

## BALANCED BLOCK DESIGNS AND GENERALIZED YOUSEN DESIGNS, I. CONSTRUCTION (PATCHWORK)<sup>1</sup>

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The elementary constructive methods for BBD's and GYD's, which have been used in optimality considerations for a number of years, are listed and illustrated. These are not detailed algebraic or geometric prescriptions for listing each entry, but rather methods for combining LS's and known BIBD's to yield the desired products. Nevertheless, there results a large class of useful and previously unpublished designs.

**1. Introduction.** The original Youden square (YS) for  $v$  varieties was a  $k \times v$  array obtained from a BIBD  $(v, b, k, r, \lambda)$  with  $b = v > k$  by considering blocks as columns, arranged to make each variety appear once per row. Generalizations by Shrikhande [9] and Agrawal [1] allowed  $b = mv$  for integral  $v$ . In the simpler setting of one-way heterogeneity, a number of authors have considered "binary" and "ternary" designs which were meant to generalize BIBD's to block size  $k > v$ .

It was noticed in design optimality proofs over the last fifteen years [3], [4], [5], [6], [7] that a restriction like  $k > v$  seemed inessential and sometimes mathematically unnatural (despite the obvious practical motivation), and the BIBD and YS were generalized in [3] to the balanced block design (BBD) and generalized Youden design (GYD), which we now define. In the block design setting with  $v$  varieties and  $b$  blocks of size  $k$ , we again think of a design as a  $k \times b$  array with blocks as columns, and let  $n_{ij}$  be the number of appearances of variety  $i$  in block  $j$ . Write  $r_i = \sum_j n_{ij}$  and  $\lambda_{ih} = \sum_j n_{ij}n_{hj}$ , and let  $\theta$  be the fractional part of  $k/v$ .

**DEFINITION 1.** A BBD is a design with all  $r_i$  equal, all  $\lambda_{ih}$  equal for  $i < h$ , and  $|n_{ij} - k/v| < 1$  for all  $i, j$ .

The last condition is conveniently thought of and described as " $n_{ij}$ 's as nearly equal as possible". Designs satisfying all but this last condition have by now been called "balanced" by some authors because they estimate every difference between two varieties, with the same variance. This nomenclature seems misleading combinatorially (the sets  $\{n_{i1}\}$  and  $\{n_{i2}\}$  may differ), and without the last condition of Definition 1 there is no relationship to optimality. We also use the notation  $(v, b, k, r, \lambda)$  or  $(v, b, k)$  for a BBD, with all  $r_i = r = bk/v$  and all  $\lambda_{ih} = \lambda$  in the above definition. The usual counting argument shows that

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$\lambda = b[k^2(v-1) - v^2\theta(1-\theta)]/v^2(v-1)$ . Other formulas and resulting divisibility restrictions are detailed in [8].

**DEFINITION 2.** A  $b_1 \times b_2$  array of the symbols  $1, 2, \dots, v$  is a GYD if it is a BBD when each of {rows} and {columns} is considered as blocks.

During the author's investigation of optimality properties of GYD's, designs of this type were constructed by piecing together other designs—what we shall call “patchwork methods”. (Example 4.4 discusses a GYD of [3] not of this type.) The resulting designs were referred to when needed, but the methods were never published. Recently, Ruiz and Seiden [8] have given elegant geometric constructions of GYD's for certain parameter values involving prime powers. Since the patchwork techniques include additional parameter values which are of practical value and which are required in the optimality considerations [7], and since these techniques are quick to use, have not appeared in the literature, and use ideas which may be applicable in other design constructions, it seems worthwhile to list them in the present note; optimality considerations, which use some of the present results but contain mostly quite different ideas, will appear in the sequel.

It will be seen that the basic techniques are elementary and largely obvious. Nevertheless, if methodically applied they yield large families of previously unpublished designs of practical size.

We require some further definitions.

As usual, we use  $m | n$  to mean that  $n/m$  is an integer.

