Bakerian Lecture

On relaxation methods:* A mathematics for engineering science

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1. By engineering, in this lecture, I intend the art whereby science is applied to useful ends; by engineering science, that corpus of knowledge—mathematics, physics, chemistry and the like—which is pursued with a view to such practical application. I am not concerned to defend these definitions, only to make my meaning clear. Such as they are, their distinction between engineering and engineering science is analogous with the distinction between clinical medicine—the art—and medicine—the science of our medical schools and research centres.

As engineering advances the scope of engineering science advances too, and roughly (I suppose) it may be said that the engineering science of any given time is the physics of fifty years before. Thus its field of study now is very much the same as that of nineteenth-century physicists like Kelvin, Stokes or Rayleigh: in 'field physics' (of which my lecture treats this afternoon) we are concerned with problems in hydrodynamics, elasticity and the like such as make up the bulk of their collected papers. There is, however, this difference in our outlook (and it arises because our science is directed to practical ends) that *we would rather have power to calculate approximately for any data than power to calculate exactly for data of a few restricted kinds.*

2. Take, as an example, Saint-Venant's well-known theory of torsion for a bar of non-circular cross-section. It is formally complete, and its equations have been satisfied exactly for various mathematical shapes—equilateral triangles, ellipses and the like. From a purely physical standpoint this is enough, and nineteenth-century physicists passed, in elasticity, on to other problems; but engineering science, in so far as it is concerned with the problem, is concerned with the torsional properties of shapes (e.g. of '⌶-girders') which are not expressible mathematically, therefore

* The name 'Relaxation Methods' is an abbreviation of the more complete description given in the titles of my 1935 papers (Refs. 1 and 2): 'Stress-Calculation in Frameworks by the "Method of Systematic Relaxation of Constraints".'

are not tractable by orthodox mathematics. It has no great enthusiasm for exact solutions: 10 % accuracy in an estimation of stresses is good enough. But it does ask for methods which can be applied without restriction, *to any shape of cross-section*; and the mathematics which sufficed the nineteenth-century physicist it finds wanting in this respect.

Here and in other problems of field physics, for some six years I have been seeking, with a zealous team of co-workers* at no time numbering more than six, to furnish engineering science with a mathematics of its own; a mathematics not exact, but on the other hand not thus hedged about with troublesome restrictions. Being a team without official standing, it has fluctuated in size and personnel because ministries from time to time have taken its members for other work; but it has never been allowed to suffer extinction, since new recruits have been found.† From time to time our work has led to solutions having a war-time interest, and in consequence our energies have been focussed on a single objective, sometimes entailing much repetitive computation. In the intervals I have sought to extend the *range* of our methods, trying to guess what problems were most likely to attain war-time importance.

My aim to-day is not to explain the details of our methods, but to show the kind of thing that they can do. Figure 1, for example, shows one of our first solutions (in 1937) of the torsion problem. The equiangular section is, for orthodox analysis, one of the easiest to treat; but I do not think that orthodox analysis can do much with a *pierced* triangular section, and moreover, here and throughout this lecture it should be remembered that *any problem we have solved for one shape of boundary we could have solved for any*.

3. Having made that point, I now show further results of our earliest work. First, in the theory of torsion Prandtl (1923) sought to determine the consequences of 'yield' whereby, when the shear stress has attained some limiting value, the corresponding strain can increase without limit (figure 2). Here too a formal solution can be stated; but the difficulties of analysis are now much greater, because of the whole cross-section some parts behave 'elastically' and others 'plastically', and *the common boundary of the elastic*

* Their names appear as co-authors in the list of references. Acknowledgement is made to the Ministry of Supply, for grants in support of researches made on behalf of certain war committees; also to the Ministry of Aircraft Production (through its Aeronautical Research Committee) and to the Department of Scientific and Industrial Research, for grants in general support of our effort.

† In this connexion I gratefully acknowledge help received on many occasions from Dr C. P. Snow of the Central Register.

and plastic portions is not known in advance. Our computations for an equilateral triangle are shown in figures 3 *a*, *b*, where the contour curves show the direction and (by the closeness of their spacing) the intensity of the shear stress on cross-sections. On account of symmetry, only one-sixth of the complete triangle is reproduced in figure 3*a*.

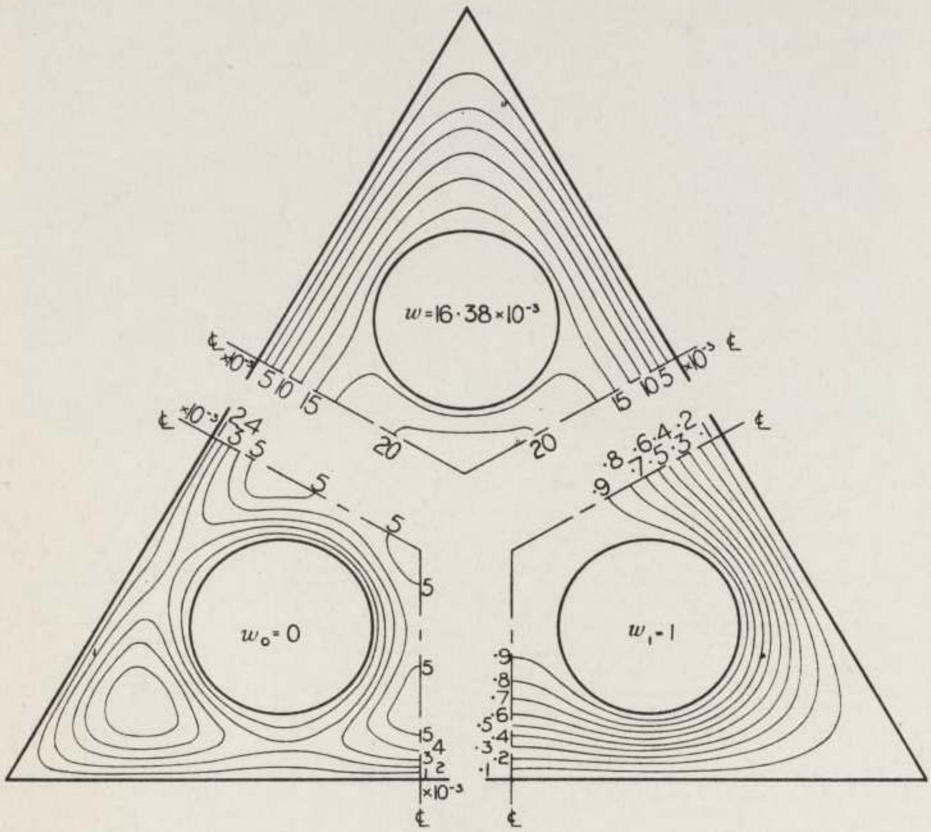


FIGURE 1. Solution of the torsion problem for a pierced triangle (Ref. 5, figure 7). The final solution (at top of diagram) was obtained by synthesis of the two solutions given below.

It will be seen that near the centre of each side the spacing of the contours is uniform so that the shear stress has a constant value. That value is the limiting stress f_Y in figure 2, and the dotted curve shows the extent of the plastic region (*not known initially*). As the twist increases, plastic strain extends over more and more of the cross-section; but however far it extends, elastic conditions are maintained in a spinal region extending from the centre to the corners (figure 3*b*). Experiment gives general confirmation of these conclusions (cf. Nadai 1931, Chap. 19).

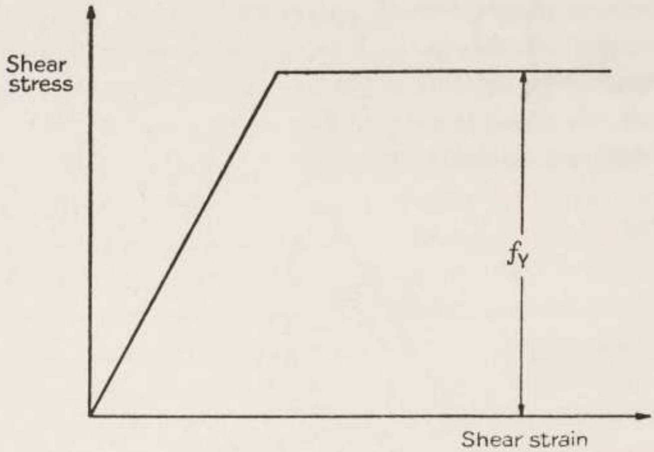
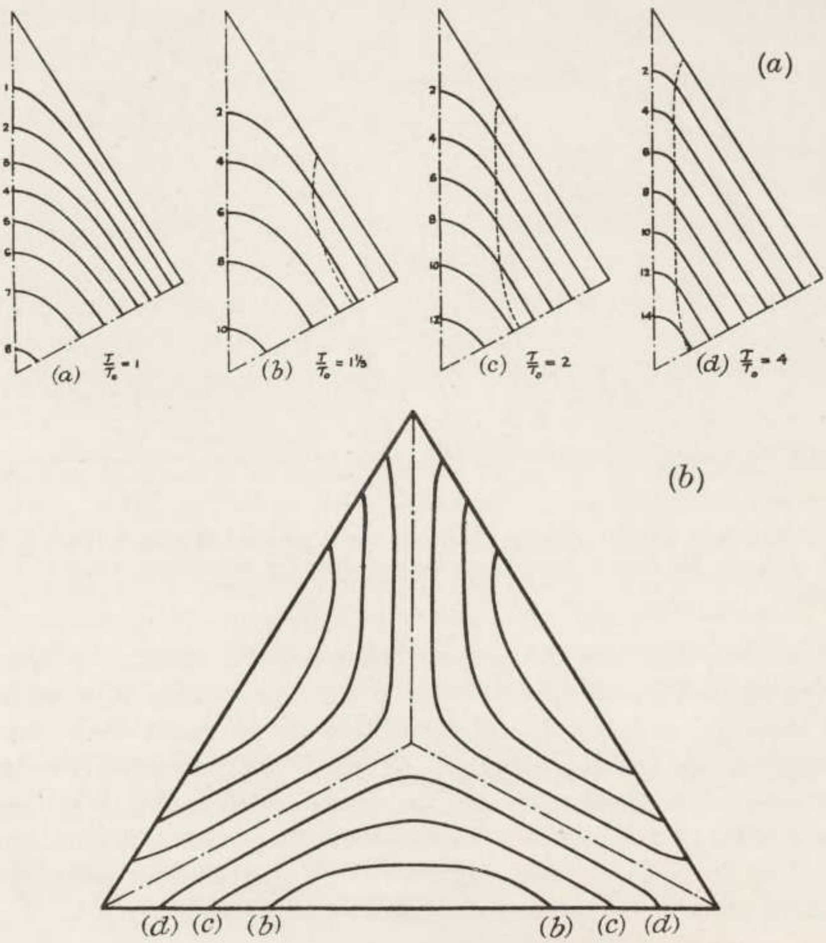


FIGURE 2. 'Plastic' stress-strain diagram (Ref. 5, figure 9).



FIGURES 3 a, b. 'Plastic torsion' of a triangular bar (Ref. 5, figures 10 and 11).

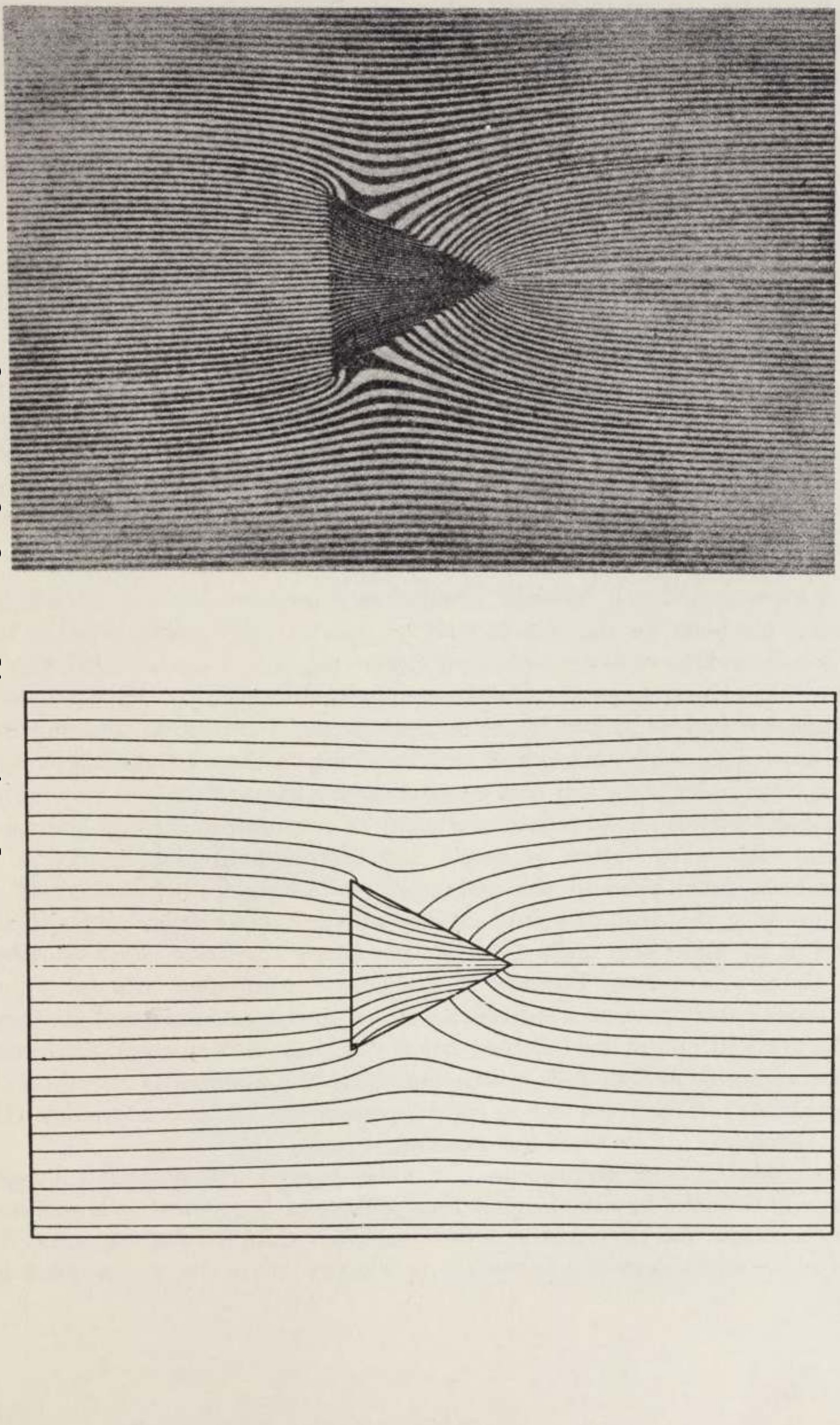


FIGURE 4. Magnetic lines (Ref. 5, figures 14 and 15), and comparison with figure 29 of Hele-Shaw & Hay (1900).

