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Baer*-Semigroups and the Logic of Quantum Mechanics *

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Abstract. The theory of orthomodular ortholattices provides mathematical constructs utilized in the quantum logic approach to the mathematical foundations of quantum physics. There exists a remarkable connection between the mathematical theories of orthomodular ortholattices and Baer *-semigroups; therefore, the question arises whether there exists a phenomenologically interpretable role for Baer *-semigroups in the context of the quantum logic approach. Arguments, involving the quantum theory of measurements, yield the result that the theory of Baer *-semigroups provides the mathematical constructs for the discussion of "operations" and conditional probabilities.

0. Introduction

An affirmative answer to the following question would be extremely useful in the quantum logic approach to the foundations of quantum physics:

Question I. Does the collection of events pertaining to a physical system, which exhibits quantum effects, admit a phenomenologically interpretable orthomodular ortholattice structure?

If the word "ortholattice" is replaced by "orthoposet", then the answer is evidently affirmative. This aspect of Question I will be reviewed in Section I.

There exists a remarkable connection between orthomodular ortholattices and Baer *-semigroups. If $(S, \circ, *, ')$ is a Baer *-semigroup, then there exists an orthomodular ortholattice $(P'(S), \leqq, ')$ with $P'(S) \subset S$. If $(L, \leqq, ')$ is an orthomodular ortholattice, then there exists a Baer *-semigroup $(S(L), \circ, *, ')$ where $S(L)$ consists of a set of mappings of from L into L and there exists an injective mapping $j : L \to S(L)$. Since orthomodular ortholattices evidently have a role in the quantum logic approach[1] and since orthomodular ortholattices and Baer *-semigroups are closely related mathematical objects, the following question arises:

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[1] For example, the set of events has the structure of an orthomodular ortholattice in von Neumann's Hilbert space model of quantum mechanics.

Question II. Do Baer *-semigroups have a phenomenologically interpretable role in the quantum logic approach ?

In Section II, this question will be answered positively provided one accepts a number of assertions of the conventional quantum theory of measurements[2]. Indeed, the theory of Baer *-semigroups will provide mathematical constructs for the discussion of operations[3] and conditional probabilities within the context of the quantum logic approach. A corollary to the affirmative answer of Question II will be the assertion that the orthoposet of events in Question I is an ortholattice. Furthermore, a new approach to the phenomenological interpretation of the lattice operations will be obtained.

Necessary definitions and theorems from the theories of orthomodular ortholattices and Baer *-semigroups are included in an Appendix.

I. Event-State Structures

The quantum logic approach to the mathematical foundations of quantum physics studies two distinguished sets, the set of events and the set of states, pertaining to a physical system. Some formulations of the quantum logic approach treat events as primitive entities and states as derived entities (see, for example, [3, 4, 12, 13, 23]). Other formulations treat the events and the states as equally primitive entities (see, for example, [7, 9, 15, 17, 18, 25, 28, 30]). Although the collection of axioms varies from one formulation to another, the following definition yields a mathematical structure which is widely utilized.

Definition I.1. An *event-state structure* is a triple $(\mathscr{E}, \mathscr{S}, P)$ where

i) $\mathscr{E}$ is a set called the *logic* of the event-state structure and an element of $\mathscr{E}$ is called an *event*,

ii) $\mathscr{S}$ is a set and an element of $\mathscr{S}$ is called a *state*,

iii) P is a function $P : \mathscr{E} \times \mathscr{S} \to [0, 1]$ called the *probability function* and if $p \in \mathscr{E}$ and $\alpha \in \mathscr{S}$, then $P(p, \alpha)$ is called *the probability of occurrence of the event p in the state α*,

iv) if $p \in \mathscr{E}$, then the subsets $\mathscr{S}_1(p)$ and $\mathscr{S}_0(p)$ of $\mathscr{S}$ are defined by

$$\mathscr{S}_1(p) = \{\alpha \in \mathscr{S} : P(p, \alpha) = 1\}$$

$$\mathscr{S}_0(p) = \{\alpha \in \mathscr{S} : P(p, \alpha) = 0\}$$

and if $\alpha \in \mathscr{S}_1(p)$ (respectively, $\alpha \in \mathscr{S}_0(p)$) then the event p is said *to occur* (respectively, *non-occur*) *with certainty in the state α*, and

v) Axioms I.1 through I.7 are satisfied.

Axiom I.1. If $p, q \in \mathscr{E}$ and $\mathscr{S}_1(p) = \mathscr{S}_1(q)$, then $p = q$.

[2] For a review of the quantum theory of measurements, see [11] and [12].

[3] The concept of operation was introduced in the algebraic approach to quantum field theory by HAAG and KASTLER [10].

Axiom I.2. There exists an event $1 \in \mathscr{E}$ such that $\mathscr{S}_1(1) = \mathscr{S}$.

Axiom I.3. If $p, q \in \mathscr{E}$ and $\mathscr{S}_1(p) \subset \mathscr{S}_1(q)$, then $\mathscr{S}_0(q) \subset \mathscr{S}_0(p)$.

Axiom I.4. If $p \in \mathscr{E}$, then there exists an event $p' \in \mathscr{E}$ such that $\mathscr{S}_1(p') = \mathscr{S}_0(p)$ and $\mathscr{S}_0(p') = \mathscr{S}_1(p)$.

Axiom I.5. If

i) $p_1, p_2, \ldots \in \mathscr{E}$ and

ii) $\mathscr{S}_1(p_i) \subset \mathscr{S}_0(p_j)$ for $i \neq j$,

then there exists a $p \in \mathscr{E}$ such that

a) $\mathscr{S}_1(p_i) \subset \mathscr{S}_1(p)$ for all i,

b) if $q \in \mathscr{E}$ and $\mathscr{S}_1(p_i) \subset \mathscr{S}_1(q)$ for all i, then $\mathscr{S}_1(p) \subset \mathscr{S}_1(q)$, and

c) if $\alpha \in \mathscr{S}$, then

$$P(p, \alpha) = \sum_i P(p_i, \alpha) .$$

Axiom I.6. If $\alpha, \beta \in \mathscr{S}$ and $P(p, \alpha) = P(p, \beta)$ for all $p \in \mathscr{E}$, then $\alpha = \beta$.

Axiom I.7. If

i) $\alpha_1, \alpha_2, \ldots \in \mathscr{S}$,

ii) $t_1, t_2, \ldots \in [0, 1]$, and

iii) $\sum_i t_i = 1$,

then there exists an $\alpha \in \mathscr{S}$ such that

$$P(p, \alpha) = \sum_i t_i P(p, \alpha_i)$$

for all $p \in \mathscr{E}$.

The phenomenological interpretation of the mathematical system, event-state structure, may be specified by selecting a collection of rules for the interpretation of the primitive entities: events, states, and probability function. The following collection is a possible (but obviously not the only) choice for these rules. An event-state structure $(\mathscr{E}, \mathscr{S}, P)$ is associated with the class of physical systems of a specified kind. A state may be identified with a "state-preparation procedure", that is, instructions for an apparatus which produces sample physical systems of the specified kind. An event may be identified with the "occurrence or non-occurrence" of a particular phenomenon pertaining to physical systems of the specified kind. More specifically, an event may be identified with an "observation procedure", that is, instructions for an apparatus which interacts with a sample physical system and indicates either yes or no corresponding to the occurrence or non-occurrence of the phenomenon[4]. The interpretation of $P(p, \alpha)$ for $p \in \mathscr{E}$ and $\alpha \in \mathscr{S}$ would then be the following. Prepare an ensemble of sample physical systems utilizing a state-preparation procedure corresponding to α. Determine

[4] For comments on state-preparation procedures and observation procedures (in the context of the algebraic approach to quantum physics), see [5].

the occurrence or non-occurrence of the event p utilizing an observation procedure for p with each sample of this ensemble. If the ensemble is sufficiently large, then the frequency of occurrence of P should be close to $P(p, \alpha)$.

The phenomenological interpretation of the general aspects of the quantum logic approach are discussed in [12], in particular, Chapters 5 and 6; however, brief comments on the specific axioms adopted above will be necessary.

Axiom I.1 asserts that if p and q are events and if the set of states in which p occurs with certainty coincides with the set of states in which q occurs with certainty, then the events p and q are identical. This axiom is stronger than the corresponding axiom adopted, for example, in [18] and [25]. Its adoption is motivated by the phenomenological interpretation of the relation $\leqq$ introduced in the following definition.

Definition I.2. If $(\mathscr{E}, \mathscr{S}, P)$ is an event-state structure, then the relation $\leqq$ on $\mathscr{E}$, called the *relation of implication*, is defined as follows: for $p, q \in \mathscr{E}$, $p \leqq q$ means $\mathscr{S}_1(p) \subset \mathscr{S}_1(q)$.

The relation $\leqq$ is evidently reflexive and transitive, since $\subset$ is a reflexive and transitive relation[5]. Axiom I.1 and the antisymmetry of $\subset$ imply that $\leqq$ is antisymmetric; hence, the relation $\leqq$ is a partial ordering of $\mathscr{E}$. The phenomenological interpretation of the relation $\leqq$ may be briefly summarized: $p \leqq q$ means if p occurs with certainty, then q occurs with certainty. Indeed, if $\alpha \in \mathscr{S}$ is any state and if p occurs with certainty in the state α, then $\alpha \in \mathscr{S}_1(p) \subset \mathscr{S}_1(q)$ when $p \leqq q$ and q occurs with certainty in the state α. This interpretation of $\leqq$ evidently corresponds to the phenomenological concept of implication (see [12, 13, 23, 24, and 27]) more closely than the relation $\lesssim$ on $\mathscr{E}$ defined as follows (see [18, 19, 25, and 31]): for $p, q \in \mathscr{E}$, $p \lesssim q$ means $P(p, \alpha) \leqq P(q, \alpha)$ for all $\alpha \in \mathscr{S}$.

