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BAIRE SPACES AND QUASICONTINUOUS MAPPINGS

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Abstract

The notion of quasicontinuity was perhaps the first time used by R. Baire in [2]. Let X, Y be topological spaces and $Q(X, Y)$ be the space of quasicontinuous mappings from X to Y. If X is a Baire space and Y is metrizable, in $Q(X, Y)$ we can approach each (x, y) in the graph Grf of f along some trajectory of the form $\{(x_k, f_{n_k}(x_k)) : k \in \omega\}$ if and only if we can approach most points along a vertical trajectory. This result generalizes Theorem 5 from [3]. Moreover in the class of topological spaces with the property QP we give a characterization of Baire spaces by the above mentioned fact. We also study topological spaces with the property QP.

1 Introduction

In what follows let X, Y be topological spaces and $\mathbb{R}$ be the space of real numbers with the usual metric.

In the paper [18], S. Kempisty introduced a notion similar to continuity for real-valued functions defined in $\mathbb{R}$. For general topological spaces this notion can be given the following equivalent formulation.

Definition 1. A function $f : X \to Y$ is called quasicontinuous at $x \in X$ if for every open set $V \subset Y, f(x) \in V$ and open set $U \subset X$, $x \in U$ there is a nonempty open set $W \subset U$ such that $f(W) \subset V$. If f is quasicontinuous at every point of X, we say that f is quasicontinuous.

The notion of quasicontinuity was perhaps the first time used by R. Baire in [2] in the study of points of continuity of separately continuous functions. As Baire indicated in his paper [2] the condition of quasicontinuity has been suggested by V. Volterra.

There is a rich literature concerning the study of quasicontinuity (see, for instance [6], [9], [17], [19], [22], [25], [26], [27]).

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It seems that quasicontinuous mappings and Baire spaces suit each other very well.

A topological space is a Baire space, provided countable collections of dense open subsets have a dense intersection (equivalently, nonempty open subsets are of second Baire category).

It is known [25] that if X is a Baire space and Y a metric space, then every quasicontinuous function $f : X \to Y$ has the set $C(f)$, of continuity points, a dense G_δ-set. The same works also for a more general space Y (see [17]).

Also (see [4]) if X is a Baire space and Y a metric space, the pointwise limit $f : X \to Y$ of a sequence from $Q(X, Y)$ has the set $C(f)$ a dense G_δ-set. Conversely, if X is a metric space and Y is a separable metric space, then each function with the dense set $C(f)$ is the pointwise limit of a sequence of quasicontinuous functions ([28], for $X = \mathbb{R}$ see [13] and [5]).

It was proved in [16] (see also the paragraph 4) that for a Baire space X, if $\{f_n : n \in \omega\}$ is a sequence in $Q(X, \mathbb{R})$ pointwise convergent to $f : X \to \mathbb{R}$, f is quasicontinuous iff $\{f_n : n \in \omega\}$ is equi-quasicontinuous. (The same works also for every metric range space Y.) Moreover in [16] we can find interesting characterizations of Baire spaces by the above mentioned fact.

It is a motivation of this paper to continue in the study of interactions between Baire spaces and quasicontinuous mappings.

Beer in his paper [3] studied relations between pointwise and topological convergence of continuous functions. In our paper we will generalize Theorem 5 from [3] for quasicontinuous functions and in the class of topological spaces with the property QP we give a new interesting characterization of Baire spaces using quasicontinuous functions defined on them.

At the end of our introduction notice that the notion of quasicontinuity recently turned out to be instrumental in the proof that some semitopological groups are actually topological ones (see [7], [8]), in the proof of some generalizations of Michael's selection theorem (see [12]) and also in characterizations of minimal usco maps and densely continuous forms via their selections (see [15], [24]).

2 Baire spaces and quasicontinuos mappings

Let X be a topological space and let $\{C_n : n \in \omega\}$ be a sequence of nonempty subsets of X. The lower and upper closed limits of $\{C_n : n \in \omega\}$ are defined as follows [20]: LiC_n (resp. LsC_n) is the set of all points x each neighbourhood of which meets all but finitely (resp. infinitely) many sets C_n.

Let Y be another topological space. We can identify every function $f : X \to Y$ with its graph $Grf = \{(x, f(x)) : x \in X\}$.

If (Y, d) is a metric space, a function $f : X \to Y$ is said to be cliquish at $x \in X$ [29], if for each $\epsilon > 0$ and each neighbourhood U of x there is a nonempty open set $G \subset U$ such that $d(f(u), f(v)) < \epsilon$ for every $u, v \in G$. A function $f : X \to Y$ is called cliquish if it is cliquish at every point $x \in X$.

If (Y,d) is a metric space, $y \in Y$ and $\epsilon > 0$, by $S(y,\epsilon)$ we denote the open ϵ-ball about the point y.

Proposition 1. *Let X, Y be topological spaces. Let $\{f_n : n \in \omega\}$ be a sequence of functions from X to Y and let $f : X \to Y$ be a quasicontinuous function. Let E be a dense set in X such that for each $x \in E$, $f(x)$ is a cluster point of $\{f_n(x) : n \in \omega\}$. Then $Grf \subset LsGrf_n$.*

Proof. Let U, V be open sets in X and Y, respectively, let $(x, f(x)) \in U \times V$ and $n_0 \in \omega$. The quasicontinuity of f at x implies that there is a nonempty open set $G \subset U$ such that $f(s) \in V$ for every $s \in U$. The density of E implies that there is $e \in E \cap G$. Thus $f(e) \in V$ and since $f(e)$ is a cluster point of $\{f_n(e) : n \in \omega\}$, there is $n \geq n_0$ such that $f_n(e) \in V$. Thus $(e, f_n(e)) \in U \times V$. We proved that $Grf \subset LsGrf_n$. □

Notice that the above proposition works also for nets of functions.

It is clear that we cannot replace the quasicontinuity of f in the above proposition by the cliquishness of f.

