_0^{n(p_0)}}$ be the morphism defined by $\varphi(x) = \hat{x}$, for $x \in \mathbb{Z}$. $Ker(\varphi)$ is a proper submodule of $\mathbb{Z}$, hence it is essential in $\mathbb{Z}$. Extend φ to $\overline{\varphi}$, an endomorphism of M, where $\overline{\varphi} = \varphi(\pi_{p_0})$, and π_{p_0} is the canonical projection of M onto $\mathbb{Z}_{p_0^{n(p_0)}}$. The kernel $Ker(\overline{\varphi}) \leq^e M$, but $Ker(\overline{\varphi}) \neq M$, hence M is not Baer, a contradiction. Hence either $t(M) = 0$ or $f(M) = 0$.

Assume $f(M) = 0$. Then $M = t(M)$; it is a finite direct sum of modules of the form $\mathbb{Z}_{p^{n(p)}}$, where p is prime and $n(p) \in \mathbb{N}$. Therefore $\mathbb{Z}_{p^{n(p)}}$ must be a Baer module, by Theorem 2.3.1. Assume there exists a prime p such that $n(p) > 1$; for this $\mathbb{Z}_{p^{n(p)}}$, we set $\varphi(\hat{x}) = p\hat{x} : \mathbb{Z}_{p^{n(p)}} \to \mathbb{Z}_{p^{n(p)}}$. Then $\varphi \neq 0$ $(p \cdot \hat{1} = \hat{p} \neq \hat{0}$ since $n(p) > 1)$; $Ker(\varphi) \neq 0$ $(p \cdot p^{n(\hat{p})-1} = p^{n\hat{(p)}} = \hat{0})$, and since $\mathbb{Z}_{p^n}$ is uniform, $Ker(\varphi)$ cannot be a summand. Thus $\mathbb{Z}_{p^{n(p)}}$ is not a Baer module, a contradiction. Hence $t(M) = \bigoplus_{\mathcal{P}} \mathbb{Z}_p$, with $\mathcal{P} \subseteq \mathbb{Z}$ a finite collection of primes (possibly in multiple instances).

Finally, assume that $t(M) = 0$; then $M = f(M) \cong \mathbb{Z}^n$ which we already know is a Baer module, by Example 2.1.2. □

Remark 2.3.4. The statement of Proposition 2.3.3 holds true for any finitely generated module over any Principal Ideal Domain instead of $\mathbb{Z}$.

Next we characterize an indecomposable Baer module over an arbitrary ring.

Theorem 2.3.5. *M is an indecomposable Baer module if and only if $\forall\ 0 \neq \varphi \in End(M)$, φ is a monomorphism.*

Proof. Let M be indecomposable and $0 \neq \varphi \in End(M)$. M being Baer, $Ker(\varphi) \leq^{\oplus} M$, hence $Ker(\varphi) = 0$ or $Ker(\varphi) = M$. As $\varphi \neq 0$ it follows that φ is a monomorphism.

Conversely, assume that M were not indecomposable, hence $M = M_1 \oplus M_2$ with $M_1,\ M_2 \neq 0$. Take $\varphi = \pi_1$ the canonical projection of M onto M_1; $Ker(\varphi) = M_2$ is proper submodule, a contradiction. Baer condition for M follows immediately, as the only summands are M and 0. □

Corollary 2.3.6. *R is a Baer ring with no proper idempotent elements if and only if R is a domain.*

Proposition 2.3.7. *Let M be an indecomposable Baer module. Then, for any $\varphi \in End(M)$, φ is uniquely defined by the image under φ of a single element $0 \neq m \in M$. Consequently, $End(M)$ embeds in the set $\{m \in M | r_R(m) \supseteq r_R(m_0)\}$, for a fixed arbitrary nonzero element $0 \neq m_0 \in M$.*

Proof. Let $0 \neq m_0 \in M$. Assume there exist $\varphi_1, \varphi_2 \in End(M)$, $\varphi_1(m_0) = \varphi_2(m_0)$. Let $\psi \in End(M)$. Then $m_0 \in Ker(\varphi_1 - \varphi_2) \neq 0$, hence, by Theorem 2.3.5, $\varphi_1 - \varphi_2 = 0$. Hence, any morphism φ is uniquely defined by the image at m_0. Since m_0 can only be mapped onto an element with a larger right annihilator in R, the last part of the conclusion follows easily. □

Remark 2.3.8. In view of Proposition 2.3.7, if there are unique elements with minimal annihilators in R, then the indecomposable Baer modules become iso-endo, namely modules for which all non-zero endomorphisms are isomorphisms.

A question that arises naturally when discussing decompositions of a module is when does a module have an indecomposable decomposition. It is known in general that if the module is either Noetherian or Artinian we can obtain such a decomposition. For Baer modules, if one such decomposition exists (and, in this case, finite) we have the following result.

Proposition 2.3.9. *If a Baer module M can be decomposed into a finite direct sum of indecomposable summands, then any other, arbitrary direct sum decomposition of M is finite.*

Proof. Let $M = M_1 \oplus M_2 \oplus \ldots \oplus M_n$, where $n \in \mathbb{N}$, be a finite direct sum of indecomposable summands. Assume now that M also decomposes as $M = \bigoplus_{i \in \mathcal{I}} N_i$. Let $0 \neq m_j \in M_j$, $j \in 1 \ldots n$. Then $m_j = \Sigma_{i \in \mathcal{I}_j} n_i^j$, where $|\mathcal{I}_j| < \infty$, $\forall\, j \in 1 \ldots n$.

But then $M_j \cap (\bigoplus_{i \in \mathcal{I}_j} N_i) \neq 0$; moreover, by Theorem 2.1.4 $M_j \cap (\bigoplus_{i \in \mathcal{I}_j} N_i) \leq^{\oplus} M_j$, which is indecomposable, hence $M_j \cap (\bigoplus_{i \in \mathcal{I}_j} N_i) = M_j \Rightarrow M_j \subseteq (\bigoplus_{i \in \mathcal{I}_j} N_i)$. But then $M = M_1 \oplus \ldots \oplus M_n \subseteq \bigoplus_{i \in \mathcal{I}_1 \cup \ldots \cup \mathcal{I}_n} N_i$, hence we actually have equality. But the union of $\mathcal{I}_1, \ldots, \mathcal{I}_n$ is a finite set, hence only finitely many N_i are non-zero, for $i \in \mathcal{I}$. □

It is of interest to know whether the properties of $\mathcal{K}$-nonsingularity and $\mathcal{K}$-cononsingularity we defined earlier, pass to direct summands. We can answer in the positive for the first.

Proposition 2.3.10. *Let M be a $\mathcal{K}$-nonsingular module. Then $\forall\ N \leq^{\oplus} M$, N is also $\mathcal{K}$-nonsingular.*

Proof. Let $\varphi \in End_R(N)$ so that $Ker\varphi \leq^e N$. Extend this morphism to M, by taking $\overline{\varphi} = \varphi\pi_N : M \Rightarrow N \subseteq M$, where π_N is the canonical projection onto N. Then $Ker\overline{\varphi} = N' \oplus Ker\varphi$, where $M = N \oplus N'$. Thus, $Ker\overline{\varphi} \leq^e M \Rightarrow \overline{\varphi} = 0$, since M is $\mathcal{K}$-nonsingular. Hence $\varphi = 0$. □

2.4 Direct sums of Baer modules

It is a well-known fact that a (finite) direct sum of extending modules is not always extending. In this section we show that a direct sum of Baer modules is not always Baer, similarly. After presenting some examples, we provide a necessary condition for direct sums of Baer modules to be Baer.

Example 2.4.1. The $\mathbb{Z}$ modules $\mathbb{Z}/p\mathbb{Z}$ and $\mathbb{Z}/p^3\mathbb{Z}$ are extending, since they are uniform (where $p \in \mathbb{Z}$ is a prime number). Yet $(\mathbb{Z}/p\mathbb{Z}) \oplus (\mathbb{Z}/p^3\mathbb{Z})$ is not extending (the submodule generated by the element $(\hat{1}, \hat{p})$ is closed but not a summand).

Example 2.4.2. Let R be a commutative domain which is not Dedekind. Let $M = R^{(I)}$, where $\mathcal{I}$ is an index set, $|\mathcal{I}| = \infty$. Then M is not a Baer R-module, while R_R is a Baer module. It can be shown (see Chapter 4) that for a free module over a domain to be Baer, the ring must be at least hereditary.

The next examples illustrate that even finite direct sums of Baer modules are not necessarily Baer modules.

Example 2.4.3. The $\mathbb{Z}$-module $M = \mathbb{Z} \oplus \mathbb{Z}_2$ is not Baer, even though $\mathbb{Z}$ and $\mathbb{Z}_2$ are both Baer $\mathbb{Z}$-modules (the map $(n, \hat{m}) \mapsto (0, \hat{n})$ has kernel $2\mathbb{Z} \oplus \mathbb{Z}_2 \neq \mathbb{Z} \oplus \mathbb{Z}_2$, which is not a summand, since $\mathbb{Z}$ is uniform).

Example 2.4.4. (Page 109 in [14]) Let R be a commutative domain that is not Prüfer. For example, $R = \mathbb{Z}[X]$, which is a commutative (noetherian) domain, hence is a Baer ring. Similar to the the example 2.4.2, a finitely generated free module is Baer only if the base ring is semihereditary. In fact, in Chapter 4 we will present a necessary and sufficient condition for a free module over a commutative Baer ring to be a Baer module (Theorem 4.1.16).

These examples provide another instance of connections of the behaviour of direct sums of Baer modules and of extending modules. Also, conditions for free modules to be Baer or extending, respectively, are similar.

As mentioned earlier, a direct sum of extending modules is not extending. There are several options for a sufficient condition for such a sum to be extending, but no significant progress has been made in finding a necessary condition or a full characterization. We suggest that using the theory of Baer modules we may be able to provide a new approach and possibly improve our chances in solving this problem.

Note that if we consider nonsingular extending modules, their direct sum is a nonsingular module, hence $\mathcal{K}$-nonsingular. Thus, if we expect that the direct sum is extending, by Theorem 2.2.2, it must also be Baer; thus, a necessary and sufficient condition for a direct sum of Baer modules to be Baer could be a starting point for a similar condition for extending modules.

We first define the following relative property of modules.

Definition 2.4.5. Let M and N be Baer modules. We say M and N are *relatively Baer* if $\forall\ \varphi : M \to N$, $Ker\varphi \leq^{\oplus} M$ and $\forall\ \psi : N \to M$, $Ker\psi \leq^{\oplus} N$.

Theorem 2.4.6. *If $M = \bigoplus_{i\in\mathcal{I}} M_i$ is a Baer module ($\mathcal{I}$ an index set), then the class $\{M_i\}_{i\in\mathcal{I}}$ satisfies the following:*

a) M_i is a Baer module, $\forall\ i \in \mathcal{I}$

b) for any pair $(i,j) \in \mathcal{I}\times\mathcal{I}$, M_i and M_j are relatively Baer

c) $\forall\ i \neq j \in \mathcal{I}$, $\forall$ monomorphisms $\varphi : M_i' \leq^{\oplus} M_i \to M_i$ and $\psi : M_j' \leq^{\oplus} M_j \to M_i$, the set

$$\mathcal{A} = \{(\varphi^{-1}(a), -\psi^{-1}(a)|a \in Im(\varphi)\cap Im(\psi)\}$$

is a direct summand of $M_i' \oplus M_j'$

Proof. The elements of the endomorphism ring of $\bigoplus_{i\in\mathcal{I}} M_i$ are matrices, for which the (i,j) entries are morphisms $M_j \to M_i$. Since $\bigoplus_{i\in\mathcal{I}} M_i$ is Baer, the kernel of every endomorphism is a direct summand, by definition.

Part (a) follows from Theorem 2.3.1.

To show (b), take the endomorphism $(\varphi_{i'j'})_{i',j'\in\mathcal{I}}$, with 1) $\varphi_{i'j'} = 0$, $\forall\ i' \neq i$ and $j' \neq j$; 2) $\varphi_{ij} = \psi$. $Ker((\varphi_{i'j'})) = (\bigoplus_{j'\in(\mathcal{I}\setminus\{j\})} M_k) \oplus Ker(\psi)$, as it is easily checked. As this must be a summand we get $Ker(\psi) \leq^{\oplus} M_j$.

To prove (c), observe that as φ is defined on $M_i' \leq^{\oplus} M_i$, it can be extended to M_i, by considering $\varphi\pi'$, where π' is the canonical projection of M_i onto M_i'; similarly with ψ. To simplify notation, we use the same symbols for these new morphisms. Take the endomorphism $(\alpha_{i'j'})_{i',j'\in\mathcal{I}}$, with: 1) $\alpha_{i'j'} = 0$, $\forall\ (i',j') \neq (i,j),(i,i)$; 2) $\alpha_{ii} = \varphi$; 3) $\alpha_{ij} = \psi$. $K = Ker((\alpha_{i'j'})) = \{(b,c)|\varphi(b)+\psi(c) = 0\}$. Note that $Ker(\varphi) \oplus Ker(\psi) \subseteq K$. Moreover, since both the kernels of φ and ψ are direct summands, we have $M_i = Ker(\varphi) \oplus M_i'$ and $M_j = Ker(\psi) \oplus M_j'$. Note that φ is mono on M_i' and ψ is mono on M_j'. We have $\varphi(b) + \psi(c) = 0$ only if $\varphi(b) = -\psi(c) \in Im(\varphi) \cap Im(\psi)$. For $(b,c) \in (M_i' \oplus M_j') \cap K$, we get $(b,c) \in \{(\varphi|_{M_i'}^{-1}(a), -\psi|_{M_j'}^{-1}(a)), a \in Im(\varphi) \cap Im(\psi)\} = \mathcal{A}$. $(Ker(\varphi) \oplus Ker(\psi)) \cap \mathcal{A} = \{(0,0)\}$, obviously. Given the fact that any pair $(b,c) \in K$ can be written uniquely as $(b,c) = (b',c') + (b'',c'')$ with $(b',c') \in Ker(\varphi) \oplus Ker(\psi)$ and $(b'',c'') \in M_i' \oplus M_j'$, we have that $K = Ker(\varphi) \oplus Ker(\psi) \oplus \mathcal{A}$. Now, K must be a summand of $M_i' \oplus M_j'$; hence $\mathcal{A} \leq^{\oplus} M_i' \oplus M_j'$. $\square$

Example 2.4.7. If M is Baer, then the family $\{M_i\}_{i\in\mathcal{I}}$, where $M_i \cong M$, $\forall\ i \in I$, satisfies the relative Baer condition. However, as Example 2.4.2 has shown, it is not enough for the direct sum $\bigoplus_{i\in\mathcal{I}} M_i$ to be Baer.