4. Secondly, we confront the phenomenon of refraction (e.g.) in the theory of magnetism when we deal with fields containing iron. Few cases of this kind yield to orthodox analysis, and Hele-Shaw & Hay (1900) devised a highly ingenious technique for obtaining solutions by experiment. Here again our methods have proved successful, and are not restricted to particular shapes of boundary. Figure 4 compares the results of computation with those of experiment, for an iron prism of triangular section.

5. A like freedom from restriction is found in dealing with the problem of conformal transformation: any of the four standard types of transformation (figure 5) can be effected by our methods for a region of any specified form. From the standpoint of engineering science, conformal transformation is a device whereby a problem hard to solve as presented can be simplified by a change of coordinates. Thus in studying the flow of a compressible fluid through a convergent-divergent nozzle we have found it advantageous to transform the 'field' of the fluid into a rectangle. Figure 6 shows our solution of this problem.

6. I have mentioned 'plastic torsion' as a problem hard to attack by orthodox methods for the reason that we cannot say *initially* what is the common boundary of the plastic and elastic regions: a similar difficulty is confronted in the treatment of fluid motion characterized by 'free surfaces'. Sometimes it can be turned by an *analytical* use of conformal transformation—as was shown by Kirchhoff and Rayleigh in their treatment of jets, etc., in two dimensions; but this elegant device has pitfalls—as was found by Davison & Rosenhead (1940) in a study of percolation through granular material (figure 7). Here, a single and simple boundary condition is imposed along the sides of the rectangular retaining wall, but the rest of the boundary (namely, the free surface of the 'water table' AE) is *not* known in advance, and on it a double boundary condition must be satisfied. Using the device, Davison & Rosenhead compelled this remaining boundary to start at A and to finish at D; but having no means of allowing for the vertical boundary DE they could not compel it to stay within the porous material, and in fact it was found (in the absence of an assumed evaporation) to pass from left to right through the vertical boundary DE before returning (as on their assumptions it must) to D.

Later, when I come to describe our methods, you will see that they are essentially *tentative* like the engineering process of 'scraping' to a surface-plate or gauge: for that reason I felt confident that they would serve in cases where some part of a boundary is initially unknown, and applied to this problem (in 1940–1) they led quickly to positive results. The true solution, as you see, involves 'seepage' of water through a part of the vertical side of the retaining wall.

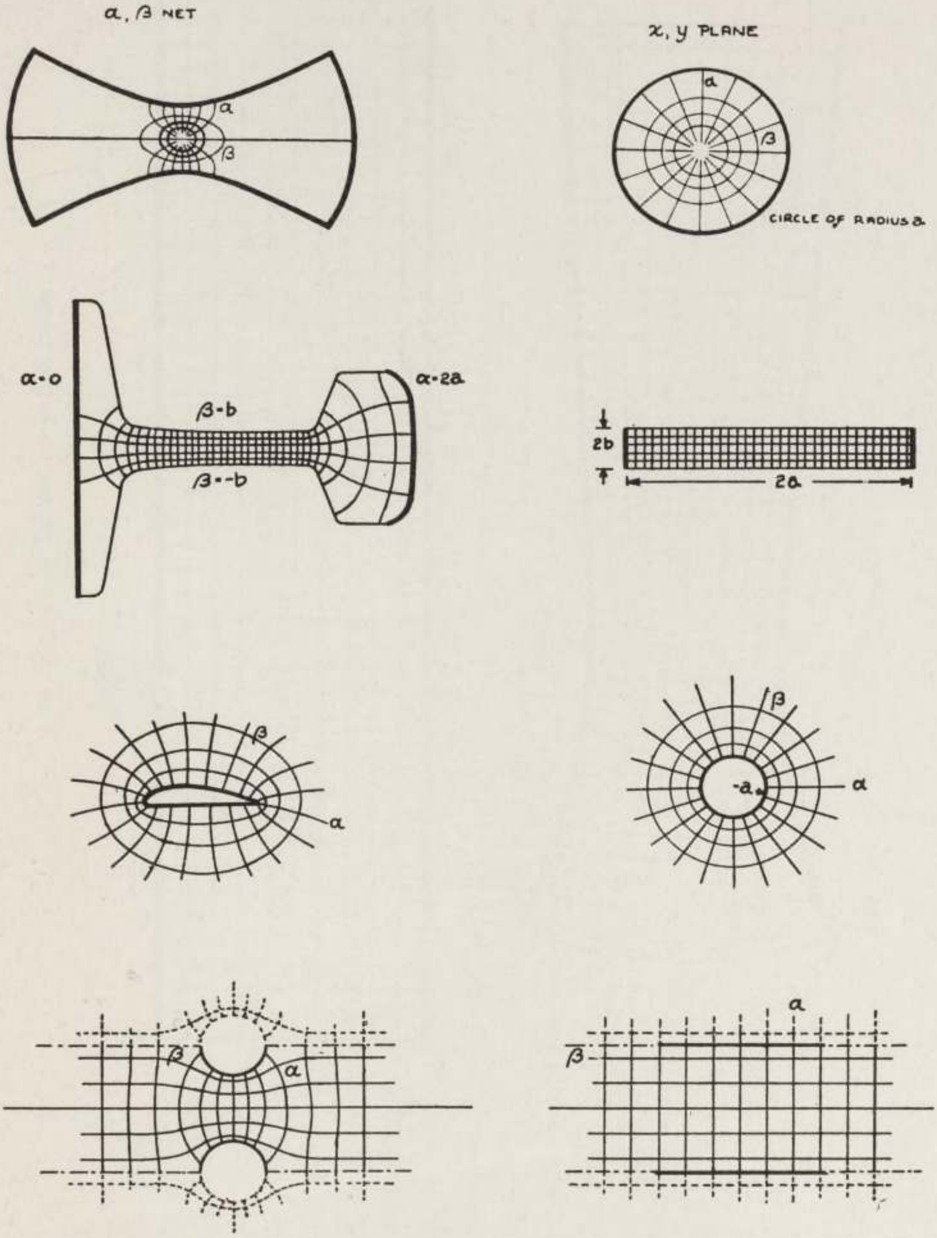
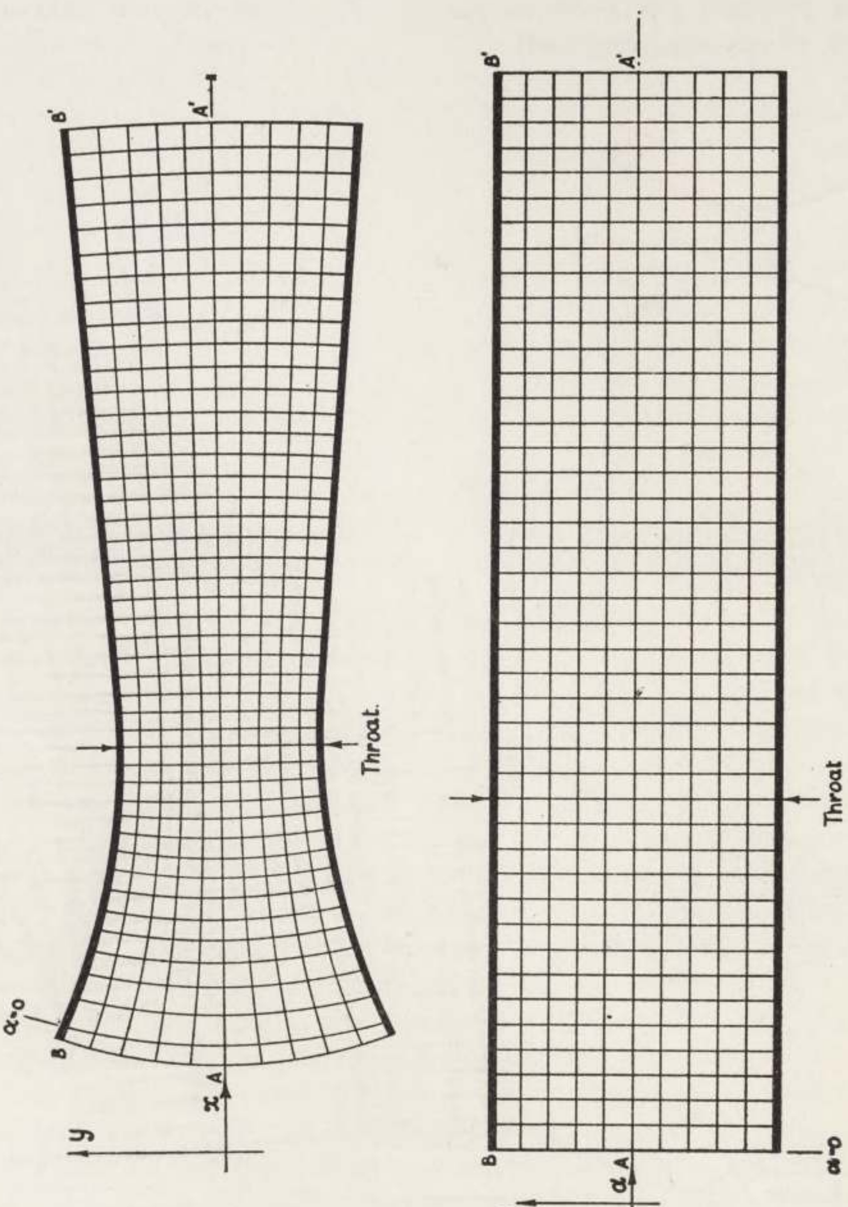


FIGURE 5. Types of conformal transformation (Ref. 7, figure 1).



Part of figure 6; legend on facing page.

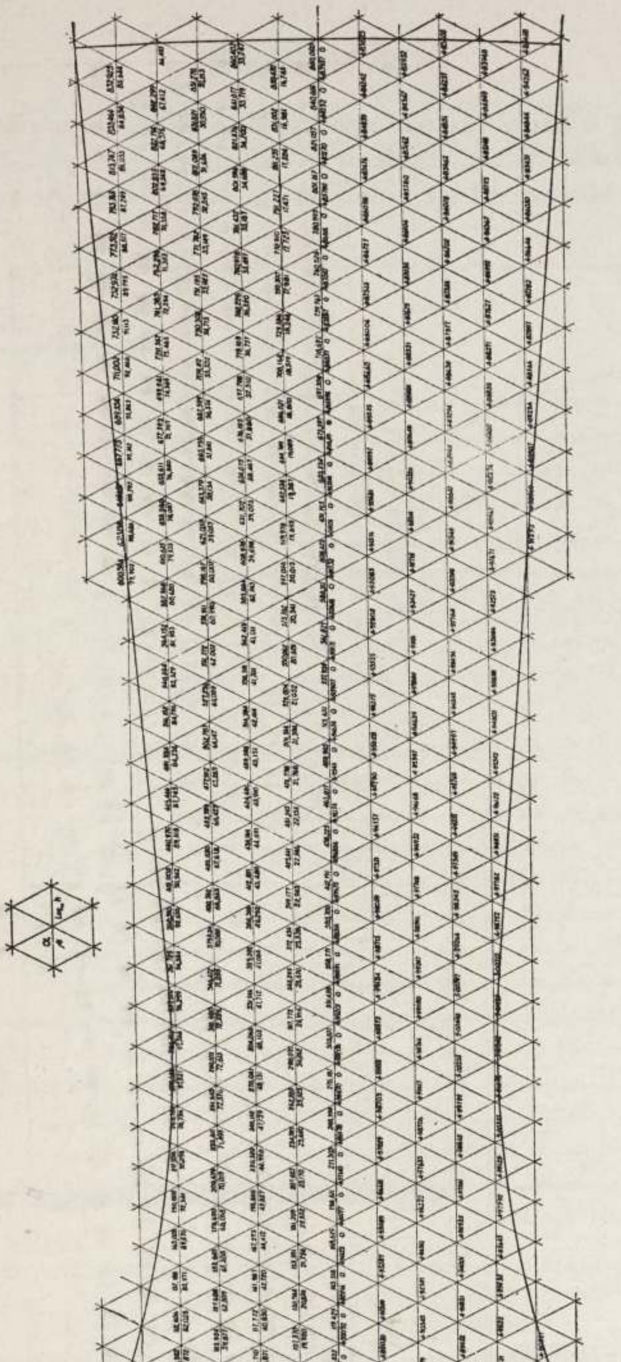


FIGURE 6. Conformal transformation of convergent-divergent nozzle (Ref. 16, figures 2 and 7*a*).