Example 1. Let $X = Y = \mathbb{R}$ and $f : X \to Y$ be defined as follows: $f(0) = 1$ and $f(x) = 0$ otherwise. Of course f is cliquish and it is not quasicontinuous at 0. For every $n \in \omega$ let f_n be a function identically equal to 0. Then for every $x \neq 0$, $\{f_n(x) : n \in \omega\}$ converges to $f(x)$ and Grf is not contained in $LsGrf_n$.

The following theorem generalizes Theorem 5 in [3].

Theorem 1. *Let X be a Baire space and let (Y,d) be a metric space. Let $\{f_n : n \in \omega\}$ be a sequence of quasicontinuous functions from X to Y and $f : X \to Y$ be a cliquish function such that*

$$Grf \subset LsGrf_n.$$

Then there is a dense G_δ-set E in X such that for each $x \in E$, $f(x)$ is a cluster point of $\{f_n(x) : n \in \omega\}$.

Proof. We use an idea from [3]. For each $n \in \omega$ and $\epsilon > 0$ put

$$B(n,\epsilon) = \cup\{V : V \text{ open}, \exists j \geq n, d(f_j(z), f(z)) < \epsilon \text{ for every } z \in V\}.$$

Of course $B(n,\epsilon)$ is open for every $n \in \omega$ and every $\epsilon > 0$. Now we prove that $B(n,\epsilon)$ is dense for every $n \in \omega$ and every $\epsilon > 0$. Let $n \in \omega$ and $\epsilon > 0$. Let G be a nonempty open subset of X. We want to prove that $G \cap B(n,\epsilon) \neq \emptyset$. The cliquishness of f implies that there is a nonempty open subset V of G such that

$$d(f(z), f(v)) < \epsilon/3 \text{ for every } z, v \in V.$$

Since $Grf \subset LsGrf_n$ there is $j \geq n$ such that

$$V \times S(f(v), \epsilon/3) \cap Grf_j \neq \emptyset, \text{ where } v \in V.$$

Let $u \in V$ be such that $d(f_j(u), f(v)) < \epsilon/3$. The quasicontinuity of f_j at u implies that there is a nonempty open set $H \subset V$ such that $d(f_j(z), f_j(u)) < \epsilon/3$ for every $z \in H$. We claim that $H \subset G \cap B(n, \epsilon)$. Let $z \in H$. Then $d(f_j(z), f(z)) \leq d(f_j(z), f_j(u)) + d(f_j(u), f(v)) + d(f(v), f(z)) < \epsilon$. Put

$$E = \cap_k \cap_n B(n, 1/k).$$

Since X is a Baire space, E must be a dense set in X. □

Thus we have the following result:

Theorem 2. *Let X be a Baire space and Y be a metrizable space. Let $\{f_n : n \in \omega\}$ be a sequence in $Q(X,Y)$ and $f \in Q(X,Y)$. The following are equivalent:*

(1) $Grf \subset LsGrf_n$;

(2) there is a dense set E in X such that for each $x \in E$, $f(x)$ is a cluster point of $\{f_n(x) : n \in \omega\}$.

Thus if X is a Baire space and Y is metrizable, in $Q(X,Y)$ we can approach each (x,y) in Grf along some trajectory of the form $\{(x_k, f_{n_k}(x_k)) : k \in \omega\}$ if and only if we can approach most points along a vertical trajectory.

We give a characterization of Baire spaces by the above theorem in some classes of spaces.

We say that a topological space X has the property CP (QP) if for every nonempty nowhere dense closed set $F \subset X$ there is a continuous (quasicontinuous) function $g : X \setminus F \to [0,1]$ such that the oscillation ω_g of g is equal to 1 for every $x \in F$.

(If A is a subset of X and $f : A \to Y$ is a function, then the function $\omega_f : \mathrm{Cl}\, A \to [0, \infty]$ defined by

$$\omega_f(x) = \inf\{\sup\{d(f(y), f(z)) : y, z \in A \cap U\} : U \text{ is a neighbourhood of } x\}$$

is called the oscillation of f at x.)

By [6], every pseudometrizable space and every perfectly normal locally connected space has the property CP. Evidently, each space with the property CP has the property QP. In the next paragraph we will show that there is a space with the property QP but not CP and that there is a Hausdorff compact space without the property QP.

Theorem 3. *Let X be a topological space with the property QP. The following are equivalent:*

(1) X is a Baire space;

(2) For every metric space Y, every sequence $\{f_n : n \in \omega\}$ in $Q(X,Y)$, every $f \in Q(X,Y)$, $Grf \subset LsGrf_n$ iff there is a dense set E in X such that for each $x \in E$, $f(x)$ is a cluster point of $\{f_n(x) : n \in \omega\}$;

(3) For every sequence $\{f_n : n \in \omega\}$ in $Q(X, \mathbb{R})$, every $f \in Q(X, \mathbb{R})$, $Grf \subset LsGrf_n$ iff there is a dense set E in X such that for each $x \in E$, $f(x)$ is a cluster point of $\{f_n(x) : n \in \omega\}$.

Proof. Only (3) $\Rightarrow$ (1) needs some explanation. Suppose X is not a Baire space. Let G be a nonempty open set in X which is of the first Baire category. Let $\{K_n : n \in \omega\}$ be a sequence of nowhere dense subsets of G such that $G = \cup\{K_n : n \in \omega\}$. For every $n \in \omega$ put $L_n = \cup\{\mathrm{Cl}\, K_i : i \leq n\} \cup (\mathrm{Cl}\, G \setminus G)$. Then $\mathrm{Cl}\, G = \cup\{L_n : n \in \omega\}$.

Let $n \in \omega$. Since X has the property QP there is a quasicontinuous function $g_n : X \setminus L_n \to [0,1]$ such that $\omega_{g_n}(x) = 1$ for every $x \in L_n$. Of course, the function $g_n^* : X \to [0,1]$ defined by $g_n^*(x) = 0$ for every $x \in L_n$ and $g_n^*(x) = g_n(x)$ otherwise, is quasicontinuous. Let $f_n : X \to [0,1]$ be a function defined as follows: $f_n(x) = g_n^*(x)$ for every $x \in \mathrm{Cl}\, G$ and $f_n(x) = 0$ otherwise. It is easy to verify that the function f_n is quasicontinuous.