If we put conditions on the endomorphism rings we obtain the following result, due to Wilson, which we rephrased in our setting (Lemma 4, [37]).

Proposition 2.4.8. *Let M be a finite direct sum of copies of some finite rank, torsion-free module whose endomorphism ring is a PID. Then M is Baer.*

Proof. In [37] it is proved that M has SSIP and also, that the kernel of any endomorphism of M is a summand of M. Hence, using our Theorem 2.1.4, M is Baer. $\square$

One sufficient condition for a finite direct sum of extending modules to be extending is that they be relatively injective (see [18], Proposition 7.10 in [14]). We prove that an analogue is true for Baer modules.

Theorem 2.4.9. *Let $\{M_i\}_{i\leq n}$ be a class of Baer modules, where $n \in \mathbb{N}$. For any $i \neq j$, M_i and M_j are relative Baer and relative injective. Then, $\bigoplus_{i\leq n} M_i$ is a Baer module.*

Proof. We prove by induction on n.

Start with $n = 2$. Let $\{\varphi_j\}_{j\in\mathcal{J}}$ be a class of endomorphisms of $M_1 \oplus M_2$, where $\mathcal{J}$ is any index set. We want to prove that $\bigcap_{j\in\mathcal{J}} Ker(\varphi_j) \leq^{\oplus} M_1 \oplus M_2$. Set $K = \bigcap_{j\in\mathcal{J}} Ker(\varphi_j)$.

We prove that we can reduce the problem to the case when K has zero intersection with either M_1 or M_2. Assume $K \cap M_1 \neq 0$. We have that $Ker(\varphi_j) \cap M_1 = Ker(\pi_1\varphi_j\iota_1) \cap Ker(\pi_2\varphi_j\iota_1)$, where π_1, π_2 are the canonical projections, and ι_1, ι_2 are the canonical inclusions (when restricting the morphism φ_j to M_1, the elements from M_1 that are in its kernel are those that have both their image components 0). But both $Ker(\pi_1\varphi_j\iota_1)$ and $Ker(\pi_2\varphi_j\iota_1)$ are summands, as the first is the kernel of the endomorphism $\pi_1\varphi_j\iota_1$ of M_1, and the second is the kernel of the morphism $\pi_2\varphi_j\iota_1$ from M_1 to M_2 (and we have relative Baer condition). Since M_1 has SSIP by Theorem 2.1.4, the left-hand side of the equality is a summand too. Hence $K \cap M_1 = (\bigcap_{j\in\mathcal{J}} Ker(\varphi_j)) \cap M_1 = \bigcap_{j\in\mathcal{J}} (Ker(\varphi_j) \cap M_1) \leq^{\oplus} M_1$. Therefore $K = (K \cap M_1) \oplus K'$ and $M_1 = (K \cap M_1) \oplus M_1'$. Similarly, we obtain $K' \cap M_2 \leq^{\oplus} M_2$ and that $K' = (K' \cap M_2) \oplus K''$ and $M_2 = (K' \cap M_2) \oplus M_2'$. In that case, K'' is the intersection of the kernels of all morphisms φ_j restricted to $M_1' \oplus M_2'$. Being summands of M_1 and M_2 respectively, $K'' \cap M_1' = 0$ and $K'' \cap M_2' = 0$; M_1', M_2' are Baer. M_1' and M_2' are relatively Baer, and are relatively injective. Hence the reduction of the problem to the situation above does not decrease its generality. Assume from now on, for the sake of simplifying notations, that $K'' = K$ and $M_1' = M_1$, $M_2' = M_2$.

Because of relative injectivity and using Lemma 1.3.14 we can embed K into a summand N_2 with the properties: $K \subseteq N_2$ and $M_1 \oplus N_2 = M_1 \oplus M_2$. $N_2 \cong M_2$ and so N_2 is Baer, and relatively Baer and relatively injective with M_1. Taking p_1 and p_2 the canonical projections onto M_1 and N_2, and i_1, i_2 the canonical inclusions into M_1 and N_2, respectively, we obtain, similar to the above argument, that $K = \bigcap_{j \in \mathcal{J}} (Ker(p_1 \varphi_j i_2) \cap Ker(p_2 \varphi_j i_2)$. As for each j both those kernels are summands in N_2 (by Baer and relative Baer assumption), and then intersection of arbitrary number of summands is again a summand (by Theorem 2.1.4), $K \leq^{\oplus} N_2 \leq^{\oplus} M_1 \oplus M_2$, which is what we wanted to prove.

Similarly, we can prove that (in the settings of the above hypothesis) $M_1 \oplus M_2$ and M_3 are relatively Baer. Take any $\varphi : M_3 \to M_1 \oplus M_2$; $Ker(\varphi) = Ker(\pi_1 \varphi) \cap Ker(\pi_2 \varphi) \leq^{\oplus} M_1 \oplus M_2$. Take now $\psi : M_1 \oplus M_2 \to M_3$. If $Ker(\psi) \cap M_1 \neq 0$, then $Ker(\psi) \cap M_1 \leq^{\oplus} M_1$ and $Ker(\psi) \cap M_1 \leq^{\oplus} Ker(\psi)$. Hence we can reduce the problem (similarly to the situation above) to the case when $Ker(\psi) \cap M_1 = 0$. But since M_1 and M_2 are relative injective, we can embed $Ker(\psi)$ into a summand L, $Ker(\psi) \leq L$, $M_1 \oplus L = M_1 \oplus M_2$ where $L \cong M_2$. From this it easily follows that $Ker(\psi) \leq^{\oplus} L$ (L is Baer, relative Baer with M_i's), which, together with the Baer property of $M_1 \oplus M_2$, gives us relative Baer property of $M_1 \oplus M_2$ and M_3.

Assuming now that a direct sum of n Baer modules M_i, $i \in 1, \ldots, n$ that are both relative Baer and relative injective, is Baer, and that this direct sum is relative Baer with respect to M_{n+1}, we go now to step $n + 1$. Since relative injectivity transfers to direct sums (that is, $\bigoplus_{i\leq n} M_i$ is relative injective with M_{n+1} - Lemma 1.3.15), we have that: $\bigoplus_{i\leq n} M_i$ and M_{n+1} are both Baer modules; they are relative Baer; they are relative injective. Hence $\bigoplus_{i\leq n} M_i \oplus M_{n+1} = \bigoplus_{i\leq n+1} M_i$ is a Baer module. $\square$

In the next result, we describe the behaviour of arbitrary summands of a direct sum of indecomposable Baer modules.

Proposition 2.4.10. *Let $(M_i)_{i\in\mathcal{I}}$ be a class of indecomposable Baer modules satisfying the relative Baer property, for $\mathcal{I}$ an index set, and let $M = \bigoplus_{i\in\mathcal{I}} M_i$. For any $N \leq^{\oplus} M$ either $M_i \subseteq N$ or $N \cap M_i = 0$, $\forall\ i \in \mathcal{I}$.*

Proof. Recall that we denote by S the endomorphism ring of M. Let $e_i^2 = e_i$ be the idempotents in S corresponding to the decomposition $M = \bigoplus_{i\in\mathcal{I}} M_i$. Let $N = fM$, for some $f^2 = f \in S$. For any $i \in \mathcal{I}$, $e_iSe_i \cong S_i = End(M_i)$.

Assume $0 \neq m \in N \cap M_i$, for a certain $i \in \mathcal{I}$. Then $e_im = m$; $fm = m$; so, $e_ife_im = m$. Since M_i is indecomposable Baer, by Proposition 2.3.7 the endomorphism e_ife_i is uniquely defined its value at m, hence $e_ife_i = e_i$. Similarly, taking $(1-e_i)fe_im = 0$, we obtain that $Ker(1-e_i)fe_i \neq 0$, yet, by relative Baer property, $Ker(1-e_i)fe_i \leq^{\oplus} M_i$, hence $Ker(1-e_i)fe_i = M_i$.

Consequently, $fe_i = e_ife_i + (1-e_i)fe_i = e_i$, hence $M_i \subseteq N$. $\square$

Corollary 2.4.11. *Let M be an indecomposable Baer module, and let $M_i \cong M$, for $i \in \mathcal{I}$, $\mathcal{I}$ an index set. Then $\forall\ N \leq^{\oplus} \bigoplus_{i\in\mathcal{I}} M_i$ we have either $M_i \cap N = 0$ or $M_i \subseteq N$.*

Lemma 2.4.12. *Let M_1 and M_2 be Baer modules, M_1 and M_2 relatively Baer. Let $N_1 \leq^{\oplus} M = M_1 \oplus M_2$. Then $N_1 \cap M_1 \leq^{\oplus} M_1$ and $N_1 \cap M_2 \leq^{\oplus} M_2$ ($N_1 \cap M_1 \leq^{\oplus} M$ and $N_1 \cap M_2 \leq^{\oplus} M$).*

Proof. Take N_2 so that $N_1 \oplus N_2 = M$. Take π_1, π_2 the canonical projections of M onto M_1 and M_2, respectively; p_1, p_2 the canonical projections of M onto N_1 and N_2, respectively.

Construct the following maps:

$$\varphi : M_1 \to M_1; \varphi = \pi_1(p_2|_{M_1})$$

and

$$\psi : M_1 \to M_2; \psi = \pi_2(p_1|_{M_1})$$

$Ker\varphi = \{m \in M_1 | \pi_1(p_2(m)) = 0\} = \{m \in M_1 | p_2(m) \in M_2\}$; $Ker\psi = \{m \in M_1 | \pi_2(p_1(m)) = 0\} = \{m \in M_1 | p_1(m) \in M_1\}$. Each of $Ker\varphi$, $Ker\psi$ is a direct summand of M (using Baer and relative Baer properties). And

$$K = Ker\varphi \cap Ker\psi = \{m \in M_1 | p_2(m) \in M_2, p_1(m) \in M_1\}$$

is also a direct summand of M_1.

But $m = p_1(m) + p_2(m)$; since the first term is in M_1 and the second in M_2 we must have that $p_1(m) = m$, $p_2(m) = 0$. Therefore we conclude that for any $m \in Ker\varphi \cap Ker\psi$, $m = p_1(m) \Rightarrow m \in M_1 \cap N_1$.

Conversely, for any $m \in M_1 \cap N_1$, $p_1(m) = m \Rightarrow \psi(m) = 0$, $p_2(m) = 0 \Rightarrow \varphi(m) = 0$. Hence $m \in Ker\varphi \cap Ker\psi$, $M_1 \cap N_1 \leq^{\oplus} M_1 \leq^{\oplus} M$.

The preceding holds similarly for $M_2 \cap N_1$. □

Remark 2.4.13. This result can be easily generalized to any direct sum of relative Baer, Baer modules - a sketch of the proof here: take $N_1 \oplus N_2 = \bigoplus M_i$; π_i canonical projection onto M_i, μ_i "canonical" projection onto $\bigoplus_{k \neq i} M_k$, p_j canonical projection onto N_j, $j = 1, 2$. Take $\varphi = \mu_i(p_2|_{M_i})$ and $\psi = \pi_i(p_1|_{M_i})$. $Ker\varphi$ and $Ker\psi$ must be summands (the first , since it is taking M_i into the rest, its kernel is an intersection of all components onto M_ks, the second simply because it's from M_i to M_i). $K = Ker\varphi \cap Ker\psi$ is a summand, too. $K = \{\mathrm{m}|p_1(m) \in M_i$ and $p_2(m) \in \bigoplus_{k\neq i} M_k$; $m = p_1(m) + p_2(m) \Rightarrow m = p_1(m)$ and $p_2(m) = 0\}$. Similarly like above, this means that the intersection of kernels is precisely $M_i \cap N_1$.

Remark 2.4.14. Another observation we can make here is the fact that we didn't use the Baer property of M_2 in Lemma 2.4.12, which means that we get the property only using the properties that M_1 is Baer and that the kernels of all morphisms from M_1 to M_2 are direct summands in M_1.

A sufficient condition for an arbitrary direct sum of Baer modules to be Baer is that each module be fully invariant in the direct sum.

Proposition 2.4.15. *Let $M = \bigoplus_{i\in\mathcal{I}} M_i$ ($\mathcal{I}$ an index set) and let $M_i \trianglelefteq M$, $\forall\ i \in \mathcal{I}$. Then $\bigoplus_{i\in\mathcal{I}} M_i$ is a Baer module if and only if M_i is a Baer module, $\forall\ i \in \mathcal{I}$.*

Proof. The necessity is clear, by Theorem 2.3.1.

To prove sufficiency, note that since $M_i \trianglelefteq M$, $\forall\ i \in \mathcal{I}$, $Hom(M_i, M_j) = 0$, $\forall\ i \neq j$, $i, j \in I$. Hence in the endomorphism ring of $M = \bigoplus_{i\in\mathcal{I}} M_i$, viewed as a matrix ring, for each endomorphism there are only elements on the 'diagonal'. Let $I \leq {}_S S$. Hence $r_M(I) = \bigoplus_{i\in\mathcal{I}} r_{M_i}(I \cap S_i)$, where $S_i = End_R(M_i)$. Since on each component, the right annihilator is a summand in M_i (since each M_i is Baer) it follows that $r_M(I) = \bigoplus_{i\in\mathcal{I}} r_{M_i}(I \cap S_i) \leq^{\oplus} \bigoplus_{i\in\mathcal{I}} M_i = M$, hence M is a Baer module. $\square$

We conclude this chapter, by showing that a ring R for which every right R-modules is Baer must be precisely a semisimple artinian ring.

Theorem 2.4.16. *Let R be a ring. The following are equivalent:*

1. R is semisimple artinian.

2. Every (right) R-module is Baer

3. Every injective right R module is Baer

Proof. (1)$\Rightarrow$(2)$\Rightarrow$(3) are obvious.