Figures 8 and 9 show further examples, and throw some light on the failure of conformal transformation in the case examined by Davison & Rosenhead. In both a rubble 'blanket' is assumed to be provided, as is

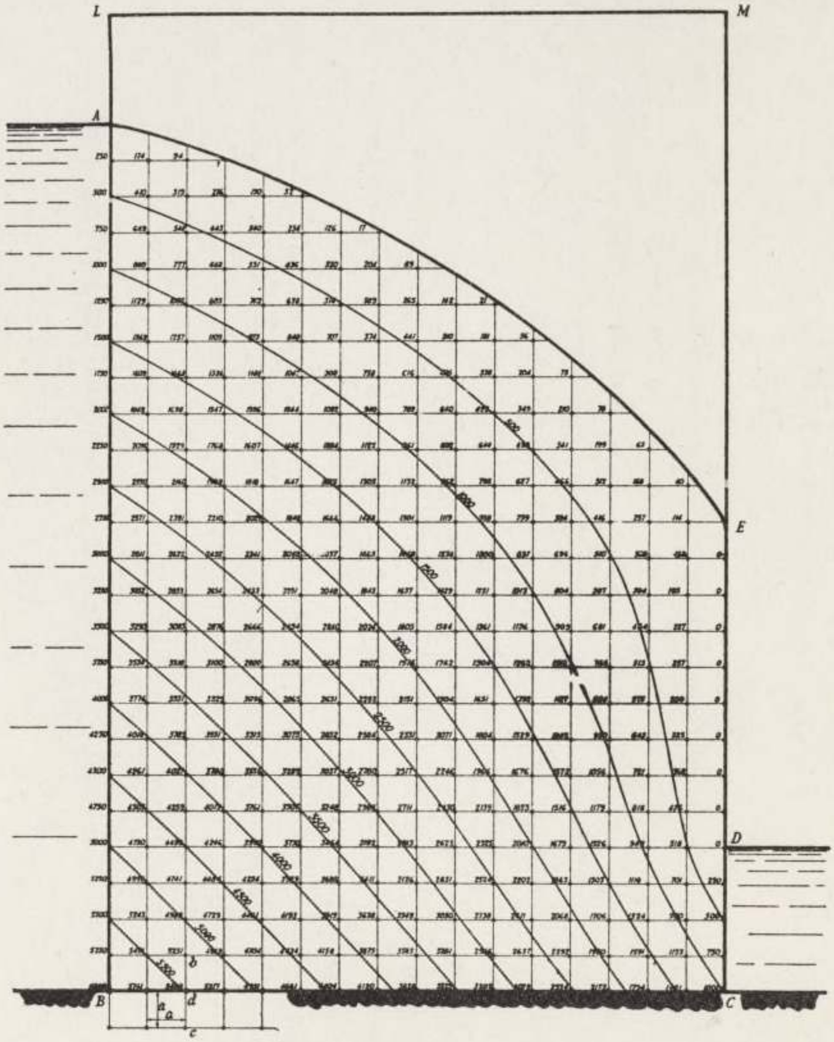


FIGURE 7. Percolation problem of Davison & Rosenhead (Ref. 9, figure 5). The curves are contours of constant pressure.

customary,* to prevent erosion by drainage down the side exposed to air. In the second (due to the assumption of two strata of different porosities) refraction enters again as a complicating factor.

* Cf. Casagrande (1937).

7. So far I have dealt with problems differing in respect of their boundary conditions but all having as governing equation either the two-dimensional form of Laplace's equation, viz.

$$\left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right]\psi = 0, \tag{1}$$

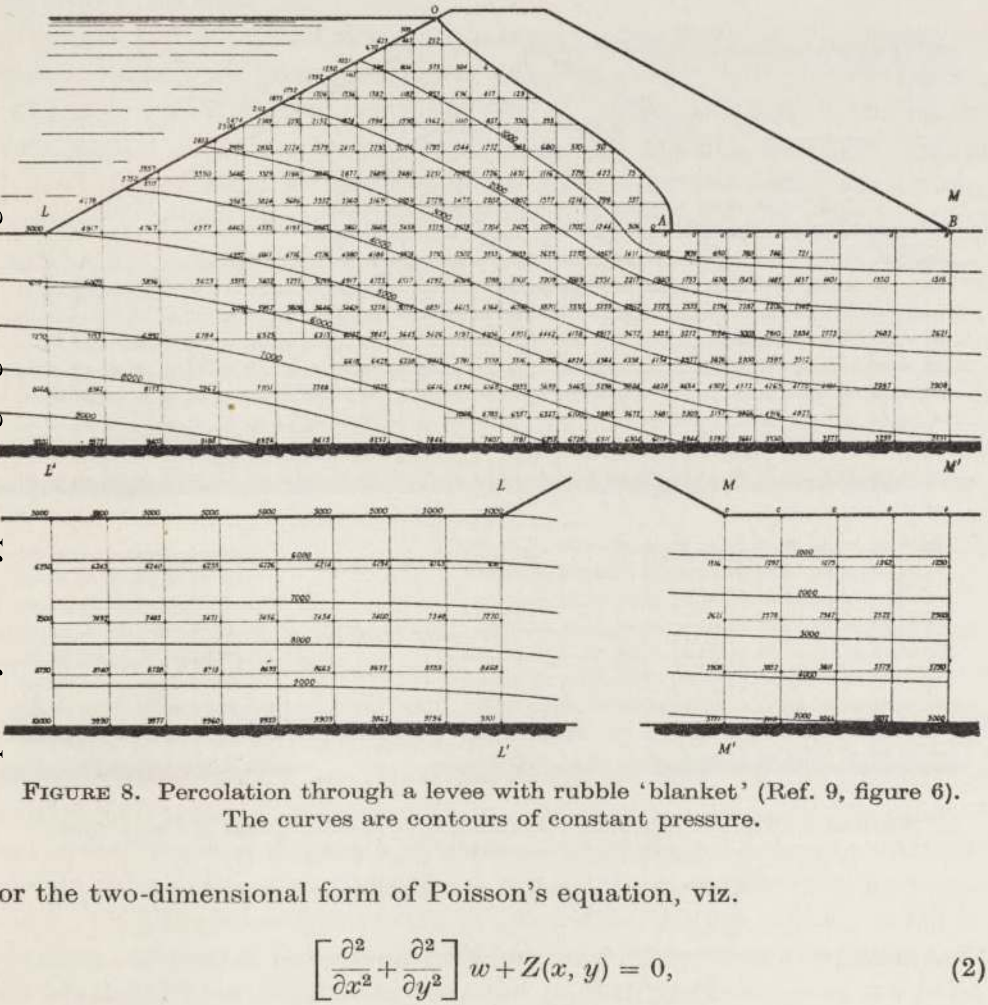


FIGURE 8. Percolation through a levee with rubble 'blanket' (Ref. 9, figure 6). The curves are contours of constant pressure.

or the two-dimensional form of Poisson's equation, viz.

$$\left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right] w+Z(x, y) = 0, \tag{2}$$

where $Z(x, y)$ is specified. I now give some account of our methods as applied to these 'plane-harmonic' equations, treating (1) as a particular case of (2).

As my slides have shown, we present our solutions in the form of numerical values of the wanted function (w) at nodal points of a uniform lattice or 'net'. The meshes (in theory) may be either hexagonal, or square, or triangular (figure 10); the values satisfy, not the governing equation (2) as it stands, but the approximation to it which results when its differentials are replaced by finite-difference approximations. Such replacement, of course, is no new device: indeed, it is hard to suggest an alternative, if the aim is to evolve a method applicable to any shape of boundary. I believe

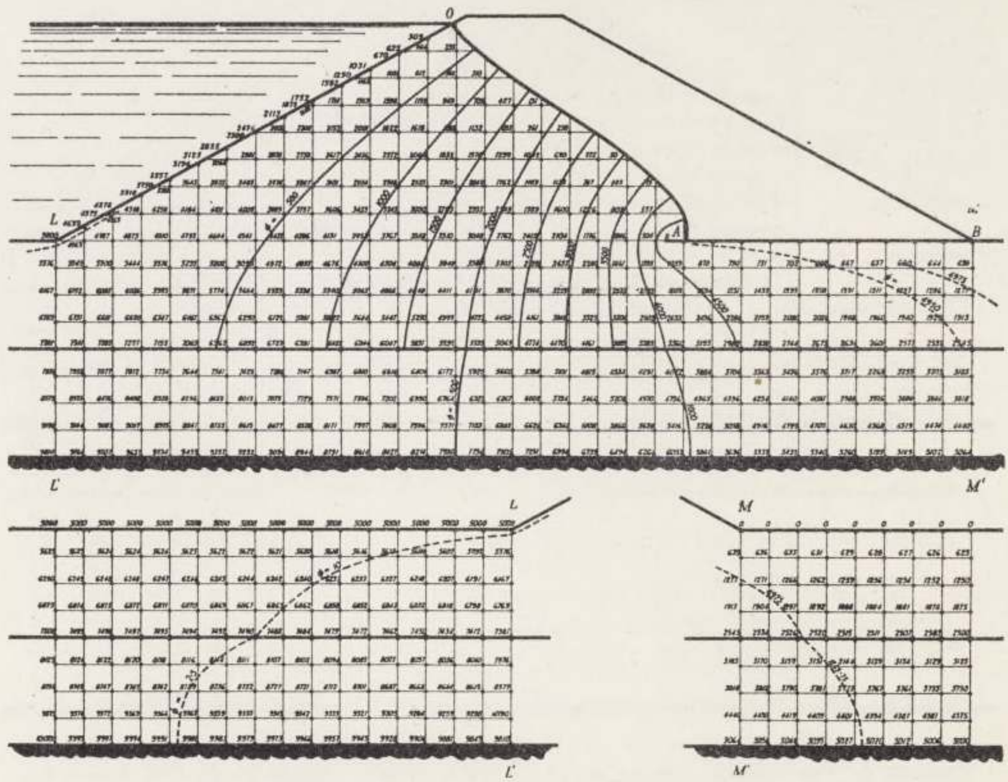


FIGURE 9. The same problem modified by the assumption of two strata having different porosities (Ref. 9, figure 7).
The curves are contours of 'velocity potential' ϕ.

that triangular nets (which have some advantages in respect of accuracy) have not been employed before; but they *could* have been employed by earlier investigators whose methods can hardly be described as 'relaxational', and it is not this feature that I want to stress.

What I regard as the essence of 'relaxation methods' is their visualization of any plane-harmonic problem as concerned with a mechanical system executing controlled displacements under the action of constraints. That notion came to me first as a means of stress-calculation in engineering (structural) frameworks, where 'redundancy' (i.e. a superfluity of members above the number which would suffice to render a framework 'just stiff') introduces difficulties and uncertainties with which every engineer is familiar. Postponing for the moment my account of our attack on these structural problems, I now give a mechanical interpretation of our square and triangular nets.

Prandtl (1903) showed that an equation of the form of (2) governs (if these are small) the transverse deflexions w of a uniformly tensioned membrane (e.g. a soap film) under the action of transverse pressure having, at every point, an intensity proportional to Z. Griffith & Taylor (1917) utilized this analogy to find experimentally, with the aid of soap films, solutions of the torsion problem for cross-sections of non-mathematical

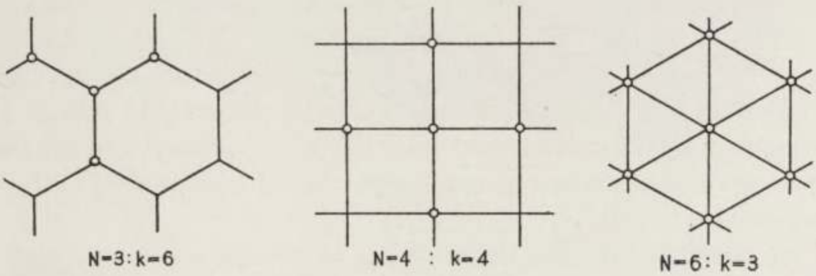


FIGURE 10. Three types of relaxation net (Ref. 5, figure 1).

shapes. We have shown (1943) that the finite-difference approximation to (2) may be interpreted similarly, as governing the transverse deflexions of nodal points of a tensioned net when the transverse pressure is concentrated at these points in accordance with the rules of statics (Ref. 14, §§ 7–9). Consequently an actual net might be used (in the manner of Griffith & Taylor) to solve the finite-difference equation by experiment; but in fact the net is tractable theoretically, and greater accuracy is attainable in calculation. (In experiment, you see, there is this dilemma—that unless considerable displacements are permitted there is nothing much to measure; whereas if they are, then the analogue is inexact,—the deflexions are governed by a different equation.)