Axioms I.1 and I.2 assert the existence of a unique event $1 \in \mathscr{E}$ such that $\mathscr{S}_1(1) = \mathscr{S}$ (and, hence, $\mathscr{S}_0(1) = \emptyset$); moreover, 1 is the greatest element of $\mathscr{E}$ with respect to $\leqq$ since $p \leqq 1$ for all $p \in \mathscr{E}$. Axioms I.1 and I.4 assert if $p \in \mathscr{E}$, then there exists a unique $p' \in \mathscr{E}$ such that $\mathscr{S}_1(p') = \mathscr{S}_0(p)$ and $\mathscr{S}_0(p') = \mathscr{S}_1(p)$. Axiom I.4 applied to the event $1 \in \mathscr{E}$ yields the unique event 0 in $\mathscr{E}$ such that $\mathscr{S}_1(0) = \emptyset$ and $\mathscr{S}_0(0) = \mathscr{S}$, namely, $1'$; moreover, 0 is the least element of $\mathscr{E}$ with respect to $\leqq$ since $0 \leqq p$ for all $p \in \mathscr{E}$. These remarks motivate introduction of the following terminology.

Definition I.3. Let $(\mathscr{E}, \mathscr{S}, P)$ be an event-state structure.

a) The unique event $1 \in \mathscr{E}$ such that $\mathscr{S}_1(1) = \mathscr{S}$ and $\mathscr{S}_0(1) = \emptyset$ is called the *certain* event.

[5] See the Appendix for definitions of terminology from the theory of orthomodular ortholattices.

b) If $p \in \mathscr{E}$, then the unique event $p' \in \mathscr{E}$ such that $\mathscr{S}_1(p') = \mathscr{S}_0(p)$ and $\mathscr{S}_0(p') = \mathscr{S}_1(p)$ is called the *negation* of p.

c) The unique event 0, namely, $1'$, of $\mathscr{E}$ such that $\mathscr{S}_1(0) = \emptyset$ and $\mathscr{S}_0(0) = \mathscr{S}$ is called the *impossible* event.

Axiom I.3 asserts if $p, q \in \mathscr{E}$ and $\mathscr{S}_1(p) \subset \mathscr{S}_1(q)$ (that is, "if p occurs with certainty, then q occurs with certainty"), then $\mathscr{S}_0(q) \subset \mathscr{S}_0(p)$ (that is, "if q non-occurs with certainty, then p non-occurs with certainty"). Consequently, in terms of $\leqq$ and $'$, Axiom I.3 asserts if $p, q \in \mathscr{E}$ and $p \leqq q$, then $q' \leqq p'$. From the defining property of p' and Axiom I.1, it is also evident that $(p')' = p$ for all $p \in \mathscr{E}$. Since $\mathscr{S}_1(p) \cap \mathscr{S}_1(p') = \emptyset$, the greatest lower bound of p and p' with respect to $\leqq$ exists and equals the impossible event 0. Since $\mathscr{S}_0(p) \cap \mathscr{S}_0(p') = \emptyset$ and Axioms I.2 and I.3 are valid, it also follows that the least upper bound of p and p' with respect to $\leqq$ exists and equals the certain event 1. These remarks are summarized by the following theorem.

Theorem I.1. If $(\mathscr{E}, \mathscr{S}, P)$ is an event-state structure, then

a) $(\mathscr{E}, \leqq)$ is a poset,

b) 0 and 1 are the least and greatest events, respectively, of the poset $(\mathscr{E}, \leqq)$,

c) $p \to p'$ is an orthocomplementation of the poset $(\mathscr{E}, \leqq)$,

d) if $p, q \in \mathscr{E}$, then the following are equivalent:

i) $p \leqq q$,

ii) $\mathscr{S}_1(p) \subset \mathscr{S}_1(q)$,

iii) $\mathscr{S}_0(q) \subset \mathscr{S}_0(p)$,

e) if $p, q \in \mathscr{E}$ then the following are equivalent:

i) $p \perp q$ (for definition, see Appendix),

ii) $\mathscr{S}_1(p) \subset \mathscr{S}_0(q)$,

iii) $p \leqq q'$.

f) and if $p \in \mathscr{E}$, then the following are equivalent:

i) $p = 0$,

ii) $\mathscr{S}_1(p) = \emptyset$,

iii) $\mathscr{S}_0(p) = \mathscr{S}$.

Proof. Only assertions e) and f) remain to be proven. The relation $\perp$ on $\mathscr{E}$ is defined by $p \perp q$ means $p \leqq q'$. $p \leqq q'$ is equivalent to $\mathscr{S}_1(p) \subset \mathscr{S}_1(q')$ and, hence, also to $\mathscr{S}_1(p) \subset \mathscr{S}_0(q)$, since $\mathscr{S}_0(q) = \mathscr{S}_1(q')$. Assertion f) follows immediately from d) by taking $q = 0$.

The following definitions are useful for the discussion of Axiom I.5, I.6, and I.7.

Definition I.4. Let $(\mathscr{E}, \mathscr{S}, P)$ be an event-state structure.

a) If $p, q \in \mathscr{E}$ and $p \perp q$, then p and q are *mutually exclusive* events.

b) If $\alpha \in \mathscr{S}$, then the function $\mu_\alpha : \mathscr{E} \to [0, 1]$ is defined by

$$\mu_\alpha(p) = P(p, \alpha), \quad p \in \mathscr{E} .$$

c) $\hat{\mathscr{S}}$ denotes the set defined by

$$\hat{\mathscr{S}} = \{\mu_\alpha : \alpha \in \mathscr{S}\}\,.$$

For $p, q \in \mathscr{E}$, p and q are mutually exclusive events if and only if "p occurs with certainty whenever q non-occurs with certainty". Consequently, $p \perp q$ is a generalization of the concept of mutually exclusive events of conventional probability theory. Axiom I.5, therefore, asserts if $p_1, p_2, \ldots$ is a countable set of pairwise mutually exclusive events, then there exists a $p \in \mathscr{E}$ such that

a) $p_i \leqq p$ for all i,

b) if $q \in \mathscr{E}$ and $p_i \leqq q$ for all i, then $p \leqq q$, and

c) if $\alpha \in \mathscr{S}$, then

$$P(p, \alpha) = \sum_i P(p_i, \alpha)$$

or

$$\mu_\alpha(p) = \sum_i \mu_\alpha(p_i)\,.$$

a) and b) express the fact that p is the least upper bound, $\bigvee_i p_i$, of the set $\{p_1, p_2, \ldots\}$ of events. Consequently, Axiom I.5 asserts the existence of the least upper bound of countable sets of pairwise mutually exclusive events and, furthermore, the law of additivity of probabilities for mutually exclusive events (see [14]).

Theorem I.2. If $(\mathscr{E}, \mathscr{S}, P)$ is an event-state structure, then

a) $(\mathscr{E}, \leqq, ')$ is an orthomodular σ-orthoposet,

b) $\hat{\mathscr{S}}$ is a strongly-order-determining, σ-convex set of probability measures on $(\mathscr{E}, \leqq, ')$,

c) $\alpha \to \mu_\alpha$ is a bijection of $\mathscr{S}$ onto $\hat{\mathscr{S}}$.

Proof. $(\mathscr{E}, \leqq, ')$ is a σ-orthoposet and μ_α is a probability measure on $(\mathscr{E}, \leqq, ')$ for each $\alpha \in \mathscr{S}$ because of Axiom I.5. Axiom I.6 and the definition of $\hat{\mathscr{S}}$ assert that $\alpha \to \mu_\alpha$ is a bijection. Axiom I.7 asserts the σ-convexity of $\hat{\mathscr{S}}$. $\hat{\mathscr{S}}$ is strongly-order-determining because for $p, q \in \mathscr{E}$,

$$\text{if } \mathscr{S}_1(p) \subset \mathscr{S}_1(q), \quad \text{then} \quad p \leqq q$$

is equivalent to

$$\text{if } \{\mu \in \hat{\mathscr{S}} : \mu(p) = 1\} \subset \{\mu \in \hat{\mathscr{S}} : \mu(q) = 1\}, \quad \text{then} \quad p \leqq q\,.$$

$(\mathscr{E}, \leqq, ')$ is orthomodular, since any orthoposet possessing a separating set of probability measures is orthomodular.

The proof of the converse of Theorem I.2 is straightforward and left to the reader.

Theorem I.3. *If*

a) $(\mathscr{X}, \lesssim, \perp)$ is a σ-orthoposet,

b) $\mathscr{M}$ is a σ-convex, strongly-order-determining set of probability measures on $\mathscr{X}$, and

c) $P: \mathscr{X} \times \mathscr{M} \to [0, 1]$ is defined by

$$P(x, m) = m(x), \quad x \in \mathscr{X}, \; m \in \mathscr{M},$$

then $(\mathscr{X}, \mathscr{M}, P)$ is an event-state structure; moreover,

i) for $x, y \in \mathscr{X}$, $x \lesssim y$ if and only if $x \leqq y$,

ii) for $x \in \mathscr{X}$, $x^{\perp} = x'$, and

iii) $\mathscr{M} = \hat{\mathscr{M}}$.

(where $x \leqq y$, x' and $\hat{\mathscr{M}}$ are defined using the definitions relating to event-state structures).

Consequently, an event-state structure may be viewed either as a triple $(\mathscr{E}, \mathscr{S}, P)$ satisfying Axioms I.1 through I.7 or a pair $(\mathscr{E}, \hat{\mathscr{S}})$ where $\hat{\mathscr{S}}$ is a σ-convex, strongly-order-determining set of probability measures on an orthomodular σ-orthoposet. Both of these points of view will be employed in the following.

Example I.1. If $\mathscr{P}(H)$ is the set of *all* orthogonal projections on a separable complex Hilbert space H of dimension greater than two, if $(\mathscr{P}(H), \mathscr{S}, P)$ is an event-state structure, if $\leqq$ coincides with the usual order of projections,

$$P \leqq Q \quad \text{if} \quad PQ = P, \; P, Q \in \mathscr{P}(H)$$

and if P' is the orthogonal complement of $P (P' = I - P)$ for $P \in \mathscr{P}(H)$, then there exists a bijection $\alpha \in \mathscr{S} \to D_{\alpha}$ of $\mathscr{S}$ onto the set $\mathscr{D}(H)$ (the set of density operators) of all positive, trace-class operators with trace equal to one such that

$$P(P, \alpha) = \mathrm{Tr}(D_{\alpha} P)$$

for all $P \in \mathscr{P}(H)$, $\alpha \in \mathscr{S}$. This, of course, is the event-state structure of von Neumann's Hilbert space model of quantum mechanics (see [29] and [18], pp. 71—81).