**DEFINITION 3.** A GYD is *regular* if  $v | b_1$  or  $v | b_2$ .

**DEFINITION 4.** A BIBD is *partly resolvable* (PR) if  $k | v$  and there are  $v/k$  blocks whose union contains each variety once.

This last clearly generalizes resolvability. There is an extension to BIBD's or BBD's in which  $tv/k$  blocks contain  $t$  appearances of each variety, which for our application must be suitably balanced in a manner which will be indicated in connection with Proposition 6.

We denote by  $r \times s$  LS( $v$ ) an  $rv \times sv$  array formed from  $r \times s$  latin squares of order  $v$ . By  $\text{int}\{x\}$  we denote the greatest integer  $\leq x$ .

The author is indebted to Esther Seiden for many helpful discussions.

**2. Regular GYD's.** We shall see that no new methods are required in the regular case. The first proposition is evident:

**PROPOSITION 1.** A BBD, after rearrangement in columns, is the union of  $b$  complete blocks in which each treatment appears  $\text{int}\{k/v\}$  times, and a BIBD.

Thus, necessary for the existence of a BBD with given parameters is existence of an associated BIBD with corresponding parameters.

The methods used to make YS's from BIBD's, as devised by Hartley and Smith [2] or Shrikhande [9] and Agrawal [1] (using also systems of distinct

representatives) can be extended to certain BBD's, as described in [4]. We state this along with a variant used later:

**PROPOSITION 2.** (i) *A BBD with  $v|b$  can be made into a  $k \times b$  GYD by rearrangement within columns (blocks).*

(ii) *If  $v|kb$ , any  $k \times b$  array (not necessarily a BBD) with equal replications of varieties can be rearranged in columns so that each variety occurs as nearly equally as possible per row (and hence equally if  $v|b$ ).*

The above simple rearrangement schemes can fail to yield a GYD if  $v \nmid b$ , even if one starts with a BBD [4]. In a sense, the nonregular GYD construction problem can be thought of as that of finding when a BBD with  $v \nmid b, k$  can be suitably rearranged within columns. But this point of view has not yet yielded any useful generalization of the methods of [2], [9], [1].

By applying Fisher's result to the BIBD of Proposition 1, we obtain

**PROPOSITION 3.** *A BBD with  $b < v$  must have  $v|k$ .*

Hence, the only nonregular GYD's have  $b_1, b_2 > v$ .

Thus, for given  $b_1, b_2, v$ , the existence of a regular GYD is by Propositions 1 and 2 equivalent to that of a corresponding BIBD; when such a BIBD is known, these propositions yield a method of constructing the desired GYD. Proposition 3 delimits the additional cases we must study.

**3. Patchwork methods.** In what follows,  $t, a_i, b_i$  and  $c_i$  denote positive integers. When  $v$  is understood from the context, we shall often describe a GYD by its  $b_i$  alone. The simplest patchwork joins a regular GYD to a general GYD in obvious fashion:

**PROPOSITION 4.** *The union of rows of an  $a_1v \times b_2$  GYD and a  $c_1 \times b_2$  GYD yield an  $(a_1v + c_1) \times b_2$  GYD.*

This is used to yield new designs when the  $c_1 \times b_2$  GYD is nonregular. The conclusion of Proposition 4 is not generally true if the  $a_1v \times b_2$  GYD is replaced by a nonregular one.

In view of Proposition 1, a necessary condition for existence of a GYD with  $b_i = a_iv + c_i$  is existence of the two BIBD's  $B_i = (v, b_i, c_{3-i})$ . The next two propositions are our main patchwork methods, which impose additional conditions on the  $B_i$  to obtain GYD's. (We also illustrate, in Example 4.4, that not all GYD's can be constructed by these methods.)