9. A great advantage of the 'net analogy' (to minds like mine) is that it enables one to judge *by intuition* the extent of the error which is *and must be* entailed by our use of finite-difference approximations; and for that reason I prefer to spend my own energies in extending the range of problems which can be treated by this simple concept, notwithstanding that

the more mathematically minded of my colleagues are bent (I am glad to say) on improving the accuracy of our approximations.

The appeal to me of the mechanical ('net') analogy may be illustrated by reference to a slide shown earlier (figure 7). Here, if we state our problem as that of determining the fluid pressure p (which is 'plane-harmonic'),

(i) the wanted function has specified values along AB, CDE, EA, and specified normal gradients along BC,

(ii) a further condition is imposed along AE,

but the form of AE is not known initially, being determined by a double boundary condition. We made no progress until we examined this double condition as relating to p interpreted mechanically, as the transverse displacement of a membrane fixed (to sloping boundaries) along AB, BC, CD, unloaded except along AE, and there loaded by edge forces having uniform horizontal line-intensity; along AED the displacement must be zero. A model (figure 11) shows the nature of the mechanical problem. Every string is maintained at the same tension by a hanging load, and other weights exert an equal pressure where they touch the string. I move each weight until it can no longer keep the string pressed against the base (or 'datum plane'): when every weight is in equilibrium, I have the wanted curve.

We set ourselves to solve this problem by computation (of course, on a net finer than the net in my model).* Then solutions came rapidly, *owing to the tentative nature of the relaxation process.*

10. I must now explain that process, and to do so I will summarize ideas by which (in 1935) I was led to a relaxational treatment of *braced frameworks*. I shall be very brief, for I have already given them in a book published late in 1940 (Ref. 19).

The standard problem in frameworks is, given the loads which come upon the joints, to deduce the resulting stresses or (what amounts to the same thing) the joint-displacements. Let us suppose that *displacements* are wanted. Then the orthodox procedure is, taking the joint-displacements as unknowns, to formulate equations of equilibrium for every joint, having the specified forces on their right-hand sides. Up to this point a firm grasp is kept on physical realities; but now there follows a series of operations, performed in accordance with definite mathematical rules, but in which all

* The net was given large meshes in the model, partly for simplicity of manufacture and adjustment, but also in order to illustrate the satisfactory representation of a wanted function which even a coarse net permits. When the weights are in adjustment, the eye gains a quite clear impression (no doubt, helped largely by imagination) of a continuous function coinciding with the net at its nodal points.

physical contact is lost. We dive, so to speak, into a surf of computations, from which we emerge, slightly breathless, with a result which should be,

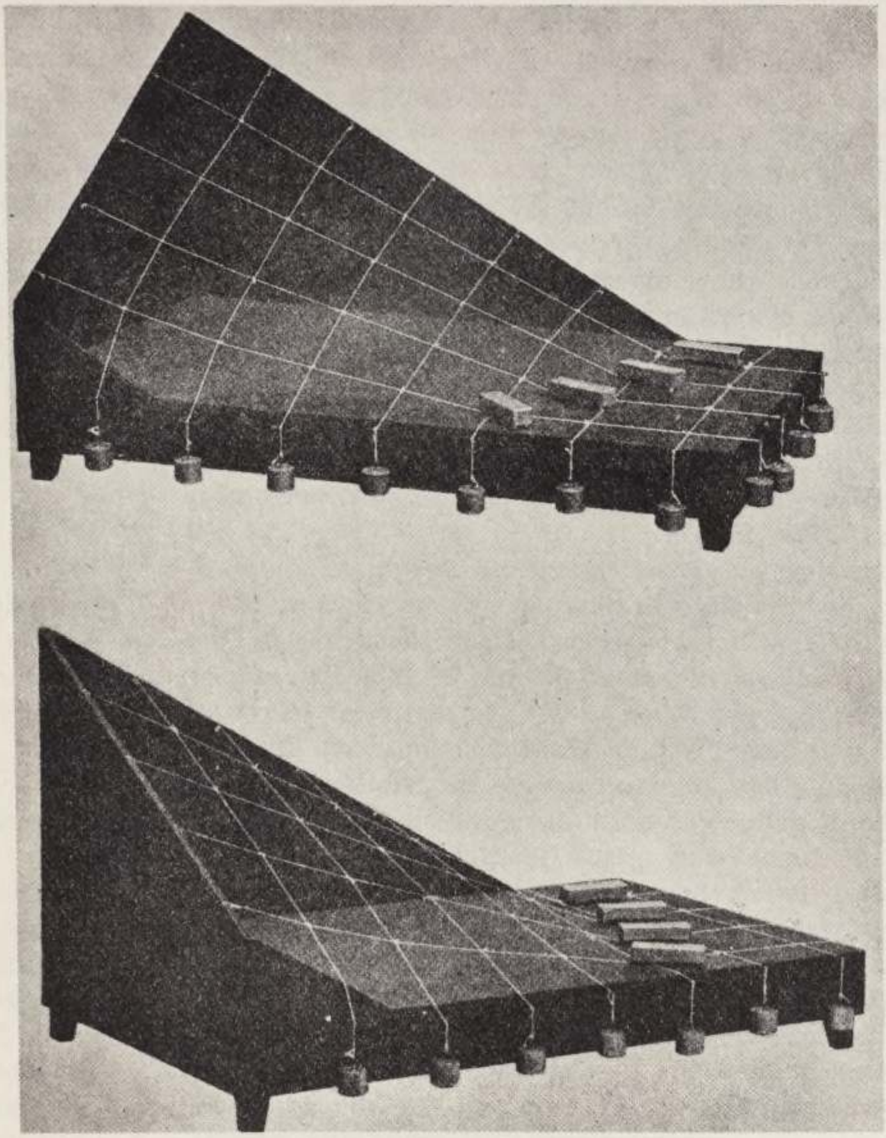


FIGURE 11. Model illustrating net analogue of figure 7.
(Photographs by Prof. C. M. White.)

and we hope is, a pearl. It is a series of displacement values, which ought to satisfy all of the original equations. We try them, and perhaps they do. If not, we must dive again.

In contrast with this process, I sought to reproduce in computation, step by step, a process which might (at least in imagination) be applied to the actual framework. "An ordinary engineering 'jack' (e.g. for automobiles) is a means whereby a controlled displacement may be imposed at any point: it is easy to imagine devices whereby this displacement can be recorded, together with the load sustained at any instant; also to visualize an arrangement in which every joint of a framework which would normally be free to move is provided with a jack of this kind to control its displacement. Suppose that initially the jacks fix the joints in positions such that the framework is not strained: then, plainly, when the external loads are applied they will be taken wholly by the jacks. Suppose that subsequently one jack is relaxed, so that one joint is permitted to travel slowly through a specified distance: then load will be transferred from that jack to adjacent jacks and to the framework, and strain-energy will be stored in the latter. If the force on the jack which is relaxed had a component in the direction of the travel, that jack will be relieved, and strain-energy will be stored in the framework at the expense of the potential energy of the external forces" (Ref. 19, § 3).

Here then is an imagined process whereby a framework could be brought from the unstrained to the fully-strained configuration; and every step is easy to follow in calculation. The governing consideration, you see, is that *indirect solutions are easy*: there is no difficulty in computing the forces brought into play when all but one particular joint are held fixed and on that joint a specified displacement is imposed. So, for every step in the loading process, we can compute the relative changes in the jack load-readings; then, in a further computation, we can follow the whole process, keeping touch with reality throughout. Two computational tables are entailed (figure 12): the first (the 'operations table') exhibits the effects on the recorded loads of displacing every joint severally, also (e.g. operations '6' and '7') of any simultaneous displacement of two or more joints which we may expect to find useful; in the second (the 'liquidation table') the first line shows the applied forces which are the loads initially recorded, subsequent lines show a steady decrease in magnitude of the recorded loads in consequence of operations which are detailed in the first and second columns. The tentative nature of the relaxation process will be apparent to anyone who examines this table carefully.

11. Now, in place of a braced framework, let us suppose that we load our tensioned net. Nothing need be altered in our imagined (physical) process, and again every step can be followed in computation. In fact

there is this simplification, that whereas in the framework every unit operation may be different, in the net (owing to its regularity) every operation is the same. We no longer need an operations table: it

TABLE 1

(Units: 1 ton weight; 1 foot.)

Operation no.	Nature of operation	(1) X_A	(2) X_B	(3) Y_A	(4) Y_B	(5) Y_C	X_C	X_D	Y_D
1 (*a*)	$u_A = 1$	*−3,980*	*0*	*312*	*0*	*768*	*1,280*	*2,700*	*−1,080*
1 (*b*)	$= 0.2513$	−1,000	0	78_4	0	193	322	678	-271_4
2 (*a*)	$u_B = 1$	*0*	*−4,100*	*0*	*720*	*0*	*3,380*	*720*	*−720*
2 (*b*)	$= 0.2439$	0	−1,000	0	175	0	825	175	−175
3 (*a*)	$v_A = 1$	*312*	*0*	*−5,393*	*4,500*	*461*	*768*	*−1,080*	*432*
3 (*b*)	$= 0.1854$	58	0	−1,000	834	85_5	142_4	-200_2	80_1
4 (*a*)	$v_B = 1$	*0*	*720*	*4,500*	*−5,220*	*0*	*0*	*−720*	*720*
4 (*b*)	$= 0.1916$	0	138	862	−1,000	0	0	−138	138
5 (*a*)	$v_C = 1$	*768*	*0*	*460.8*	*0*	*−2,480.8*	*−768*	*0*	*2,020*
5 (*b*)	$= 0.4031$	309_6	0	186	0	−1,000	-309_6	0	814
6 (*a*)	$v_A = v_B = 1$	*312*	*720*	*−893*	*−720*	*461*	*768*	*−1,809*	*1,152*
6 (*b*)	$= 0.6200$	193	446	−554	−446	286	476	−1,116	714
7 (*a*)	$v_A = -v_B = 1$	*312*	*−720*	*−9,893*	*9,720*	*461*	*768*	*−360*	*−288*
7 (*b*)	$= 0.6510$	16	−37	−504	496	23	39	−18	−15

Operation	Multiplier × 10²	X_A	X_B	Y_A	Y_B	Y_C
	(Initial forces)→	15	7.07	..	7.07	..
1 (*b*)	15	−15	0	1.18	0	2.89
		0	7.07	1.18	7.07	2.89
2 (*b*)	7	0	−7.00	0	−1.22	0
		0	0.07	1.18	8.29	2.89
6 (*b*)	10	1.93	4.46	−5.54	−4.46	2.86
		1.93	4.53	−4.36	3.83	5.75
7 (*b*)	−8	−0.13	0.30	4.03	−3.97	−0.18
		1.80	4.83	−0.33	−0.14	5.57
5 (*b*)	7	2.18	0	1.30	0	−7.00
		3.98	4.83	0.97	−0.14	−1.43
2 (*b*)	6	0	−6.00	0	1.05	0
		3.98	−1.17	0.97	+0.91	−1.43
1 (*b*)	5	−5.00	0	0.39	0	+0.97
		−1.02	−1.17	1.36	+0.91	−0.46
6 (*b*)	2.5	0.48	+1.12	−1.39	−1.11	0.72
		−0.54	−0.05	−0.03	−0.20	0.26
5 (*b*)	0.20	0.06	0	0.04	0	−0.20
		−0.48	−0.05	+0.01	−0.20	0.06
1 (*b*)	−0.5	0.50	0	−0.04	0	−0.10
		0.02	−0.05	−0.03	−0.20	−0.04
6 (*b*)	−0.25	−0.05	−0.11	+0.14	0.11	−0.07
		−0.03	−0.16	+0.11	−0.09	−0.11
2 (*b*)	−0.2	0	0.20	0	-0.03_5	0
		−0.03	0.04	0.11	-0.12_5	−0.11
7 (*b*)	0.2	0	−0.01	−0.10	0.10	0
		−0.03	0.03	0.01	-0.02_5	−0.11
5 (*b*)	−0.12	−0.04	0	−0.02	0	0.12
		−0.07	0.03	−0.01	-0.02_5	+0.01
1 (*b*)	−0.06	0.06	0	0	0	−0.02
		−0.01	0.03	−0.01	-0.02_5	−0.01
6 (*b*)	−0.04	0	−0.02	0.02	0.020	−0.01
		−0.01	0.01	0.01	0	−0.02