Example I.2. The event-state structure $(\mathscr{E}, \hat{\mathscr{S}})$ where $\mathscr{E}$ is a σ-algebra of subsets of a set X and $\hat{\mathscr{S}}$ is a σ-convex, strongly-order-determining set of probability measures on $\mathscr{E}$ corresponds to the Kolmogorov model of probability theory (see [14 and 21]) with the additional feature that many probability measures are considered instead of one distinguished probability measure.

The formulations of the quantum logic approach to the foundations of quantum physics presented in [7, 18, 25, 28, 31] are evidently more general than the formulation adopted here. Indeed, these formulations replace the strongly-order-determining property of $\hat{\mathscr{S}}$ by at least one of the following consequences of this property: (1) $\hat{\mathscr{S}}$ is order-determining and (2) if $p \in \mathscr{E}$ and $p \neq 0$, then there exists an $\alpha \in \mathscr{S}$ such that $\mu_{\alpha}(p) = 1$. Discussions of the condition of strong-order-determining may be found in [8, 17, and 30].

II. Event-State-Operation Structures

Heuristic arguments have motivated the study of mathematical constructs corresponding to a number of physical concepts. For example, the notion of compatibility (or simultaneous observability) of events corresponds to a distinguished relation C on $\mathcal{E}$ (see [13, 18, and 25]). There exists at most one relation C on $\mathcal{E}$ with the following properties:

a) if $p, q \in \mathcal{E}$ and $p \leqq q$, then $p \mathrel{\mathsf{C}} q$;

b) if $p, q \in \mathcal{E}$ and $p \mathrel{\mathsf{C}} q$, then

i) $p \mathrel{\mathsf{C}} q'$,

ii) $q \mathrel{\mathsf{C}} p$,

iii) $p \wedge q$ and $p \vee q$ exist in $\mathcal{E}$, and

c) if $p_1, p_2, q \in \mathcal{E}$, $p_1 \mathrel{\mathsf{C}} p_2$, $p_1 \mathrel{\mathsf{C}} q$, and $p_2 \mathrel{\mathsf{C}} q$, then

i) $p_1 \wedge p_2 \mathrel{\mathsf{C}} q$,

ii) $(p_1 \wedge p_2) \vee q = (p_1 \vee q) \vee (p_2 \vee q)$.

Indeed, the relation C is determined by the following property: for $p, q \in \mathcal{E}$, $p \mathrel{\mathsf{C}} q$ if and only if there exists a Boolean sublogic $\mathcal{B} \subset \mathcal{E}$ such that $p, q \in \mathcal{B}$. The existence of a relation C satisfying a), b), and c) is not asserted; however, there always exists a relation C which satisfies properties a), b), and the following:

c') if $p_1, p_2, q \in \mathcal{E}$, $p_1 \mathrel{\mathsf{C}} p_2$, $p_1 \mathrel{\mathsf{C}} q$, $p_2 \mathrel{\mathsf{C}} q$ and $(p_1 \vee q) \wedge (p_2 \vee q)$ exists in $\mathcal{E}$, then

i) $p_1 \wedge p_2 \mathrel{\mathsf{C}} q$,

ii) $(p_1 \wedge p_2) \vee q = (p_1 \vee q) \wedge (p_2 \vee q)$.

This relation C may be defined as follows: for $p, q \in \mathcal{E}$, $p \mathrel{\mathsf{C}} q$ means there exists $p_0, q_0, r \in \mathcal{E}$ such that

i) $p_0 \perp q_0$,

ii) $p_0 \perp r$ and $p = p_0 \vee r$,

iii) $q_0 \perp r$ and $q = q_0 \vee r$.

If $p, q \in \mathcal{E}$ and $p \mathrel{\mathsf{C}} q$, then $p \wedge q$ exists in $\mathcal{E}$. The lattice property of $(\mathcal{E}, \leqq, ')$ discussed in Question I, therefore, becomes the following: if $p, q \in \mathcal{E}$ and $p \mathrel{\mathsf{C}} q$ does not hold, then does $p \wedge q$ exist in $\mathcal{E}$? The corresponding phenomenological question is evidently the following (see [12], pp. 74—78): if observation procedures for two incompatible (i.e., non-simultaneously observable) events are given, then how does one describe the observation procedure for the "and" (or conjunction) of these two events? Although answers to this question have been attempted in [1, 2, 12, 17, 23 and 24], no completely adequate answer is currently available. For example, in the context of an event-state structure, the arguments of [12, 23, and 24] reduce to the assertion of the universal validity of the hypothesis of the following theorem.

Theorem II.1. *Let $(\mathcal{E}, \mathcal{S}, P)$ be an event-state structure. If $p_1, p_2 \in \mathcal{E}$ and there exists an event $p \in \mathcal{E}$ such that*

$$\mathcal{S}_1(p) = \mathcal{S}_1(p_1) \cap \mathcal{S}_1(p_2)\,,$$

then the greatest lower bound $p_1 \wedge p_2$ of p_1 and p_2 with respect to $\leqq$ exists and equals p.

Proof. Since p satisfies $\mathscr{S}_1(p) = \mathscr{S}_1(p_1) \cap \mathscr{S}_1(p_2)$ by hypothesis, $\mathscr{S}_1(p) \subset \mathscr{S}_1(p_1)$ and $\mathscr{S}_1(p) \subset \mathscr{S}_1(p_2)$; hence $p \leqq p_1$ and $p \leqq p_2$. Let $q \in \mathscr{E}$, $q \leqq p_1$ and $q \leqq p_2$. It follows that $\mathscr{S}_1(q) \subset \mathscr{S}_1(p_1)$ and $\mathscr{S}_1(q) \subset \mathscr{S}_1(p_2)$; hence $\mathscr{S}_1(q) \subset \mathscr{S}_1(p_1) \cap \mathscr{S}_1(p_2) = \mathscr{S}_1(p)$ and $q \leqq p$. Consequently, if $q \in \mathscr{E}$, $q \leqq p_1$ and $q \leqq p_2$, then $q \leqq p$. Therefore, the greatest lower bound of p_1 and p_2 exists and equals p. Q.E.D.

One result of this section will be to provide a new approach to Question I by introducing the theory of Baer *-semigroups into the context of the quantum logic approach.

The introduction of the concept of conditional probability in conventional probability theory greatly enhances the utility of the theory and deepens the mathematical structure of the theory (see [14 and 21]). The concept of conditional probability is expressed as a mathematical object defined constructively in terms of the primitive entities of the theory in an intuitively obvious fashion. In the case of a general event-state structure, there apparently exists no manifestly evident way of defining a mathematical construct corresponding to conditional probability in terms of the primitive objects of the theory in a constructive fashion. However, there exists a mathematical construct in von Neumann's Hilbert space model of quantum mechanics which is widely employed to represent the concept of conditional probability. These remarks provide the initial motivation for considering *event-state-operation structures*, event-state structures equipped with an additional primitive entity corresponding essentially to conditional probability. A role for Baer *-semigroups in the quantum logic approach will emerge from the study of these event-state-operation structures.

Definition II.1. Let $(\mathscr{E}, \mathscr{S}, P)$ be an event-state structure.

a) Σ denotes the set of all maps $x : \mathscr{D}_x \to \mathscr{R}_x$ with domain $\mathscr{D}_x \subset \mathscr{S}$ and range $\mathscr{R}_x \subset \mathscr{S}$. If $x \in \Sigma$ and $\alpha \in \mathscr{D}_x$, then $x(\alpha)$ (or $x\alpha$, for brevity) denotes the image of α under x.

b) If $x, y \in \Sigma$, then $x = y$ means

i) $\mathscr{D}_x = \mathscr{D}_y$ and

ii) $x\alpha = y\alpha$ for all $\alpha \in \mathscr{D}_x$.

c) $0 : \mathscr{D}_0 \to \mathscr{R}_0$ is defined by $\mathscr{D}_0 = \emptyset$.

d) $1 : \mathscr{D}_1 \to \mathscr{D}_1$ is defined by

i) $\mathscr{D}_1 = \mathscr{S}$ and

ii) $1\alpha = \alpha$ for all $\alpha \in \mathscr{D}_1$.

e) If $x, y \in \Sigma$, then $x \circ y : \mathscr{D}_{x \circ y} \to \mathscr{R}_{x \circ y}$ is defined by

i) $\mathscr{D}_{x \circ y} = \{\alpha \in \mathscr{D}_y : y\alpha \in \mathscr{D}_x\}$ and

ii) $(x \circ y)\alpha = x(y\alpha)$ for all $\alpha \in \mathscr{D}_{x \circ y}$.

In all manipulations with the elements of Σ care must be taken to examine the domains of definition (as, for example, domains of definitions must be checked for unbounded operators on Hilbert space). It is evident that $(\Sigma, \circ)$ is a semigroup with a unit element 1 and a zero element 0.

