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BAER SUBPLANES IN FINITE PROJECTIVE AND AFFINE PLANES

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1. Introduction. Let π be a projective or an affine plane; a configuration **C** of π is a subset of points and a subset of lines in π such that a point P of **C** is incident with a line $\mathfrak{l}$ of **C** if and only if P is incident with $\mathfrak{l}$ in π. A configuration of a projective plane π which is a projective plane itself is called a projective subplane of π, and a configuration of an affine plane π' which is an affine plane with the improper line of π' is an affine subplane of π'.

Let π be a finite projective (respectively, an affine) plane of order n and π_0 a projective (respectively, an affine) subplane of π of order n_0 different from π; then $n_0 \leqq \sqrt{n}$. If $n_0 = \sqrt{n}$, then π_0 is called a *Baer subplane* of π. Thus, Baer subplanes are the "biggest" possible proper subplanes of finite planes. Their importance is underlined by the following theorem of Baer [**1**]: If a finite projective (respectively, an affine) plane π admits a collineation α of order 2 fixing pointwise a projective (respectively, an affine) subplane of π, then the subplane is a Baer subplane; the collineation α is called a Baer involution.

The existence of Baer subplanes is well known in the case when π is a desarguesian projective or affine plane of square order $n = m^2$. Then the following results are true:

RESULT 1. *If π is a desarguesian projective plane of order $n = m^2$, then:*

(a_1) *Every quadrangle (i.e., set of 4 points, no 3 of which are collinear) of π is contained in exactly one Baer subplane of π, and:*

(b_1) *Any two distinct Baer subplanes containing 3 common points on a line $\mathfrak{l}$ contain on $\mathfrak{l}$ exactly $m + 1$ common points.*

RESULT 2. *If π is a desarguesian affine plane of order $n = m^2$, then:*

(a_2) *Every affine triangle (i.e., set of 3 non-collinear affine points) of π is contained in exactly one affine Baer subplane of π, and:*

(b_2) *Any two distinct affine Baer subplanes containing 2 common affine points contain exactly m common collinear affine points.*

In a previous note [**2**], we have considered finite projective planes with Baer subplanes admitting Baer involutions. The purpose of the present paper is to investigate Baer subplanes in finite planes without considering collineations. We shall show that finite projective planes satisfying assumptions (a_1) and (b_1) of Result 1 are desarguesian. This statement is a corollary to the following

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result obtained for affine planes: Finite affine planes satisfying assumptions (a_2') and (b_2') of Result 2 are translation planes. The proofs of the above statements are given in § 3. In § 2, the assertions of Results 1 and 2 are verified.

2. Baer subplanes in finite desarguesian planes. In this section, we shall recall well known properties of finite desarguesian projective and affine planes.

Let π be a desarguesian projective plane of square order $n = m^2$. It is known that (1) the order of π is a prime power $n = m^2 = p^{2r}$; (2) the ternary ring coordinatizing π with respect to any quadrangle is the Galois field K of order p^{2r}; (3) the collineation group of π is transitive on the ordered quadrangles of π and is triply transitive on the points of any line in π; (4) the perspectivities of π with a given axis form a group of order $n^2(n-1)$ (see, for instance, [**4** p. 401]).

Let A, B, C, D be an arbitrary quadrangle in π. The Galois field K coordinatizing π with respect to A, B, C, D has a subfield K_0 of order p^r; the points of π with coordinates from K_0 form a Baer subplane π_0 of π containing A, B, C, D. Suppose that there exists a Baer subplane π_0' different from π_0 containing A, B, C, D; then the coordinates of the points in π_0' form a subfield K_0' of K or order p^r. However, the Galois field or order p^{2r} contains a unique subfield of order p^r. Thus, $K_0 = K_0'$ and π_0 and π_0' coincide. In other words:

(a_1) *Any quadrangle of π is contained in a unique Baer subplane.*

Denote by N the number of quadrangles in π and by N_0 the number of quadrangles in any Baer subplane of π. From (a_1'), it follows that the number b of the Baer subplanes in π is

$$b = \frac{N}{N_0} = \frac{(n^2+n+1)(n^2+n)n^2(n-1)^2/4!}{(n+\sqrt{n}+1)(n+\sqrt{n})n(\sqrt{n}-1)^2/4!}$$
$$= (n-\sqrt{n}+1)n\sqrt{n}(n+1)(\sqrt{n}+1).$$