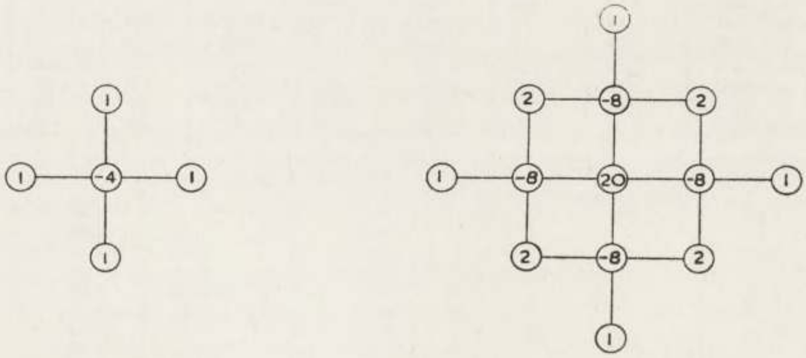
FIGURE 12. Typical 'operations' and 'liquidation' table (Ref. 19, tables II and III).

is replaced by a standard 'relaxation pattern' (figure 13). And liquidation, too, no longer calls for a table: it can be effected on the 'relaxation net'

12. The 'patterns' on the right of figure 13 relate, similarly, to the finite-difference approximation to the biharmonic equation

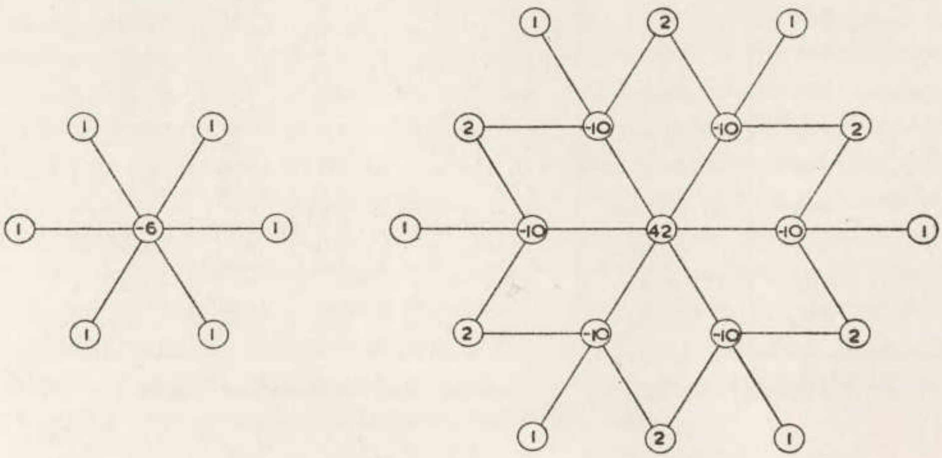
$$\left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right]^2 w = Z(x, y), \tag{3}$$

in which $Z(x, y)$, as before, is specified. This equation governs (*inter alia*) the transverse displacement under pressure, also the 'Airy stress-function' χ which gives the stresses induced by extension, of a uniform elastic plate. I need not here develop a mechanical analogue of the finite-difference approximation: enough, that this leads (by similar reasoning) to the patterns shown.



(*a*) (*b*)

'Relaxation patterns' for the operators ∇^2 and ∇^4: square net ($N=4$)



(*c*) (*d*)

'Relaxation patterns' for the operators ∇^2 and ∇^4: triangular net ($N=6$)

FIGURE 13. 'Relaxation patterns' (Ref. 10, figures 5).

Our paper dealing with biharmonic analysis (Ref. 10) will not have open publication during the war, but I am allowed to show one solution—for the stresses in a standard 'cement briquette'. The problem is indicated in the left-hand diagram of figure 14: the 'jaws' apply a measured pull to the specimen, and calculation of stresses is needed to decide how far the assumption is justified, that uniform and simple tension is imposed across the waist. But here a question is presented as to the coefficient of friction between the jaws and the briquette: to leave it open, in calculation we have separated (as the photo-elastic method cannot do) the effects of (*a*) the longitudinal pull and (*b*) the transverse squeeze which the jaws exert.

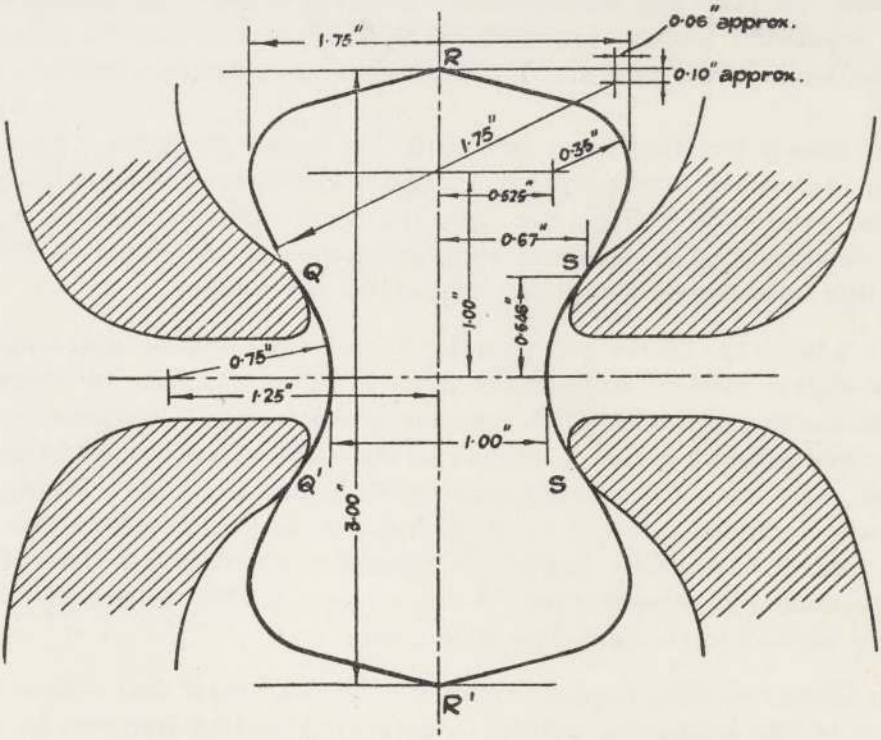
Our results for the second system (*b*) are shown in figures 14 and 15: figure 14 shows the cross-tension X_x, figure 15 shows on the left the longitudinal tension Y_y and on the right the shear stress-component X_y. It may be concluded that the squeeze (and hence the frictional coefficient) has little influence on the tension (Y_y) across the waist.

13. I leave the biharmonic equation (3) with the remark, that while the photo-elastic method will always have value as a means to *qualitative* understanding of a wanted stress-system, computation can now provide closer quantitative accuracy, and at no increased cost in respect of labour. It can, moreover, deal not only with problems in which edge tractions, but also with problems in which the edge displacements are specified, and even with 'mixed' boundary conditions (tractions specified at some points, displacements at others): Coker & Filon (1931, § 4.39) restrict the photo-elastic method to the first class of problem.

14. Before passing to other types of equation I must deal with another aspect of the relaxation process—whether applied to frameworks or to nets. *Is that process always convergent?*

This is a question for mathematicians, and I, from the standpoint of engineering science, would first propound another: *What is an 'exact' solution?* I have said that the orthodox procedure, if displacements are wanted, is to formulate equations in these as unknowns, with the specified loadings on their right-hand sides. But when we turn to reality from the rather artificial atmosphere of the examination hall, we are faced with the consideration that *in fact loadings are never known exactly*. To them, if to any of the data, 'tolerances' should be attached: $5 \pm 0{\cdot}1$ tons here, $10 \pm 0{\cdot}3$ tons there. But this is rarely done, even in cases confronted as problems of research; *because the doing of it would have no consequences for orthodox methods of attack.*

In our methods we fix attention, always, not on the wanted quantities (stresses or displacements) but on the data of the problem (on the loads). We assume these, always, to have some specified 'margin of uncertainty', and we 'whittle' away at them, accounting for more and more, until every 'residual' of a datum quantity has been brought within that margin. At that point we stop, asserting that to 'whittle' further would be not only waste of time, but *meaningless*. If you cannot tell the loads exactly



Part of figure 14; legend on facing page.

(and in practice you never can), then you have no claim to an exact solution. We need not apologize—we may even boast—that ours is 'mathematics with a fringe'

15. It may not seem an important feature of the relaxation method, but in fact I believe it to be both fundamental *and important philosophically*, that in it we fix attention not on the wanted quantities but on the data of a problem. Ours is a tentative process, like the scraping process whereby, in engineering, a part is brought to coincide with surface-plate or gauge; and our 'margin of uncertainty' (in the loads) is akin to the margin of tolerance between a 'go' and a 'not go' gauge. 'Exact' measurement has

no meaning in metrology,—we can only work to some limiting accuracy; and so, I contend, exact solution has no meaning in a mathematics aimed at practical ends.

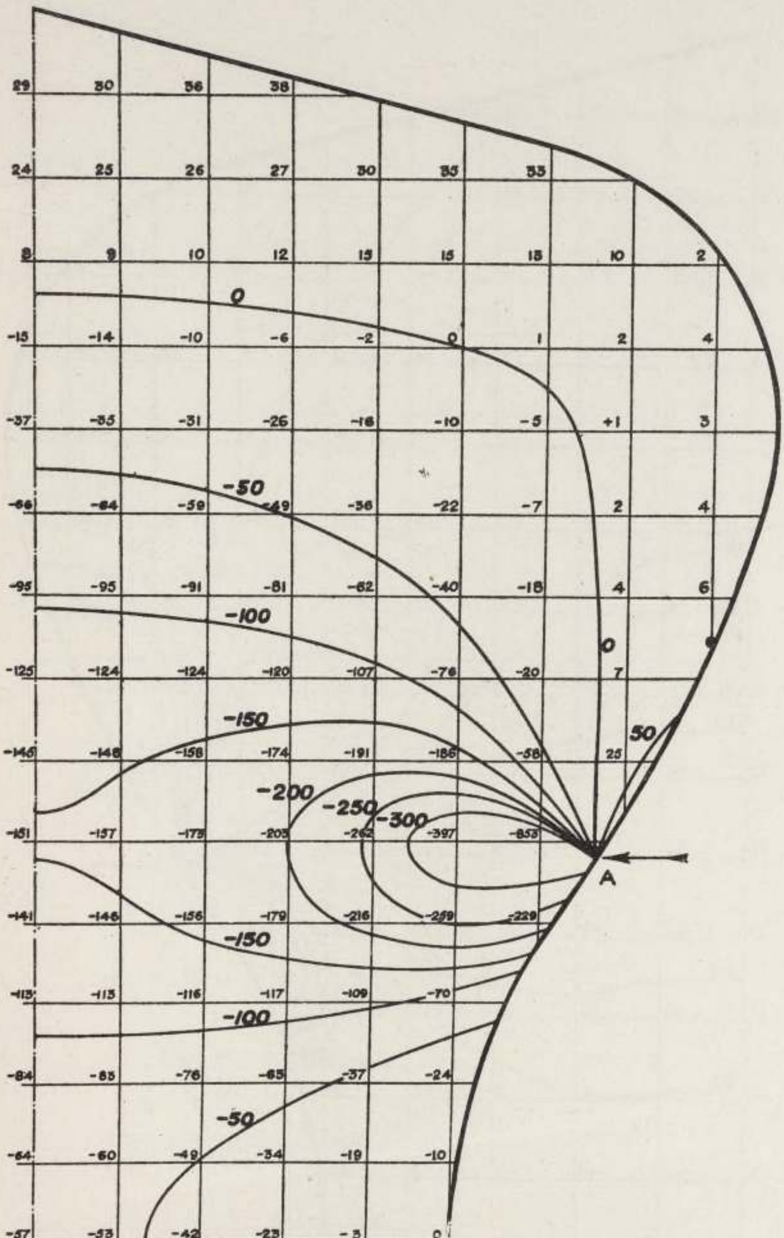
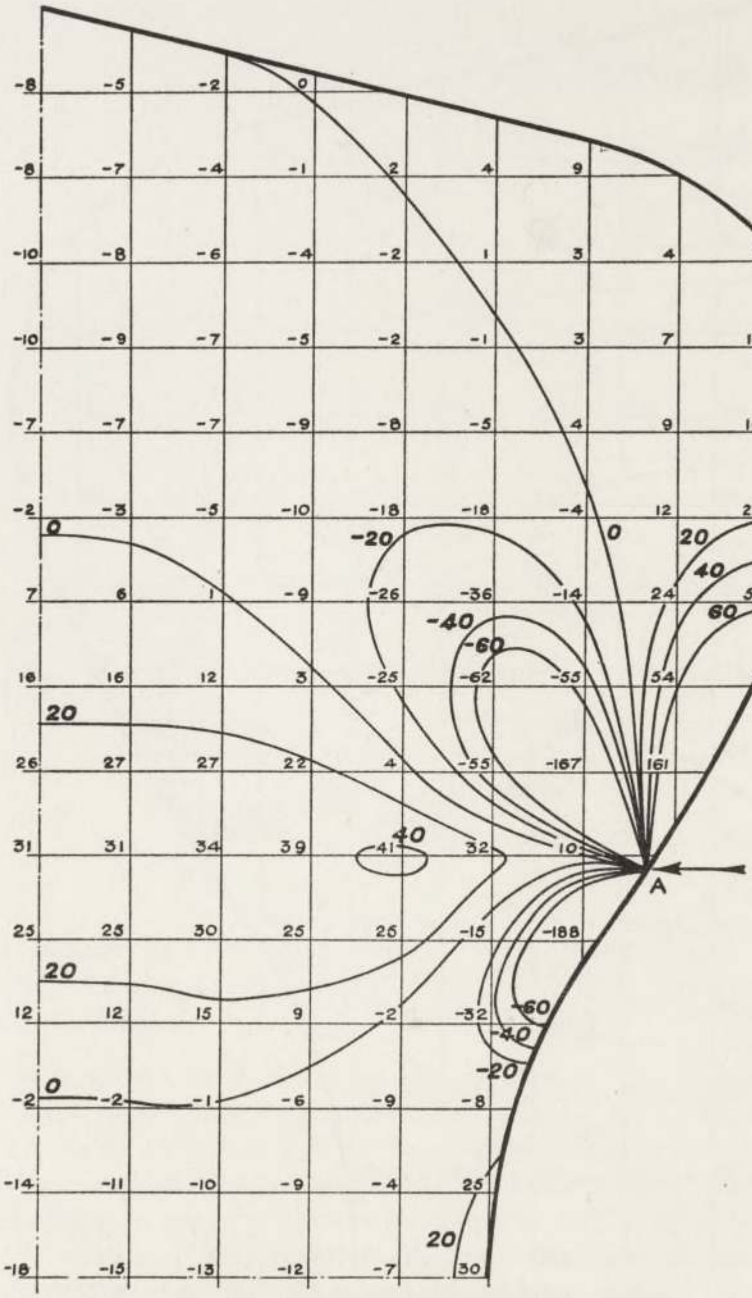


FIGURE 14. Concrete briquette problem, and computed values of X_x (Ref. 10, figures 21, 25*a*).