**PROPOSITION 5.** *Suppose there are PRBIBD's with parameters  $(v, a_1v + c_1, c_2)$  and  $(v, a_2v + c_2, c_1)$  where  $c_1c_2 = v$ . Then there is an  $(a_1v + c_1) \times (a_2v + c_2)$  GYD.*

*Construction.* We label the required array

$$G = \begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix}.$$

Here  $G_{11}$  is any  $a_1v \times a_2v$  GYD, for example (but not necessarily)  $a_1 \times a_2$  LS( $v$ ). Next,  $(G_{21}, G_{22})$  is the PRBIBD with block size  $c_1$ , the  $c_2$  blocks which contain

each treatment once being  $G_{22}$ . After a renumbering of varieties in the other PRBIBD, to make the  $c_1$  blocks containing each treatment once be  $G'_{22}$ , that PRBIBD is taken to be  $(G'_{12}, G'_{22})$ . Finally, the columns of  $G_{21}$  and of  $G'_{12}$  are rearranged as described in the second part of Proposition 2 so that each row of  $G_{21}$  contains each variety  $a_2$  times, and similarly for  $G'_{12}$ .

The dual role of  $G_{22}$  in the above construction makes clear a generalization of the notion of partial resolvability of the two designs used above: We require  $c_1 c_2 = tv$  and the existence of BBD's  $(v, a_i v + c_i, c_{3-i})$  as before (no longer necessarily BIBD's), but with  $G_{22}$  containing each variety  $t$  times, having columns which are  $c_2$  of the blocks of the first BBD and rows which are  $c_1$  of the blocks of the other. It is no longer automatic, as it was when  $t = 1$ , that existence in these BBD's of  $c_{3-i}$  blocks of size  $c_i$  containing each variety  $t$  times suffices to permit a renumbering of varieties which allows the two BIBD's to fit consistently in  $G_{22}$ . Illustrations of how this generalization works are contained in Section 4. We now state the principle formally, and then some simple sufficient conditions which will be useful for large  $b_i$ . We denote the columns of  $(G_{21}, G_{22})$  by  $B_2$ , and those of  $(G'_{12}, G'_{22})$  by  $B_1$ .

**PROPOSITION 6.** *Assume  $v | c_1 c_2$ . Suppose (i) there are BBD's  $B_i = (v, b_i, c_{3-i})$  with  $v | b_i - c_i$  and (ii) there are  $c_2$  blocks of  $B_2$ , whose union contains each variety exactly  $t$  times, and such that (iii) the  $c_1 \times c_2$  array formed with these blocks as columns has rows which are blocks of  $B_1$ . Then there is a  $b_1 \times b_2$  GYD. Moreover, (iii) is satisfied if (ii) is satisfied and (iv)  $B_1$  is composed of all the blocks of at least  $c_1$  BBD's; and (ii) and (iii) are satisfied if, in addition to (iv),  $B_2$  is composed of all the blocks of at least  $c_2$  BBD's.*

*Construction.* The GYD is constructed from (i), (ii), (iii) as in Proposition 5. Next, assuming (ii), rearrange within the columns of  $G_{22}$  so that varieties are as nearly equally replicated as possible in each row (Proposition 2). Each of these  $c_1$  rows can be taken as a block of a different (relabelled) component BBD of  $B_1$ , proving that (iv) implies (iii). If also  $B_2$  has  $c_2$  component BBD's this device also produces the columns of  $G_{22}$ .

The number of component BBD's needed for  $B_2$  to satisfy (ii) can often be reduced from  $c_2$  without much knowledge of  $B_2$ . For example, if  $c_2$  is even,  $B_2$  contains at least  $c_2/2$  copies of the same sub-BIBD, and one knows two blocks of the latter with an even number of common varieties, that clearly suffices.

There are various BIBD operations which can be extended to BBD's, such as *derivation* and *residuation*. Because of Proposition 1, these yield nothing new for BBD's, and additional argument is needed to yield new GYD's. (It is well known that certain operations, such as identification of varieties in sets of  $q$  to form a design with  $v/q$  varieties, do not work at all except for very special parameter values.) As an illustration of the type of additional argument that is needed to obtain new GYD's, after an obvious consideration of the operation of *complementation*, we then give an application of it (Proposition 8).