Let $f : X \to [0,1]$ be a function defined as follows: $f(x) = 1$ for every $x \in \mathrm{Cl}\, G$ and $f(x) = 0$ otherwise. Of course f is quasicontinuous and $Grf \subset LsGrf_n$. (Let $x \in G$. Let $U \times V$ be a neighbourhood of $(x, f(x))$ and $n \in \omega$. There is $k \geq n$ such that $x \in L_k$. Since $\omega_{f_k}(x) = 1$, there must exist $y \in U$ with $f_k(y) \in V$. Thus $(y, f_k(y)) \in U \times V$. Now let $x \in \mathrm{Cl}\, G \setminus G$. Since $\mathrm{Cl}\, G \setminus G \subset L_n$ for every $n \in \omega$, $\omega_{f_n}(x) = 1$ for every $n \in \omega$.) It is easy to verify that for every $x \in G$ we have $|f_n(x) - f(x)| = 1$ eventually. □

Notice that the above characterization does not hold in general topological spaces.

Example 2. Let X be a countable set with the cofinite topology. Then X is of the first Baire category. Every quasicontinuous function on X must be constant. Thus the conditions (2) and (3) are always satisfied on X. The same holds for $X = \mathbb{R}$ equipped with the topology $\tau = \{X, \emptyset\} \cup \{(a, \infty) : a \in \mathbb{R}\}$.

If a sequence $\{f_n : n \in \omega\}$ is equi-quasicontinuous (see Definition 2) then the assumptions on a space X and a function f in Theorem 1 can be omitted.

Theorem 4. *Let X be a topological space, let $\{f_n : n \in \omega\}$ be sequence of real-valued functions equi-quasicontinuous at points of some dense set D and let $f : X \to \mathbb{R}$ be a function such that $Grf \subset LsGrf_n$. Then there is a dense set $E \subset X$ such that for each $x \in E$, $f(x)$ is a cluster point of $\{f_n(x) : n \in \omega\}$.*

Proof. Suppose by way of contradiction that there is an open set $U \subset X$ such that for each $x \in U$, $f(x)$ is not a cluster point of $\{f_n(x) : n \in \omega\}$. Let $x_0 \in U \cap D$. The equi-quasicontinuity of $\{f_n : n \in \omega\}$ at x_0 implies that there is an open set $G \subset U$ and $n_0 \in \omega$ such that $| f_n(y) - f_n(x_0) | < 1$ for every $y \in G$ and for every $n > n_0$. Let y_0 be an arbitrary point in G. The sequence $\{f_n(x_0) : n \in \omega\}$ has not a cluster point, so there is $n_1 > n_0$ such that $|f_n(x_0)| > |f(y_0)| + 2$ for every $n > n_1$. For each $y \in G$ we have $|f_n(y) - f(y_0)| \geq |f_n(y)| - |f(y_0)| > |f_n(x_0)| - 1 - |f_n(x_0)| + 2 > 1$. Therefore, $G \times (f(y_0) - 1, f(y_0) + 1)$ is a neighbourhood of $(y_0, f(y_0))$ disjoint from Grf_n for each $n > n_1$, a contradiction to $Grf \subset LsGrf_n$. □

Theorem 5. *Let X be a topological space. Let $\{f_n : n \in \omega\}$ be a sequence of real-valued functions equi-quasicontinuous at points of some dense set in X and let $f : X \to \mathbb{R}$ be a quasicontinuous function. The following are equivalent:*

(1) $Grf \subset LsGrf_n$*;*

(2) There is a dense set $E \subset X$ *such that for each* $x \in E$*,* $f(x)$ *is a cluster point of* $\{f_n(x) : n \in \omega\}$*.*

3 Topological spaces with the property QP

Let X be a developable space with a development $\mathcal{G} = \{\mathcal{G}_n : n \in \omega\}$. For a mapping $\varphi : \omega \to \omega$ denote $U(\mathcal{G}, \varphi) = \{x \in X : st(st(x, \mathcal{G}_{\varphi(n)}), \mathcal{G}_{\varphi(n)}) \subset st(x, \mathcal{G}_n)\}$. We say that a developable space X has the property U if there are a development $\mathcal{G} = \{\mathcal{G}_n : n \in \omega\}$ and a mapping $\varphi : \omega \to \omega$ such that the set $U(\mathcal{G}, \varphi)$ is dense in X and $\mathcal{G}_{n+1}$ refines $\mathcal{G}_n$ for every $n \in \omega$.

Theorem 6. *Let* X *be a Baire Moore space with the property U. Let* F *be a nonempty nowhere dense closed subset of* X*. Then there is a closed nowhere dense set* H *containing* F *and there is a family* $\mathcal{K} = \bigcup_{n\in\omega} \mathcal{K}_n$ *of nonempty open sets such that*

(a) $\forall K \in \mathcal{K} : \mathrm{Cl}\, K \cap H = \emptyset$*;*

(b) $\forall K, L \in \mathcal{K} : \mathrm{Cl}\, K \cap \mathrm{Cl}\, L = \emptyset$ *for* $K \neq L$*;*

(c) $\forall x \in X \setminus H\ \exists V$ *a neighbourhood of* x *such that the set* $\{K \in \mathcal{K} : V \cap K \neq \emptyset\}$ *has at most one element;*

(d) $\forall x \in H\ \forall U$ *a neighbourhood of* x $\exists k \in \omega\ \forall n \geq k\ \exists K \in \mathcal{K}_n$ *such that* $U \cap K \neq \emptyset$*.*