16. And now, as to the convergence of our computations for frameworks and the like. When we seek a stable equilibrium configuration for some mechanical system, we are (by a general theorem in mechanics) seeking



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that configuration in which the total potential energy takes its minimum value; consequently any process which continually 'whittles at' (so as to reduce) the total potential energy must bring the configuration nearer

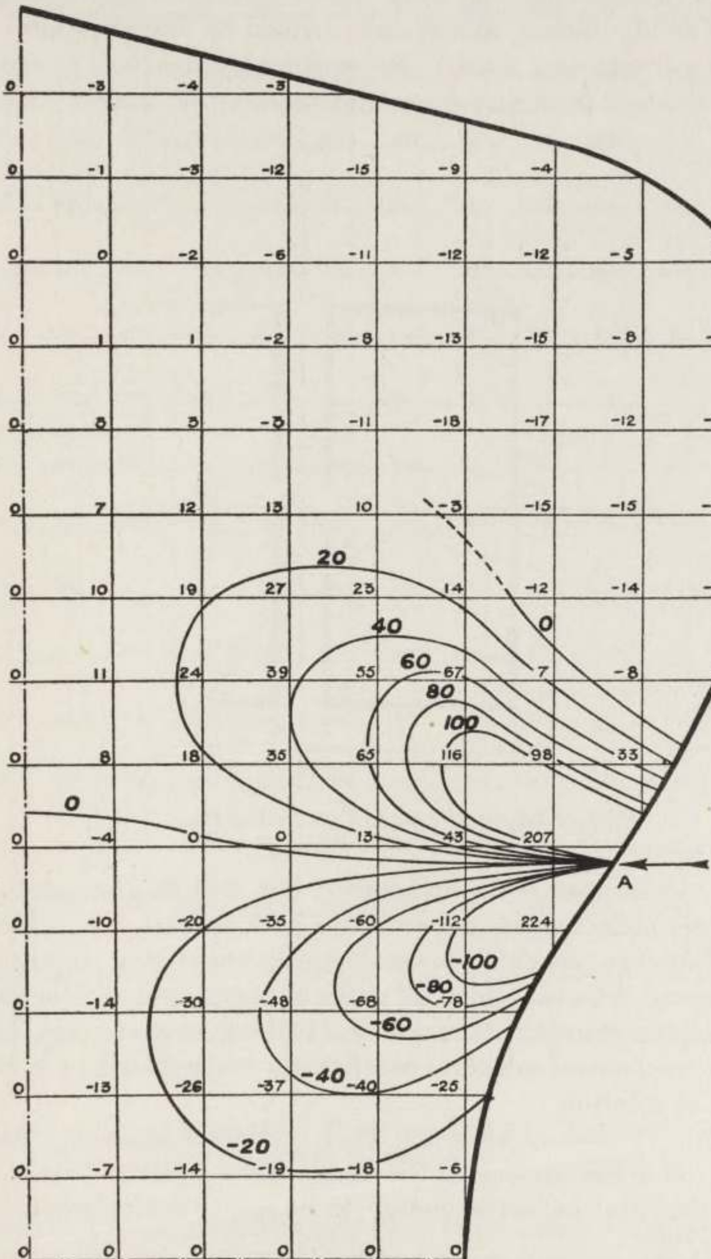
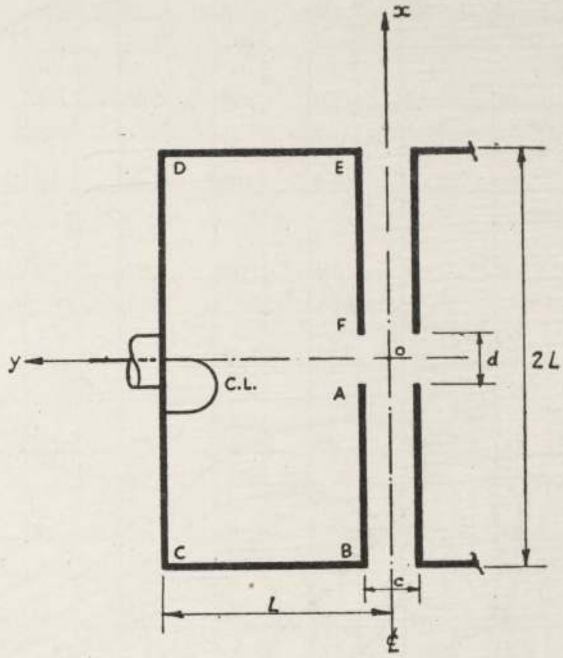


FIGURE 15. Concrete briquette problem: values of Y_y and X_y (Ref. 10, figures 25 *b*, *c*).

to what is wanted. An analogy has been suggested by Prof. Temple, which I quote for mathematicians to whom my picture of jacks and workmen makes a fainter appeal than to engineers:—In a valley devoid of friction the place at which a ball will come to rest is its lowest point: that point is the 'wanted configuration' in a system defined by two variables—x (east and west) and y (north and south). Proceeding by the rules of our relaxation method, we start from anywhere and proceed by 'stages': sometimes



Part of figure 16; legend on facing page.

east-and-west, sometimes north-and-south; but always proceeding downhill, and in every stage continuing until our path is level.* It is clear that we shall tend always towards our goal, unless the valley contains other 'stationary points' (hills) at which all paths are level; but this would mean that more than one point exists at which a ball can come to rest, and that possibility (in mechanical systems) can usually be excluded by a theorem of uniqueness of solution.

You will observe that in adopting Prof. Temple's analogy I have imposed no restriction except this on the shape of the valley,—that is, I have not assumed the total potential energy to be a quadratic function of the

* Slope is here the analogue of 'residual force'. Friction will give a 'margin of uncertainty'.

displacements: consequently my argument is *not* restricted to systems in which Hooke's law is satisfied,—it only requires that the wanted con-

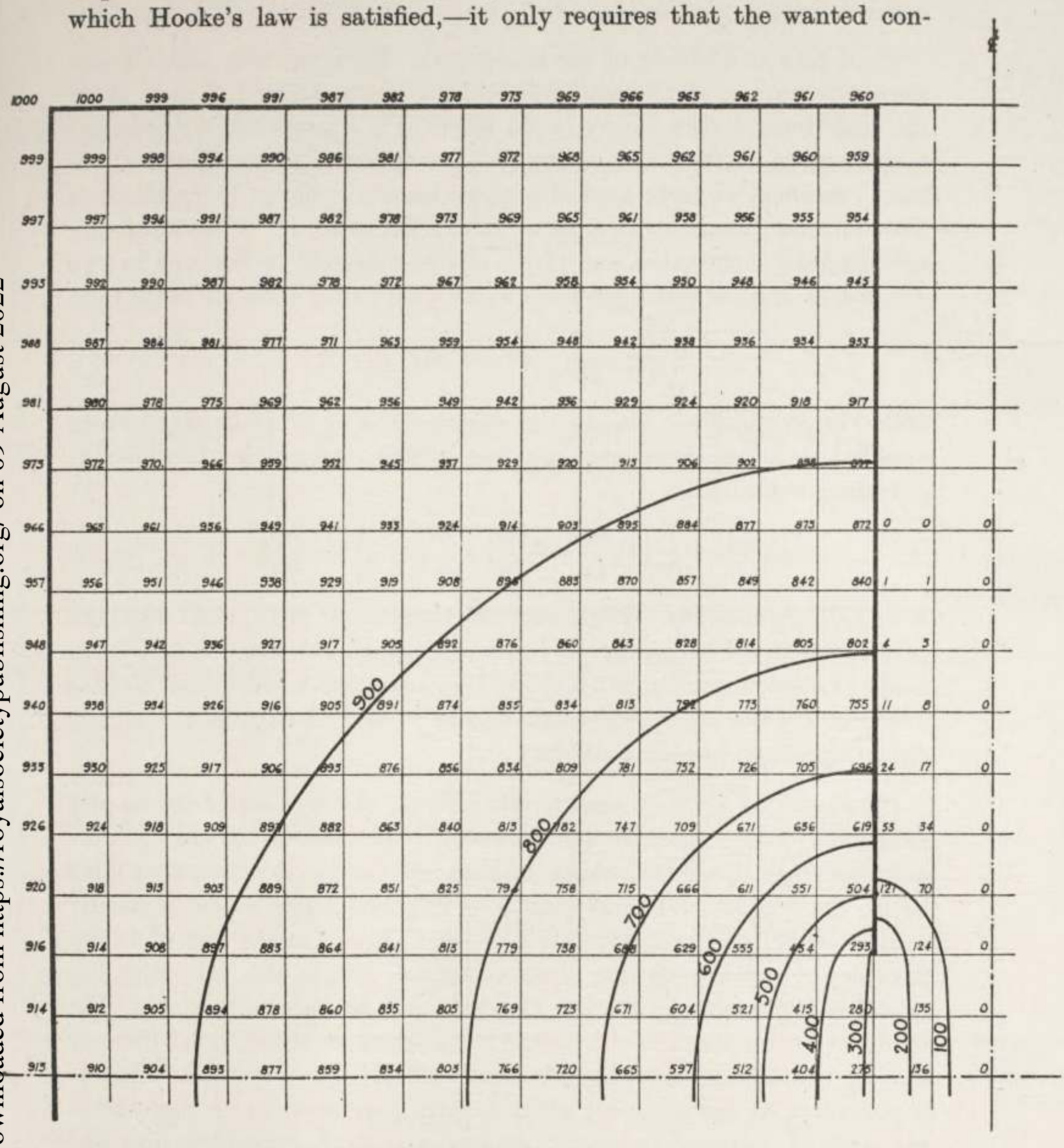


FIGURE 16. 'Wave guide' solution (Ref. 12, figures 3 and 4). The contours may be interpreted as relating to free transverse vibrations of an unloaded membrane.

figuration shall be unique. The point has importance, because we are using relaxation methods, now, on non-linear systems. In relating to linear systems (the main concern of my book) Temple has already undertaken a mathematical defence of our methods: I now state my belief that a wider defence is possible.

17. I pass to a review of our latest work. First, not only statical, but *eigenwerte* (characteristic number) problems can be treated: I have given the underlying theory in my book, where it is exemplified by torsional oscillations in shafting and by the elastic stability of struts. The methods there described we have applied without essential change to problems in 'field physics', treated with a use of 'relaxation nets'; e.g. (1) to transverse vibrations of membranes and (2) to electro-magnetic oscillations in two dimensions, both of which problems entail a governing equation of the form

$$\left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right] w+\lambda w \,.\, F(x, y) = 0, \tag{4}$$

with $F(x, y)$ specified; also to the elastic stability of plane frameworks or of flat plating to transverse 'buckling',—this last a problem of which the governing equation is

$$\lambda D \nabla^4 w+\frac{\partial}{\partial x}\left(P_x \frac{\partial w}{\partial x}-S \frac{\partial w}{\partial y}\right)+\frac{\partial}{\partial y}\left(P_y \frac{\partial w}{\partial y}-S \frac{\partial w}{\partial x}\right) = 0, \tag{5}$$

with P_x, S, P_y specified.* *Eigenwerte* solutions to (4) or (5) must yield not only the 'natural frequency' or 'critical loading' but also the associated mode of distortion. Figure 16 shows the computed mode for an electro-nagnetic ('wave guide') problem; figures 17 and 18 explain a distinctly difficult problem in elastic stability.

18. Figure 17 shows a rectangular strip of plating, loaded (in its own plane) by forces uniformly distributed. The plotted curves are contours of the principal *compressive* stress, and are confined (as obviously they must be) to the bottom half. This system of stresses (chosen as being exactly calculable) is closely representative of what obtains in the web of a deep ⌶-girder or in the wing spar of an aeroplane; and in this connexion the question is presented, whether the web will be elastically stable. The tendency to buckle, since it is due to compressive stress, must increase sharply towards the right-hand bottom corner;† but close to the edges it is prevented by the clamping which is there presumed, so we may expect the mode of distortion to have its greatest amplitude somewhere near this corner, and elsewhere to involve no serious deflexions. This expectation is realized (figure 18).