**PROPOSITION 7.** *If a  $b_1 \times a_2 v$  GYD  $\bar{G}$  contains a  $b_1 \times b_2$  GYD  $G$  as a rectangular subarray, then the complement  $\bar{G} - G$  of the latter in the former is a GYD.*

If the  $a_2 v$  is replaced by a number not divisible by  $v$ , the above conclusion is generally false.

The operation of complementation is in a sense the opposite of that of union, used in Proposition 4. Similarly, there is an opposite of the operation of Proposition 5, consisting of the removal of  $G_{21}$ ,  $G_{22}$ ,  $G_{12}$  from an  $a_1 v \times a_2 v$  GYD  $G$ , under appropriate conditions.

It is not clear which GYD parameter values are obtainable from Proposition 7 but not from Propositions 4, 5, and 6, but the use of Proposition 7 sometimes avoids the more complex verifications needed in using Proposition 6.

**PROPOSITION 8.** *Suppose  $v = c_1 c_2$  and that there exist*

- (i) BIBD  $(v, a_3 v - c_2, c_1)$ ,
- (ii) PRBIBD  $(v, a_2 v + c_2, c_1)$ ,
- (iii) PRBIBD  $(v, a_1 v + c_1, c_2)$ .

*Then there is an  $(a_1 v + c_1) \times (a_3 v - c_2)$  GYD.*

*Construction.* Let  $G$  be the  $(a_1 v + c_1) \times (a_2 + c_2 v)$  GYD obtained by the method of Proposition 5; we shall use the same notation for the  $G_{ij}$ . Since each variety occurs once in  $G_{22}$ , so that rearrangement of its columns amounts merely to relabeling varieties, it follows from the second part of Proposition 2 that a reordering of the columns of the BIBD (i) can be used to produce an array  $G_{23}$  such that each of the rows of the  $c_1 \times a_3 v$  array  $(G_{22}, G_{23})$  has equal replication of each variety. Let  $H$  be an  $a_1 v \times (v - c_2)$  array whose rows are complements in  $\{1, 2, \dots, v\}$  of the corresponding rows of  $G_{12}$ . By the second part of Proposition 2, we can reorder within the rows of  $H$  so that each of the columns of  $H$  has equal replication of each variety. Let  $G_{13}$  be the union of the columns of this reordered  $H$  with those of an  $a_1 v \times (a_3 - 1)v$  GYD (or with nothing, if  $a_3 = 1$ ). Then  $\bar{G} = [G, \begin{pmatrix} G_{13} \\ G_{23} \end{pmatrix}]$  is seen to be a regular GYD, and hence (Proposition 7) so is  $\begin{pmatrix} G_{13} \\ G_{23} \end{pmatrix}$ .

Proposition 8 can be extended using Proposition 6.

We note that a useful condition implying (i) and (ii) is existence of a PRBIBD  $(v, b, c_1)$  with  $b \mid a_3 v - c_2, a_2 v + c_2$ .

Note that the construction cannot be simplified to mere application of Proposition 6 to  $\begin{pmatrix} G_{12} & G_{13} \\ G_{22} & G_{23} \end{pmatrix}$  in place of  $\bar{G}$ ; for,  $\begin{pmatrix} G_{12} \\ G_{22} \end{pmatrix}$  will not be a GYD unless  $c_1 = v$  (Proposition 3).

**4. Examples of nonregular GYD's.** We do not attempt an exhaustive list of GYD's, but rather illustrate the patchwork methods in a representative selection of cases, including ones which have turned out to be important in optimality considerations [7]. Designs for many of the parameter values we consider can be obtained by using any of several methods (Propositions 4–8), but we give only one construction here in each case.