Definition II.2. An *event-state-operation structure* is a 4-tuple $(\mathscr{E}, \mathscr{S}, P, \Omega)$ where $(\mathscr{E}, \mathscr{S}, P)$ is an event-state structure and Ω is a mapping $\Omega : \mathscr{E} \to \Sigma (p \in \mathscr{E} \to \Omega_p \in \Sigma)$ which satisfies Axioms II.1 through II.7. If $p \in \mathscr{E}$, then Ω_p is called *the operation corresponding to the event p* (relative to Ω). If $p \in \mathscr{E}$ and $\alpha \in \mathscr{D}_{\Omega_p}$, then $\Omega_p \alpha$ is called *the state conditioned on the event p and the state* α (relative to Ω). If, moreover, $q \in \mathscr{E}$, then $P(q, \Omega_p \alpha)$ is called *the probability of q conditioned on the event p and the state* α (relative to Ω). S_Ω denotes the subset of Σ defined by

$$S = \{\Omega_{p_1} \circ \Omega_{p_2} \circ \cdots \circ \Omega_{p_n} : p_1, p_2, \ldots, p_n \in \mathscr{E}\} .$$

An element of S_Ω is called an *operation.*

Axiom II.1. If $p \in \mathscr{E}$, then the domain $\mathscr{D}_{\Omega_p}$ of Ω_p coincides with the set $\mathscr{D}_p$ defined by

$$\mathscr{D}_p = \{\alpha \in \mathscr{S} : P(p, \alpha) \neq 0\} .$$

Axiom II.2. If $p \in \mathscr{E}$, $\alpha \in \mathscr{D}_p$ and $P(p, \alpha) = 1$, then

$$\Omega_p \alpha = \alpha .$$

Axiom II.3. If $p \in \mathscr{E}$ and $\alpha \in \mathscr{D}_p$, then $P(p, \Omega_p \alpha) = 1$.

Axiom II.4. If $p_1, p_2, \ldots, p_n, q_1, q_2, \ldots, q_m \in \mathscr{E}$ and

$$\Omega_{p_1} \circ \Omega_{p_2} \circ \cdots \circ \Omega_{p_n} = \Omega_{q_1} \circ \Omega_{q_2} \circ \cdots \circ \Omega_{q_m} ,$$

then

$$\Omega_{p_n} \circ \Omega_{p_{n-1}} \circ \cdots \circ \Omega_{p_1} = \Omega_{q_m} \circ \Omega_{q_{m-1}} \circ \cdots \circ \Omega_{q_1} .$$

Axiom II.5. If $x \in \mathscr{S}_\Omega$, then there exists a $q_x \in \mathscr{E}$ such that

$$\mathscr{S}_1(q_x) = C\mathscr{D}_x = \{\alpha \in \mathscr{S} : \alpha \notin \mathscr{D}_x\} .$$

Axiom II.6. If $p, q \in \mathscr{E}$, $q \leq p$ and $\alpha \in \mathscr{D}_p$, then

$$P(q, \Omega_p \alpha) = \frac{P(q, \alpha)}{P(p, \alpha)} .$$

Axiom II.7. If $p, q \in \mathscr{E}$, $p \mathrel{C} q$ and $\alpha \in \mathscr{D}_p$, then

$$P(q, \Omega_p, \alpha) = P(p \wedge q, \Omega_p \alpha) .$$

The rules of interpretation for an event-state structure must be augmented to include the concept of operation. The rule of interpretation adopted here depends upon the following phenomenological assertion. *If $p \in \mathscr{E}$, then an observation procedure for p can be selected to fulfill the following "gentleness" requirement: after utilizing this observation procedure with a sample physical system to determine the occurrence or non-occurrence of p, the resulting physical system is again a member of the class of physical* *systems corresponding to* $(\mathscr{E}, \mathscr{S}, P)$. A critical discussion of this assertion may be found in [12 and 19]. If $p \in \mathscr{E}$ and $\alpha \in \mathscr{S}$ with $P(p, \alpha) \neq 0$, then the following describes a state-preparation procedure:

Step A. Produce a sample physical system utilizing a state-preparation procedure corresponding to α.

Step B. Determine the occurrence or non-occurrence of the event p utilizing an observation procedure corresponding to p.

Step C. If the event p occurred, then accept the physical system resulting from this observation procedure as a sample physical system; if the event p did not occur, then do not accept the resultant physical system as a sample.

There should exist a state in $\mathscr{S}$ corresponding to this state-preparation procedure. The rule of interpretation for Ω_p adopted here is the assertion that this state is $\Omega_p \alpha$. The terminology *operation* is employed since this rule of interpretation corresponds essentially to a special case of the "operations" utilized by Haag and Kastler in the algebraic approach to quantum field theory [10].

Example II.1. The event-state structure $(\mathscr{P}(H), \mathscr{S}, P)$ of von Neumann's Hilbert space model of quantum mechanics admits an operation map Ω. Indeed, for $P \in \mathscr{P}(H)$, Ω_P may be defined as follows: if $\alpha \in \mathscr{S}$ is the state with density operator $D_\alpha \in \mathscr{D}(H)$ and

$$P(P, \alpha) = \mathrm{Tr}(D_\alpha P) \neq 0 ,$$

then $\Omega_P \alpha$ is the state $\alpha' \in \mathscr{S}$ with the density operator $D_{\alpha'} \in \mathscr{D}(H)$ given by

$$D_{\alpha'} = \frac{P D_\alpha P}{\mathrm{Tr}(D_\alpha P)} .$$

This is the usual way of introducing "conditional probability" in quantum mechanics (see, for example, [20], p. 333 and [16]). The verification of all the axioms except Axiom II.5 is straightforward. If $x \in \mathscr{S}_\Omega$ and $x = \Omega_{P_1} \circ \cdots \circ \Omega_{P_n}$, where $P_1, P_2, \ldots, P_n \in \mathscr{P}(H)$, then the projection Q on the null space of $P_1 P_2 \ldots P_n$ satisfies Axiom II.5.

Example II.2. The event-state structure $(\mathscr{E}, \hat{\mathscr{S}})$ of Example I.2 also admits an operation map. For $p \in \mathscr{E}$, Ω_p is defined as follows: if $\mu \in \hat{\mathscr{S}}$ and $\mu(p) \neq 0$, then $\Omega_p \mu$ is the element of $\hat{\mathscr{S}}$ defined by

$$(\Omega_p \mu)(q) = \frac{\mu(p \wedge q)}{\mu(p)}, \quad q \in \mathscr{E} .$$

This is the usual formulation of conditional probability from the Kolmogorov model of probability theory. The verification of the axioms is straightforward in this case.

The motivation for Axioms II.1 through II.7 will now be discussed utilizing the previously adopted rule of interpretation. If $p \in \mathscr{E}$ and $\alpha \in \mathscr{S}$, then the rule of interpretation yields a state in the case $P(p, \alpha) \neq 0$;

however, the rule of interpretation does not yield a state when $P(p, \alpha) = 0$, since the samples will not satisfy the condition that the event p occurs. Consequently, the domain $\mathscr{D}_{\Omega_p}$ of Ω_p should, indeed, be the set $\mathscr{D}_p = \{\alpha \in \mathscr{S} : P(p, \alpha) \neq 0\}$. Axioms II.2 and II.3 evidently assert that the observation procedure corresponding to $p \in \mathscr{E}$ may be selected to be a measurement of the first kind in the sense of Pauli (see [12 and 22]). Axioms II.6 and II.7 are immediate consequences of the assertion: if $p, q \in \mathscr{E}$ and $p \mathrm{C} q$, then the observation of the event q should not disturb the results of the observation of the event p (since p and q are compatible events) and, hence, the arguments about frequencies of occurrence of conventional probability theory should be applicable.

Consequently, Axioms II.1, II.2, II.3, II.6, and II.7 are explicitly part of the conventional quantum theory of measurements. Axioms II.4 and II.5 are implicitly part of the conventional quantum theory of measurements since Example II.1 satisfies these axioms; however, the role of these axioms has evidently not been previously discussed.

If $x \in S_\Omega$, then there exist $p_1, p_2, \ldots, p_n \in \mathscr{E}$ such that

$$x = \Omega_{p_1} \circ \Omega_{p_2} \circ \cdots \circ \Omega_{p_n} .$$

The element x of S_Ω, therefore, represents the experimental procedure of first executing the operation Ω_{p_n}, then executing the operation $\Omega_{p_{n-1}}$, and so on until finally executing the operation Ω_{p_1}. The experimental procedure obtained by executing these operations in the reverse order yields an element of S_Ω also, namely, $\Omega_{p_n} \circ \Omega_{p_{n-1}} \circ \cdots \circ \Omega_{p_1}$. It, therefore, seems desirable to introduce a mapping $*$ of S_Ω into S_Ω which corresponds to this reversal of the order of the execution of operations. Consequently, x^* would be the element $\Omega_{p_n} \circ \Omega_{p_{n-1}} \circ \cdots \circ \Omega_{p_1}$ of S_Ω. However $x \to x^*$ might not be a well-defined mapping. Indeed, there might also exist $q_1, q_2, \ldots, q_m \in \mathscr{E}$ such that $x = \Omega_{q_1} \circ \Omega_{q_2} \circ \cdots \circ \Omega_{q_m}$, ($x$ also represents the experimental procedure of first executing Ω_{q_m} then executing $\Omega_{q_{m-1}}$, and so on until finally executing Ω_{q_1}) but such that the "reversal" of this experimental procedure does not coincide with the "reversal" of the experimental procedure corresponding to the p_i's; that is,

$$\Omega_{p_1} \circ \Omega_{p_2} \circ \cdots \circ \Omega_{p_n} = \Omega_{q_1} \circ \Omega_{q_2} \circ \cdots \circ \Omega_{q_m}$$

but

$$\Omega_{p_n} \circ \Omega_{p_{n-1}} \circ \cdots \circ \Omega_{p_1} \neq \Omega_{q_m} \circ \Omega_{q_{m-1}} \circ \cdots \circ \Omega_{q_1} .$$

Axiom II.4 asserts that this does *not* happen; consequently, the following mapping $* : S_\Omega \to S_\Omega$ is well-defined.