To prove (3)$\Rightarrow$(1), consider the module: $B = E(M) \oplus E(E(M)/M)$, where M is an arbitrary right R-module. B is injective (being the direct sum of two injective modules) and hence is Baer by hypothesis. Let $\varphi : E(M) \to E(E(M)/M)$ be defined by $\varphi(x) = x + M$, $\forall\ x \in E(M)$. Then $Ker\varphi = M$ is a direct summand of $E(M)$, by Theorem 2.4.6. Since $M \leq^e E(M)$ we get $M = E(M)$. Since M arbitrarily chosen, we get that all right R-modules are injective, hence R must be semisimple Artinian. $\square$

CHAPTER 3

QUASI-BAER MODULES

In this chapter we define the concept of quasi-Baer modules and provide examples. The notions of FI-$\mathcal{K}$-nonsingularity and FI-$\mathcal{K}$-cononsingularity are introduced to obtain the connections of quasi-Baer modules to the FI-extending modules, analogous to the result of Chatters-Khuri for Baer rings ([12]). We show that direct summands of quasi-Baer modules are quasi-Baer and that direct sums of copies of a quasi-Baer module are always quasi-Baer. In particular, any projective module over a quasi-Baer ring is a quasi-Baer module. Some results relating to direct sums of arbitrary quasi-Baer modules are also provided. We remark that the quasi-Baer property for rings is a Morita invariant property, and hence provides nice results.

3.1 Definitions

Definition 3.1.1. A right R-module M is called a *quasi-Baer module* if for all $N \trianglelefteq M$, $l_S(N) = Se$, with $e^2 = e \in S = End(M)$. Equivalently, a module is quasi-Baer if and only if $\forall\ J \trianglelefteq S$, $r_M(J) = fM$ for $f^2 = f \in S$

Note. The equivalence between the two conditions in the definition follows by an application of Lemma 1.3.16.

Example 3.1.2. All semisimple modules are quasi-Baer; all Baer and quasi-Baer rings are quasi-Baer modules, viewed as modules over themselves. The Baer modules shown in Chapter 2 are obviously quasi-Baer modules. All finitely generated abelian groups are quasi-Baer. A finitely generated abelian group that is not semisimple is an example of a quasi-Baer module that is not a Baer module.

Lemma 3.1.3. *Let $N_i \trianglelefteq^{\oplus} M$, where $1 \leq i \leq n$, $n \in \mathbb{N}$. Then $\bigcap_{1 \leq i \leq n} N_i \trianglelefteq^{\oplus} M$.*

Proof. The intersection of any family of fully invariant submodules is a fully invariant submodule, as it is easily checked. By Lemma 1.3.17, $N_1 \cap N_2 \leq^{\oplus} N_1 \Rightarrow N_1 \cap N_2 \leq^{\oplus} M$. Intersecting successively with $N_3, \ldots, N_n$ we obtain at every step a direct summand. □

Proposition 3.1.4. *Let M be a module with the property that all ideals of S are generated (as left ideals) by finitely many principal two-sided ideals. Then M is quasi-Baer if and only if for any right semicentral endomorphism φ, $Ker\varphi \leq^{\oplus} M$.*

Proof. Necessity is clear, since the set of principal ideals of S is included in the set of all ideals of S.

For sufficiency, take an ideal $I \trianglelefteq S$. Then, by our assumption, $I = S(\varphi_k | k \in \mathcal{K})$, where $\mathcal{K}$ is a finite index set. For each φ_k, $Ker\varphi_k \trianglelefteq^{\oplus} M$: $Ker\varphi_k \leq^{\oplus} M$ by hypothesis; $Ker\varphi_k$ is also invariant due to the fact that $S\varphi_k \trianglelefteq S$. In this case, $r_M(I) = \bigcap_{k \in \mathcal{K}} Ker\varphi_k$. By Lemma 3.1.3, $\bigcap_{k \in \mathcal{K}} Ker\varphi_k \leq^{\oplus} M$. □

Similar to the Baer modules case, quasi-Baer modules do indeed have a certain kind of nonsingularity. For the quasi-Baer case, we shall have to employ fully invariant submodules of M and ideals in S for such a definition to be meaningful. We introduce the following two definitions.

Definition 3.1.5. A module M is called *FI-$\mathcal{K}$-nonsingular* if, for any $I \trianglelefteq S$ so that $r_M(I) \leq^e eM$ for $e^2 = e \in S$, $r_M(I) = eM$.

Definition 3.1.6. A module M is called *FI-$\mathcal{K}$-cononsingular* if, for every $N \trianglelefteq^{\oplus} M$ and $N' \trianglelefteq N$ so that $\varphi(N') \neq 0$, $\forall\, \varphi \in End(N)$, we get that $N' \leq^e N$.

We first provide a characterization of FI-$\mathcal{K}$-nonsingularity and FI-$\mathcal{K}$-cononsingularity. As a consequence, we show that the $\mathcal{K}$-nonsingularity and $\mathcal{K}$-cononsingularity concepts we studied in Chapter 2 imply these new ones, respectively: $\mathcal{K}$-nonsingularity implies FI-$\mathcal{K}$-nonsingularity and $\mathcal{K}$-cononsingularity implies FI-$\mathcal{K}$-cononsingularity.

Proposition 3.1.7. *Let M be an R-module.*

(i) M is FI-$\mathcal{K}$-nonsingular if and only if, for all $I \trianglelefteq S$, $r_M(I) \leq^e eM$ for $e^2 = e \in S$, implies $I \cap Se = 0$;

(ii) M is FI-$\mathcal{K}$-cononsingular if and only if, for all $N \trianglelefteq M$, $r_M(l_S(N)) \leq^{\oplus} M$ implies $N \leq^e r_M(l_S(N))$.

Proof. (i) Let $I \trianglelefteq S$ so that $r_M(I) \leq^e eM \Rightarrow S(1-e) \subseteq I \Rightarrow I = S(1-e) \oplus Ie = S(1-e) \oplus I \cap Se$ (since I is an ideal of S). Assuming that $I \cap Se \neq 0 \Rightarrow \exists$ $0 \neq s \in I$ so that $se = s$, by taking $s \in I \cap Se$. But, by FI-$\mathcal{K}$-nonsingularity, $r_M(I) = eM \Rightarrow sM = seM = 0 \Rightarrow s = 0$, absurd. Hence $I \cap Se = 0$.

Conversely, to show FI-$\mathcal{K}$-nonsingularity of M, let $I \leq {}_S S$ such that $r_M(I) \leq^e eM$. Then, by hypothesis, we have that $I \cap Se = 0 \Rightarrow I \subseteq S(1-e)$ since $I \trianglelefteq S$. Thus $eM \subseteq r_M(I)$ (by Lemma 1.3.16), hence $eM = r_M(I)$.

(ii) $r_M(l_S(N)) = eM$ for $e^2 = e \in S$ implies $l_S(N) \subseteq S(1-e)$. $N \trianglelefteq M \Rightarrow eM \trianglelefteq M \Rightarrow S(1-e) \trianglelefteq S$. Using now the fact that $N \subseteq r_M(l_S(N)) = eM$, and since $l_S(N) \subseteq S(1-e) \Rightarrow l_{eSe}N = 0 \Rightarrow N \leq^e eM = r_M(l_S(N))$ by FI-$\mathcal{K}$-cononsingularity.

Conversely, to show FI-$\mathcal{K}$-cononsingularity, let $N = eM \trianglelefteq M$ where $e^2 = e \in S$, and $N' \trianglelefteq eM$ so that $l_{eSe}(N') = 0$. Note that, since $eM \trianglelefteq M \Rightarrow S(1-e) \trianglelefteq S \Rightarrow eS \trianglelefteq S \Rightarrow se = ese$. Then $l_S(N') = S(1-e)$ (since $(1-e)N' \subseteq (1-e)N = (1-e)eM = 0$; taking $s = se \in Se$ so that $sN' = 0 \Rightarrow eseN' = 0 \Rightarrow ese = se = 0$). Then $r_M(l_S(N')) = eM \Rightarrow N' \leq^e r_M(l_S(N')) = eM = N$. □

Note. The characterizations in Proposition 3.1.7 are similar to those in Proposition 2.1.12, differing only by the choice of ideals of S and fully invariant submodules of M instead of left ideals of S and submodules of M, respectively, in Proposition 3.1.7.

These concepts indeed properly generalize the notions of $\mathcal{K}$-nonsingularity and $\mathcal{K}$-cononsingularity, respectively.

Corollary 3.1.8. *We have the following implications:*

a) if M is $\mathcal{K}$-nonsingular, then M is FI-$\mathcal{K}$-nonsingular;

b) if M is $\mathcal{K}$-cononsingular, then M is FI-$\mathcal{K}$-cononsingular.

Proof. Easily follows from the Proposition 3.1.7 and Proposition 2.1.12. □

Example 3.1.9. Any prime ring R with a nonzero singular ideal has the property that R_R is FI-$\mathcal{K}$-nonsingular (since it is quasi-Baer; see Theorem 3.2.2, which we will prove shortly), but not $\mathcal{K}$-nonsingular (see [30]; also Examples 4.3, 4.4 in [11]).

Any module which is Baer and FI-extending but not extending has the property that it is FI-$\mathcal{K}$-cononsingular but not $\mathcal{K}$-cononsingular. We exhibit one such module in Example 2.2.9 (a domain R which is not right Ore domain): R is Baer and not right extending, but it is right FI-extending.

3.2 Connections to FI-extending modules

Our starting point is the following result.

Theorem 3.2.1. *(Proposition 4.4 [9]) Let R be right nonsingular. Then R is right FI-extending if and only if R is quasi-Baer and $A \leq^e r(l(A))$, for all $A \trianglelefteq R$.*

In the main theorem of this section we establish connections between quasi-Baer modules and FI-extending modules, which are similar to the Baer and extending module case, but which require the newly defined concepts of FI-$\mathcal{K}$-nonsingularity and FI-$\mathcal{K}$-cononsingularity for a complete characterization.

Theorem 3.2.2. *A module M is FI-extending and FI-$\mathcal{K}$-nonsingular if and only if M is quasi-Baer and FI-$\mathcal{K}$-cononsingular.*

Lemma 3.2.3. *Let M be FI-extending. Then M is FI-$\mathcal{K}$-cononsingular.*

Proof. Let $N \trianglelefteq^{\oplus} M$. Then, by Proposition 1.2, [9], N is FI-extending. Take $N' \trianglelefteq N$ such that $\varphi(N') \neq 0$, $\forall\, \varphi \in End(N)$. By the FI-extending property $N' \leq^e \overline{N'} \leq^{\oplus} N$. Assume $\overline{N'} \oplus N_2 = N$ where $N_2 \neq 0$. Then π_2, the canonical projection of N onto N_2 has the property that $\pi_2(N') = 0$, a contradiction. Hence $N_2 = 0$, and $N' \leq^e N$. □

Lemma 3.2.4. *Let M be an FI-$\mathcal{K}$-nonsingular, FI-extending module. Then M is a quasi-Baer module.*

Proof. Let $I \trianglelefteq S$. We want to show that $r_M(I) \leq^{\oplus} M$. We have that $r_M(I) \trianglelefteq M$, and by FI-extending property we get $r_M(I) \leq^e eM$, $e^2 = e \in S$. By FI-$\mathcal{K}$-nonsingularity we get that $r_M(I) = eM$. □

Lemma 3.2.5. *Let M be a quasi-Baer module. Then M is FI-$\mathcal{K}$-nonsingular.*

Proof. Let $I \trianglelefteq S$, with $r_M(I) \leq^e eM$, $e^2 = e \in S$. Then, by the quasi-Baer property, $r_M(I) \leq^\oplus M$. As $r_M(I) \subseteq eM$ it follows that $r_M(I) \leq^\oplus eM$. Since it is also essential, $r_M(I) = eM$. $\square$

Lemma 3.2.6. *Let M be an FI-$\mathcal{K}$-cononsingular quasi-Baer module. Then M is FI-extending.*

Proof. Let $N \trianglelefteq M$, and $l_S(N) = Se$ (by quasi-Baer property). Hence, $N \subseteq (1-e)M$. Moreover, since $N \trianglelefteq M$, $Se \trianglelefteq S$ hence $(1-e)M \trianglelefteq^\oplus M$. Now let $\varphi \in End((1-e)M)$, thus $\varphi = (1-e)\varphi(1-e) \in S$. Suppose $\varphi(N) = 0 \Rightarrow (1-e)\varphi(1-e) \in l_S(N) = Se$. But then $(1-e)\varphi(1-e) \in [(1-e)S](1-e) \cap Se = 0$. So, by the FI-$\mathcal{K}$-cononsingularity of M we get that $N \leq^e (1-e)M$, hence M is FI-extending. $\square$

Remark 3.2.7. In the above proof we also get that $(1-e)M \trianglelefteq M$ ($N \trianglelefteq M \Rightarrow Se = l_S(N) \trianglelefteq S \Rightarrow (1-e)M = r_M(l_S(N)) \trianglelefteq M$), and so we obtain that M is, in fact, strongly FI-extending.

Proof. The proof of Theorem 3.2.2 follows from Lemma 3.2.3, Lemma 3.2.4, Lemma 3.2.5 and Lemma 3.2.6. $\square$

Next, we characterize the FI-$\mathcal{K}$-nonsingularity in the ring case. First, a definition.

Definition 3.2.8. For a ring R, we call the following set the *right FI-singular set* of R: $\tilde{Z}(R_R) = \{r \in R|$ either $r = 0$ or $\exists\ e^2 = e, r = re \neq 0$ and $rI = 0$ for some $I \trianglelefteq R, I \leq^e eR\}$

Proposition 3.2.9. *The ring R is FI-$\mathcal{K}$-nonsingular (as a right R-module) if and only if $\tilde{Z}(R_R) = 0$.*

Proof. For necessity, assume $\tilde{Z}(R_R) \neq 0$, hence there exists $0 \neq t \in \tilde{Z}(R_R) \Rightarrow \exists$ $e^2 = e \in R$, with $t = te \neq 0$ and $I \trianglelefteq R$, $I \leq^e eR$ so that $tI = 0$. Let $J = l_R(I)$. Since $I \trianglelefteq R$, $J \trianglelefteq R$; also, since $I \subseteq eR \Rightarrow J \supseteq R(1-e)$. Since $tI = 0$ we have $t \in J$. Consider now $r_R(J)$. Now $1 - e \in J \Rightarrow r_R(J) \subseteq eR$ and $I \subseteq r_R(J)$, since $I \subseteq r_R(l_R(I))$; hence $r_R(J) \leq^e eR$. But $t = te \neq 0$, hence $e \notin r_R(J) \Rightarrow r_R(J) \neq eR$. Contradiction, because R is right FI-$\mathcal{K}$-nonsingular.