In optimality considerations, the values of  $\theta_i =$  fractional part of  $b_i/v$  are important. In a sense, the cases where  $\theta_i = \frac{1}{2}$  represent maximum departure of a GYD from the regularity of  $a_1 \times a_2 LS(v)$ , and the best chance for a GYD not to be optimum.

In these examples  $t, q, J_i,$  and  $J_i'$  will be positive integral parameters of design series.  $J_i'$  generally represents the number of replications of a sub-BIBD making up  $B_i,$  and is expressed in terms of the unrestricted  $J_i.$

EXAMPLE 4.1. *The  $\theta_1 = \theta_2 = \frac{1}{2}$  series.* It is easily seen that these designs can be represented in terms of three integer parameters  $t > 0, J_1 > 0, J_2 > 0,$  with

$$(4.1) \quad v = 4t, \quad b_i = 2t(4t - 1)(2J_i - 1).$$

The  $J_i'$  defined above will be seen to be  $(2J_i - 1)t.$

EXAMPLE 4.1.1. *The  $t = 1$  series.* A  $6(2J_1 - 1) \times 6(2J_2 - 1)$  GYD with  $v = 4$  is obtained by using Proposition 5 with  $(G_{21}, G_{22})$  equal to  $(2J_2 - 1)$  copies of the  $(4, 6, 2)$  BIBD and  $(G'_{12}, G'_{22})$  equal to  $(2J_1 - 1)$  copies.

EXAMPLE 4.1.2. *General considerations for  $t > 1.$*  We must now use Proposition 6, since each variety occurs  $t$  times in  $G_{22}(2t \times 2t).$  The required BIBD's are most easily obtained as  $(2J_1 - 1)t$  replicates of the BIBD

$$(4.2) \quad (4t, 2(4t - 1), 2t, 4t - 1, 2t - 1),$$

if it exists. Professor Seiden points out that this is always the case if an  $8t \times 8t$  Hadamard matrix exists.

PROPOSITION 9. *For  $t > 1,$  if either  $J_i \geq 2$  and the BIBD (4.2) exists, then the GYD exists.*

*Construction.* Suppose (4.2) exists and  $J_1 \geq 2,$  so that  $(2J_1 - 1)t \geq c_1.$  Then (iv) of Proposition 6 is satisfied, and we need only verify (ii). In the design (4.2), the number  $\binom{8t-2}{2}$  of pairs of different blocks is odd, and the sum of all block intersection numbers is  $4t\binom{4t-1}{2},$  which is even. Hence, at least one intersection number is even, and the corresponding pair of blocks can be used as described in the paragraph following "Construction" of Proposition 6, since  $(2J_2 - 1)t \geq c_2/2.$

In the remainder of the discussion of the  $\theta_1 = \theta_2 = \frac{1}{2}$  series, assuming (4.2) exists, it is thus only necessary to consider the values  $J_1 = J_2 = 1.$

EXAMPLE 4.1.3. *The case  $t = 2.$*  Here there is a resolvable design (4.2) which easily yields

$$G_{22} = \left\| \begin{array}{cccc} \infty & 6 & 3 & 4 \\ 0 & 2 & 5 & 1 \\ 1 & 5 & 2 & 0 \\ 3 & 4 & \infty & 6 \end{array} \right\|$$

as a  $G_{22}$  whose rows and columns are blocks of (4.2); one does not even need the 2 copies of (4.2) available to us, in obtaining  $G_{22}.$