Proof. Let $\mathcal{G} = \{\mathcal{G}_n : n \in \omega\}$ be a development for X such that $\mathcal{G}_{n+1}$ refines $\mathcal{G}_n$ and let $\varphi : \omega \to \omega$ be a mapping such that the set $U(\mathcal{G}, \varphi)$ is dense in X. Put

$$H = \bigcap_{n\in\omega} \mathrm{Cl}\, st(F, \mathcal{G}_n).$$

Evidently, H is closed and $F \subset H$. We have

$$H = \bigcap_{n\in\omega} \mathrm{Cl}\, st(F, \mathcal{G}_n) = \bigcap_{n\in\omega} ((\mathrm{Cl}\, st(F, \mathcal{G}_n) \setminus st(F, \mathcal{G}_n)) \cup st(F, \mathcal{G}_n)) \subset$$
$$\bigcap_{n\in\omega} st(F, \mathcal{G}_n) \cup \bigcup_{n\in\omega} (\mathrm{Cl}\, st(F, \mathcal{G}_n) \setminus st(F, \mathcal{G}_n)).$$

If $x \in \bigcap_{n\in\omega} st(F, \mathcal{G}_n)$ then there are $y_n \in F$ such that $x \in st(y_n, \mathcal{G}_n)$. Then $y_n \in st(x, \mathcal{G}_n)$ and the sequence $\{y_n : n \in \omega\}$ converges to x. Since F is closed, we have $x \in F$. Therefore $\bigcap_{n\in\omega} st(F, \mathcal{G}_n)$ is a nowhere dense set and the sets $\mathrm{Cl}\, st(F, \mathcal{G}_n) \setminus st(F, \mathcal{G}_n)$ are nowhere dense, too. Now H is a closed set of the first category and since X is a Baire space, H is nowhere dense.

Assume that for each $i < n$ we have constructed families of nonempty open sets $\mathcal{K}_i$ such that for each $i < n$

(1) $\mathrm{Cl}\, K \cap st(\mathrm{Cl}\, L, \mathcal{G}_{\varphi(\varphi(i))}) = \emptyset$ for each $K, L \in \mathcal{K}_i$ with $K \neq L$;

(2) the sets $\bigcup_{S\in\mathcal{S}} \operatorname{Cl} S$ are closed whenever $\mathcal{S} \subset \mathcal{K}_i$;

(3) $\operatorname{Cl} K \subset M_i$ for each $K \in \mathcal{K}_i$;

(4) S_i is a maximal element in $\mathcal{T}_i$ and for each $K \in \mathcal{K}_i$ there is $s \in S_i$ such that $\operatorname{Cl} K \subset st(s, \mathcal{G}_{\varphi(\varphi(i))})$,

where $K_0 = \emptyset$, $K_i = \bigcup_{K\in\mathcal{K}_i} \operatorname{Cl} K$, $M_i = st(F, \mathcal{G}_i) \setminus (H \cup \bigcup_{j<i} K_j)$ and

$$\mathcal{T}_i = \{T \subset M_i \cap U(\mathcal{G}, \varphi) : a \notin st(b, \mathcal{G}_i) \text{ for each } a, b \in T \text{ with } a \neq b\}.$$

Put

$$M_n = st(F, \mathcal{G}_n) \setminus (H \cup \bigcup_{i<n} K_i).$$

If $st(F, \mathcal{G}_n) \subset H \cup \bigcup_{i<n} K_i$ then $st(F, \mathcal{G}_n) \setminus \bigcup_{i<n} K_i \subset H$ and hence by (2), $st(F, \mathcal{G}_n) \setminus \bigcup_{i<n} K_i$ is an open nowhere dense set and hence it is empty. On the other hand, by (3), $F \cap K_i = \emptyset$ for $i < n$ and hence $\emptyset \neq F \subset st(F, \mathcal{G}_n) \setminus \bigcup_{i<n} K_i$, a contradiction. Hence M_n is an open nonempty set.

Put

$$\mathcal{T}_n = \{T \subset M_n \cap U(\mathcal{G}, \varphi) : a \notin st(b, \mathcal{G}_n) \text{ for each } a, b \in T \text{ with } a \neq b\}.$$

Since the union of each chain in $\mathcal{T}_n$ belongs to $\mathcal{T}_n$, according to Zorn lemma there is a maximal element S_n in $\mathcal{T}_n$.

For each $s \in S_n$ the set M_n is a neighbourhood of s and hence there is an open set L_s such that $s \in L_s \subset \operatorname{Cl} L_s \subset M_n \cap st(s, \mathcal{G}_{\varphi(\varphi(n))})$.

Put

$$\mathcal{K}_n = \{L_s : s \in S_n\} \text{ and } K_n = \bigcup_{K\in\mathcal{K}_n} \operatorname{Cl} K.$$

We will show that $\mathcal{K}_n$ and S_n satisfy (1)-(4).

(1): Assume that there is $y \in \operatorname{Cl} K \cap st(\operatorname{Cl} L, \mathcal{G}_{\varphi(\varphi(n))})$ for some $K, L \in \mathcal{K}_n$, $K \neq L$. Then there are $a, b \in S_n$ such that $\operatorname{Cl} K \subset st(a, \mathcal{G}_{\varphi(\varphi(n))})$ and $\operatorname{Cl} L \subset st(b, \mathcal{G}_{\varphi(\varphi(n))})$. Since $a, b \in U(\mathcal{G}, \varphi)$ we have $y \in \operatorname{Cl} K \subset st(st(a, \mathcal{G}_{\varphi(\varphi(n))}), \mathcal{G}_{\varphi(\varphi(n))}) \subset st(a, \mathcal{G}_{\varphi(n)})$ and hence $a \in st(y\ \mathcal{G}_{\varphi(n)})$.

Further, $y \in st(\operatorname{Cl} L, \mathcal{G}_{\varphi(\varphi(n))}) \subset st(st(b, \mathcal{G}_{\varphi(\varphi(n))}), \mathcal{G}_{\varphi(\varphi(n))}) \subset st(b, \mathcal{G}_{\varphi(n)})$. Therefore $a \in st(y, \mathcal{G}_{\varphi(n)}) \subset st(st(b, \mathcal{G}_{\varphi(n)}), \mathcal{G}_{\varphi(n)}) \subset st(b, \mathcal{G}_n)$, a contradiction to the construction of S_n.