For sufficiency, assume that $\exists\ I \trianglelefteq R$ so that $r_R(I) \leq^e eR$ for some $e^2 = e \in R$, but $r_R(I) \neq eR$. Hence there exists an element $0 \neq t \in I$ so that $teR \neq 0$. Thus we have $0 \neq te = (te)e$; $r_R(I) \leq eR \Rightarrow er_R(I) = r_R(I)$, thus $(te)r_R(I) = t(er_R(I)) = tr_R(I) = 0$. Since $r_R(I) \trianglelefteq R$ and $r_R(I) \leq^e eR$, $te \in \tilde{Z}(R_R) = 0 \Rightarrow te = 0$, contradiction. $\square$

Proposition 3.2.10. *Let R be a ring. Then the following hold:*

a) the set $\tilde{Z}(R_R)$ is closed under left multiplication with elements of R

b) let $A \trianglelefteq R$ with $A \leq^e eR$ where $e^2 = e \in R$; then $eR + \tilde{Z}(R_R)$ is a set closed under left multiplication with elements of R

Proof. (a) Taking $0 \neq t \in \tilde{Z}(R_R)$, then $\exists\ e^2 = e, t = te \neq 0$ and $tI = 0$ for some $I \trianglelefteq R, I \leq^e eR$. In this case, for an arbitrary $r \in R$, if $0 \neq rt = rte$, $rtI = 0$ and hence $rt \in \tilde{Z}(R_R)$.

(b) We already know that $\tilde{Z}(R_R)$ is closed under left multiplication, from part (a). To show that $eR + \tilde{Z}(R_R)$ is left R-closed too we only need to show that, $\forall r \in R$, $re \in eR + \tilde{Z}(R_R)$.

We have: $re = ere + (1-e)re$, and $ere \in eR$; we show that $(1-e)re \in \tilde{Z}(R_R)$: $(1-e)reA = (1-e)rA$ (since $A \subseteq eR$) $\subseteq (1-e)A$ (since $A \trianglelefteq R$) $\subseteq A \cap (1-e)R \subseteq eR \cap (1-e)R = 0$. Since $(1-e)re \in Re$, $A \trianglelefteq R$ and $A \leq^e eR$ we obtain that

$(1-e)re \in \tilde{Z}(R_R)$ (by definition). Therefore, $eR + \tilde{Z}(R_R)$ is closed under the left multiplication with elements of R. $\square$

Corollary 3.2.11. *Let R be FI-$\mathcal{K}$-nonsingular. If $A \trianglelefteq R$ and $A \leq^e eR$, then $eR \trianglelefteq R$, and hence e is a left semicentral idempotent of R.*

Proof. Since R is FI-$\mathcal{K}$-nonsingular $\Rightarrow \tilde{Z}(R_R) = 0$ and so the set eR is closed under left and right multiplication with elements of $R \Rightarrow eR \trianglelefteq R$. $\square$

Remark 3.2.12. By the definition, note that FI-$\mathcal{K}$-cononsingularity for a ring R can be viewed as a restricted cononsingularity for R, only applied to all ideals of ring direct summands of R, rather than to all right ideals of R.

We can now state the connection between quasi-Baer rings and FI-extending rings, which extends the result in [9]

Theorem 3.2.13. *A ring R is right FI-extending ring and $\tilde{Z}(R_R) = 0$ if and only if R is a quasi-Baer ring and is FI-$\mathcal{K}$-cononsingular.*

Proof. Let R be right FI-extending and $\tilde{Z}(R_R) = 0$ Then R is right FI-$\mathcal{K}$-cononsingular and quasi-Baer, by Proposition 3.2.9 and Theorem 3.2.2. For the converse, the ring R that is quasi-Baer and FI-$\mathcal{K}$-cononsingular is FI-extending and FI-$\mathcal{K}$-nonsingular, by Theorem 3.2.2, and $\tilde{Z}(R_R) = 0$ by Proposition 3.2.9. $\square$

3.3 Direct summands and direct sums of quasi-Baer modules

Given the connections of quasi-Baer modules to FI-extending modules, one would expect that direct summands of quasi-Baer modules do not behave well, similar to the case of the FI-extending modules. On the contrary, we can prove that direct summands of a quasi-Baer module do, in fact, inherit the property.

Theorem 3.3.1. *Let M be a quasi-Baer module. Then for any $N \leq^{\oplus} M$, N is also a quasi-Baer module.*

Proof. Since $N \leq^{\oplus} M$, there exists $e^2 = e \in S$ so that $N = eM$, and let $F \trianglelefteq N$. By Lemma 1.3.19, there exists $G \trianglelefteq (1-e)M$ so that $F \oplus G \trianglelefteq M$. Since M is quasi-Baer, $I = l_S(F \oplus G) \trianglelefteq^{\oplus} S$. The endomorphism ring of $N = eM$ is eSe, and since $I \trianglelefteq S$, $eIe = eSe \cap I$ (one inclusion is obvious, while the other one results from the following argument: $i \in I \cap eSe \Rightarrow i = ese = e^2se^2 = eie \in eIe$). Note also that, $I = Sf$ for some $f^2 = f \in S$, and so $eIe = eSfe$. But, since $Sf \trianglelefteq S$, $fe \in Sf \Rightarrow fe = fef$; we can write hence $eIe = eSfe = eSfef = eSfefe = (eSfe)(efe)$. Since $(efe)^2 = efeefe = efefe = efee = efe$; we have $(eSfe)(efe) \subseteq (eSe)(efe)$. On the other hand, let $(ese)(efe) \in (eSe)(efe)$; $eseefe = esefe = esefef = esefefe = e((se)f)efe = e((se)f)eefe = (e((se)f)e)(efe) \in (eSfe)(efe)$. Hence we have that $eIe \leq^{\oplus} eSe$ (in fact, it is a fully invariant direct summand because efe is a semicentral idempotent in eSe: $(efe)(ese) = efese = efesef = efesefe = (efe)(ese)(efe)$).

Now we only have to show that $eIe = l_{eSe}(F)$. It is clear that $(eIe)(F) = 0$: $eie(F) = ei(F) = e(0) = 0$. Assume there exists $0 \neq eje \in eSe$, $eje \notin eIe$ so that $eje(F) = 0$. But $ejeG \subseteq eje(1-e)M = 0$, and so $eje \in l_S(F \oplus G) = I$. But then $eje = eejee = e(eje)e \in eIe$, a contradiction. Hence $l_{eSe}(F) = eIe \leq^{\oplus} eSe$. Since F was arbitrarily chosen, hence N is quasi-Baer. □

Theorem 3.3.2. *Let M_1 and M_2 be quasi-Baer modules. If we have the property that $\psi(x) = 0 \ \forall\ \psi \in Hom(M_i, M_j)$ implies $x = 0$ ($i \neq j$, $i, j = 1, 2$) then $M_1 \oplus M_2$ is quasi-Baer.*

Proof. Let $S = End(M_1 \oplus M_2)$, and let $I \trianglelefteq S$. Then $r_{M_1 \oplus M_2}(I) \trianglelefteq M_1 \oplus M_2$, hence, using Lemma 1.3.17, $r_{M_1 \oplus M_2}(I) = N_1 \oplus N_2$, where $N_i \trianglelefteq M_i$, $i = 1, 2$. As mentioned,

$$S = \begin{pmatrix} S_1 & Hom(M_2, M_1) \\ Hom(M_1, M_2) & S_2 \end{pmatrix}.$$

Since $I \trianglelefteq S$ we have the following ideals in S_1 and S_2, respectively.

- $I_1 = \{\varphi \in S_1 | \varphi = \xi_{11} \text{ with } (\xi_{ij})_{i,j=1,2} \in I\} \trianglelefteq S_1$

- $I_2 = \{\varphi \in S_2 | \varphi = \xi_{22} \text{ with } (\xi_{ij})_{i,j=1,2} \in I\} \trianglelefteq S_2$

Define $I_{12} = \{\psi \in Hom(M_1, M_2) | \psi = \xi_{12} \text{ with } (\xi_{ij})_{i,j=1,2} \in I\}$ and $I_{21} = \{\psi \in Hom(N, M) | \psi = \xi_{21} \text{ with } (\xi_{ij})_{i,j=1,2} \in I\}$.

Let $N_1' = r_{M_1}(I_1)$. We have that $N_1 = N_1' \cap (\bigcap_{\psi \in I_{12}} Ker\psi)$. Since M_1 is quasi-Baer, we know that $r_{M_1}(I_1) \leq^{\oplus} M_1$.

We also have that $\psi(N_1')$ satisfies $\chi(\psi(N_1')) = 0 \ \forall \ \chi \in Hom(M_2, M_1)$, since $\chi(\psi) \in I_1$ (for $\psi \in I_{12}$).

Since we have the property that $\chi(x) = 0 \ \forall \ \chi \in Hom(M_2, M_1)$ implies $x = 0$ then we have that $\psi(N_1') = 0$, $\forall \ \psi \in I_{12}$, and so $N_1 = N_1' \leq^{\oplus} M_1$. $\square$

The next result provides another rich source of examples for quasi-Baer modules, as we will see in Corollary 3.3.4.

Proposition 3.3.3. *$M = \bigoplus_{i \in \mathcal{I}} M_i$ is quasi-Baer if M_i is quasi-Baer and subisomorphic to (i.e. isomorphic to a submodule of) M_j, $\forall \ i \neq j$; $i, j \in \mathcal{I}$, where $\mathcal{I}$ is an index set.*

Proof. Let S_i be the endomorphism ring of M_i, $\forall \ i \in \mathcal{I}$. The endomorphism ring of M, S, is a ring of matrices, with elements of S_i in the ii-position, and maps $M_j \to M_i$

in the ij-position, $\forall\, i,j \in \mathcal{I}$, $i \neq j$. We need to show, $\forall\, I \trianglelefteq S$, $r_M(I) \leq^{\oplus} M$. But since $r_M(I) \trianglelefteq M$, $r_M(I) = \bigoplus_{i\in\mathcal{I}} r_M(I) \cap M_i$. We only have to analyze, hence, the column morphisms (i.e. matrices) taking M_i into M, for an $i \in \mathcal{I}$. Similar to our previous theorem's proof, we have that the i-th column of $I \trianglelefteq S$ has elements from an ideal $I_i \trianglelefteq S_i$ in the i-th position, and certain elements from $Hom(M_i, M_j)$ in the remaining places (call the union of all these sets $\mathcal{A}$). $r_M(I) \cap M_i = r_{M_i}(I_i) \cap (\cap_{\varphi\in\mathcal{A}}(Ker(\varphi)))$. But $M_i' = r_{M_i}(I_i) \leq^{\oplus} M_i$, since M_i is a quasi-Baer module. If we take a $\varphi \in \mathcal{A}$, for example $\varphi : M_i \to M_j$, $i,j \in \mathcal{I}$, $i \neq j$, then $\psi_{ji}\varphi \in I_i$, where $\psi_{ji} : M_j \to M_i$ is the monomorphism taking M_j into M_i; we obtain this by noting that if we multiply a morphism in I, having φ in the ji-position, with the morphism $(\chi_{kl})_{k,l\in\mathcal{I}}$, where $\chi_{kl} = 0$ for $(k,l) \neq (i,j)$ and $\chi_{ij} = \psi_{ji}$, then we get a morphism in I with $\psi_{ji}\varphi : M_i \to M_i$ in the ii-position. This means that $\psi_{ji}\varphi(M_i') = 0$; as ψ_{ji} is a monomorphism, hence $\varphi(M_i') = 0$, thus $M_i' \subseteq Ker(\varphi)$. Since $\varphi \in \mathcal{A}$ was arbitrarily chosen, $r_M(I) \cap M_i = r_{M_i}(I_i) \cap (\cap_{\varphi\in\mathcal{A}}(Ker(\varphi))) = M_i' \leq^{\oplus} M_i$.

Using this argument for all $i \in \mathcal{I}$ we obtain that

$$r_M(I) = \bigoplus_{i\in\mathcal{I}} M_i' \leq^{\oplus} \bigoplus_{i\in\mathcal{I}} M_i = M.$$

□

Corollary 3.3.4. *A projective module over a quasi-Baer ring is a quasi-Baer module.*

Proof. A free module over a quasi-Baer ring is quasi-Baer module, based on the Theorem 3.3.3 above. A summand of a quasi-Baer is a quasi-Baer module, by Theorem 3.3.1. Hence the result follows. □

Example 3.3.5. An infinitely generated free module M over a non-Dedekind commutative domain R is not a Baer R-module, as was observed in Chapter 2. On the other hand, it is a free R-module (since R is, in fact, quasi-Baer), thus M is a quasi-Baer module.

Example 3.3.6. The module $M = \mathbb{Z} \oplus \mathbb{Z}_2$ is extending, FI-extending (hence $\mathcal{K}$-cononsingular, FI-$\mathcal{K}$-cononsingular), but is not Baer, nor quasi-Baer (hence not K-nonsingular, nor FI-K-nonsingular).

Proof. The fact that M is extending is known. It's is hence also FI-extending. By our results (Theorem 2.12) we have that M is $\mathcal{K}$-cononsingular and hence also FI-$\mathcal{K}$-cononsingular.

M is not Baer since the kernel of the endomorphism: $\varphi(m, \hat{n}) = \hat{m}$ is $2\mathbb{Z} \oplus \mathbb{Z}_2$ which is not a summand in M. Hence, by Theorem 2.12, it cannot be $\mathcal{K}$-nonsingular.

Next we show that M is not quasi-Baer, which will also prove that it is not FI-$\mathcal{K}$-nonsingular.

Take the submodule $2\mathbb{Z} \oplus 0$. It is fully invariant, due to the following: $2\mathbb{Z} \trianglelefteq \mathbb{Z}$; there are no morphisms from $\mathbb{Z}_2$ to $\mathbb{Z}$; any morphism from $\mathbb{Z}$ to $\mathbb{Z}_2$ will bring $2\mathbb{Z}$ in 0; $0 \trianglelefteq \mathbb{Z}_2$.