Definition II.3. Let $(\mathscr{E}, \mathscr{S}, P, \Omega)$ be an event-state-operation structure. The mapping $* : S_\Omega \to S_\Omega$ is defined as follows: if $x \in S_\Omega$, then select $p_1, p_2, \ldots, p_n \in \mathscr{E}$ such that

$$x = \Omega_{p_1} \circ \Omega_{p_2} \circ \cdots \circ \Omega_{p_n}$$

and define x^* to be the element

$$x^* = \Omega_{p_n} \circ \Omega_{p_{n-1}} \circ \cdots \circ \Omega_{p_1}\,.$$

Theorem II.2. *If* $(\mathscr{E}, \mathscr{S}, P, \Omega)$ *is an event-state-operation structure, then* S_Ω *is a subsemigroup of* Σ*; moreover,*

a) $\Omega_1 = 0,\ \Omega_1 = 1$,

b) *if* $p \in \mathscr{E}$, *then*

$$\Omega_p \circ \Omega_p = \Omega_p$$

and the range of Ω_p *equals* $\mathscr{S}_1(p)$*;*

c) $* : S_\Omega \to S_\Omega$ *is the unique mapping at* S_Ω *into* S_Ω *such that*

i) $*$ *is an involution for the semigroup* $(S_\Omega, \circ)$,

ii) $(\Omega_p)^* = \Omega_p$ *for all* $p \in \mathscr{E}$*;*

d) if $p, q \in \mathscr{E}$, *then the following are equivalnet properties:*

i) $p \leqq q$,

ii) $\mathscr{S}_1(p) \subset \mathscr{S}_1(q)$,

iii) $\mathscr{S}_0(q) \subset \mathscr{S}_0(p)$,

iv) $\Omega_q \circ \Omega_p = \Omega_p$,

v) $\Omega_p \circ \Omega_q = \Omega_p$.

Proof. S_Ω is obviously a subsemigroup of Σ relative to the composition $\circ$ since

$$S_\Omega = \{\Omega_{p_1} \circ \cdots \circ \Omega_{p_n} : p_1, \ldots, p_n \in \mathscr{E}\}\,.$$

Since

$$\mathscr{D}_0 = \{\alpha \in \mathscr{S} : P(0, \alpha) \neq 0\} = \emptyset\,,$$

the domain of Ω_0 is $\emptyset$ and, hence, $\Omega_0 = 0$. Since

$$\mathscr{D}_1 = \{\alpha \in \mathscr{S} : P(1, \alpha) \neq 0\} = \mathscr{S}\,,$$

the domain of Ω_1 is $\mathscr{S}$. If $\alpha \in \mathscr{S}$, then $P(1, \alpha) = 1$ by Axiom I.3 and, hence, $\Omega_1 \alpha = \alpha$ by Axiom II.2; consequently, $\Omega_1 = 1$.

Assertion b) is a consequence of Axioms II.1, II.2, and II.3. Let $p \in \mathscr{E}$. If $\alpha \in \mathscr{D}_p$, then $P(p, \Omega_p \alpha) = 1$ by Axiom II.3. Hence, $\Omega_p \alpha \in \mathscr{D}_p$ for $\alpha \in \mathscr{D}_p$ and since

$$\mathscr{D}_{\Omega_p \circ \Omega_p} = \{\alpha \in \mathscr{D}_p : \Omega_p \alpha \in \mathscr{D}_p\}\,,$$

it follows that $\mathscr{D}_{\Omega_p \circ \Omega_p} = \mathscr{D}_{\Omega_p} = \mathscr{D}_p$. Since $P(p, \Omega_p \alpha) = 1$ for $\alpha \in \mathscr{D}_p$,

$$(\Omega_p \circ \Omega_p)\,\alpha = \Omega_p(\Omega_p \alpha) = \Omega_p \alpha$$

by Axiom II.2; hence, $\Omega_p \circ \Omega_p = \Omega_p$. Since $P(p, \Omega_p \alpha) = 1$ for $\alpha \in \mathscr{D}_p$, the range of Ω_p is contained in $\mathscr{S}_1(p)$. If $\alpha \in \mathscr{S}_1(p)$, then $\Omega_p \alpha = \alpha$ by Axioms II.1 and II.2; hence, the range of Ω_p contains $\mathscr{S}_1(p)$.

It is evident that $x \to x^*$ is an involution such that $(\Omega_p)^* = \Omega_p$ for every $p \in \mathscr{E}$ and, moreover, it is the only such involution.

The equivalence of i), ii), and iii) of assertion d) is a general property of event-state structures. The equivalence of iv) and v) is an obvious consequence of assertion c). Assume $\mathscr{S}_1(p) \subset \mathscr{S}_1(q)$. If $\alpha \in \mathscr{D}_p$, then

$\Omega_p \alpha \in \mathscr{S}_1(p)$ by Axiom II.3, hence, if $\alpha \in \mathscr{D}_p$, then $\Omega_p \alpha \in \mathscr{D}_q$ and

$$\mathscr{D}_{\Omega_q \circ \Omega_p} = \{\alpha \in \mathscr{D}_p : \Omega_p \alpha \in \mathscr{D}_q\} = \mathscr{D}_p .$$

If $\alpha \in \mathscr{D}_p$, then $\Omega_p \alpha \in \mathscr{S}_1(q)$ and by Axiom II.2

$$(\Omega_q \circ \Omega_p)\alpha = \Omega_q(\Omega_q \alpha) = \Omega_p \alpha .$$

Thus, $\Omega_q \circ \Omega_p = \Omega_p$ if $\mathscr{S}_1(p) \subset \mathscr{S}_1(q)$. If $\Omega_p \circ \Omega_q = \Omega_p$, then

$$\mathscr{D}_p = \mathscr{D}_{\Omega_p \circ \Omega_q} = \{\alpha \in \mathscr{D}_q : \Omega_q \alpha \in \mathscr{D}_p\} \subset \mathscr{D}_q ;$$

hence, $\mathscr{S}_0(q) \subset \mathscr{S}_0(p)$ since $\mathscr{D}_p = C\mathscr{S}_0(p)$ and $\mathscr{D}_q = C\mathscr{S}_0(q)$. Hence, i) implies iii). Q.E.D.

Consequently, Axiom II.4 provides the semigroup $(S_\Omega, \circ)$ with an involution. The existence of this involution then yields a characterization of the partial order $\leqq$ of $(\mathscr{E}, \leqq, ')$ in terms of product $\circ$ of $(S_\Omega, \circ)$. In terms of the theory of involution semigroups (see Appendix), the theorem asserts: $(S_\Omega, \circ, *)$ is an involution semigroup such that

i) For each $p \in \mathscr{E}$, Ω_p is a *projection*, that is, Ω_p is an element of

$$P(S_\Omega) = \{e \in S_\Omega : e \circ e = e^* = e\}$$

ii) $p \in \mathscr{E} \to \Omega_p \in P(S_\Omega)$ is an order preserving map of $(\mathscr{E}, \leqq)$ into $(P(S_\Omega), \leqq)$ where

$$e \leqq f \quad \text{means} \quad e \circ f = e$$

for $e, f \in P(S_\Omega)$.

If $x \in S_\Omega$, then there exist $p_1, p_2, \ldots, p_n \in \mathscr{E}$ such that

$$x = \Omega_{p_1} \circ \Omega_{p_2} \circ \cdots \circ \Omega_{p_n} .$$

Let $p_{n+1} = 1$. $\alpha \in \mathscr{S}$ is an element of the domain, $\mathscr{D}_x$, of x if and only if α is an element of the domain of $\Omega_{p_1} \circ \Omega_{p_2} \circ \cdots \circ \Omega_{p_n}$. Consequently, $\alpha \in \mathscr{D}_x$ if and only if

$$\Omega_{p_{j+1}} \circ \Omega_{p_{j+2}} \circ \cdots \circ \Omega_{p_{n+1}} \alpha \in \mathscr{D}_{p_j}$$

for $j = n, n-1, \ldots, 1$. Therefore, $\alpha \notin \mathscr{D}_x$ if and only if there exists an i, $n \geqq i \geqq 1$, such that

$$\Omega_{p_{j+1}} \circ \cdots \circ \Omega_{p_{n+1}} \alpha \in \mathscr{D}_{p_j}$$

for $j = i+1, \ldots, 1$ and $\Omega_{p_{i+1}} \circ \cdots \circ \Omega_{p_{n+1}} \alpha \notin \mathscr{D}_{p_i}$. Because of Axiom II.1, this characterization of $C\mathscr{D}_x$ may be expressed as follows: $\alpha \in C\mathscr{D}_x$ if and only if there exists an i, $n \geqq i \geqq 1$, such that

$$P(p_j, \Omega_{p_{j+1}} \circ \cdots \circ \Omega_{p_{n+1}} \alpha) \neq 0$$

for $j = i+1, \ldots, 1$ and

$$P(p_i, \Omega_{p_{i+1}} \circ \cdots \circ \Omega_{p_{n+1}} \alpha) = 0 .$$

This characterization of $C\mathscr{D}_x$ evidently provides an experimental procedure for determining whether a state belongs to $C\mathscr{D}_x$. Axiom II.5

asserts the existence of an event q such that q occurs with certainty in the state α if and only if $\alpha \in C\mathscr{D}_x$, that is,

$$\mathscr{S}_1(q) = C\mathscr{D}_x .$$

If $q_1 \in \mathscr{E}$ and $\mathscr{S}_1(q_1) = C\mathscr{D}_x$ also, then $\mathscr{S}_1(q_1) = \mathscr{S}_1(q)$ and, hence $q_1 = q$.

Definition II.4. If $(\mathscr{E}, \mathscr{S}, P, \Omega)$ is an event-state-operation structure, then the mapping $' : S_\Omega \to P(S_\Omega)$ is defined as follows: for $x \in S_\Omega$, x' is the element Ω_{q_x} of $P(S_\Omega)$, where $q_x \in \mathscr{E}$ is the unique element of $\mathscr{E}$ such that $\mathscr{S}_1(q_x) = C\mathscr{D}_x$.

The mapping $' : S_\Omega \to P(S_\Omega)$, provided by Axiom II.5, gives the involution semigroup $(S_\Omega, \circ, *)$ the structure of a Baer *-semigroup (see Appendix).