(2): Let $\mathcal{S} \subset \mathcal{K}_n$, $z_k \in \bigcup_{S\in\mathcal{S}} \operatorname{Cl} S$ and let $\{z_k : k \in \omega\}$ converge to z. Then there are $S_k \in \mathcal{S}$ with $z_k \in \operatorname{Cl} S_k$. Further there is $m \in \omega$ such that $z_k \in st(z, \mathcal{G}_{\varphi(\varphi(n))})$ for each $k \geq m$. This yields $z \in st(z_m, \mathcal{G}_{\varphi(\varphi(n))})$ and hence there is $p > m$ such that $z_k \in st(z_m, \mathcal{G}_{\varphi(\varphi(n))})$ for $k \geq p$. Therefore $z_k \in \operatorname{Cl} S_k \cap st(z_m, \mathcal{G}_{\varphi(\varphi(n))}) \subset \operatorname{Cl} S_k \cap st(\operatorname{Cl} S_m, \mathcal{G}_{\varphi(\varphi(n))})$. However, by (1) we have $\operatorname{Cl} S_k \cap st(\operatorname{Cl} S_m, \mathcal{G}_{\varphi(\varphi(n))}) = \emptyset$ for $S_m \neq S_k$. Therefore $S_k = S_m$ for $k \geq p$. This yields $z_k \in \operatorname{Cl} S_m$ for $k \geq p$ and hence $z \in \operatorname{Cl} S_m \subset \bigcup_{S\in\mathcal{S}} \operatorname{Cl} S$.

(3) and (4) follow from the construction.

Now, we will show that the family $\mathcal{K} = \bigcup_{n\in\omega} \mathcal{K}_n$ of nonempty open sets satisfies (a)-(d).

(a) follows from (3) and (b) follows from (1) and (3).

(c): Let $x \in X \setminus H$. Then there is $n \in \omega$ such that $x \notin \mathrm{Cl}\, st(F, \mathcal{G}_n)$. Let U be an open neighbourhood of x such that $U \subset X \setminus \mathrm{Cl}\, st(F, \mathcal{G}_n)$. Since $\mathcal{G}_{n+1}$ refines $\mathcal{G}_n$ we have $U \cap K_i = \emptyset$ for each $i \geq n$.

If $x \notin \bigcup_{i \in \omega} K_i$ then $V = U \setminus \bigcup_{i<n} K_i$ is a neighbourhood of x disjoint from each $K \in \mathcal{K}$. Otherwise, $x \in \mathrm{Cl}\, K$ for some $K \in \mathcal{K}_j$, $j < n$. By (b), $x \notin \mathrm{Cl}\, S$ for each $S \in \mathcal{K}$ with $S \neq K$. By (2), the sets K_i for $i < n$ and $\bigcup_{S \in \mathcal{K}_j \setminus \{K\}} \mathrm{Cl}\, S$ are closed and hence $V = U \setminus (\bigcup_{i<n, i \neq j} K_i \cup \bigcup_{S \in \mathcal{K}_j \setminus \{K\}} \mathrm{Cl}\, S)$ is a neighbourhood of x such that $\{L \in \mathcal{K} : V \cap L \neq \emptyset\} = \{K\}$.

(d) Let $x \in H$ and let U be a neighbourhood of x. Then there is $n \in \omega$ such that $x \in st(x, \mathcal{G}_n) \subset U$. We will show that $st(x, \mathcal{G}_n) \cap K_m \neq \emptyset$ for each $m \geq \varphi(n)$.

Assume that $st(x, \mathcal{G}_n) \cap K_m = \emptyset$ for some $m \geq \varphi(n)$. Since $H \cap K_i = \emptyset$ for $i < m$, by (2) the set $st(x, \mathcal{G}_m) \setminus \bigcup_{i<m} K_i$ is an open neighbourhood of x. Since $x \in \mathrm{Cl}\, st(F, \mathcal{G}_m)$, the set $(st(x, \mathcal{G}_m) \setminus \bigcup_{i<m} K_i) \cap st(F, \mathcal{G}_m)$ is open and nonempty. Now, the set $W_m = (st(x, \mathcal{G}_m) \setminus \bigcup_{i<m} K_i) \cap st(F, \mathcal{G}_m) \setminus H$ is open and nonempty, too.

The set $U(\mathcal{G}, \varphi)$ is dense a hence there is $p \in W_m \cap U(\mathcal{G}, \varphi)$. We will show that $p \notin st(U(\mathcal{G}, \varphi) \setminus st(x, \mathcal{G}_n), \mathcal{G}_m)$. If namely $p \in st(U(\mathcal{G}, \varphi) \setminus st(x, \mathcal{G}_n), \mathcal{G}_m)$ then there is $z \in U(\mathcal{G}, \varphi) \setminus st(x, \mathcal{G}_n)$ such that $p \in st(z, \mathcal{G}_m)$. Then $p \in st(z, \mathcal{G}_{\varphi(n)})$ and hence $st(p, \mathcal{G}_{\varphi(n)}) \subset st(st(z, \mathcal{G}_{\varphi(n)}), \mathcal{G}_{\varphi(n)}) \subset st(z, \mathcal{G}_n)$. We have $p \in W_m \subset st(x, \mathcal{G}_{\varphi(n)})$ and hence $x \in st(p, \mathcal{G}_{\varphi(n)}) \subset st(z, \mathcal{G}_n)$. On the other hand, $z \notin st(x, \mathcal{G}_n)$ and hence $x \notin st(z, \mathcal{G}_n)$, a contradiction.

By assumption, we have $st(x, \mathcal{G}_n) \cap K_m = \emptyset$ and hence $st(x, \mathcal{G}_n) \cap S_m = \emptyset$, i.e. $S_m \subset U(\mathcal{G}, \varphi) \setminus st(x, \mathcal{G}_n)$. Since $p \notin st(U(\mathcal{G}, \varphi) \setminus st(x, \mathcal{G}_n), \mathcal{G}_m)$ we have $p \notin st(s, \mathcal{G}_m)$ for each $s \in S_m$. However, $p \in W_m \cap U(\mathcal{G}, \varphi)$ and $p \notin S_m$, a contradiction to the maximality of S_m.