Consider the following incidence structure $\mathscr{I}$: the points of $\mathscr{I}$ are the lines of π, the blocks of $\mathscr{I}$ the Baer subplanes of π, and a point of $\mathscr{I}$ is incident with a block of $\mathscr{I}$ exactly when the corresponding line is contained in the corresponding Baer subplane in $\mathscr{I}$. The number of points in $\mathscr{I}$ is $v = n^2 + n + 1$, the number of the blocks is b; any block is incident with $k = n + \sqrt{n} + 1$ points of $\mathscr{I}$, and in view of (3), every point of $\mathscr{I}$ is incident with the same number r of blocks. Counting the number of incident point-block pairs in $\mathscr{I}$, we obtain

$$r = \frac{bk}{v} = n\sqrt{n}(n+1)(\sqrt{n}+1).$$

Take any line $\mathfrak{l}$ of π and consider the incidence structure $\mathscr{I}'$ whose points are the points of $\mathfrak{l}$ and blocks the Baer subplanes of π containing $\mathfrak{l}$; a point of

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$\mathscr{I}'$ is incident with a block of $\mathscr{I}'$ if and only if the corresponding point of π is contained in the corresponding Baer subplane of π. The structure $\mathscr{I}'$ contains $v' = n + 1$ points and $b' = r$ blocks. Every block of $\mathscr{I}'$ is incident with $k' = \sqrt{n} + 1$ points of $\mathscr{I}'$, and in view of (3), any 3 distinct points of $\mathscr{I}'$ are incident with a common number λ of blocks. Thus, $\mathscr{I}'$ is a

$$3 - (n + 1, \sqrt{n} + 1, \lambda) \text{ - design.}$$

It follows (see, e.g., [**3**, §2]) that

$$\lambda = \frac{b'k'(k' - 1)(k' - 2)}{v'(v' - 1)(v' - 2)} = n(\sqrt{n} + 1).$$

Hence:

(*) *Any 3 distinct points of $\mathfrak{l}$ are contained in exactly $n(\sqrt{n} + 1)$ Baer subplanes.*

Let A, B, C, be three arbitrary distinct points on $\mathfrak{l}$ contained in a Baer subplane π_0. Consider the images of π_0 under the group Δ of perspectivities of π with axis $\mathfrak{l}$. It is easy to verify that a perspectivity α of Δ maps π_0 onto itself if and only if the centre of α is in π_0 and, for an arbitrary point $X \in \pi_0$, its image $X\alpha \in \pi_0$; in this case, α induces a perspectivity in π_0. There are n $(\sqrt{n} - 1)$ perspectivities in Δ mapping π_0 onto itself. They form a subgroup Δ_0 of Δ and the number of the images of π_0 under Δ is equal to the index $[\Delta : \Delta_0]$ which is, according to (4), exactly $n(\sqrt{n} + 1)$. The points of $\mathfrak{l}$ are fixed under any element of Δ; therefore, each of the $n(\sqrt{n} + 1)$ images of π_0 under Δ contains the $\sqrt{n} + 1$ points of π_0 on $\mathfrak{l}$. In view of (*), the images of π_0 under Δ are the only Baer subplanes through A, B, C. This implies:

(b_1) *Any two distinct Baer subplanes of π containing 3 common points on a line $\mathfrak{l}$ of π contain on $\mathfrak{l}$ exactly $m + 1$ common points.*

Thus, Result 1 is verified.

In order to verify Result 2, consider a desarguesian affine plane π of order $n = m^2$. The plane π can be extended to a projective plane $\bar{\pi}$ by adjoining its improper points and its improper line. Let A, B, C, be any affine triangle of π; the lines AB and AC intersect the improper line in two improper points B^* and C^*, respectively. An affine subplane of π containing A, B, C must contain the improper points B^* and C^*. Since $\bar{\pi}$ is a desarguesian projective plane, according to (a_1), the quadrangle B, C, B^*, C^* is contained in exactly one Baer subplane of $\bar{\pi}$.