Assume now $l_S(2\mathbb{Z} \oplus 0) \leq^{\oplus} S$. Then $r_M(l_S(2\mathbb{Z} \oplus 0)) \trianglelefteq^{\oplus} M$. Moreover, $2\mathbb{Z} \oplus 0 \leq r_M(l_S(2\mathbb{Z} \oplus 0))$. The only fully invariant summand of M that satisfies this condition is M itself (fully invariant submodules project onto a direct sum decomposition by respective intersection with summands; we only have four choices, given uniformity of $\mathbb{Z}$ and $\mathbb{Z}_2$: $0 \oplus 0$, $\mathbb{Z} \oplus 0$, $0 \oplus \mathbb{Z}_2$ and M; the second choice is not fully invariant, though). But we have that the endomorphism corresponding to the matrix with zero everywhere except the lower right corner, where we have $1_{\mathbb{Z}_2}$, is in $l_S(2\mathbb{Z} \oplus 0)$, and so $r_M(l_S(2\mathbb{Z} \oplus 0)) \neq M$. □

In [10] it was shown that a nonsingular summand of an FI-extending module is FI-extending, thus answering in the positive, for this particular case, the question whether direct summands of an FI-extending module are FI-extending. We can generalize the result, by proving that direct summands of an FI-$\mathcal{K}$-nonsingular FI-extending module are FI-extending.

Proposition 3.3.7. *Let M be a FI-$\mathcal{K}$-nonsingular FI-extending module. Then $\forall$ $N \leq^{\oplus} M$, N is FI-extending.*

Proof. If M is FI-extending and FI-$\mathcal{K}$-nonsingular, by Remark 3.2.7 M is in fact strongly FI-extending. It is known [10] that summands of strongly FI-extending modules are strongly FI-extending, hence FI-extending. $\square$

Definition 3.3.8. Let M_1 and M_2 be quasi-Baer modules. We call them *relative quasi-Baer modules* iff

$$\bigcap_{\varphi \in Hom(M_i, M_j)} Ker\varphi \leq^{\oplus} M_i$$

for $i, j \in \{1, 2\}$, $i \neq j$.

Theorem 3.3.9. *Let $\{M_i\}_{i \in \mathcal{I}}$ be a class of modules. If $M = \bigoplus_{i \in \mathcal{I}} M_i$ is quasi-Baer then M_i is quasi-Baer and M_i, M_j are relative quasi-Baer, $\forall$ $i, j \in \mathcal{I}$, $i \neq j$ ($\mathcal{I}$ is an index set).*

Proof. We already know that direct summands of quasi-Baer modules are quasi-Baer (Theorem 3.3.1). We have to show that each pair of summands M_i, M_j in the above decomposition are relative quasi-Baer, for $i \neq j$.

For the sake of simplifying notation we consider two elements of the decomposition, which we label M_1 and M_2 and we use the fact that $M_1 \oplus M_2$ is quasi-Baer, by Theorem 3.3.1. Let $K_i = \bigcap_{\varphi \in Hom(M_i, M_j)} Ker\varphi$, $i, j \in \{1, 2\}$, $i \neq j$.

We show first that $K_1 \oplus K_2 \trianglelefteq M_1 \oplus M_2$. Take $\alpha \in End(M_1 \oplus M_2)$; i.e.

$$\alpha = \begin{pmatrix} \varphi_{11} & \varphi_{12} \\ \varphi_{21} & \varphi_{22} \end{pmatrix}$$

where $\varphi_{ij} : M_j \to M_i$, for $i, j \in \{1, 2\}$. Obviously $\varphi_{12}(K_2) = 0$ and $\varphi_{21}(K_1) = 0$ (by definition of K_1, K_2). Consider $\varphi_{11}(K_1)$. Taking any $\psi \in Hom(M_1, M_2)$, $\psi(\varphi_{11}(K_1)) = (\psi(\varphi_{11}))(K_1) = 0$ as $(\psi(\varphi_{11})) \in Hom(M_1, M_2)$ and by the definition of K_1. Hence $\varphi_{11}(K_1) \subseteq K_1$. Similarly, $\varphi_{22}(K_2) \subseteq K_2$. Putting the above together we obtain that $\alpha(K_1 \oplus K_2) \subseteq K_1 \oplus K_2$. Since α was arbitrarily chosen, we get that $K_1 \oplus K_2 \trianglelefteq M_1 \oplus M_2$.

Note that this also implies that $K_1 \trianglelefteq M_1$, $K_2 \trianglelefteq M_2$.

Let's now show that $K_1 \oplus K_2 \leq^{\oplus} M_1 \oplus M_2$. Take $l_{S_{12}}(K_1 \oplus K_2)$, where $S_{12} = End(M_1 \oplus M_2)$.

Considering $\alpha \in l_{S_{12}}(K_1 \oplus K_2)$, α a matrix as above, and $k_1 + k_2 \in K_1 \oplus K_2$, we notice the following: $\varphi_{11}(k_1) + \varphi_{12}(k_2) = \varphi_{11}(k_1) = 0 \Rightarrow \varphi_{11} \in l_{S_1}(K_1)$ and $\varphi_{21}(k_1) + \varphi_{22}(k_2) = \varphi_{22}(k_2) = 0 \Rightarrow \varphi_{22} \in l_{S_2}(K_2)$, $S_1 = End(M_1)$, $S_2 = End(M_2)$. At the same time, $\alpha \in End(M_1 \oplus M_2)$ so that $\varphi_{11} \in l_{S_1}(K_1)$, $\varphi_{22} \in l_{S_2}(K_2)$ and φ_{12}, φ_{21} arbitrary in their respective Homs will have the property $\alpha \in l_{S_{12}}(K_1 \oplus K_2)$. Hence

$$l_{S_{12}}(K_1 \oplus K_2) = \begin{pmatrix} l_{S_1}(K_1) & Hom(M_2, M_1) \\ Hom(M_1, M_2) & l_{S_2}(K_2) \end{pmatrix}$$

Take now $r_{M_1 \oplus M_2}(l_{S_{12}}(K_1 \oplus K_2))$. Since $l_{S_{12}}(K_1 \oplus K_2) \trianglelefteq S_{12}$, $r_{M_1 \oplus M_2}(l_{S_{12}}(K_1 \oplus K_2)) \trianglelefteq M_1 \oplus M_2$, hence it decomposes onto the two components, $r_{M_1 \oplus M_2}(l_{S_{12}}(K_1 \oplus K_2)) = K_1' \oplus K_2'$, where $K_1' = r_{M_1 \oplus M_2}(l_{S_{12}}(K_1 \oplus K_2)) \cap M_1$, $K_2' = r_{M_1 \oplus M_2}(l_{S_{12}}(K_1 \oplus K_2)) \cap M_2$. We analyze the two components separately. Take $\alpha \in l_{S_{12}}(K_1 \oplus K_2)$ (a matrix as above); $\alpha(k_1') = 0 \Rightarrow \varphi_{11}(k_1') = 0$ and $\varphi_{21}(k_1') = 0$, for $k_1' \in K_1' \Rightarrow$

$K_1' = r_{M_1}(l_{S_1}(K_1) \cap (\bigcap_{\psi \in Hom(M_1,M_2)} Ker\psi)$. Since $\bigcap_{\psi \in Hom(M_1,M_2)} Ker\psi = K_1$, and $r_{M_1}(l_{S_1}(K_1)) \supseteq K_1$, $K_1' = K_1$. Similarly for $K_2' = K_2$.

As a result, we obtain that $r_{M_1 \oplus M_2}(l_{S_{12}}(K_1 \oplus K_2)) = K_1 \oplus K_2$. In addition to this, since $M_1 \oplus M_2$ is quasi-Baer, $l_{S_{12}}(K_1 \oplus K_2) \leq^{\oplus} {}_{S_{12}}S_{12} \Rightarrow r_{M_1 \oplus M_2}(l_{S_{12}}(K_1 \oplus K_2)) \leq^{\oplus} M_1 \oplus M_2$. Hence $K_1 \oplus K_2 \leq^{\oplus} M_1 \oplus M_2$.

In conclusion (since the indexes were chosen arbitrarily), if M is quasi-Baer, then M_i is quasi-Baer and M_i, M_j are relative quasi-Baer, $\forall\, i, j \in I$, $i \neq j$. $\square$

Remark 3.3.10. Let M_1 and M_2 be two quasi-Baer modules that are relatively quasi-Baer. Then we can decompose $M_i = K_i \oplus M_i'$, where $K_i = \bigcap_{\varphi \in Hom(M_i,M_j)} Ker\varphi \leq^{\oplus} M_i$, for $i \neq j$, $i, j = 1..2$. By Theorem 3.3.1 K_1, K_2, M_1', M_2' are quasi-Baer. We have that M_1' and M_2' satisfy the condition in Theorem 3.3.2, hence $\overline{M} = M_1' \oplus M_2'$ is quasi-Baer. At the same time, $\overline{K} = K_1 \oplus K_2$ is quasi-Baer, since $K_1, K_2 \trianglelefteq K_1 \oplus K_2$ for obvious reasons (they consist of elements that are mapped into zero). We need, thus, find condition for $\overline{M} \oplus \overline{K}$ be quasi-Baer, for which the endomorphism ring will have only zero maps between $\overline{K}$ and $\overline{M}$.

CHAPTER 4

ENDOMORPHISM RINGS

Since the Baer and quasi-Baer properties of modules have been defined by us in terms of idempotent endomorphisms, in the endomorphism ring of a module, it is of interest to investigate connections of properties of the endomorphisms rings with those of the module. In this chapter, we will study these connections. In particular, we will investigate the transfer of some properties between a module and its endomorphism ring. A characterization for the module to be Baer is provided in terms of its endomorphism ring. Kaplansky ([22]) introduced a type theory for Baer rings. Berberian ([2]) expanded the type theory to further details. The type theory was developed and extended by Goodearl and Boyle ([16], [17]) for self-injective regular rings and for nonsingular injective modules, respectively. Recently, in [34] the authors provided a generalized approach to type theory decompositions, applicable to nonsingular injective modules. In this chapter, we use the properties of idempotents of the endomorphism ring of a Baer module to provide a type theory for Baer modules, similar to the type theory for Baer rings. Some finiteness conditions related to Baer modules are studied, and applications included.

4.1 (Quasi-) Baer modules and endomorphism rings

Our first result shows that the ring of endomorphisms of a Baer or a quasi-Baer module inherits the property without any additional conditions.

Theorem 4.1.1. *Let M be a Baer (respectively, quasi-Baer) module. Then $S = End(M)$ is a Baer (respectively, quasi-Baer) ring.*

Proof. Let $I \leq S$ be a left (respectively, two-sided) ideal. Since M is Baer (respectively, quasi-Baer), $r_M(I) \leq^{\oplus} M$, thus there exists $e^2 = e \in S$ such that $r_M(I) = eM$. We claim that $r_S(I) = eS$ also holds. For any $e\psi \in eS$, we observe that $Ie\psi = 0$, as for $\forall\ x \in M$, $Ie\psi(x) \subseteq IeM = 0$. Therefore $IeS = 0$, and $eS \subseteq r_S(I)$. Next, let $\varphi \in r_S(I)$ be any element; then we can write $\varphi = e\varphi + (1-e)\varphi$. Since $I\varphi = 0$, $I\varphi(M) = 0 \Rightarrow I(\varphi(M)) = 0$. Hence $\varphi(M) \subseteq r_M(I) = eM$. Let $m \in M$ be arbitrary; then $\varphi(m) = em' \Rightarrow e\varphi(m) = em' = \varphi(m) \Rightarrow e\varphi = \varphi$. Hence $\varphi \in eS$ which yields $eS = r_S(I)$. $\square$

As an application, we can prove the following result, from [31].

Proposition 4.1.2. *Let M be an extending module such that its endomorphism ring S is a regular ring. Then M is a Baer module, and subsequently S is a Baer ring.*

Proof. In view of Theorem 2.2.2 we only have to show that M is $\mathcal{K}$-nonsingular. Take $\varphi \in S$ so that $r_M(S\varphi) = Ker(\varphi) \leq^{e} M$. Since S is regular, there exists $\psi \in S$ so that $\varphi = \varphi\psi\varphi$, hence $\psi\varphi = (\psi\varphi)(\psi\varphi)$ is an idempotent with the property that $S\varphi = S\psi\varphi$; but then $r_M(S\varphi) = r_M(\psi\varphi) = (1-\psi\varphi)M \leq^{\oplus} M$. Hence $Ker(\varphi) = r_M(S\varphi) = M \Rightarrow \varphi = 0$. $\square$

Converse of Theorem 4.1.1 is not true in general. Namely, the fact that the endomorphism ring of a module is Baer or quasi-Baer does not imply that the module itself is Baer or quasi-Baer, as the next example shows.

Example 4.1.3. Let $M = \mathbb{Z}_{p^\infty}$, considered as a $\mathbb{Z}$-module. Then it is well-known that $End_{\mathbb{Z}}(M)$ is the ring of p-adic integers (Example 3, page 216 in [15]). Since the ring of p-adic integers is a commutative domain, it is a (quasi-) Baer ring. However $M = \mathbb{Z}_{p^\infty}$ is not a (quasi-) Baer module.

We recall the following definition.

Definition 4.1.4. A module M is called *retractable* if $Hom(M, N) \neq 0$, $\forall\ 0 \neq N \leq M$ (or, equivalently, $\exists\ 0 \neq \varphi \in S$ with $Im(\varphi) \subseteq N$).

There is also a weaker version of retractability, defined below (see [24]).

Definition 4.1.5. A module is called *e-retractable* if $Hom(M, N) \neq 0$, $\forall\ 0 \neq N \leq^c M$.

Proposition 4.1.6. *Let M be retractable. Then M is Baer if and only if S is a Baer ring.*

Proof. The direct implication has already been shown, in Theorem 4.1.1 . We now prove the reverse implication. Let $I \leq {}_S S$; since S is Baer, $r_S(I) = eS$ for $e^2 = e \in S$. Hence, $r_M(I) \supseteq eM$. Assume $\exists\ m \in M \setminus eM$ so that $Im = 0$; $m = em + (1-e)m$ with $(1-e)m \neq 0$. Since $em \in r_M(I)$, take $0 \neq (1-e)m = m - em \in r_M(I) \cap (1-e)M$. By retractability, there exists $0 \neq \varphi \in S$, $Im(\varphi) \subseteq (1-e)mR$. But in this case, $I\varphi M \subseteq I(1-e)mR = 0$, hence $\varphi \in r_S(I)$. But $\varphi = (1-e)\varphi \in eS \cap (1-e)S = 0$, absurd. Hence $r_M(I) = eM$, implying that M is a Baer module. $\square$

Same holds for quasi-Baer modules and their endomorphism rings.