Therefore for each $k \geq \varphi(n)$ there is $K \in \mathcal{K}_k$ such that $U \cap \mathrm{Cl}\, K \neq \emptyset$. Since U and K are open, we have $U \cap K \neq \emptyset$. □

Theorem 7. *Let X be a topological space such that for each nonempty nowhere dense closed set F there is a nowhere dense set H containing F and a family $\mathcal{K} = \bigcup_{n \in \omega} \mathcal{K}_n$ of open nonempty sets satisfying (a)-(d) of Theorem 6. Then X has the property QP.*

Proof. Let F be a nowhere dense closed set. Let H and $\mathcal{K} = \bigcup_{n \in \omega} \mathcal{K}_n$ satisfy (a)-(d). Put $A = \bigcup_{i \in \omega} K_{2i}$, where $K_i = \bigcup_{K \in \mathcal{K}_i} \mathrm{Cl}\, K$. Define a function $f : X \setminus F \to [0,1]$ as $f(x) = 0$ for $x \in A$ and $f(x) = 1$ otherwise.

Let $x \in F$ and let U be a neighbourhood of x. Then there are an even i and an odd j such that $U \cap K_i \neq \emptyset$ and $U \cap K_j \neq \emptyset$.Therefore there are points $y, z \in U \setminus F$ such that $f(y) = 0$ and $f(z) = 1$. This yields that $\omega_f(x) = 1$.

Now, we will show that f is quasicontinuous. Let $x \in X \setminus F$ and let U be an open neighbourhood of x. If $x \in X \setminus H$ then there is an open neighbourhood V of x such that $\{K \in \mathcal{K} : V \cap K \neq \emptyset\}$ has at most one element. If this set is empty, then $f(x) = f(y) = 1$ for each $y \in V$. If $\{K \in \mathcal{K} : V \cap K \neq \emptyset\} = \{L\}$ and $x \in \mathrm{Cl}\, L$, then $G = U \cap V \cap L$ is an open nonempty subset of U such that $f(y) = f(x)$ for each $y \in G$. If $x \notin \mathrm{Cl}\, L$, we can argue as above.

Finally, let $x \in H\backslash F$. Then by (d) there is an odd j such that $U \cap K \neq \emptyset$ for some $K \in K_j$ Now, $U \cap K$ is an open nonempty subset of U such that $f(y) = f(x) = 1$ for each $y \in U \cap K$. □

Proposition 2. *The Niemytzki plane is the space with the property QP but not CP.*

Proof. Let X be the Niemytzki plane. It means, $X = P \cup L$, where $P = \{(x, y) \in \mathbb{R}^2 : y > 0\}$ and $L = \{(x, y) \in \mathbb{R}^2 : y = 0\}$. The basis of open neighbourhoods of points from P are disc in the plane which are small enough to lie within P. The basis of open neighbourhoods of points from L are sets $\{(p, 0)\} \cup A$, where A is an open disc in P which is tangent to the x-axis at $(p, 0)$. It is easy to see that X is a Baire Moore space with the property U. By Theorem 6 and 7 X has the property QP.

Assume that X has the property CP. Let $\emptyset \neq F \subset L$. Then F is a nonempty nowhere dense closed set in X. The space P as a subspace of X is second countable, therefore there are only $\mathfrak{c}$ continuous functions on P to $[0, 1]$. However, we have $2^{\mathfrak{c}}$ subsets of L, therefore there are nonempty sets $F_1, F_2 \subset L$, $F_1 \neq F_2$, and continuous functions $f_1 : X \setminus F_1 \to [0, 1]$, $f_2 : X \setminus F_2 \to [0, 1]$ and $g : P \to [0, 1]$ such that $\omega_{f_1}(z) = 1$ for each $z \in F_1$, $\omega_{f_2}(z) = 1$ for each $z \in F_2$ and $f_1(z) = f_2(z) = g(z)$ for each $z \in P$. Since $F_1 \neq F_2$, there is $w \in F_1 \setminus F_2$ (or $w \in F_2 \setminus F_1$). We have $w \in X \setminus F_2$ and hence f_2 is continuous at w, so $0 \leq \omega_g(w) \leq \omega_{f_2}(w) = 0$. On the other hand, $w \in F_1$ and it is easy to see that $\omega_g(w) = \omega_{f_1}(w) = 1$, a contradiction. Therefore the Niemytzki plane has not the property CP. □

Proposition 3. *There is a Hausdorff compact space which has not the property QP.*

Proof. Let $X = \beta\omega$ (the Čech-Stone compactification of ω) and let $z \in \beta\omega \setminus \omega$. Then $\{z\}$ is a nonempty closed nowhere dense set in X. Assume that there is a quasicontinuous function $g : X \setminus \{z\} \to [0, 1]$ such that $\omega_g(z) = 1$. Put $A = g^{-1}([0, 1/3))$ and $B = g^{-1}((2/3, 1])$. Then A and B are disjoint semi-open sets in $X \setminus \{z\}$ and (since $X \setminus \{z\}$ is open) also in X. (A set $A \subset X$ is semi-open, if $A \subset \mathrm{Cl}(\mathrm{Int}\, A)$ [21].)

There is a continuous function $f : X \to [0, 1]$ such that $f(x) = 0$ for $x \in A \cap \omega$ and $f(x) = 1$ for $x \in B \cap \omega$. Let U be an open neighbourhood of z in X. Since $\omega_g(z) = 1$, there are points $z_1, z_2 \in U \backslash \{z\}$ such that $g(z_2) - g(z_1) > 5/6$. This yields that $z_2 \in B$ and $z_1 \in A$. Therefore $z \in \mathrm{Cl}\, A \cap \mathrm{Cl}\, B$ and, since A and B are semi-open, we have $\mathrm{Cl}\, A = \mathrm{Cl}\,\mathrm{Int}\, A$ and $\mathrm{Cl}\, B = \mathrm{Cl}\,\mathrm{Int}\, B$. Hence the set $U \cap \mathrm{Int}\, A \neq \emptyset$, and, since ω is dense in X, there is $y_1 \in U \cap \mathrm{Int}\, A \cap \omega$, and $f(y_1) = 0$. Similarly, there is $y_2 \in U \cap \mathrm{Int}\, B \cap \omega$, and $f(y_2) = 1$. However then $\omega_f(z) = 1$, a contradiction to the continuity of f. □

4 Some comments to the pointwise convergence of quasicontinuous mappings

Of course the pointwise limit of a sequence of even continuous functions need not be quasicontinuous.