This implies:

(a_2) *Any affine triangle of π is contained in exactly one affine Baer subplane of π.*

Take any two distinct affine Baer subplanes π_0 and π_0' intersecting in at least 2 common affine points A, B. Both subplanes contain the improper point

C^* of AB. Thus, the corresponding projective planes $\bar{\pi}_0$ and $\bar{\pi}_0'$ contain 3 common points on the line AB of $\bar{\pi}$ and, according to (b_1), they have precisely $m + 1$ points of AB in common. In view of (a_2), the planes π_0 and π_0' cannot contain any common affine point $X \notin AB$.
Thus, it follows that:

(b_2) Any two distinct affine Baer subplanes containing at least 2 *common affine points contain exactly m common collinear affine points.*

This verifies Result 2.

3. Main results

3.1. Affine planes. Let $\mathfrak{A}$ be an affine plane of square order $n = m^2$ with affine Baer subplanes satisfying conditions (a_2) and (b_2) of Results 2. Denote the collection of the affine Baer subplanes in $\mathfrak{A}$ by $\mathfrak{B}$.

First of all, we shall prove the following Lemma:

LEMMA 1. *If, in an affine plane $\mathfrak{A}$ of order $n = m^2$ containing affine Baer subplanes, conditions* (a_2) *and* (b_2) *are satisfied, then any two affine Baer subplanes intersecting in m affine points contain m common mutually parallel affine lines.*

Proof. Let π_1 and π_2 be two distinct affine Baer subplanes of $\mathfrak{A}$ intersecting in a set $\mathfrak{l}_0$ of m common affine points. According to (a_2), two Baer subplanes cannot contain non-collinear affine points. Therefore, $\mathfrak{l}_0$ is contained in a line $\mathfrak{l}$ of $\mathfrak{A}$ and $\mathfrak{l}_0 = \pi_1 \cap \mathfrak{l} = \pi_2 \cap \mathfrak{l}$. Moreover, $\mathfrak{l}_0$ contains all common affine points of π_1 and π_2.

Take a point $X \in \pi_2 - \pi_1$. Since π_1 is a Baer subplane, there is exactly one line x of π_1 through X (see, for instance, [**4**, p. 398]). Denote the improper point of x by X^* and the improper point of $\mathfrak{l}$ by L^*. There are two possibilities.

Case 1. $X^* = L^*$. Since L^* is an improper point of π_2, the line x is also contained in π_2. Take a point $Y \in \pi_2 - (\pi_1 \cup x)$. Denote the unique line of π_1 through Y by y and the improper point of y by Y^*. Suppose that $Y^* \neq L^*$. Then y interesects $\mathfrak{l}_0$ in a point, say D. Since $D \in \pi_2$, it follows that $y \in \pi_2$. Thus, the affine point $E = y \cap x$ is contained in $\pi_1 \cap \pi_2$. In other words, π_1 and π_2 contain a common point $E \notin \mathfrak{l}_0$, contradicting (a_2). Therefore $Y^* = L^*$. There are $m^2 - m$ affine points in $\pi_2 - \pi_1$. Through each point in $\pi_2 - \pi_1$ there is a unique line of π_1; denote the set of these lines by $\mathscr{P}$. According to the above investigations, any line from $\mathscr{P}$ intersects the improper line of $\mathfrak{A}$ in L^*; therefore, it is also contained in π_2. Since each line of $\mathscr{P}$ carries m points of π_2, the number of the lines in $\mathscr{P}$ is $(m^2 - m)/m = m - 1$. The lines of $\mathscr{P}$ together with $\mathfrak{l}$ form a set of m mutually parallel affine lines common to π_1 and π_2.

Case 2. $X^* \neq L^*$. In this case, x intersects $\mathfrak{l}_0$ in a point $D \in \pi_1 \cap \pi_2$. Since x contains two points X and D of π_2, the line x is contained in $\pi_1 \cap \pi_2$ and also its improper point $X^* \in \pi_1 \cap \pi_2$. Take a point $Y \in \pi_2 - (\pi_1 \cup x)$, and denote the unique line of π_1 through Y by y. Suppose that the improper point Y^* of y is different from X^*. Clearly, $Y^* \neq L^*$; hence, y intersects $\mathfrak{l}_0$ in a point E; this implies that $y \in \pi_1 \cap \pi_2$ and $Y^* \in \pi_1 \cap \pi_2$. If $E \neq D$, then $F = x \cap y \notin \mathfrak{l}_0$ and $\pi_1 \cap \pi_2$ contains a point $F \notin \mathfrak{l}_0$, contradicting (a_2). Assume that $E = D$. Then $\mathfrak{l}_0$ contains a point $F \neq D$. The line Y^*F is contained in both planes π_1 and π_2 and it intersects x in an affine point $G \in \pi_1 \cap \pi_2$. Since $G \notin \mathfrak{l}_0$, the triangle D, F, G is contained in two distinct affine Baer subplanes, contradicting (a_2). Hence, any line of π_1 through a point of $\pi_2 - \pi_1$ contains the improper point X^*. The $m^2 - m$ points of $\pi_2 - \pi_1$ determine m lines through X^*, each of them carrying $m - 1$ points $\pi_2 - \pi_1$. These m lines form a pencil of mutually parallel affine lines common to π_1 and π_2.