* This problem was exemplified in the lecture by a model of a 'through girder' bridge.

† Cf. the values of stress-intensity which are attached to the contours in figure 17.

19. The mode exhibited in figure 18 was far from simple to compute, owing to the nature of the loading and of the boundary conditions: even

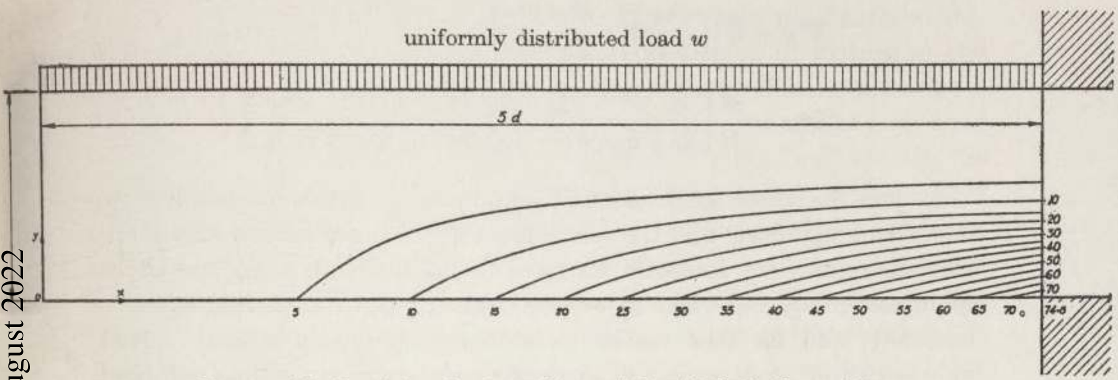


FIGURE 17. Loaded web of cantilever girder (Ref. 11, figure 11). Contours exhibit the variation of the principal compressive stress.

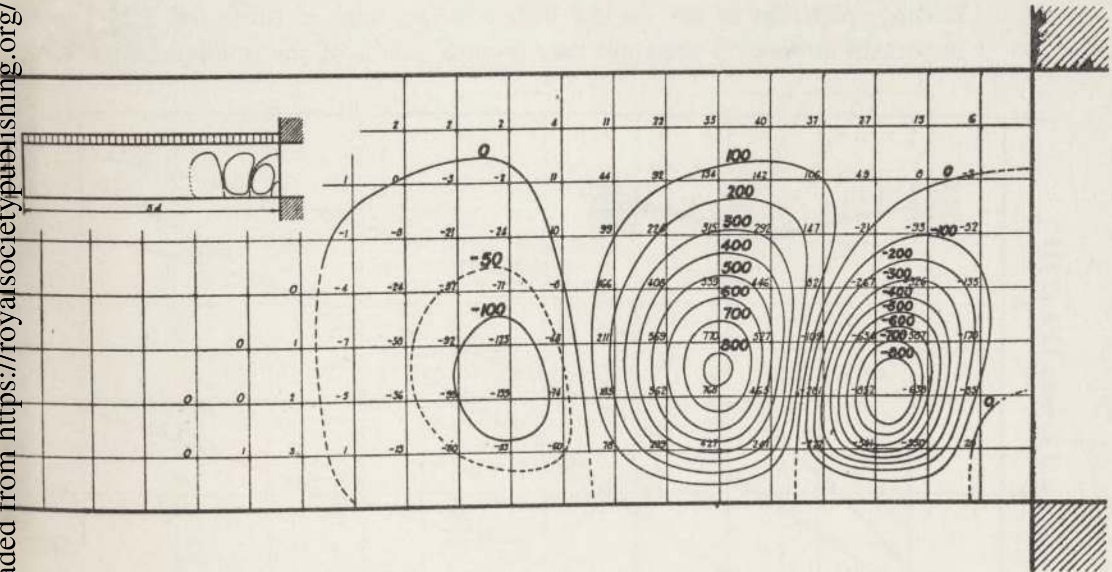


FIGURE 18. Mode of distortion under critical loading of cantilever girder web (Ref. 11; figure 12).
(The absolute magnitude of the distortion is indefinite.)

in simple cases the same is true when the mode is characterized by 'large deflexions'—'well-developed buckling'. H. Wagner has emphasized (1929) the importance for aeronautics of this latter class of problems. They are very difficult, as may be judged from the governing equations which follow (v. Karman 1910):

$$\left.\begin{aligned}\nabla^4\chi &= E\left\{\left(\frac{\partial^2 w}{\partial x\,\partial y}\right)^2 - \frac{\partial^2 w}{\partial x^2}\cdot\frac{\partial^2 w}{\partial y^2}\right\},\\ \nabla^4 w &= \frac{2h}{D}\left[\frac{Z}{2h} + \frac{\partial^2\chi}{\partial y^2}\cdot\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2\chi}{\partial x^2}\cdot\frac{\partial^2 w}{\partial y^2} - 2\,\frac{\partial^2\chi}{\partial x\,\partial y}\cdot\frac{\partial^2 w}{\partial x\,\partial y}\right].\end{aligned}\right\} \tag{6}$$

As you see, these are non-linear simultaneous equations relating two dependent variables, and the few solutions which are known entail very elaborate analysis. Recently we have found that—at considerable cost in labour—they too yield to relaxational attack. We wanted a test of accuracy, and for that reason have treated problems already solved: figures 19 and 20 show that in these our accuracy is more than sufficient, and again I would emphasize that we could have treated *any* shape of boundary.

Figure 19 shows, to a base (μ) proportional to the intensity of transverse loading, plottings of the central deflexion (w_0) and of three practically important stresses. I need not here go into details of the problem: what

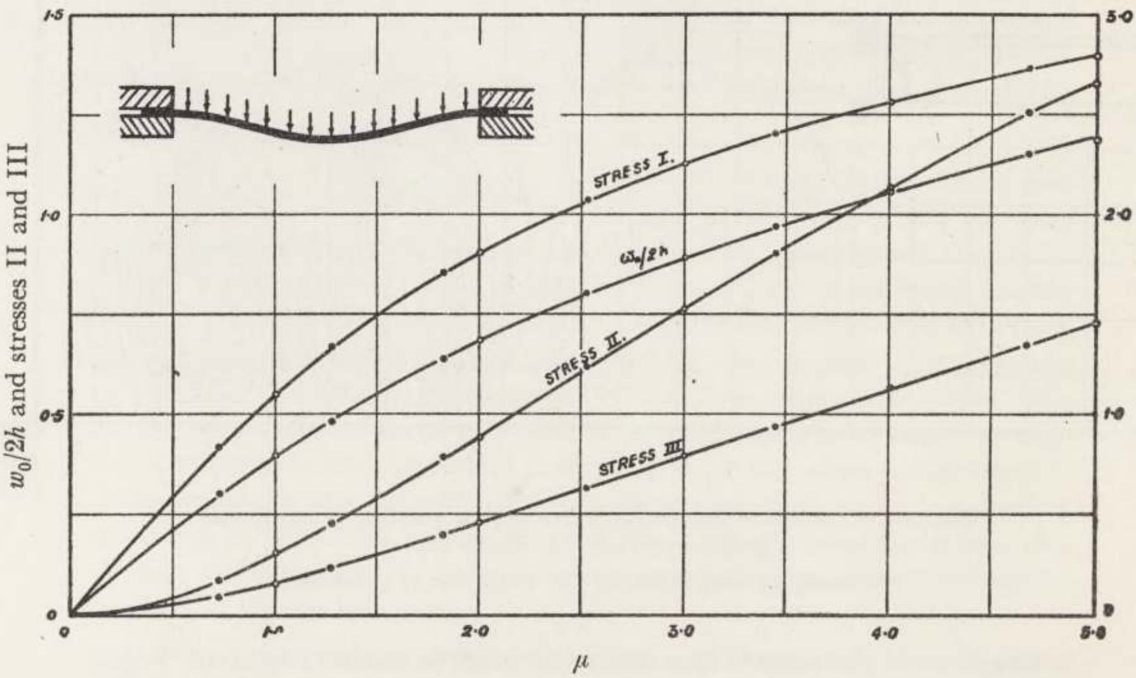


FIGURE 19. Transversely-loaded circular plate: comparison with exact solution by S. Way (Ref. 15, figure 4).

matters now is the order of the agreement between our results (the open circles) and Way's (the points shown black).

20. Clearly the agreement is satisfactory, and equally good agreement is revealed, in figure 20, between our results and those of Friedrichs and Stoker for a circular plate sustaining edge thrust. (The comparison is between our results—the broken curve—and the open circles.) A word should be said regarding the parabolas lettered A, B, C, D, E, because the method used to obtain them (an extension of 'Rayleigh's principle') seems likely to have great value for these difficult problems of 'well-developed buckling'. Having the 'critical load for small deflexions' ($\lambda = 1{\cdot}1$ in figure 20), together with the associated mode, without additional computation beyond a single integration we were able to construct the parabola A, and we could show that *the wanted* (*broken-line*) *curve must lie below this.* Consequently a relatively short exploration (indicated by the points numbered 1, 2, 3, 4, 5 on the relevant ordinate) led to the required solution for $\mu = 2$; having this we could similarly construct the parabola C, thus shortening the exploration for $\mu = 3$; and so on. The resulting (dotted) curve extends well beyond the elastic range of ordinary material.

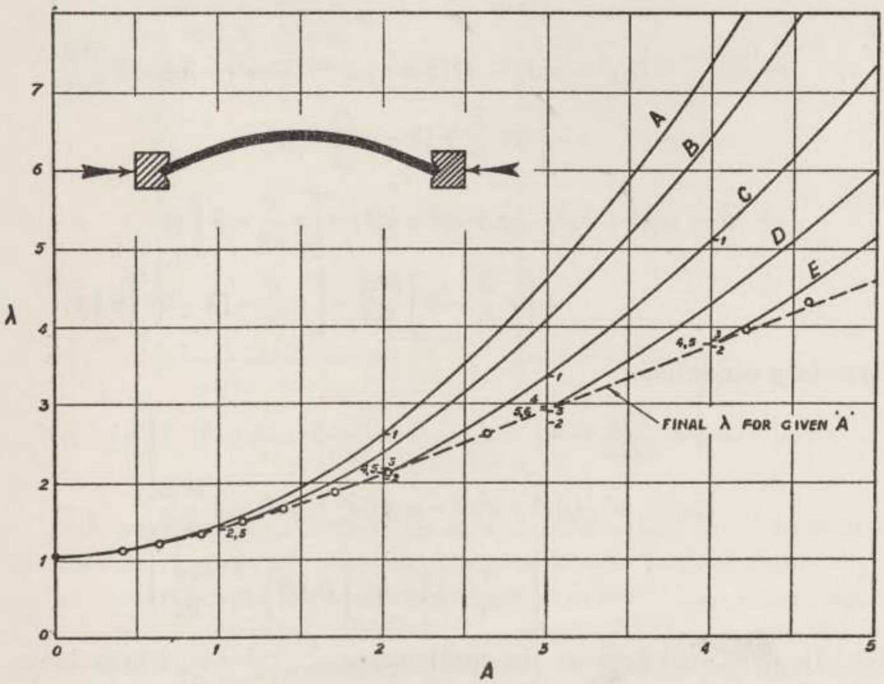


FIGURE 20. Elastic stability of compressed circular plate: comparison with exact solution of Friedrichs and Stoker (Ref. 15, figure 8).

21. Also complicated, although now the governing equations are linear, are problems of stress-calculation for solids of revolution. We have dealt with several in a paper recently communicated, and I show results for the hardest of them—stress-determination for a toroidal hook. Here, strictly treated, not one of the six stress-components is zero; all can be expressed in terms of two related functions ϕ and ψ, and the boundary conditions are *three* in number, therefore hard to satisfy.