However it is known that the pointwise limit of an equicontinuous sequence of functions is continuous. Of course equicontinuity is too strong; it is not necessary to guarantee continuity of the pointwise limit of a sequence of continuous functions.

There is a rich literature concerning necessary and sufficient conditions for continuity of the pointwise limit of a net of continuous functions (see [10] for a survey).

Of course mathematicians studied also the pointwise convergence of quasicontinuous mappings (see [4], [5], [12], [13], [16], [28]).

In the paper [16], the notion of equi-quasicontinuity was defined, a sufficient condition under which the pointwise limit of a sequence of quasicontinuous functions is quasicontinuous. The following definition of equi-quasicontinuity is given (for simplicity) for real-valued functions, however we can extend it for functions with values in metric spaces and also for nets.

Definition 2. Let $\{f_n : n \in \omega\}$ be a sequence of real-valued functions defined on a topological space X. We say that the sequence $\{f_n : n \in \omega\}$ is equi-quasicontinuous at $x \in X$ if for every $\epsilon > 0$ and every open neighbourhood U of x there is $n_0 \in \omega$ and a nonempty open set $W \subset U$ such that $\mid f_n(z) - f_n(x) \mid < \epsilon$ for every $z \in W$ and for every $n \geq n_0$. We say that $\{f_n : n \in \omega\}$ is equi-quasicontinuous if it is equi-quasicontinuous at every $x \in X$.

Of course every equicontinuous sequence is also equi-quasicontinuous and there are easy examples of equi-quasicontinuous sequences which are not equicontinuous. Of course members of an equi-quasicontinuous sequence need not be quasicontinuous functions.

In [16] the following proposition and theorem were proved.

Proposition 4. *Let $\{f_n : n \in \omega\}$ be a sequence of real-valued functions defined on a topological space X pointwise convergent to a real-valued function f defined on X. If $\{f_n : n \in \omega\}$ is equi-quasicontinuous at $x \in X$, then f is quasicontinuous at x.*

Theorem 8. *Let X be a Baire space. Let $\{f_n : n \in \omega\}$ be a sequence of real-valued quasicontinuous functions defined on X pointwise convergent to a function $f : X \to \mathbb{R}$. Then the following are equivalent:*
(1) f is quasicontinuous;
(2) $\{f_n : n \in \omega\}$ is equi-quasicontinuous.

In [16], Theorem 8 was proved using the Choquet game for Baire spaces. Here we offer an easy direct proof:

Proof. (1) $\Rightarrow$ (2) Let $x_0 \in X$, U be a neighbourhood of x_0 and $\epsilon > 0$. The quasicontinuity of f at x_0 implies the existence of a nonempty open subset $U_1 \subset U$

such that $| f(x) - f(x_0) | < \epsilon/4$ for every $x \in U_1$. Put

$$A_n = \{x \in U_1 : \text{ for every } m \geq n : | f_m(x) - f(x) | < \epsilon/4\}.$$

The pointwise convergence of $\{f_n\}$ to f implies that $U_1 = \cup_n A_n$. Since X is a Baire space, there is $n_0 \in \omega$ such that A_{n_0} is not nowhere dense in U_1. There is a nonempty open set $V \subset U_1$ such that A_{n_0} is dense in V. There is $n_1 > n_0$ such that $| f_n(x_0) - f(x_0) | < \epsilon/4$ for every $n \geq n_1$.

We claim that for every $n \geq n_1$ and for every $x \in V$ we have $| f_n(x) - f_n(x_0) | < \epsilon$. Let $n \geq n_1$ and $x \in V$. The quasicontinuity of f_n at x implies the existence of a nonempty open subset $G \subset V$ such that $| f_n(t) - f_n(x) | < \epsilon/4$ for every $t \in G$. A_{n_0} is dense in V, thus there is $z \in A_{n_0} \cap G$. Thus

$$| f_n(x) - f_n(x_0) | \leq | f_n(x) - f_n(z) | + | f_n(z) - f(z) | + | f(z) - f(x_0) | + | f(x_0) - f_n(x_0) | < \epsilon.$$

□

Moreover in [16] a characterization of Baire spaces by the above mentioned fact in the class of metrizable spaces and in the class of quasi-regular T_1 topological spaces with locally countable π-base was given. However such a characterization does not hold in general topological spaces as Example 2 shows. Example 2 answers negatively the question in [16] for T_1 topological spaces.

5 Miscellanea

Definition 3. ([1]) Let X be a topological space and (Y, d) be a metric space. Let $\{f_n : n \in \omega\}$ be a sequence of functions from X to Y and let $f : X \to Y$. Then $\{f_n : n \in \omega\}$ is called Alexandroff convergent to f on X, provided it pointwise converges to f, and for every $\epsilon > 0$ and $n_0 \in \omega$ there exist a countable open covering $\{\Gamma_0, \Gamma_1, ...\}$ of X and a sequence $\{n_k : k \in \omega\}$ in ω such that $n_k > n_0$ for every k and for each $x \in \Gamma_k$ we have $d(f_{n_k}(x), f(x)) < \epsilon$.

The following proposition is probably known.

Proposition 5. *Let X be a topological space and (Y, d) be a metric space. Let $\{f_n : n \in \omega\}$ be a sequence of quasicontinuous functions from X to Y Alexandroff convergent to a function $f : X \to Y$. Then f is quasicontinuous too.*

Let (Z, d) be a metric space. If $E \subset Z$ and $\epsilon > 0$, let $S(E, \epsilon)$ denote the union of all open ϵ-balls whose centers run over E.