This finishes the proof of Lemma 1.

Take any two distinct affine points A, B and let C be an arbitrary affine point not on the line AB. In view of (a_2), the triangle A, B, C is contained in exactly one subplane, say π_0, of $\mathfrak{B}$. According to (b_2), for any affine point $C' \notin \pi_0$, the unique subplane π_0' containing A, B, C' intersects π_0 in exactly m affine points of the line AB. These points are all the points of π_0 on AB. Thus, the following statement is true:

(i) *Any two distinct affine points A, B determine a unique set $\overline{AB}$ of m points on the line AB of $\mathfrak{A}$. The set $\overline{AB}$ consists of the points common to all subplanes of $\mathfrak{B}$ containing A and B.*

The sets $\overline{AB}$ will be called segments.

Consider the incidence structure $\mathscr{I}$ whose points are the affine points of $\mathfrak{A}$, whose blocks are the segments in $\mathfrak{A}$, and in which incidence is defined as set-theoretical inclusion.

A linear manifold of $\mathscr{I}$ is a subset $\mathscr{L}$ of points in $\mathscr{I}$ such that for any two distinct points A, B of $\mathscr{L}$, the points of the block $\overline{AB}$ are contained in $\mathscr{L}$. Let $\mathscr{J}$ be any set of points in $\mathscr{I}$; the linear manifold $\langle \mathscr{J} \rangle$ generated by $\mathscr{J}$ is the intersection of all linear manifolds in $\mathscr{I}$ containing $\mathscr{J}$. A linear manifold $\langle A, B, C \rangle$ generated by 3 points A, B, C of $\mathscr{I}$ not on a common block of $\mathscr{I}$, is called a *plane* of $\mathscr{I}$.

(ii) *Any plane of $\mathscr{I}$ is an affine plane of order m whose lines are blocks of $\mathscr{I}$.*

For, if C does not belong to the line AB of $\mathfrak{A}$, then in view of (a_2), the points A, B, C are contained in a unique subplane of $\mathfrak{B}$ which is an affine plane of order m; it coincides with the linear manifold $\langle A, B, C \rangle$. On the other hand, if $C \in AB$, consider the line AB. This line contains $n = m^2$ points, any block of $\mathscr{I}$ on AB contains m points, and any two distinct points of AB are contained in a unique block. It follows easily that AB is an affine plane of order m whose lines are certain blocks of $\mathscr{I}$. This affine plane is again the linear manifold $\langle A, B, C \rangle$.

Thus, $\mathscr{I}$ contains two families of planes: planes of type **B** which are Baer subplanes in $\mathfrak{A}$ and planes of type **L** which are lines in $\mathfrak{A}$.

Our aim is to show that $\mathscr{I}$ is a four-dimensional affine space or order m with the blocks as lines.

Any two distinct points of $\mathscr{I}$ are contained in exactly one block of $\mathscr{I}$ and every block contains at least two distinct points. Hence, according to [**5**], $\mathscr{I}$ is an affine space if the following properties are satisfied:

(a) between the blocks an equivalence relation, called parallelism, is defined;

(b) for any point-block pair (P, b), there exists a unique block b' incident with P and parallel to b;

(c) for any four distinct points A, B, C, D such that $\overline{AB}$ is parallel to $\overline{CD}$ and any point $P \in \overline{AC}$, either $P \in \overline{CD}$ or $\overline{AB}$ and $\overline{PD}$ have a point in common.