Theorem II.3. *If $(\mathscr{E}, \mathscr{S}, P, \Omega)$ is an event-state-operation structure, then $(S_\Omega, \circ, *, ')$ is a Baer *-semigroup; moreover, the mapping $p \in \mathscr{E} \to \Omega_p \in P(S_\Omega)$ is an isomorphism of the orthomodular orthoposet $(\mathscr{E}, \leqq, ')$ onto the orthomodular orthoposet $(P'(S_\Omega), \leqq, ')$ (see Appendix for a discussion of $P'(S_\Omega)$).*

Proof. Let $x \in S_\Omega$. $(S_\Omega, \circ, *, ')$ is a Baer *-semigroup provided: if $y \in S_\Omega$, then $x \circ y = 0$ is equivalent to $x' \circ y = y$. $x \circ y = 0$ is equivalent to

$$\emptyset = \mathscr{D}_{x \circ y} = \{\alpha \in \mathscr{D}_y : y\alpha \in \mathscr{D}_x\}$$

or to the assertion: (A) if $\alpha \in \mathscr{D}_y$, then $y\alpha \in C\mathscr{D}_x$. Consequently, if $\alpha \in \mathscr{D}_y$, then $y\alpha \in \mathscr{D}_{x'}$, since $\mathscr{D}_{x'} = C\mathscr{D}_x$, and $\alpha \in \mathscr{D}_{x' \circ y}$. Since $\mathscr{D}_{x' \circ y} \subset \mathscr{D}_y$, it follows that $\mathscr{D}_{x' \circ y} = \mathscr{D}_y$ when assertion (A) holds. Since $C\mathscr{D}_x = \mathscr{S}_1(q_x)$, assertion (A) is equivalent to the following assertion by Axioms II.2 and II.3: (B) If $\alpha \in \mathscr{D}_y$, then $\Omega_{q_x}(y\alpha) = y\alpha$. Consequently, assertion (A) is equivalent to the assertion: (C) $\mathscr{D}_y = \mathscr{D}_{x' \circ y}$ and if $\alpha \in \mathscr{D}_y$, then $(\Omega_{q_x} \circ y)\alpha = y\alpha$. Since $\Omega_{q_x} = x'$, $x \circ y = 0$ is equivalent to $x' \circ y = y$.

If $p \in \mathscr{E}$, then $\Omega_p \in P(S_\Omega)$; moreover, $(\Omega_p)' = \Omega_{p'}$. Indeed, if $p \in \mathscr{E}$, then

$$\mathscr{S}_1(p') = \mathscr{S}_0(p) = CC\mathscr{S}_0(p)$$
$$= C\mathscr{D}_p = C\mathscr{D}_{\Omega_p} .$$

and p' satisfies the criterion of Axiom II.5 for the case $x = \Omega_p$; hence, $(\Omega_p)' = \Omega_{p'}$. If $p \in \mathscr{E}$, then

$$(\Omega_p)'' = (\Omega_{p'})' = \Omega_{p''} = \Omega_p$$

and, hence, Ω_p is a closed projection, that is, $\Omega_p \in P'(S_\Omega)$. The mapping $p \in \mathscr{E} \to \Omega_p \in P'(S_\Omega)$ preserves order, since

$$p \leqq q \quad \text{if and only if} \quad \Omega_p \circ \Omega_q = \Omega_p ,$$

and preserves orthocomplementation, since $(\Omega_p)' = \Omega_{p'}$. This mapping is injective and it is surjective, since $P'(S_\Omega) = \{x' : x \in S_\Omega\}$ and $x' = \Omega_{q_x}$ for $x \in S_\Omega$. Consequently, $p \in \mathscr{E} \to \Omega_p \in P'(S_\Omega)$ is an isomorphism of the orthoposet $(\mathscr{E}, \leqq, ')$ onto the orthoposet $(P'(S_\Omega), \leqq, ')$. Q.E.D.

III. On the Lattice Structure of $(\mathscr{E}, \leqq, ')$

The event-state structure may be viewed as a *passive* picture for the description of physical systems since it considers only the probability of occurrence of events. The introduction of the concept of operation provides an *active* picture; indeed, the operations in S_Ω correspond to *filtering* experiments. The orthoposet $(\mathscr{E}, \leqq, ')$ of events is isomorphic to the orthoposet $(P'(S_\Omega), \leqq, ')$ under the mapping $p \in \mathscr{E} \to \Omega_p \in P'(S_\Omega)$. In $(P'(S_\Omega), \leqq, ')$, the order relation $\leqq$ is defined in terms of the composition $\circ$ of operations; indeed, for $p, q \in \mathscr{E}$,

$$p \leqq q \quad \text{if and only if} \quad \Omega_p \circ \Omega_q = \Omega_p .$$

The question, therefore, arises whether the greatest lower bound $p \wedge q$ of p and q in $\mathscr{E}$, an order theoretic construct in $(\mathscr{E}, \leqq, ')$, can be interpreted in terms of the composition $\circ$ of the Baer *-semigroup $(S_\Omega, \circ, *, ')$.

Theorem III.1. *If $(\mathscr{E}, \mathscr{S}, P, \Omega)$ is an event-state-operation structure, then $(\mathscr{E}, \leqq, ')$ is an ortholattice; moreover, if $p, q \in \mathscr{E}$, then*

$$\Omega_{p \wedge q} = (\Omega_{p'} \circ \Omega_q)' \circ \Omega_q .$$

Proof. $(P'(S_\Omega), \leqq, ')$ is an orthomodular ortholattice such that if $e, f \in P'(S_\Omega)$, then

$$e \wedge f = (e' \circ f)' \circ f$$

(see Appendix). The theorem follows immediately from the fact that $p \in \mathscr{E} \to \Omega_p \in P'(S_\Omega)$ is an isomorphism of $(\mathscr{E}, \leqq, ')$ onto $(P'(S_\Omega), \leqq, ')$. Q.E.D.

Consequently, $(\mathscr{E}, \leqq, ')$ is an ortholattice for an event-state-operation structure; however, the greatest lower bound $p \wedge q$ in $\mathscr{E}$ is represented in $P'(S_\Omega)$ utilizing not only the composition $\circ$ of operations but also the mapping $' : S_\Omega \to P'(S_\Omega)$.

Since the compatibility relation C discussed at the beginning of Section II involves only the order and orthocomplementation of $(\mathscr{E}, \leqq, ')$, it must also be expressible in terms of the order and orthocomplementation of the isomorphic ortholattice $(P'(S_\Omega), \leqq, ')$.

Theorem III.2. *If $(\mathscr{E}, \mathscr{S}, P, \Omega)$ is an event-state-operation structure and $p, q \in \mathscr{E}$, then the following are equivalent:*

a) $p \mathrel{\mathrm{C}} q$,

b) $\Omega_p \circ \Omega_q = \Omega_q \circ \Omega_p$;

moreover, if $p \mathrel{\mathrm{C}} q$, then

$$\Omega_{p \wedge q} = \Omega_p \circ \Omega_q .$$

Proof. The relation $\bar{C}$ may be defined in the ortholattice $(P'(S_\Omega), \leqq, ')$ as follows: for $e, f \in P'(S_\Omega)$, $e\bar{C}f$ means there exists a triple $e_0, f_0, g \in P'(S_\Omega)$ such that

i) $e_0 \perp f_0$,

ii) $e_0 \perp g$ and $e = e_0 \vee g$,

iii) $f_0 \perp g$ and $f = f_0 \vee g$.

It is a fact from the theory of Baer *-semigroup that for $e, f \in P'(S_\Omega)$, $e\,\bar{C}\,f$ is equivalent to $e \circ f = f \circ e$ and, moreover, if $e\,\bar{C}\,f$, then $e \wedge f = e \circ f$. The assertion of the theorem then follows from the fact that $(\mathscr{E}, \leqq, ')$ and $(P'(S_\Omega), \leqq, ')$ are isomorphic under $p \to \Omega_p$. Q.E.D.

Consequently, the compatibility of events corresponds to commutativity of the associated operations. Furthermore, in the case of compatibility, the greatest lower bound $p \wedge q$, of p and q (which is interpreted as the conjunction or "and" of p and q) corresponds to the composition of the associated operations Ω_p and Ω_q. This, of course, is an intuitively reasonable result.

IV. Comments

Although Axioms II.6 and II.7 have not been utilized, they are included in the definition of an event-state-operation structure because of their equivalence to the conventional expression for conditional probabilities involving compatible events.

Theorem IV.1. *If $(\mathscr{E}, \mathscr{S}, P, \Omega)$ is a 4-tuple which satisfies Axiom II.1, then Axioms II.6 and II.7 are equivalent to the following: if $p, q \in \mathscr{E}$, $p\,C\,q$, and $\alpha \in \mathscr{D}_p$, then*

$$P(q, \Omega_p\alpha) = \frac{P(q \wedge p, \alpha)}{P(p, \alpha)}\,.$$

Proof. Assume Axioms II.6 and II.7 and let $p, q \in \mathscr{E}$, $p\,C\,q$, and $\alpha \in \mathscr{D}_p$. Since $p\,C\,q$,

$$P(q, \Omega_p\alpha) = P(q \wedge p, \Omega_p\alpha)$$

by Axiom II.7. Since $q \wedge p \leqq p$,

$$P(q \wedge p, \Omega_p\alpha) = \frac{P(q \wedge p, \alpha)}{P(p, \alpha)}$$

by Axiom II.6; hence,

$$P(a, \Omega_p\alpha) = \frac{P(q \wedge p, \alpha)}{P(p, \alpha)}\,.$$

Conversely, assume the validity of

$$P(q, \Omega_p\alpha) = \frac{P(q \wedge p, \alpha)}{P(p, \alpha)}$$

for $p, q \in \mathscr{E}$, $p\,C\,q$ and $\alpha \in \mathscr{D}_p$. If $p, q \in \mathscr{E}$, $q \leqq p$ and $\alpha \in \mathscr{D}_p$, then

$$P(q, \Omega_p\alpha) = \frac{P(q \wedge p, \alpha)}{P(p, \alpha)} = \frac{P(q, \alpha)}{P(p, \alpha)}\,;$$

hence, Axiom II.6 is valid. If $p, q \in \mathscr{E}$, $p \, \mathrm{C} \, q$, and $\alpha \in \mathscr{D}_p$, then

$$P(q \wedge p, \Omega_p \alpha) = \frac{P((q \wedge p) \wedge p, \alpha)}{P(p, \alpha)} = \frac{P(q \wedge p, \alpha)}{P(p, \alpha)} = P(q, \Omega_p \alpha);$$

hence, Axiom II.7 is valid. Q.E.D.