EXAMPLE 4.1.4. *The case  $t = 3$ .* A resolvable design (4.2) can be developed mod 11 from the blocks  $\beta_1 = (1, 2, 4, 5, 6, 10)$  and  $\beta_2 = (3, 7, 8, 9, 0, \infty)$ . There are many possible constructions of  $G_{22}$ . For example, to get equal replication numbers, add 0, 1 and 3 to  $\beta_1$  and 0, 1, 3 to  $\beta_2$ . The resulting  $G_{22}$ , whose columns are  $\beta_2 + 2, 4, 7$  and  $\beta_1 + 2, 4, 7$ , is

$$G_{22} = \begin{vmatrix} 2 & 4 & 10 & 1 & 5 & 6 \\ 0 & 7 & 5 & 3 & 6 & 2 \\ 5 & 2 & 7 & 4 & 8 & 9 \\ 9 & 0 & \infty & 7 & 3 & 8 \\ 10 & \infty & 4 & 8 & 9 & 1 \\ \infty & 1 & 3 & 6 & 10 & 0 \end{vmatrix}$$

The author is again grateful to Esther Seiden, this time for pointing out the existence of this resolvable design on Preece's list [7a]; the more common non-resolvable design on most BIBD lists had not led to a successful construction.

EXAMPLE 4.1.5. *The case  $t = 4$ .* There is a resolvable design (4.2), and  $G_{22}$  is easily constructed in a manner similar to that used in Example 4.1.3.

The above examples give constructions of the  $\theta_1 = \theta_2 = \frac{1}{2}$  series for all  $v \leq 16$ , which probably includes most "practical" cases.

EXAMPLE 4.2. *The series  $v = c_1 c_2, b_i = a_i v + c_i$ .* To avoid trivialities, we assume  $c_1 \geq 2, c_2 \geq 3$ ; the case  $c_1 = c_2 = 2$  falls under Example 4.1.1. We first consider a subseries for which the calculations are particularly simple.

EXAMPLE 4.2.1. *Assume each  $c_i - 1$  relatively prime to  $c_1 c_2 - 1$*  (always true if  $c_1 = 2$ ). For a PRBIBD  $(G_{21}, G_{22})(v, b', k', r', \lambda')$  to exist, we need  $r' = \lambda'(v - 1)/(k' - 1) = \lambda'(c_1 c_2 - 1)/(c_1 - 1)$ ; hence,  $r'$  is of the form  $J_2'(c_1 c_2 - 1)$  and  $b' = vr'/k' = c_2 J_2'(c_1 c_2 - 1)$ . Also, in order to use Proposition 5 we require  $v | b' - c_2$ , which yields  $J_2' = J_2 c_1 - 1$ . Thus, finally, the parameters of  $(G_{21}, G_{22})$  are necessarily of the form

$$(4.3) \quad (c_1 c_2, c_2(J_2 c_1 - 1)(c_1 c_2 - 1), c_1, (J_2 c_1 - 1)(c_1 c_2 - 1), (J_2 c_1 - 1)(c_1 - 1))$$

with  $J_2 > 0$ , and  $a_2 = J_2 c_1 c_2 - c_2 - J_2$ . Interchanging subscripts 1 and 2 in (4.3), we obtain the form (4.3)' (say) of  $(G'_{21}, G'_{22})$ .

The design (4.3) can be obtained as  $J_2 c_1 - 1$  copies of a  $(c_1 c_2, c_2(c_1 c_2 - 1), c_1)$  BIBD if the latter exists, and similarly for (4.3)'. One must still check that PR designs result.

If

$$(4.4) \quad J_2 c_1 - 1 \geq c_2 \quad \text{and} \quad J_1 c_2 - 1 \geq c_1,$$

we can use the tool of Proposition 6(iv) in this simpler  $t = 1$  setting: There are enough copies of  $(c_1 c_2, c_2(c_1 c_2 - 1), c_1)$  to take a relabeled block from each of  $c_2$  different copies, as the columns of  $G_{22}$ ; similarly for rows. Thus, a GYD always exists under (4.4), if the two BIBD's  $(c_1 c_2, c_i(c_1 c_2 - 1), c_{3-i})$  exist.