Proposition 4.1.7. *Let M be retractable. Then M is quasi-Baer if and only if S is a quasi-Baer ring.*

Proof. The direct implication is true, by Theorem 4.1.1.

For the converse let $I \trianglelefteq S$; since S is quasi-Baer, $r_S(I) = eS$ for $e^2 = e \in S$. Hence, $r_M(I) \supseteq eM$. Assume $\exists\ m \in M \setminus eM$ so that $Im = 0$; without loss of generality we can assume $0 \neq m \in (1-e)M$. By retractability, there exists $0 \neq \varphi \in S$, $Im(\varphi) \subseteq mR$. But in this case, $I\varphi M \subseteq ImR = 0$, hence $\varphi \in r_S(I)$. But $\varphi = (1-e)\varphi \in eS \cap (1-e)S = 0$, absurd. Hence $r_M(I) = eM$, implying that M is a quasi-Baer module. □

Example 4.1.8. Free modules are retractable, as the following will prove. If $M = R^{(I)}$, where R is a ring, then $\forall\ 0 \neq N \leq M$, we can take $0 \neq n \in N$. Since morphisms are defined by the values the 1s, in every summand, respectively, are mapped into, construct the morphism $\varphi : M \to M$ which maps the 1 from only one of the summands to n, and the other 'units' to 0. The image of this morphism is $0 \neq nR \leq N$, hence M is retractable.

Remark 4.1.9. If M is e-retractable, but not retractable, the Baer property does not pass from the endomorphism ring to the module, in general. In Example 4.1.3, $\mathbb{Z}_{p^\infty}$ is a uniform module, hence the only essentially closed submodules are 0 and itself, hence it is e-retractable by default. Yet, while its endomorphism ring is Baer, the module is not Baer module.

To obtain a full characterization, we note that Baer modules have an intrinsic 'retractability'.

Definition 4.1.10. A module M is called *quasi-retractable* if $Hom(M, r_M(I)) \neq 0$, $\forall\ 0 \neq r_M(I)$, $I \leq {}_S S$ (or, equivalently, if $r_M(I) \neq 0$ then $r_S(I) \neq 0$, $\forall\ I \leq {}_S S$).

In the next result we drop the requirement of retractability from Proposition 4.1.6 and provide a complete characterization of a Baer module in terms of its endomorphism ring.

Theorem 4.1.11. *A module M is a Baer module if and only if its endomorphism ring S is a Baer ring and M is quasi-retractable.*

Proof. For the necessity we only need prove that M is quasi-retractable if M is Baer. As M is Baer, $r_M(I) = eM$ for $e^2 = e \in S$. Assuming that $eM \neq 0 \Rightarrow I(eM) = 0 \Rightarrow (Ie)M = 0 \Rightarrow Ie = 0$. Thus $0 \neq e \in r_S(I)$.

For sufficiency, take $I \leq_S S$, and we have, since S is a Baer ring, $r_S(I) = eS$ where $e^2 = e \in S$ is an idempotent. This implies that $I \subseteq l_S(r_S(I)) = S(1-e)$. Hence $eM \subseteq r_M(I)$, since $\varphi e = 0 \Rightarrow \varphi(eM) = 0$, $\forall\ \varphi \in I$. Assume that $\exists\ 0 \neq m_0 = (1-e)m_0 \in r_M(I)$. Taking now the left left ideal $J = I + Se \leq {}_S S$, since S is Baer we have $r_S(J) = r_S(I) \cap r_S(e) = eS \cap (1-e)S = 0$. But $m_0 \in r_M(J)$, since $Im_0 = 0$ and $em_0 = e(1-e)m_0 = 0$, a contradiction since M is quasi-retractable. $\square$

This new concept is a generalization of retractability.

Lemma 4.1.12. *A retractable module is quasi-retractable.*

Proof. Let $I \leq {}_S S$ so that $r_M(I) \neq 0$. By retractability, $\exists\ \varphi \in S$ so that $0 \neq \varphi M \subseteq r_M(I)$. But in this case $I(\varphi(M)) = 0 \Rightarrow (I\varphi)M = 0 \Rightarrow I\varphi = 0 \Rightarrow 0 \neq \varphi \in r_S(I)$. $\square$

Each of the following examples exhibits an R-module M which satisfies quasi-retractability but which is not retractable (thus the class of retractable modules is a proper subclass of the class of quasi-retractable modules).

Example 4.1.13. This example is due to Chatters and is also included in [24], Example 3.4. (see also Example 5.17, in [23]).

Let K be a subfield of complex numbers $\mathbb{C}$. Let R be the ring $\begin{bmatrix} K & 0 \\ \mathbb{C} & \mathbb{C} \end{bmatrix}$. The R is left nonsingular left extending ring. Consider the module $M = Re$ where $e = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Then M is projective, extending and nonsingular (as it is summand of R) hence is Baer - thus it is quasi-retractable (it is obviously also e-retractable). But M is not retractable, since the endomorphism ring of M, which is isomorphic to K, consists isomorphisms and the zero endomorphism; on the other hand, M is not simple, and so by retractability it should have endomorphisms which are not onto. See also Theorem 4.10 in [35].

Example 4.1.14. ([23], Example 5.12) Let $R = \begin{bmatrix} \mathbb{C} & \mathbb{C} & \mathbb{C} \\ \mathbb{C} & \mathbb{R} & \mathbb{C} \\ \mathbb{C} & \mathbb{C} & \mathbb{C} \end{bmatrix}$. Let $M = Rf$ where $f = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Then M is a nonsingular, projective extending left R-module, hence quasi-retractable, but M is not retractable and $End(M) = fRf$ is not a left extending ring.

As an application, we provide a number of results.

A necessary and sufficient condition for a matrix ring over a commutative integral domain to be Baer is given by Corrolary 15 in [38] and page 17 in [22].

Theorem 4.1.15. *If R is a commutative integral domain, then $M_n(R)$ is a Baer ring (for $n > 1$) if and only if every finitely generated right ideal of R is invertible, i.e. if R is a Prüfer domain.*

We can show the following as a consequence.

Theorem 4.1.16. *Let R be a commutative integral domain. Then any finite rank (strictly greater than 1) free module M over R is Baer if and only if R is a Prüfer domain.*

Proof. If R is a Prüfer domain, then the endomorphism ring of M, ring isomorphic to $M_n(R)$, is Baer. A free module is retractable, and using Proposition 4.1.6 we obtain that M is a Baer module.

If M is a Baer module, then its endomorphism ring is Baer, hence $M_n(R)$ for $n > 1$ is a Baer ring, thus R must be a Prüfer domain. $\square$

We have a result in a more general setting.

Theorem 4.1.17. *Let R be a domain (hence a Baer ring). Then any (finitely generated) free module over R is a Baer module implies that R is right (semi-) hereditary.*

Proof. Let $I \leq R$ be a (finitely generated) right ideal of R. Then there exists $\mathcal{K}$ a (finite) index set with the property that $\exists\ \varphi : R^{(\mathcal{K})} \to I$ epimorphism $\Rightarrow I \cong R^{(\mathcal{K})}/Ker\varphi$. But φ can be viewed as an endomorphism of $R^{(\mathcal{K})}$, $\varphi : R^{(\mathcal{K})} \to I \leq R \leq R^{(\mathcal{K})}$, hence, since $R^{(\mathcal{K})}$ is Baer, $Ker\varphi \leq^{\oplus} R^{(\mathcal{K})}$, hence I is isomorphic to a summand of a free module, thus I is projective. $\square$

Proposition 4.1.18. *Let M be a module with semisimple artinian endomorphism ring S. Then M is a Baer module.*

Proof. Note that a semisimple artinian ring is Baer, hence S is Baer.

Since every left ideal $I \leq {}_SS$ is a summand in ${}_SS$ (semisimple artinian ring is, in particular, semisimple left module over itself), $I = Se$ with $e^2 = e \in S$, $r_M(I) = (1-e)M \leq^{\oplus} M$, hence M is Baer. $\square$

Note. The module might not be semisimple artinian, in the hypothesis of the above Proposition, and hence it might not be retractable (for example, take $\mathbb{Q}_{\mathbb{Z}}$). On the other hand, if the module is retractable, M is semisimple artinian (a straightforward proof, which takes into account Wederburn-Artin's Theorem; see Remark 4.4.4).

4.2 Transfer of properties between the module and its endomorphism ring

Several authors, among whom we particularly mention Khuri (whose work we use), have discussed the transfer of properties from a module M to its endomorphism ring S, and from the ring S to the module M. These properties include: extending property, quasi-continuity, continuity, absolute direct summand property, nonsingularity; see, for example, [27]. Motivated by this, we extend our study that started with the transfer of Baer property between the module and its endomorphism ring in Section 4.1.

We define the Utumi property in the module-theoretic setting. Note that in literature, the notion of Utumi module was defined only in the presence of nonsingularity (see [25]); we refine this definition.

Definition 4.2.1. We call M a $\mathcal{K}$-Utumi module if it is $\mathcal{K}$-nonsingular and $\mathcal{K}$-cononsingular.

We can restate Theorem 3.6 in [24] as follows.

Theorem 4.2.2. *Let M be nonsingular and retractable. Then M is $\mathcal{K}$-Utumi if and only if S is a right Utumi ring. When these equivalent conditions hold, S is a Baer ring if and only if M is extending.*

Proof. In [24] it is shown that S is Utumi if and only if, for each submodule $U \leq M$, $l_S(U) = 0$ implies $U \leq^e M$, which is the definition of $\mathcal{K}$-cononsingularity. Since the module M is nonsingular, hence $\mathcal{K}$-nonsingular, M is $\mathcal{K}$-Utumi if and only if M is $\mathcal{K}$-cononsingular.

By Theorem 2.2.2, if S is an Utumi ring and M is $\mathcal{K}$-Utumi, S is a Baer ring if and only if M is extending. $\square$

Next we show that the $\mathcal{K}$-nonsingularity property transfers between the module M and the endomorphism ring S if the module is retractable.

Proposition 4.2.3. *Let M be retractable. Then M is $\mathcal{K}$-nonsingular if and only if S is right nonsingular.*

Proof. Let M be a $\mathcal{K}$-nonsingular module. Let $\varphi \in S$, so that $r_S(\varphi) \leq^e S_S$. Assume $r_M(\varphi) = Ker(\varphi)$ not essential in M; hence, there exists a non-zero complement $N \leq M$, $N \cap Ker(\varphi) = 0$. By retractability, $\exists\ 0 \neq \psi \in S$, $Im\psi \subseteq N$. But $\varphi\psi \neq 0$ (as the image of ψ has zero intersection with the kernel of φ), thus $\psi S \cap r_S(\varphi) = 0$, since the image of any $\psi\psi'$ with $\psi' \in S$ is also a subset of N. This contradicts essentiality of $r_S(\varphi)$, hence $r_M(\varphi) \leq^e M \Rightarrow \varphi = 0$, by $\mathcal{K}$-nonsingularity of M.

For the converse, assume S right nonsingular. Let $\varphi \in S$, $r_M(\varphi) = Ker\varphi \leq^e M$. Assume there exists $\psi \in S$, $\psi S \cap r_S(\varphi) = 0$. This implies that $\varphi\psi S \neq 0$. But since $Ker\varphi \leq^e M$, $Im\psi \cap Ker\varphi \neq 0$. By retractability, $\exists\ 0 \neq \psi' \in S$, $Im\psi' \subseteq \psi^{-1}(Ker\varphi)$ $(0 \neq \psi^{-1}(Ker\varphi))$; $\varphi\psi\psi' = 0$ with $\psi\psi' \neq 0$, contradiction. Hence $r_S(\varphi) \leq^e S_S \Rightarrow \varphi = 0$. $\square$

Note. For the direct implication one needs only e-retractability, as the proof shows.

In the following result, we introduce a condition which allows for a similar transfer, for the $\mathcal{K}$-cononsingularity property this time, from the ring of endomorphisms S to the module M. The condition is stronger than retractability.

Proposition 4.2.4. *Let M be a module with the property that $\forall\ m \in M$, $\exists\ \varphi \in S$ with $\varphi M = mR$. If S is a right cononsingular ring then M is a $\mathcal{K}$-cononsingular module.*

Proof. Assume S is right cononsingular. Let $N \leq M$ with the property that $\varphi N \neq 0$, $\forall\ 0 \neq \varphi \in S$. Assume N is not essential in M, hence there exists $0 \neq P \leq M$ with $N \cap P = 0$. Take $I = \{\psi \in S | \psi M \subseteq N\}$. I is not zero, since it follows easily that M is retractable. Also, $I \leq S_S$, since $\psi\psi' M \subseteq \psi M \subseteq N$, $\forall\ \psi \in I$, $\psi' \in S$. Finally, $\varphi I \neq 0$, $\forall\ 0 \neq \varphi \in S$, since $\forall\ \varphi \in S$, $\exists\ n \in N$ with $\varphi n \neq 0$ (by hypothesis); $\exists\ \psi \in I$ with $\psi M = nR \Rightarrow \varphi\psi \neq 0$. Hence, since S is right cononsingular, $I \leq^e S_S$. But, if we take $p \in P$, $\exists\ \alpha \in S$ with $\alpha M = pR$; $\alpha S \cap I = 0$, as it is easily checked, and since $\alpha \neq 0$ we obtain a contradiction. Hence $N \leq^e M$, thus obtaining that M is $\mathcal{K}$-cononsingular. $\square$

The following example provides a case when $\mathcal{K}$-cononsingularity of M does not imply cononsingularity of S.

Example 4.2.5. Example 3.3 in [12] presents a module M that is nonsingular, e-retractable and extending, but whose endomorphism ring S is not extending. By Theorem 4.1.1, S is a Baer ring; since it is not extending, an application of Theorem 2.2.2 gives that S is not cononsingular. On the other hand, M is $\mathcal{K}$-Utumi, since it is nonsingular, hence $\mathcal{K}$-nonsingular, and extending, which implies that M is Baer and $\mathcal{K}$-cononsingular.