If E and F are nonempty subsets of Z and for some $\epsilon > 0$ both $E \subset S(F, \epsilon)$ and $F \subset S(E, \epsilon)$, then the Hausdorff pseudometric h_d between them is given by

$$h_d(E, F) = \inf\{\epsilon > 0 : F \subset S(E, \epsilon), E \subset S(F, \epsilon)\}.$$

Otherwise we put $h_d(E, F) = \infty$.

Now let $(X, d_X), (Y, d_Y)$ be metric spaces. On $X \times Y$ consider the box metric ρ of d_X, d_Y, defined as follows:

$$\rho[(x_1, y_1), (x_2, y_2)] = \max\{d_X(x_1, y_1), d_Y(y_1, y_2)\}.$$

As above we can identify every function $f : X \to Y$ with its graph and consider h_ρ-convergence of functions.

The following example shows that the convergence in the Hausdorff pseudometric preserves neither quasicontinuity nor cliquishness.

Example 3. Let $X = Y = [0, 1]$ with the usual Euclidean metric. Let D be a dense set in X. Let $f : X \to Y$ be defined as follows: $f(x) = 1$ if $x \in D$ and $f(x) = 0$ otherwise. Of course f is not cliquish, however it can be appoximated in the Hausdorff pseudometric (generated by the box metric on $X \times Y$) by the sequence of quasicontinuous functions $f_n : X \to Y$ defined as follows. Let $n \in \omega, n \geq 1$. Put $f_n(x) = 1$ for $x \in [\frac{i}{2^n}, \frac{i+1}{2^n}]$ for i odd, $1 \leq i \leq 2^n - 1$ and $f_n(x) = 0$ otherwise.

Definition 4. Let X be a topological space and (Y, d) be a metric space. A sequence $\{f_n : n \in \omega\}$ of functions from X to Y is equi-cliquish at $x \in X$ if for every $\epsilon > 0$ and every neighbourhood $U \subset X$ of x there is a nonempty open set $G \subset U$ and $n_0 \in \omega$ such that $d(f_n(y), f_n(z)) < \epsilon$ for every $n \geq n_0$ and for every $y, z \in G$.

It is very easy to verify that the following proposition holds.

Proposition 6. *Let X be a topological space and (Y, d) be a metric space. Let $\{f_n : n \in \omega\}$ be a sequence of functions from X to Y pointwise convergent to a function $f : X \to Y$. If $\{f_n : n \in \omega\}$ is equi-cliquish at $x \in X$, then $f : X \to Y$ is cliquish at x.*

However an analogy with Theorem 2.5 in [16] does not hold as the following example shows.

Example 4. Let $X = Y = \mathbb{R}$ with the Euclidean metric. For every $n \in \omega$, let $f_n : X \to Y$ be defined as follows: $f_n(x) = \frac{1}{q-n}$ if $x = \frac{p}{q}$ in the basic form and $q - n > 0$ and $f_n(x) = 0$ otherwise. Let $f : X \to Y$ be the function identically equal to 0. Then of course X is a Baire space, f_n is cliquish for every $n \in \omega$ and f is cliquish too. However the sequence $\{f_n : n \in \omega\}$ is not equi-cliquish. Let G be a nonempty open set in X and $m \in \omega, m \geq 1$. There is $q \in \omega, q > m$ and there is an integer p such that $\frac{p}{q} \in G$ and $\frac{p}{q}$ is in the basic form. Then for $n = q - 1 \geq m$ and for $y = \frac{p}{q}$ and for an irrational $z \in G$ we have $f_n(y) = \frac{1}{q-n} = 1$ and $f_n(z) = 0$. Thus the sequence $\{f_n : n \in \omega\}$ is not equi-cliquish.

Proposition 7. *Let X be a topological space and (Y, d) be a metric space. Let $\{f_n : n \in \omega\}$ be an equi-cliquish sequence of functions from X to Y and $f : X \to Y$ be a function such that $Grf \subset LiGrf_n$. Then f is cliquish too.*

Proof. Let $x \in X$. We prove that f is cliquish at x. Let $\epsilon > 0$ and let U be an open neighbourhood of x. There is $n_0 \in \omega$ and a nonempty open set $V \subset U$ such that $d(f_n(u), f_n(v)) < \epsilon/3$ for every $n \geq n_0$ and for every $u, v \in V$. We claim that for every $s, t \in V$, $d(f(s), f(t)) < \epsilon$. Let $s, t \in V$. Since $Grf \subset LiGrf_n$, there is $n_1 \in \omega$ such that $n_1 \geq n_0$ and for every $n \geq n_1$ we have $V \times S(f(s), \epsilon/3) \cap Grf_n \neq \emptyset$ and $V \times S(f(t), \epsilon/3) \cap Grf_n \neq \emptyset$. Let $n \geq n_1$ and $s_1, t_1 \in V$ be such that $d(f(s), f_n(s_1)) < \epsilon/3$ and $d(f(t), f_n(t_1)) < \epsilon/3$. Thus we have

$$d(f(s), f(t)) \leq d(f(s), f_n(s_1)) + d(f_n(s_1), f_n(t_1)) + d(f_n(t_1), f(t)) < \epsilon.$$

□

The following example shows that an equi-quasicontinuous sequence need not converge in the Hausdorff pseudometric to a quasicontinuous function.

Example 5. Let $X = Y = [0, 1]$ with the usual Euclidean metric. For every $n \in \omega$, $n \geq 2$ let $f_n : X \to Y$ be a function defined as follows: $f_n(x) = 1$ for every $x \in [\frac{1}{n} - \frac{1}{2^n}, \frac{1}{n} + \frac{1}{2^n}]$ and $f_n(x) = 0$ otherwise. It is easy to verify that the sequence $\{f_n : n \in \omega\}$ is equi-quasicontinuous and it converges in the Hausdorff pseudometric (generated by the box metric) to a function $f : X \to Y$ defined as follows: $f(0) = 1$ and $f(x) = 0$ otherwise, which is not quasicontinuous at 0.

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