The relation of the operations in S_Ω to the operation discussed in [10] may be examined by considering Example II.1. If $\mathscr{L}_p(H)$ is the set of finite products of projections in $\mathscr{P}(H)$,

$$\mathscr{L}_p(H) = \{P_1 P_2 \dots P_n : P_1, P_2, \dots, P_n \in \mathscr{P}(H)\},$$

then $(\mathscr{L}_p(H), \circ, *, ')$ is a Baer *-semigroup contained in the Baer *-semigroup $(\mathscr{L}_c(H), \circ, *, ')$ (see Appendix). Each $A \in \mathscr{L}_p(H)$ yields an element of S_Ω for the Example II.1. If

$$A = P_1 P_2 \dots P_n, P_1, P_2, \dots, P_n \in \mathscr{P}(H),$$

then $x_A = \Omega_{p_1} \circ \dots \circ \Omega_{p_n}$ is an element of S_Ω. A simple calculation proves: the domain of x_A is

$$\mathscr{D}_{x_A} = \{\alpha \in \mathscr{S} : \mathrm{Tr}(D_\alpha A^* A) \neq 0\}$$

and if $\alpha \in \mathscr{D}_{x_A}$, with density operator D_α, then $\alpha' = x_A \alpha$ has density operator $D_{\alpha'}$

$$D_{\alpha'} = \frac{A D_\alpha A^*}{\mathrm{Tr}(D_\alpha A^* A)}.$$

However, if $B \in \mathscr{L}_c(H)$ and $B = \lambda A$ where $\lambda \in \mathrm{C}$ (the field of complex numbers) and $\lambda \neq 0$, then

$$\frac{B D_\alpha B^*}{\mathrm{Tr}(D_\alpha B^* B)} = \frac{A D_\alpha A^*}{\mathrm{Tr}(D_\alpha A^* A)}.$$

Consequently, the Baer *-semigroup $(\mathscr{L}_c(H)/\equiv, \circ, *, ')$ is evidently the relevant semigroup in the approach adopted here instead of $(\mathscr{L}_c(H), \circ, *, ')$. $\equiv$ is the relation defined on $\mathscr{L}_c(H)$ as follows: for $A, B \in \mathscr{L}_c(H)$, $A \equiv B$ means there exists a $\lambda \in \mathrm{C}$, $\lambda \neq 0$, such that $A = \lambda B$. $\equiv$ is an equivalence relation which respects the Baer *-semigroup structure of $(\mathscr{L}_c(H), \circ, *, ')$ (see remark after Thm. A.2.); hence, $(\mathscr{L}_c(H)/\equiv, \circ, *, ')$ is also a Baer *-semigroup. However $\equiv$ does not respect the additive structure of $\mathscr{L}_c(H)$; indeed, if $A_1 \equiv B_1$ and $A_2 \equiv B_2$, then $A_1 + A_2 \not\equiv B_1 + B_2$. This remark indicates that *operations* and *observables* are evidently quite different kinds of entities. For example, there exists a phenomenological interpretation for the multiplication of operations but there exists a phenomenological interpretation for the addition of observables. It is evidently a property of examples like

Example II.1 that both operations and observables have simple descriptions in terms of the same mathemathical object, namely, an operator on a Hilbert space.

The connection between the mathematical theories of orthomodular ortholattices and Baer *-semigroups is explicit:

a) If $(S, \circ, *, ')$ is a Baer *-semigroup, then there exists an orthomodular ortholattice $(P'(S), \leqq, ')$ with

$$P'(S) = \{x \in S : x \circ x = x^* = x'' = x\}.$$

b) If $(L, \leqq, ')$ is an orthomodular ortholattice, then there exists a Baer *-semigroup $(S(L), \circ, *, ')$ where $S(L)$ consists of a set of mappings from L into L and there exists an injective mapping $j : L \to S(L)$.

The orthomodular orthoposet $(\mathscr{E}, \leqq, ')$ associated with an event-state structure $(\mathscr{E}, \mathscr{S}, P)$ is not necessarily an ortholattice. However, the introduction of operations to form an event-state-operation structure $(\mathscr{E}, \mathscr{S}, P, \Omega)$ makes $(\mathscr{E}, \leqq, ')$ into an orthomodular ortholattice and provides a Baer *-semigroup S_Ω which admits a phenomenological interpretation. S_Ω is a set of mappings of the space $\mathscr{S}$ into itself. Hence, S_Ω is not the Baer *-semigroup $S(\mathscr{E})$ mentioned in part b) of the connection between orthomodular ortholattices and Baer *-semigroups (when we take $(\mathscr{E}, \leqq, ')$ for the $(L, \leqq, ')$ of part b)). $S(\mathscr{E})$ is a collection of mappings of E into $\mathscr{E}$. The role of $S(\mathscr{E})$ will be discussed in [26].

Finally, the question arises whether the introduction of Baer *-semigroups yields any useful contributions to the quantum logic approach to the foundations of quantum physics. In general, a given mathematical construct in the theory of orthomodular ortholattices has a corresponding mathematical construct in the theory of Baer *-semigroups and vice versa. There exist a number of lattice-theoretic constructs which are extremely useful mathematical tools for the quantum logic approach but which do not possess a phenomenological interpretation. In several cases the associated construct in the theory of Baer *semigroups, indeed, possesses an intuitively reasonable phenomenological interpretation. For example, the semimodularity of $(\mathscr{E}, \leqq, ')$ is a critical property in the proof of the "concrete representation" theorems in [17] and [23]; however, no phenomenological interpretation of this lattice-theoretic concept is available. In [26], it will be shown that the semimodularity of $(\mathscr{E}, \leqq, ')$ when $\mathscr{E}$ is atomic is equivalent to the following requirement: every $x \in S_\Omega$ is a *pure* operation [10], that is, if $\alpha \in \mathscr{D}_x$ and α is a pure state (an extreme point of the convex set $\mathscr{S}$), then $x\alpha$ is a pure state.

Appendix

The first part of this appendix is review of concepts from the theory of orthomodular ortholattices while the remainder presents the necessary aspects of the theory of Baer *-semigroups.

Definition A.1. A relation R on a set $\mathscr{X}$ is a subset R of the Cartesian product $\mathscr{X} \times \mathscr{X}$; notation: xRy means $(x, y) \in R$.

Definition A.2. A relation R on a set $\mathscr{X}$ is said to be

a) *symmetric:* if $x, y \in \mathscr{X}$ and xRy, then yRx,

b) *anti-symmetric:* if $x, y \in \mathscr{X}$, xRy and yRx, then $x = y$.

c) *reflexive:* if $x \in \mathscr{X}$, then xRx.

d) *transitive:* if $x, y, z \in \mathscr{X}$, xRy and yRz, then xRz.

Definition A.3. A *poset* is a pair $(\mathscr{X}, \leqq)$ where $\mathscr{X}$ is a set and $\leqq$ is an anti-symmetric, reflexive, transitive relation (a *partial ordering*) on $\mathscr{X}$.

Definition A.4. Let $(\mathscr{X}, \leqq)$ be a poset and $\mathscr{Y} \subset \mathscr{X}$.

a) $x \in \mathscr{X}$ is an *upper bound* for $\mathscr{Y}$ provided: if $y \in \mathscr{Y}$, then $y \leqq x$.

b) $x \in \mathscr{X}$ is a *least upper bound* for $\mathscr{Y}$ provided:

i) x is an upper bound for $\mathscr{Y}$,

ii) if z is an upper bound for $\mathscr{Y}$, then $x \leqq z$.

c) The least upper bound of $\mathscr{Y}$, if it exists, is denoted by $\vee \mathscr{Y}$; in case $\mathscr{Y} = \{y_1, y_2\}$, $\vee \mathscr{Y}$ is denoted by $y_1 \vee y_2$.

d) *Lower bound, greatest lower bound,* $\wedge \mathscr{Y}$ and $y_1 \wedge y_2$ are defined dually.

e) An element $0 \in \mathscr{X}$ (respectively $1 \in \mathscr{X}$) such that $0 \leqq x$ (respectively, $x \leqq 1$) for all $x \in \mathscr{X}$ is called a *least* (respectively, *greatest*) element of $\mathscr{X}$.

f) $(\mathscr{X}, \leqq)$ is a lattice if $x_1, x_2 \in \mathscr{X}$ implies $x_1 \wedge x_2$ and $x_1 \vee x_2$ exist.

The set R of real numbers has a partial ordering, the usual ordering of real numbers. The collection 2^X of all subsets of a set X has a partial order, namely, the set-theoretic relation of inclusion. If H is a complex Hilbert space and $\mathscr{P}(H)$ is the set of all projection operators in H, then the relation $\leqq$ is a partial ordering where

$$P \leqq Q \text{ means } PQ = P, \qquad P, Q \in \mathscr{P}(H)\,.$$

Each of these examples is a lattice.

Definition A.5. Let $(\mathscr{X}, \leqq)$ be a poset with 0 and 1.

a) A mapping $' : \mathscr{X} \to \mathscr{X}$ is an *orthocomplementation* provided:

i) if $x \in \mathscr{X}$, then $(x')' = x$,

ii) if $x, y \in \mathscr{X}$ and $x \leqq y$, then $y' \leqq x'$,

iii) if $x \in \mathscr{X}$, then $x \wedge x'$ and $x \vee x'$ exist and equal 0 and 1, respectively.

b) If $' : \mathscr{X} \to \mathscr{X}$ is an orthocomplementation, the relation $\perp$, the relation of *orthogonality*, is defined as follows: for $x, y \in \mathscr{X}$, $x \perp y$ means $x \leqq y'$.

c) An *orthoposet* $(\mathscr{X}, \leqq, ')$ is a poset $(\mathscr{X}, \leqq)$ together with an orthocomplementation of $(\mathscr{X}, \leqq)$ such that if $x, y \in \mathscr{X}$ and $x \perp y$, then $x \vee y$ exists.

d) An orthoposet $(\mathscr{X}, \leqq, ')$ is a *σ-orthoposet* provided: if $x_1, x_2, \ldots \in \mathscr{X}$ and $x_i \perp x_j$ for $i \neq j$, $i, j = 1, 2, \ldots$, then $\bigvee_i x_i$ exists.

e) An orthoposet $(\mathscr{X}, \leqq, ')$ is *orthomodular* provided: if $x, y \in \mathscr{X}$ and $x \leqq y$, then $y = x \vee (x' \wedge y)$.