We shall define parallelism in the following way:

(**) *Two distinct blocks of $\mathscr{I}$ are parallel if and only if they are disjoint and belong to a common plane of $\mathscr{I}$. Each block is parallel to itself.*

In view of (ii), all planes of $\mathscr{I}$ are affine planes; hence, properties (b) and (c) are automatically satisfied. It remains to prove that parallelism is an equivalence relation. It is easy to see that parallelism is reflexive and symmetric. However, the transitivity of the parallellism defined by (**) has to be verified. Thus, we have to show that:

(iii) *If b_1, b_2, and b_3 are three blocks of $\mathscr{I}$ such that b_1 is parallel to b_2 and b_2 is parallel to b_3, then b_1 is parallel to b_3.*

For the proof of (iii), the following cases can be distinguished.

Case 1. b_1, b_2, b_3 are contained in a common plane of $\mathscr{I}$.

Case 2. b_1 and b_2 are contained in a plane of type **B** and b_2 and b_3 are contained in a plane of type **L**.

Case 3. b_1 and b_2 are contained in a plane of type **B** and b_2 and b_3 also belong to a plane of type **B**.

Since the planes of $\mathscr{I}$ are affine planes there is nothing to be proved in Case 1.

Consider Case 2. Denote the plane through b_1 and b_2 by π_0 and the plane through b_2 and b_3 by $\mathfrak{l}$. The plane π_0 is a subplane of $\mathfrak{B}$ and $\mathfrak{l}$ is a line of $\mathfrak{A}$. Take any point $X \in \mathfrak{l}$, not on b_2. According to (a_2'), the block b_1 and the point X generate a unique plane π_1 of type **B** in $\mathscr{I}$. Since b_1 and b_2 are parallel lines in an affine Baer subplane of $\mathfrak{A}$, the lines of $\mathfrak{A}$ containing b_1 and b_2, respectively, are parallel in $\mathfrak{A}$. Denote by B* the common improper point of these lines. The plane π_1 contains the point X and the improper point B* of $\mathfrak{l}$; therefore, π_1 intersects $\mathfrak{l}$ in m affine points. In view of (a_2), the common points of π_0 and π_1 are exactly the points of b_1; hence the m affine points of π_1 on $\mathfrak{l}$ form a block c_1 disjoint from b_2. For all possible choices of X in the set $\mathfrak{l} - \pi_0$, by the same investigations, we obtain $(n - m)/m = m - 1$ distinct planes π_i $(i = 1, \ldots, m - 1)$ generated by b_1 and X, intersecting $\mathfrak{l}$ in $m - 1$ distinct blocks c_i, respectively, and disjoint from b_2. However, $\mathfrak{l}$ is an affine plane of order m; the blocks of $\mathfrak{l}$ disjoint from b_2 are parallel to b_2. In the affine plane $\mathfrak{l}$ there are exactly $m - 1$ blocks distinct from b_2 and parallel to it. Thus, b_3 must be one of the blocks c_i, say c_j. But this implies that b_3 and b_1 are contained in the plane π_j of $\mathscr{I}$. Moreover, b_3 and b_1 are disjoint since the lines of $\mathfrak{A}$ carrying b_3 and b_1, respectively, are parallel in $\mathfrak{A}$. Hence, b_1 is parallel to b_3.

In Case 3, denote the plane containing b_1 and b_2 by π_1 and the plane containing b_2 and b_3 by π_2. In view of Lemma 1, the plane $\mathfrak{A}$ contains a set $\mathscr{P}^-$ of m mutually parallel lines each of which is incident with exactly m affine points of π_1 and m affine points of π_2. This implies that every affine point of π_1 or π_2 is incident with some line of $\mathscr{P}^-$. Denote the common improper point of the lines from $\mathscr{P}^-$ by X^* and the improper point of the lines containing b_1, b_2 and b_3, respectively; by L^*.

Suppose that $X^* = L^*$. Then $b_2 \in \mathscr{P}^-$. The line $\mathfrak{l}$ of $\mathfrak{A}$ containing b_3 contains a block b_4 from π_1 and, obviously, b_1 is parallel to b_4. The blocks b_4 and b_3 are contained in the plane $\mathfrak{l}$ of type $\mathbf{L}$ and are disjoint, in view of (a_2). Hence, b_4 is parallel to b_3. Applying the investigations for Case 2, it follows that b_1 is parallel to b_3.