Stress-expressions in the hook problem (Ref. 20, equations (61), (75), (76)):

$$\left.\begin{aligned} r^2.\widehat{zz} &= \frac{\partial \phi}{\partial r}, \\ r^2.\widehat{zr} &= \frac{\partial}{\partial z}\left(\frac{1}{r}\frac{\partial \psi}{\partial r} - \phi\right), \\ r^2.\widehat{\theta z} &= \frac{\partial}{\partial z}\left(\left[\frac{\partial^2}{\partial r^2} - \frac{2}{r}\frac{\partial}{\partial r}\right]\psi + \phi\right), \end{aligned}\right\} \tag{7}$$

$$\left.\begin{aligned} r^3.\widehat{rr} = {} & Q(r^2+\sigma z^2) - \sigma A(zr^2 + \tfrac{1}{3}z^3) - \left[r\frac{\partial}{\partial r} - 3\right]\phi - 2\frac{\partial^2\psi}{\partial z^2} \\ & - \left[r\frac{\partial}{\partial r} - (3-\sigma)\right]\vartheta\,{}_3^2\psi, \\ r^3.\widehat{\theta\theta} = {} & -Q(r^2+\sigma z^2) - A\{(2+\sigma)\,zr^2 - \tfrac{1}{3}\sigma z^3\} - 3\phi + 2\frac{\partial^2\psi}{\partial z^2} \\ & - \left[\sigma r\frac{\partial}{\partial r} + (3-\sigma)\right]\vartheta\,{}_3^2\psi, \\ -r^3.\widehat{r\theta} = {} & Q(r^2+\sigma z^2) - \sigma A(zr^2 + \tfrac{1}{3}z^3) - \left[r\frac{\partial}{\partial r} - 3\right]\phi \\ & + \left[r\frac{\partial}{\partial r} - 2\right]\frac{\partial^2\psi}{\partial z^2} - \left[r\frac{\partial}{\partial r} - (3-\sigma)\right]\vartheta\,{}_3^2\psi. \end{aligned}\right\} \tag{8}$$

Boundary conditions:

$$\left.\begin{aligned} & \phi = \frac{\partial^2\psi}{\partial z\,\partial r} = 0, \\ & \cos(r,\nu)\left\{Q(r^2+\sigma z^2) - \sigma A(zr^2 + \tfrac{1}{3}z^3) - 2\frac{\partial^2\psi}{\partial z^2}\right. \\ & \qquad \left. - \left[r\frac{\partial}{\partial r} - (3-\sigma)\right]\vartheta\,{}_3^2\psi\right\} = r\frac{\partial\phi}{\partial\nu}. \end{aligned}\right\} \tag{9}$$

Equations (7)–(9) present the mathematical problem, which looks distinctly 'fierce': by contrast, figure 21 shows how simple may be the implications of a complex mathematical solution. The contours of tensile stress $\widehat{\theta\theta}$, for the heavily loaded section AB, are not straight as assumed in the customary theory, but they are very flat curves. Figure 21 relates to hooks of standard B.S.I. sections, and is based on a paper recently communicated (Ref. 18).

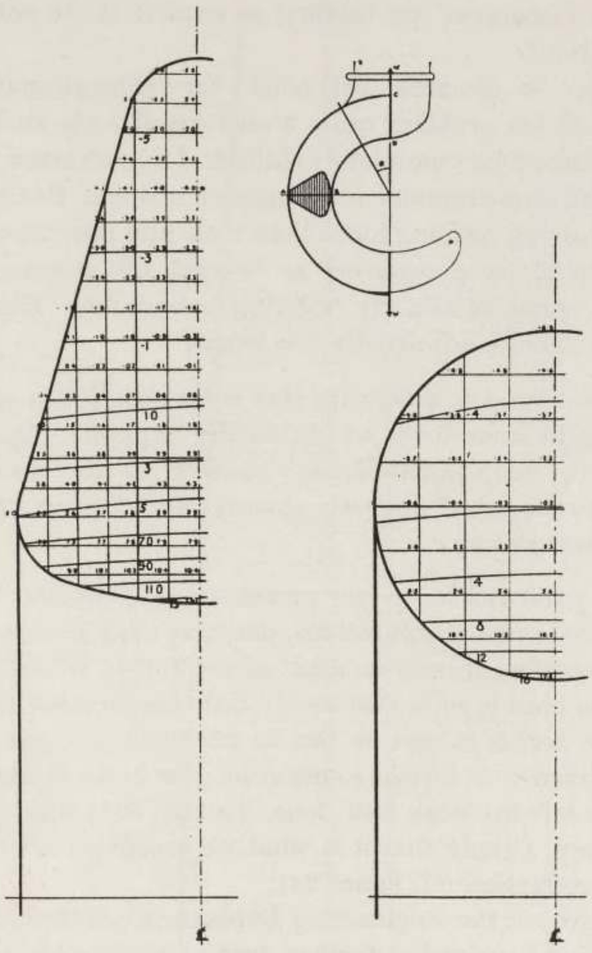


FIGURE 21. Tensile stress-distribution in two B.S.I. standard hooks (Ref. 18, figures 3, 4).

22. This work on large deflexions and on hooks will serve to illustrate yet another advantage of the relaxation method. I have shown by my examples that its errors are made apparent at every stage; I have emphasized its tentative quality, which has enabled us to overcome the difficulties of boundaries not defined initially, and of data characterized by discontinuities like the stress-strain diagram of a plastic material; and I might have mentioned in addition, that it can utilize just as easily experi-

mental data, presented in curves, as data expressible mathematically —e.g. a pressure-density relation found experimentally for steam. My last point is that by its freedom from restriction, and by its tentative quality, it lends itself ideally to *iterative* methods of solution. This, more and more, is a feature we are tending to exploit as we enter the field of *non-linear equations*.

In fact, when he discards 'orthodox' for 'relaxational' methods an investigator finds his problem quite transformed,—the incidence of computational difficulties has completely shifted. I do not mean that they have disappeared: full scope remains for inventive artifice. But he can take for granted (as involving nothing more than time and labour) operations that formerly he would have regarded as beyond his power—e.g. potential solutions for a series of slightly differing boundaries. His tools become more powerful, though admittedly less exact.

23. I have not time to elaborate this point, but I can exemplify it by recent solutions in some fields as yet hardly explored. One such field of study relates to *compressible* fluids, another to the strains in *plastic* material. I have touched on both already, and figures 22 and 23 show some of our latest results.

24. My last point relates to our presentation of results. What have we when we terminate our computations, deeming their accuracy sufficient?

No formal solution, merely a 'grid' of computed values, coarse or fine as our effort has been brief or sustained. But this we must derive from any formal solution before it can be put to practical use: the pure analyst, when having given us a formal solution he goes home to his tea, from our standpoint has left his work half done. To any who object that it is not the whole picture, I reply that it is what we accept without question in a 'half-tone' reproduction (cf. figure 24).

Mr A. N. Black, of the Engineering Department at Oxford, has pursued this line of thought somewhat further, using a microscope to analyse part of a half-tone reproduction taken from a newspaper. Each elemental square was viewed in turn, and an estimate made and recorded of the fraction of its area which was 'black' (i.e. inked); then, from the record of these estimates, a diagram was constructed on squared paper to have the same fractions inked of its corresponding squares. No particular convention was adopted,—the inked fraction might have any shape or position in its square; but viewed from a suitable distance the resulting diagrams (figure 25) reproduce with striking fidelity the general quality of the original photograph. Few 'wanted functions' are likely to have more elaborate features!

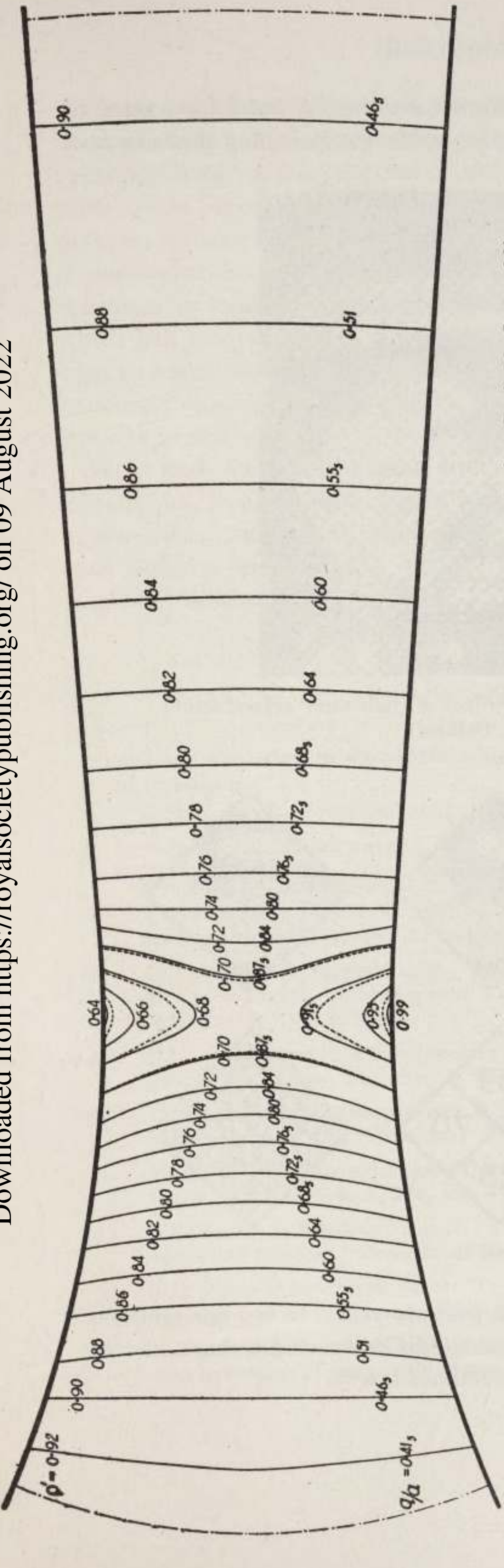


FIGURE 22. High-speed flow of gas through a nozzle: contours of density and velocity (Ref. 16, figure 11).

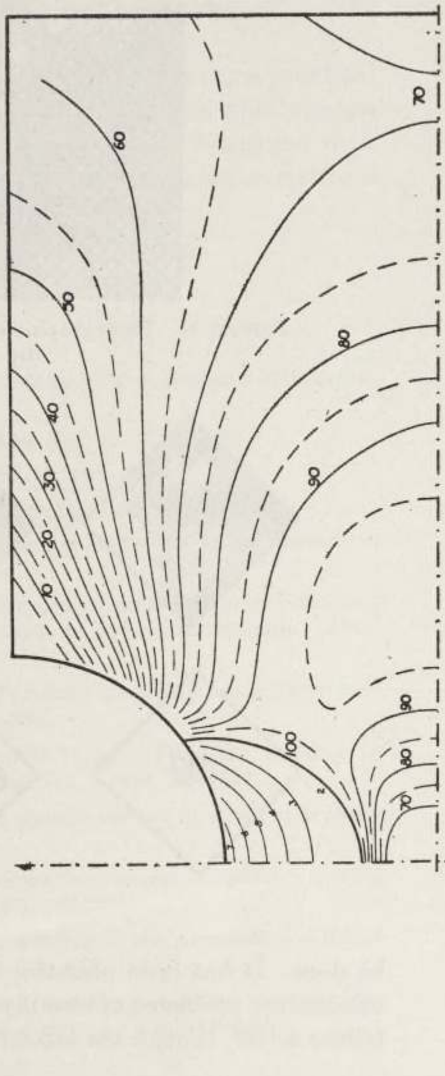


FIGURE 23. Plastic stresses in a notched bar: contours of principal shear stress (Ref. 17).

25. I have been compelled this afternoon to hurry, but I have tried to avoid detail and to give some impression of the kind of thing that can now

FIGURE 24. Photographic enlargement of 'half-tone' reproduction (by Mr V. Belfield).

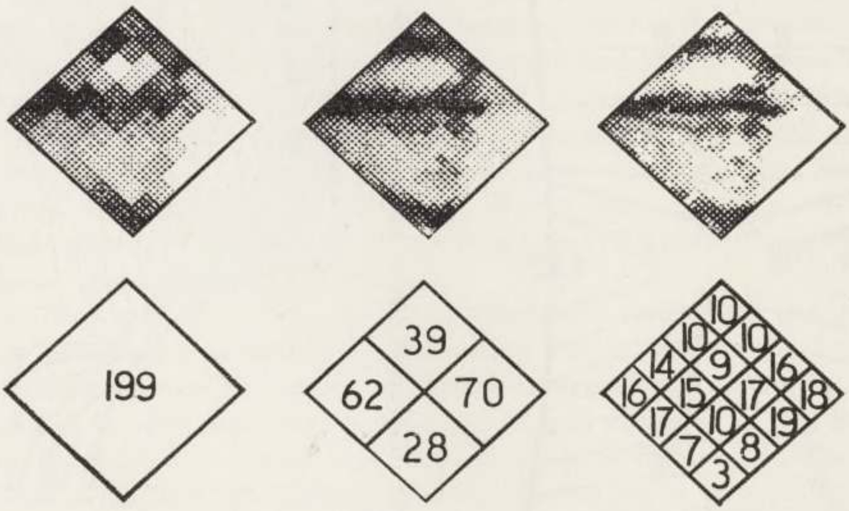


FIGURE 25

be done. It has been pleasant, in the past six years, to try our methods out against problems of steadily increasing difficulty, and to have met no failure so far, though the labour is sometimes heavy.

To exploit them fully—as is my natural desire—a problem should be 'broken down', like any problem of engineering construction, into jobs appropriate to varying degrees of skill. A novice soon acquires power to solve plane-potential problems; with longer experience other types of problem become familiar; and so on, up to the level at which real acumen is demanded for the planning of new research. This of course implies considerable facilities—a mathematical laboratory, or institute, of a kind which has more appeal in America than in this country; but it would be vain to hope for such developments in war-time, and instead I have tried to extend our *range* to the utmost, even at the cost of leaving attractive byways unexplored.

What once we regarded as an alternative method having some practical advantages, in our recent studies we are finding an indispensable weapon of research. I hope that some of my audience, who have heard me with such patience this afternoon, may perhaps see ways in which our methods may help them too.

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* Not yet communicated (July 1945).
† Not yet published (July 1945).