We are interested in connections between $\mathcal{K}$-nonsingularity and retractability. We obtain a generalization of Theorem 2.2 and Corollary 2.3 in [26].

Proposition 4.2.6. *Let M be $\mathcal{K}$-nonsingular. Then M is retractable if and only if "$U \leq^e V \leq M \iff \{\varphi \in S | \varphi M \subseteq U\} \leq^e \{\varphi \in S | \varphi M \subseteq V\}$".*

Proof. If $\psi \neq 0$, then $Ker\psi$ is not essential in M. Hence we can always take a nonzero submodule of M which is intersection complement of the $Ker\psi$, which we will call N_ψ.

Let us show now the direct implication. Let $U \leq^e V$. Let $\psi \in S$, $\psi M \subseteq V$. Take $\psi^{-1}(U) \cap N_\psi \neq 0$ (since $U \leq^e V$, there must exist elements from M that are mapped into U but not in zero; take a nonzero element in the image of ψ, multiplied conveniently so that it is still not zero while becoming an element of U). By retractability, there exists $\psi' \in S$, mapping M into $\psi^{-1}(U) \cap N_\psi$. $0 \neq \psi\psi'(M) \subseteq U$, implying the desired essentiality.

Conversely, if $\{\varphi \in S | \varphi M \subseteq U\} \leq^e \{\varphi \in S | \varphi M \subseteq V\}$ for certain $U, V \leq M$, assume U is not essential in V. Then there exists a complement of U in V, and a map $\psi \in S$ that maps M into this complement. $\psi S \cap \{\varphi \in S | \varphi M \subseteq U\} = 0$, yet $\psi S \subseteq \{\varphi \in S | \varphi M \subseteq V\}$, contradiction.

The sufficiency: let $0 \neq U \subseteq M$. Take a complement of U in M, U'. Then $U \oplus U' \leq^e M$. By hypothesis, $\{\varphi \in S | \varphi M \subseteq U \oplus U'\} \leq^e \{\varphi \in S | \varphi M \subseteq M\} = S$, hence $\{\varphi \in S | \varphi M \subseteq U \oplus U'\} \neq 0$. If $U' = 0$, take $0 \neq \varphi \in \{\varphi \in S | \varphi M \subseteq U\}$. If $U' \neq 0$, $\{\varphi \in S | \varphi M \subseteq U'\} \leq \{\varphi \in S | \varphi M \subseteq U \oplus U'\}$, but the inclusion cannot be essential, thus in particular we don't have equality. Take $0 \neq \psi \in \{\varphi \in S | \varphi M \subseteq$

$U \oplus U'\} \setminus \{\varphi \in S | \varphi M \subseteq U'\}$, and set $\varphi = \pi\psi$, where π is the canonical projection of $U \oplus U'$ onto U. The endomorphism $0 \neq \varphi$ has the property that $\varphi M \subseteq U$. $\square$

4.3 Type theory for Baer modules and nonsingular extending modules

Type theory for Baer rings was introduced by Kaplansky [22]. Goodearl and Boyle ([16], [17]) extended this theory to nonsingular injective modules. In this section we introduce type theory for Baer modules. In particular, this holds for nonsingular extending modules, and provides a decomposition of such modules into various types.

Definition 4.3.1. A module M is called *abelian* if all idempotent endomorphisms are central (i.e. commute with any endomorphism). An idempotent endomorphism e is called abelian if eM is an abelian module.

We characterize abelian Baer modules in the following.

Proposition 4.3.2. *For any M Baer module, the following conditions are equivalent:*

1. *M is abelian;*
2. *all direct summands of M are fully invariant;*
3. *isomorphic summands of M are equal;*
4. *if N_1, N_2 are summands of M and $N_1 \cap N_2 = 0$ then $Hom(N_1, N_2) = 0$.*

Proof. (1) $\Rightarrow$ (2): for $N \leq^{\oplus} M \Rightarrow N = eM$ with $e^2 = e \in S$; but e is central, hence $\varphi(eM) = (e\varphi)M \subseteq eM \ \forall\ \varphi \in S$. Thus $eM \trianglelefteq M$.

(2) $\Rightarrow$ (3): let $N_1, N_2 \leq^{\oplus} M$, so that $\exists\ \alpha : N_1 \to N_2$ isomorphism. $M = N_1 \oplus N_1'$; assume $N_2 \cap N_1' \neq 0 \Rightarrow N_2 \cap N_1' \leq^{\oplus} M \Rightarrow N_2 \cap N_1' \leq^{\oplus} N_2$. But then

$\alpha^{-1}(N_2 \cap N_1') \leq^{\oplus} N_1$, and we can construct a non-zero morphism from N_1 to N_1', equal to α on $\alpha^{-1}(N_2 \cap N_1')$ and 0 on its summand complement; but this is a contradiction, as $N_1 \trianglelefteq M$ (any map from N_1 to M can be extended to a map from M to M, and this map should invary N_1). Hence, $N_2 \cap N_1' = 0$.

Assume that $N_2 \not\subseteq N_1$, hence $\exists\, n \in N_2$ with $\pi_1'(n) \neq 0$, where π_1' is the canonical projection of M onto N_1'. Hence we can construct a nonzero map $\varphi : \pi_1(N_2) \to \pi_1'(N_2)$, where π_1 is the canonical projection of M onto N_1, by the following definition: $\varphi(\pi_1(m)) = \pi_1'(m)$. This map is well-defined, as zero can only be mapped into 0 since $\nexists\, m \in N_2 \cap N_1'$. Since $N_2 \cong N_1$ and $N_2 \cong \pi_1(N_2)$, there exists an isomorphism β, from N_1 to $\pi_1(N_2)$. Then $\varphi\beta : N_1 \to N_1'$, nonzero, hence a contradiction with the fact that N_1 is fully invariant. Thus $N_2 \subseteq N_1$.

By a similar argument we can show that $N_1 \subseteq N_2$, thus obtaining $N_1 = N_2$.

(3) $\Rightarrow$ (4) Assume $\exists\, 0 \neq \varphi : N_1 \to N_2$. For $M = N_1 \oplus N_1'$, π_1' the canonical projection of M onto N_1', we have that $\pi_1'\varphi \neq 0$. Construct the following submodule of M: $P = \{n + \pi_1'\varphi n | n \in N_1\}$. Note that $P \cap N_1' = 0$ and $P + N_1' = M$ since $N_1 \subseteq P + N_1'$; subsequently, $M = P \oplus N_1' \Rightarrow N_1 \cong P$. But this implies that $N_1 = P \Rightarrow \pi_1'\varphi = 0$ absurd. Hence $\nexists\, 0 \neq \varphi : N_1 \to N_2$.

(4) $\Rightarrow$ (1) For $N_1 \oplus N_2 = M$, $N_1 \cap N_2 = 0 \Rightarrow Hom(N_i, N_j) = 0$, $i \neq j$, $i, j = 1, 2$. But this is equivalent to $N_i \trianglelefteq M$, $i = 1, 2$, which in turn implies that e for which $N_1 = eM$ is central. $\square$

Proposition 4.3.3. *Let M be a Baer module.*

1. *If $N \leq^{\oplus} M$, and M is abelian, then N is an abelian Baer module;*

2. *Let M_i, $i \in \mathcal{I}$ be a family of modules. Then $\bigoplus_{i\in\mathcal{I}} M_i$ is abelian Baer module if and only if each M_i is abelian Baer module and $Hom(M_i, M_j) = 0$, $\forall\, i \neq j$, $i, j \in \mathcal{I}$.*

Proof. (1) (As an observation, since $N \leq^{\oplus} M \Rightarrow N \trianglelefteq^{\oplus} M$). For $P \leq^{\oplus} N \Rightarrow P \leq^{\oplus} M \Rightarrow P \trianglelefteq M$. But then $P = P \cap N \trianglelefteq N$.

(2) If $\bigoplus_{i\in I} M_i$ is an abelian Baer module then $M_i \leq^{\oplus} \bigoplus_{i\in\mathcal{I}} M_i$ is abelian Baer module (use part (1) and Theorem 2.3.1). Also, we obtain by Proposition 4.3.2, (4), $Hom(M_i, M_j) = 0$, $\forall\, i \neq j$, $i, j \in \mathcal{I}$.

Assume now that each M_i is abelian Baer module and $Hom(M_i, M_j) = 0$, $\forall\, i \neq j$, $i, j \in I$. Hence $M_i \trianglelefteq \bigoplus_{i\in\mathcal{I}} M_i$. By Proposition 2.4.15, $\bigoplus_{i\in\mathcal{I}} M_i$ is a Baer module. Also, for any summand $N \leq^{\oplus} \bigoplus_{i\in\mathcal{I}} M_i$, $N = \bigoplus_{i\in\mathcal{I}} N \cap M_i$ (application of Lemma 1.3.18). Hence, $\forall\, i \in \mathcal{I}$, $N \cap M_i \leq^{\oplus} M_i$ and so $N \cap M_i \trianglelefteq M_i \trianglelefteq \bigoplus_{i\in\mathcal{I}} M_i$. Moreover, $N = \bigoplus_{i\in\mathcal{I}} N \cap M_i \trianglelefteq \bigoplus_{i\in\mathcal{I}} M_i$, as it is easily checked. □

Definition 4.3.4. A ring R is called *directly finite* if $xy = 1 \Rightarrow yx = 1$, $\forall\, x, y \in R$. A module M is called directly finite if $End(M)$ is a directly finite ring. An idempotent endomorphism e is called directly finite if eM is a directly finite module.

Definition 4.3.5. A module that is not directly finite will be called *directly infinite.*

We have a characterization of directly finite modules in terms of direct summands (see Theorem 3.1 in [17]).

Proposition 4.3.6. *A module M is directly finite if and only if M is not isomorphic to any proper summand of itself.*

Proof. Suppose that φ is an isomorphism of M onto a summand $N \leq^{\oplus} M$. The inverse map φ^{-1} can be extended to the map $\varphi^{-1}\pi_N : M \to M$, where π_N is the canonical projection of M onto N. We get that $(\varphi^{-1}\pi_N)\varphi = 1$. Since M is directly finite, $\varphi\varphi^{-1}\pi_N = 1 \Rightarrow N = M$.

Assume for the converse that M is not isomorphic to any proper summand of itself. Let $x, y \in S$ so that $xy = 1$ (of course, $x, y \neq 0$). Then $yxyx = y \cdot 1 \cdot x = yx \Rightarrow yx$ is an idempotent. Then yx is the canonical projection of M onto yxM. Since $xy = 1$, x is an epimorphism, and y is a monomorphism. Thus $xM = M$, and $y(xM) \cong xM = M$. But yxM is a summand of M, isomorphic to M, hence it cannot be proper. yx cannot be 0 (otherwise $y = yxy = 0$ and $x = xyx = 0$), hence $yx = 1$. $\square$

Proposition 4.3.7. *Let M be a Baer module. Then the following hold:*

1. *if M is an abelian module, then M is directly finite;*
2. *$N \leq^{\oplus} M$ and M directly finite; then N is directly finite Baer module;*
3. *Let (M_i), $i \in \mathcal{I}$ a family of modules with $Hom(M_i, M_j) = 0 \ \forall\, i \neq j$, $i, j \in \mathcal{I}$ ($\mathcal{I}$ an index set); then $\bigoplus_{i \in \mathcal{I}} M_i$ is directly finite Baer module if and only if M_i is a directly finite Baer module, $\forall\, i \in \mathcal{I}$.*

Proof. (1) Let $x, y \in S$ with $xy = 1 \Rightarrow (yx)(yx) = y(xy)x = yx$, thus yx is an idempotent. Since M is abelian, yx commutes with every endomorphism, hence $1 = (xy)(xy) = x(yx)y = (yx)(xy) = yx$.

(2) For any endomorphisms $x, y \in End(N)$ with $xy = 1$, we can extend x and y to endomorphisms of $M = N \oplus N'$ to $x \oplus id_{N'}$ and $y \oplus id_{N'}$, where $id_{N'}$ is the identity map on N'. Note that $x \oplus id_{N'} \cdot y \oplus id_{N'} = xy \oplus id_{N'} = 1$, thus $y \oplus id_{N'} \cdot x \oplus id_{N'} = yx \oplus id_{N'} = 1 \Rightarrow yx = 1$.

(3) If $\bigoplus_{i \in \mathcal{I}} M_i$ is a directly finite Baer module, by Theorem 2.3.1 and part (2), M_i is a directly finite Baer module, $\forall\, i \in \mathcal{I}$.

To show the converse, note that $\bigoplus_{i \in \mathcal{I}} M_i$ is a Baer module, by Theorem 2.4.15. To prove direct finiteness, take two endomorphisms $x, y \in End(\bigoplus_{i \in \mathcal{I}} M_i)$. Since $Hom(M_i, M_j) = 0$, $\forall\, i \neq j$, $i, j \in \mathcal{I}$, $xM_i \subseteq M_i$, $\forall\, i \in \mathcal{I}$, and similarly for y. Hence we can decompose $x = \oplus_{i \in \mathcal{I}} x_i$ and $y = \oplus_{i \in \mathcal{I}} y_i$, with $x_i, y_i \in End(M_i)$. We obtain $1 = xy = \oplus_{i \in \mathcal{I}} x_i y_i \Rightarrow x_i y_i = id_{M_i} \Rightarrow y_i x_i = id_{M_i}$, $\forall\, i \in \mathcal{I}$. Thus $yx = \oplus_{i \in \mathcal{I}} y_i x_i = 1$ $\square$

Definition 4.3.8. An idempotent e in a Baer ring is called *faithful* if 0 is the only central idempotent orthogonal to e. Equivalently, e is faithful if the smallest central idempotent v in S satisfying $ve = e$ is 1.

Note. The existence of the smallest central idempotent v is verified in a Baer ring (see [22]). This v is called the *central cover* of e.

At this point we can present the description of the three main types which occur in the decomposition theory of Baer rings.

Definition 4.3.9. A Baer ring is of *type I* if it has a faithful abelian idempotent. A Baer ring is of *type II* if it has a faithful finite idempotent, but no non-zero abelian idempotents. A Baer ring is of *type III* if it has no nonzero finite idempotents. A Baer ring is *purely infinite* if it has no nonzero central finite idempotents.

In [22] it is proven that any Baer ring can be decomposed into ring direct summands of these three, main, types. Also, by decomposing the type I and, respectively, type II summands into a sum between a directly finite and a purely infinite part, we obtain a total of five types. A Baer ring decomposes, thus, uniquely into a sum of five components, as described below.