Finally, suppose that $X^* \neq L^*$. Take two arbitrary distinct points B, C on b_1. Each line of $\mathscr{P}^-$ contains exactly one point of b_3; let $D = B\,X^* \cap b_3$ and $E = CX^* \cap b_3$. The triangle B, C, D is contained in a unique Baer subplane of $\mathfrak{B}$, say π_3. Clearly, the improper points X^* and L^* of DB and BC, respectively, belong to π_3. Hence, $L^*D \in \pi_3$, $X^*C \in \pi_3$, and $X^*C \cap L^*D \in \pi_3$. However, $X^*C \cap L^*D \in b_3$; thus, $X^*C \cap L^*D = E$.

The plane π_3 contains two distinct points D and E of b_3. Therefore, π_3 contains b_3. The blocks b_1 and b_3 are disjoint and belong to a common subplane π_3 Therefore, b_1 is parallel to b_3.

This completes the proof of (iii)

Together with (iii) we have proved that $\mathscr{I}$ is an affine space whose lines are the blocks in $\mathscr{I}$. The order of $\mathscr{I}$ is m. Since $\mathscr{I}$ contains $n^2 = m^4$ points, it follows that $\mathscr{I}$ is a four-dimensional affine space:

(iv) *$\mathscr{I}$ is a four-dimensional affine space of order m.*

$\mathscr{I}$ can be completed to a four-dimensional projective space $\mathscr{I}^-$ of order m by adjoining its improper elements. The improper points of $\mathscr{I}$ form a three-dimensional projective subspace $\mathscr{I}_\infty$ of $\mathscr{I}^-$. The $m^2 + 1$ pencils of parallel lines in $\mathfrak{A}$ represent $m^2 + 1$ pencils of parallel planes in $\mathscr{I}$. Thus, the affine plane $\mathfrak{A}$ is represented in $\mathscr{I}$ in the following way: the points of $\mathfrak{A}$ are the affine points of $\mathscr{I}$, and the lines of $\mathfrak{A}$ are the affine planes of type $\mathbf{L}$ in $\mathscr{I}$; incidence is set-theoretical inclusion. The structure $\mathscr{I}^-$ is a four-dimensional projective space; therefore, it is desarguesian. As such, $\mathscr{I}^-$ admits m^4 elations with axis $\mathscr{I}_\infty$. It is easy to verify that the m^4 elations induce $m^4 = n^2$ distinct translations in $\mathfrak{A}$. Thus, $\mathfrak{A}$ is a translation plane. In other words the following theorem has been proved:

THEOREM 1. *Let $\mathfrak{A}$ be an affine plane of order $n = m^2$ containing affine Baer subplanes. If:*

Every affine triangle of $\mathfrak{A}$ is contained in exactly one affine Baer subplane of $\mathfrak{A}$, and if:

Any two distinct affine Baer subplanes containing 2 common affine points contain exactly m common affine points,

then $\mathfrak{A}$ is a translation plane.

3.2. Projective planes. Let π be a projective plane of square order $n = m^2$ with Baer subplanes satisfying conditions (a_1) and (b_1) of Result 1.

Take an arbitrary line $\mathfrak{l}$ in π and consider the affine plane $\pi_{\mathfrak{l}}$ obtained by eliminating the line $\mathfrak{l}$ together with all of its points. Denote by $\mathfrak{B}_{\mathfrak{l}}$ the set of all Baer subplanes in π containing $\mathfrak{l}$. From conditions (a_1) and (b_1) for π, it follows that the subplanes from $\mathfrak{B}_{\mathfrak{l}}$ satisfy conditions (a_2) and (b_2) of Result 2 for the affine plane $\pi_{\mathfrak{l}}$. This implies, according to Theorem 1, that:

(v) *The affine plane $\pi_{\mathfrak{l}}$ is a translation plane.*

Since $\mathfrak{l}$ is an arbitrary line of π, the plane π is desarguesian (see, for instance, [**4**. p. 403]).

Thus, for finite projective planes the following theorem can be proved:

THEOREM 2. *Let π be a projective plane of order $n = m^2$ containing Baer subplanes. If:*

Every quadrangle of π is contained in exactly one Baer subplane of π, and if:

Any two distinct Baer subplanes containing 3 common points on a line $\mathfrak{l}$, contain on the line $\mathfrak{l}$ exactly $m + 1$ common points,

then π is desarguesian.

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