2^X admits an orthocomplementation, namely, the set-theoretic complementation. The mapping $P \in \mathscr{P}(H) \to P' = I - P \in \mathscr{P}(H)$ is an orthocomplementation of $\mathscr{P}(H)$.

Definition A.6. Let $(\mathscr{X}, \leqq, ')$ be a σ-orthoposet.

a) A probability measure μ on $\mathscr{X}$ is a function $\mu : \mathscr{X} \to [0, 1]$ such that

i) $\mu(0) = 0$, $\mu(1) = 1$,

ii) if $x_1, x_2, \ldots \in \mathscr{X}$ and $x_i \perp x_j$ for $i \neq j$, then

$$\mu\left(\bigvee_i x_i\right) = \sum_i \mu(x_i)\,.$$

Let $\mathscr{M}$ be a set of probability measures on $\mathscr{X}$.

b) $\mathscr{M}$ is *order-determining* provided: if $x, y \in \mathscr{X}$ and $\mu(x) \leqq \mu(y)$ for all $\mu \in \mathscr{M}$, then $x \leqq y$.

c) $\mathscr{M}$ is *strongly-order-determining* provided: if $x, y \in \mathscr{X}$ and

$$\{\mu \in \mathscr{M} : \mu(x) = 1\} \subset \{\mu \in \mathscr{M} : \mu(y) = 1\}\,,$$

then $x \leqq y$.

d) $\mathscr{M}$ is *separating* provided: if $x, y \in \mathscr{X}$ and $\mu(x) = \mu(y)$ for all $\mu \in \mathscr{M}$, then $x = y$.

e) $\mathscr{M}$ is σ-convex provided: if $\mu_1, \mu_2, \ldots \in \mathscr{M}$, $t_1, t_2, \ldots \in [0, 1]$ and $\sum_i t_i = 1$, then there exists a $\mu \in \mathscr{M}$ such that

$$\mu(x) = \sum_i t_i \mu_i(x) \quad \text{for all} \quad x \in \mathscr{X}\,.$$

Theorem A.1. *Let $\mathscr{M}$ be a set of probability measures on a σ-orthoposet $(\mathscr{X}, \leqq, ')$. If $\mathscr{M}$ is separating, then $(\mathscr{X}, \leqq, ')$ is orthomodular. If $\mathscr{M}$ is order-determining and $(\mathscr{X}, \leqq, ')$ is orthomodular, then $\mathscr{M}$ is separating. If $\mathscr{M}$ is strongly-order-determining and $(\mathscr{X}, \leqq, ')$ is orthomodular, then $\mathscr{M}$ is order-determining.*

Proof. See, for example, [25].

For additional material on posets and lattices, see [2].

Definition A.7. a) A *semigroup* $(S, \circ)$ is a set S with a mapping $\circ : S \times S \to S((x, y) \in S \times S \to x \circ y \in S)$ such that if $x, y, z \in S$, then

$$(x \circ y) \circ z = x \circ (y \circ z)$$

i.e., $\circ$ is *associative*.

b) If $(S, \circ)$ is a semigroup, then an element $0 \in S$ (respectively, $1 \in S$) is a *zero* (respectively, *unit*) provided $0 \circ x = x \circ 0 = 0$ (respectively $1 \circ x = x \circ 1 = x$) for all $x \in S$.

c) An *involution semigroup* $(S, \circ, *)$ is a semigroup $(S, \circ)$ together with a mapping called an *involution*, $* : S \to S(x \in S \to x^* \in S)$, such that

i) if $x \in S$, then $(x^*)^* = x$,

ii) if $x, y \in S$, then $(x \circ y)^* = y^* \circ x^*$.

d) If $(S, \circ, ^*)$ is an involution semigroup, then an element of $P(S)$ is called a *projection* where

$$P(S) = \{e \in S : e \circ e = e^* = e\} .$$

e) If $(S, \circ, ^*)$ is an involution semigroup, then the relation $\leq$ on $P(S)$ is defined as follows: for $e, f \in P(S)$, $e \leq f$ means $e \circ f = e$.

If H is a complex Hilbert space, then $(\mathscr{L}_c(H), \circ)$ is a semigroup where $\mathscr{L}_c(H)$ is the set of all continuous (i.e., bounded) linear operators on H and $\circ$ is operator multiplication, if $A, B \in \mathscr{L}_c(H)$, then $A \circ B = A B$. The usual operator adjoint, $A \to A^*$, is an involution for $(\mathscr{L}_c(H), \circ)$; moreover, in this case, $P(H)$, the set of projection operators in H, coincides with $P(\mathscr{L}_c(H))$. The relation $\leq$ of e) is just the conventional partial ordering of projection operators. This illustrates the following theorem.

Theorem A.2. *If $(S, \circ, ^*)$ is an involution semigroup, then $(P(S), \leq)$ is a poset; moreover, if S has a zero 0 (respectively, unit 1), then 0 (respectively, 1) is the least (respectively, greatest) element of $P(S)$.*

Define the relation $\equiv$ on $\mathscr{L}_c(H)$ as follows: for $A, B \in \mathscr{L}_c(H)$, $A \equiv B$ means there exists a $\lambda \in \mathbb{C}$ (the complex number field) such that $\lambda \neq 0$ and $A = \lambda B$. $\equiv$ is obviously an equivalence relation (i.e., $\equiv$ is reflexive, symmetric and transitive). If $A \in \mathscr{L}_c(H)$, let C_A denote the equivalence class containing A,

$$C_A = \{B \in \mathscr{L}_c(H) : B \equiv A\}$$

and let $\mathscr{L}_c(H)/\equiv$ denote the set of all these equivalence classes. If $A, A_1, B, B_1 \in \mathscr{L}_c(H)$, $A_1 \equiv A$ and $B_1 \equiv B$, then $A_1 \circ B_1 \equiv A \circ B$; hence, $\circ$ induces a composition in $\mathscr{L}_c(H)/\equiv$ by

$$C_A \circ C_B = C_{A \circ B},\ A, B \in \mathscr{L}_c(H) .$$

Similarly, if $A, B \in \mathscr{L}_c(H)$ and $A \equiv B$, then $A^* \equiv B^*$; hence, * induces an involution in $\mathscr{L}_c(H)/\equiv$ by $(C_A)^* = C_{A^*}$, $A \in \mathscr{L}_c(H)$. $(\mathscr{L}_c(H)/\equiv, \circ, ^*)$ is an involution semigroup such that $A \to C_A$ is a homomorphism. However, if $A_1, A, B_1, B \in \mathscr{L}_c(H)$, $A_1 \equiv A$, and $B_1 \equiv B$, then $A_1 + B_1 \not\equiv A + B$, in general; indeed, if $A_1 = \lambda A$ and $B_1 = \mu B$, $\lambda, \mu \in \mathbb{C}$, $\lambda, \mu \neq 0$, then, in general, there will exist no $\nu \in \mathbb{C}$ such that

$$A_1 + B_1 = \lambda A + \mu B = \nu(A + B) .$$

Definition A.8. a) A *Baer *-semigroup* $(S, \circ, ^*, ')$ is an involution semigroup $(S, \circ, ^*)$ with a zero 0 and a mapping $' : S \to P(S)$ such that if $x \in S$, then

$$\{y \in S : x \circ y = 0\} = \{z \in S : z = x' \circ z\} .$$

b) If $(S, \circ, *, ')$ is a Baer *-semigroup, then an element of

$$P'(S) = \{e \in P(S) : (e')' = e\}$$

is called a *closed* projection.

If $A \in \mathscr{L}_c(H)$, the *null space* of A is denoted by $\mathscr{N}_A$,

$$\mathscr{N}_A = \{\psi \in H : A\psi = 0\}$$

and the projection with range $\mathscr{N}_A$ is denoted by A'. The mapping $A \to A'$ makes $(\mathscr{L}_c(H), \circ, *)$ into a Baer *-semigroup. Furthermore, if $A, B \in \mathscr{L}_c(H)$ and $A \equiv B$, then $A' \equiv B'$; consequently, both $(\mathscr{L}_c(H), \circ, *, ')$ and $(\mathscr{L}_c(H)/\equiv, \circ, *, ')$ are Baer *-semigroups where $(C_A)' = C_{A'}$ for $A \in \mathscr{L}_c(H)$.

Theorem A.3. *Let $(S, \circ, *, ')$ be a Baer *-semigroup.*

a) $P'(S) = \{x' : x \in S\}$.

b) *If $e \in P'(S)$, then $e' \in P'(S)$.*

c) *$(P'(S), \leqq, ')$ is an orthomodular ortholattice where $\leqq$ is the relation $\leqq$ on $P(S)$ restricted to $P'(S)$ and $'$ is the restriction of $' : S \to P(S)$ to $P'(S)$; moreover, if $e, f \in P'(S)$, then $e \wedge f = (e' \circ f)' \circ f$.*

d) *If $e, f \in P'(S)$, then the following are equivalent:*

i) *there exist $e_0, f_0, g \in P'(S)$ such that*

$$e_0 \perp f_0,\ e_0 \perp g,\ f_0 \perp g,\ e = e_0 \vee g \quad \text{and} \quad f = f_0 \vee g\,,$$

ii) $e \circ f = f \circ e$;

moreover, if $e \circ f = f \circ e$, then $e \wedge f = e \circ f$.

The proofs of these theorems together with further details of the theory of Baer *-semigroups may be found in [6].

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