Theorem 4.3.10. *A Baer ring decomposes uniquely into a ring direct sum of Baer rings of types: I, directly finite (I_f); I, purely infinite (I_∞); II, directly finite (II_f); II, purely infinite (II_∞); III.*

We define the five types of Baer modules in terms of the types of their endomorphism rings.

Definition 4.3.11. We call a Baer module M of type (T) if $S = End(M)$ is of type (T) (where T is one of the five types described above: I_f; I_∞; II_f; II_∞; III).

Note. The definition is valid, as the endomorphism ring of a Baer module is Baer, by Theorem 4.1.1.

We get the decomposition theory as a consequence of the decomposition at the endomorphism ring level.

Theorem 4.3.12. *A Baer module decomposes uniquely into a sum of fully invariant summands of types I_f; I_∞; II_f; II_∞; III.*

Proof. Let M be a Baer module. Then $S = End(M)$ is a Baer ring, by Theorem 4.1.1. Hence S decomposes uniquely, as a ring direct decomposition, into $S = Se_{I_f} \oplus Se_{I_\infty} \oplus Se_{II_f} \oplus Se_{II_\infty} \oplus Se_{III}$. Since this is a ring direct decomposition, $M = e_{I_f}M \oplus e_{I_\infty}M \oplus e_{II_f}M \oplus e_{II_\infty}M \oplus e_{III}M$ has the property that each of the 5 summands is fully invariant in M (each idempotent occuring is central). The endomorphism ring of $e_{I_f}M$ is $e_{I_f}Se_{I_f} = Se_{I_f}$, hence $e_{I_f}M$ is of type I_f; the endomorphism ring of $e_{I_\infty}M$ is $e_{I_\infty}Se_{I_\infty} = Se_{I_\infty}$, hence $e_{I_\infty}M$ is of type I_∞. Similarly for the remaining three summands.

To prove uniqueness, assume $M = f_{I_f}M \oplus f_{I_\infty}M \oplus f_{II_f}M \oplus f_{II_\infty}M \oplus f_{III}M$ is another decomposition with each summand fully invariant, and each summand of, respectively, type I_f; I_∞; II_f; II_∞; III. Then $S = Sf_{I_f} \oplus Sf_{I_\infty} \oplus Sf_{II_f} \oplus Sf_{II_\infty} \oplus Sf_{III}$ is a ring direct decomposition; since the type decomposition is unique, we get that $Se_{I_f} = Sf_{I_f} \Rightarrow e_{I_f}M = f_{I_f}M$, and similarly equalities for the remaining 4 types. □

Goodearl in [17] obtained a number of characterizations for nonsingular injective modules of various types. In our general setting, we extend these characterizations partially.

Proposition 4.3.13. *Let M be a Baer module. Then M is of type I if every nonzero summand of M contains a nonzero abelian summand.*

Proof. By Theorem 4.3.12 and Propositions 4.3.3 and 4.3.7, the summands of M are solely of type I, hence M is of type I, since its decomposition cannot include other types. □

Proposition 4.3.14. *Let M be a Baer module. Then M is of type II if every nonzero summand of M contains a directly finite summand, but M has no nonzero abelian summands.*

Proof. By Theorem 4.3.12 and Propositions 4.3.3 and 4.3.7, summands of M are either of type I or II. Since M does not have nonzero abelian summands, summands cannot include faithful abelian summands, hence its decomposition into types will only include type II summands. Thus, M is of type II. □

Proposition 4.3.15. *A Baer module M is of type III if $N \cong N \oplus N$, $\forall\ N \leq^\oplus M$.*

Proof. Take a nonzero summand $N \leq^\oplus M$. Since N is isomorphic to a proper summand of itself, by hypothesis, N is not directly finite, by Proposition 4.3.6. Hence M does not contain nonzero finite idempotents, which implies that M is of type III. □

4.4 Finiteness conditions

In [28] it is analyzed how the structure of a Baer ring is determined by the cardinality of the set of its idempotents. This result motivates us to analyze Baer modules in terms of cardinality of their summands.

Theorem 4.4.1. *(Theorem 2, Theorem 3 [28]) If R is a Baer ring with only countably many idempotents, then R has no infinite sets of orthogonal idempotents. If, in addition, R is a regular ring, then R is a semisimple Artinian ring.*

In [14] it is shown (Proposition 10.4) that any module which does not have infinitely many direct summands has a finite direct sum decomposition. In our first result in this section we show that if the module is Baer, we have the same conclusion under weaker assumptions.

Proposition 4.4.2. *If M is a Baer module, with only countably many direct summands, then M is a finite direct sum of indecomposable summands.*

Proof. Since M is Baer, S is Baer by Theorem 4.1.1. Since M has countably many direct summands, then S has only countably many idempotents. By Theorem 4.4.1, S has no infinite sets of orthogonal idempotents, hence any direct sum decomposition of M must be finite, thus M has a finite indecomposable direct sum decomposition. $\square$

On the same topic, we have a number of conditions which ensure that a Baer module is a semisimple Artinian module.

Theorem 4.4.3. *Let M be a Baer module with only countably many direct summands. Then M is semisimple Artinian if any of the following conditions hold:*

(i) M is retractable and S is a regular ring;

(ii) every cyclic submodule of M is a direct summand of M; or

(iii) $\forall\ m \in M$, $\exists\ f \in Hom(M, R_R)$ such that $m = mfm$ (Zelmanowitz [39] calls such a module a regular *module).*

Proof. Suppose (i) holds. By Theorem 4.1.1 and in view of Proposition 4.4.2, S becomes a regular Baer ring with only countably many idempotents. Then S is a semisimple Artinian ring, by Theorem 3 [28].

Since S is a semisimple Artinian ring, it can be decomposed into a finite, ring direct sum of simple Artinian rings, $S = \bigoplus_{1 \leq i \leq n} Se_i$, where $n \in \mathbb{N}$ (by Wederburn-Artin Theorem, for example in [1]). Note that all e_i are central idempotents. Hence we obtain the following module direct decomposition: $M = \bigoplus_{1 \leq i \leq n} e_i M$, where $e_i M$ are fully invariant submodules, and summands, of M. It is easy to see that fully invariant summands of a retractable module are also retractable.

If we can show that, for any $1 \leq i \leq n$, $e_i M$ is semisimple Artinian, we're done. To simplify notation, and without losing generality, we can assume S is simple Artinian.

Again, by Wederburn-Artin Theorem we know that a simple Artinian ring is isomorphic to a finite, $m \times m$ matrix ring over a field K, where $m \in \mathbb{N}$. Denote by ϵ_{ij} elements of the form: 1 in the position (ij) and 0 everywhere else ($1 \leq i, j \leq m$). The idempotents corresponding to ϵ_{ii} (via the ring isomorphism) produce a direct sum decomposition of $M = \bigoplus_{1 \leq i \leq m} M_i$, in which all summands have an endomorphism ring isomorphic to the field K. Moreover, for $1 \leq i \neq j \leq m$, $\epsilon_{ij}\epsilon_{ji} = \epsilon_{ii}$ and $\epsilon_{ji}\epsilon_{ij} = \epsilon_{jj}$ imply that the morphisms corresponding to ϵ_{ij}, ϵ_{ji} are isomorphisms between M_i, M_j.

Chose an arbitrary submodule $N \leq M_i$, for some $1 \leq i \leq m$. Since M is retractable, there exists an endomorphism $\varphi \in S$ with the property that $0 \neq \varphi(M) \subseteq$

N. Since $\varphi \neq 0$, there exists $1 \leq j \leq m$ so that $\varphi(M_j) \neq 0$. If $j = i$, $\varphi|_{M_i}$ is a non-zero endomorphism of M_i, hence invertible, and so surjective. This implies that $N = M_i$. Assume $j \neq i$. Taking ψ the morphism corresponding to ϵ_{ji}, $\varphi|_{M_j}(\psi) : M_i \to M_i$ is a non-zero endomorphism of M_i, and hence, by the above argument, $N = M$.

Since N was arbitrarily chosen, it implies that there are no proper submodules of M_i, hence M_i is a simple module. i was arbitrarily chosen, hence M is semisimple Artinian.

Suppose (ii) holds. Then every finitely generated submodule N of M is a direct summand of M and is a direct sum of cyclic submodules of M. We show this by induction. This obviously holds true for cyclic submodules. Next, assume that every submodule generated by $n-1$ elements is a direct summand and a direct sum of cyclic submodules. Take now N generated by n elements, $\{m_i\}_{i=1,\ldots,n}$; let $N' = \Sigma_{1 \leq i \leq n-1} M_i$ which is a direct summand and a direct sum of cyclic submodules. We have $N' \oplus N'' = M$, with π' and π'' the canonical projections of M onto N' and N'', respectively. Then $N = N' + m_n R = N' \oplus \pi''(m_n R)$ as it is easily checked; but $\pi''(m_n)R \leq^{\oplus} M$ and $\pi''(m_n)R \leq N'' \leq^{\oplus} M \Rightarrow N = N' \oplus \pi''(m_n)R \leq^{\oplus} N' \oplus N'' = M$, hence N is a direct summand and a direct sum of cyclic submodules. (See also Corollary 1.3 [39]). Since there are only finitely many disjoint direct summands of M (Proposition 4.4.2), the class of finitely generated submodules of M has ACC. Hence M is semisimple artinian.

Suppose (iii) holds. Then, by Theorem 1.6 [39], conditions (ii) holds. Thus M is semisimple artinian. $\square$

Remark 4.4.4. Our proof shows, in fact, a more general property, which states that any retractable module whose endomorphism ring is semisimple Artinian, is a semisimple Artinian module.

Example 4.4.5. Let $R = \mathbb{Z}$ and $M = \mathbb{Q}$. M is obviously (indecomposable) Baer, as all its endomorphisms are monomorphisms (Theorem 2.23 [35]). $S \approx \mathbb{Q}$ is regular (being a field). Yet M is not simple over R.

Another, more elaborate counterexample, is the following: let $R = \begin{pmatrix} K & K \\ 0 & K \end{pmatrix}$ and $M = \begin{pmatrix} K & K \\ 0 & 0 \end{pmatrix}$. Then $S = End(M) \approx K$, as it can be easily shown, and so M is an indecomposable Baer module. But M is not simple, since $0 \neq N = \begin{pmatrix} 0 & K \\ 0 & K \end{pmatrix} \leq M$.

APPENDIX A

TOPOLOGICAL PROPERTIES OF BAER MODULES

As we know, a Baer module M has the SSIP property. Hence, the direct summand submodules have the property that any intersection will give another direct summand (including 0). This gives the idea of using the direct summands as a base for closed sets, namely to define the closed subsets of M to be finite unions of direct summands. The open sets will be the set-complements of those finite unions of direct summands. An important issue is to show that the endomorphisms of M are also continuous, based on this topology. We prove that this fact is true.

Definition A.1. For a right R-module M, we define on the underlying set M the topology $\mathcal{T}_M^B$, having as base for the topology all the set complements of direct summands of M. Equivalently, the close sets of $\mathcal{T}_M^B$ are finite unions of direct summands of M.

Remark A.2. It is easy to see that the topology is well defined, given Theorem 2.1.4, which proves that arbitrary intersection of direct summands are direct summands.

Proposition A.3. *Let M be a Baer module, endowed with the topology described above. Then $\forall\ \varphi \in S$, φ is continuous.*

Proof. To show that φ is continuous, we need to prove that the inverse image through φ of any open set is an open set, or equivalently, the inverse image through φ of any closed set is a closed set. For our case, it is sufficient to show that inverse image through φ of any direct summand of M is again a direct summand. Let $N \oplus N' = M$. Let $K = Ker\varphi = \varphi^{-1}(\{0\}) \subseteq \varphi^{-1}(N)$. Since $K \leq^{\oplus} M$, by Baer property, $K \leq^{\oplus} \varphi^{-1}(N)$. $\varphi^{-1}(N) = P \oplus K$. Similarly, $K \leq^{\oplus} \varphi^{-1}(N')$; $\varphi^{-1}(N') = P' \oplus K$.

Let's prove that $P \oplus K \oplus P' = M$.

Take $m \in M$, then $\varphi(m) = n + n'$ with $n \in N$, $n' \in N'$. $\varphi^{-1}(n) \subseteq P \oplus K$, but taking any two elements so that $\varphi(p + k) = \varphi(p' + k')$, $\varphi(p - p' + k - k') = 0 \Rightarrow p - p' + k - k' \in K \Rightarrow p - p' \in K \cap P = 0 \Rightarrow p = p'$. Hence n returns to a unique "projection" onto P, which we call p. Similarly for n'. We have then that $\varphi^{-1}(n + n') = p + K + p'$ ($\subseteq$ follows from the previous argument, $\supseteq$ is easily checked). Hence $m \in p + K + p' \Rightarrow m = p + k + p'$. Hence $M \subseteq P + K + P' \Rightarrow M = P + K + P'$.

Check uniqueness: assume, for some $p \in P$, $p' \in P'$, $k \in K$, $p + k + p' = 0 \Rightarrow p + p' \in K \Rightarrow \varphi(p + p') = 0 \Rightarrow \varphi(p) = -\varphi(p') \in N \cap N' = 0 \Rightarrow \varphi(p) = \varphi(p') = 0 \Rightarrow p \in P \cap K = 0$ and $p' \in P' \cap K = 0$, hence $p = p' = k = 0$.

Hence $\varphi^{-1}(N) = P \oplus K \leq^{\oplus} M$, which is what we wanted to show.

In conclusion, all endomorphisms are also continuous with respect to the topology on M. □

Note. (a) There are continuous maps in the topology $\mathcal{T}_M^B$ of M which are not endomorphisms of M (e.g. constant, non-zero maps).

(b) The topology on M is not linear, since, for example, the neighbourhoods of $m \in M$ are not the neighbourhoods of 0 shifted in m (0 is contained in a single open set, namely M, because it is contained in any direct summand; there exist elements

M that are contained in at least one more open set than just M). Based on this fact we can also state that this topology is not Hausdorff, since 0 cannot be separated by any other element. There is an apparent parallel between the Zarisky topology on an A^n - closed sets are unions of zeroes of polynomials in $A[X]$ - and the topology defined on a Baer module - closed sets are unions of summands, which are in fact kernels of endomorphisms.

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