If one (or both) of the inequalities (4.4) is violated but the corresponding design(s) (4.3) or (and) (4.3)' is PR, of course we still obtain a GYD. This is the case when  $c_1 = 2$ , since the BIBD  $(2c_2, c_2(2c_2 - 1), 2)$  is resolvable and  $J_1c_2 - 1 \geq 2$  for  $c_2 \geq 3$ . Thus, we need only check the existence of the BIBD  $(2c_2, 2(2c_2 - 1), c_2)$  to know that a GYD exists for all  $J_i > 0$ , if  $c_1 = 2$ . As in Example 4.12, if a  $4c_2 \times 4c_2$  Hadamard matrix exists, so does this BIBD.

If  $c_1 = 3$  and  $c_2$  is even, we are still in the framework of Example 4.2.1. On the other hand, if  $c_1 = 3$  and  $c_2$  is odd, we obtain an illustration of the fact that the present example gives sufficient conditions for the existence of a design even if the  $c_i - 1$  are not both relatively prime to  $c_1c_2 - 1$ , but there may exist GYD's for other parameter values. Rather than attempt an exhaustive study of the relevant divisibility considerations for general  $c_1, c_2$ , we treat only the cited illustration:

EXAMPLE 4.2.2. The series  $c_1 = 3, c_2 = 2q + 1$ . The possible values of  $r'$  in the development of Example 4.2.1 (and, hence, of  $b'$  and  $\lambda'$ ) are seen to be multiplied by  $\frac{1}{2}$  in the present setting, for both  $(G_{21}, G_{22})$  and  $(G'_{12}, G'_{22})$ . In the subsequent treatment of the requirement  $v \mid b' - c_i$ , we obtain that any even  $J'_i$  is of the form  $2(J_i c_{3-i} - 1)$ . The parameter values of (4.3) and (4.3)' are unchanged, but one tries to achieve them as  $2(3J_2 - 1)$  copies of  $(6q + 3, (2q + 1)(3q + 1), 3)$  and  $2[(2q + 1)J_1 - 1]$  copies of  $(6q + 3, 3(3q + 1), 2q + 1)$ . If these latter BIBD's exist, we see that there are indeed some GYD's which were obtainable as indicated in italics in the previous paragraph.

If  $J'_i$  is odd, we obtain that it is of the form  $c_{3-i}(2J_i - 1) - 2$ . We are led to seek  $6J_2 - 5$  copies of  $(6q + 3, (2q + 1)(3q + 1), 3)$  and  $2(2q + 1)J_1 - (2q + 3)$  copies of  $(6q + 3, 3(3q + 1), 2q + 1)$ . Of course, the  $J'_i$  can have opposite parity.

The analogue of (4.4) is

$$(4.5) \quad \begin{array}{ll} J'_i \text{ even:} & 2(c_{3-i}J_i - 1) \geq c_i; \\ J'_i \text{ odd:} & c_{3-i}(2J_i - 1) \geq 2 + c_i. \end{array}$$

For  $i = 1$  and  $q > 1$ , these inequalities are valid for all  $J_1$ . For  $q = 1$ , we have  $c_1 = c_2$  and the same problem in both directions. Thus, as in the paragraph following (4.4) (with obvious modifications), we are led to seek a PRBIBD  $(6q + 3, (3q + 1)(2q + 1), 3)$ .

This is the resolvable Steiner triple design, which is known to exist for all  $q > 0$ . We conclude:

*If the BIBD  $(6q + 3, 3(3q + 1), 2q + 1)$  exists, then a GYD exists for all  $J_1, J_2$  (with  $J'_i$  of both odd and even forms).*

For example, for  $q = 1$  we obtain the  $v = 9, \theta_1 = \theta_2 = \frac{1}{3}$  series for  $b_i$  of the form  $12(6J_i - 2)$  or  $12(6J_i - 5)$ ; that is, of the form  $12(3J_i - 2)$ . For  $q = 2$ , the design  $(15, 21, 5)$  does not exist, and one would have to try to work with a  $(15, 21h, 5)$  design for some  $h > 1$ ; such a modification of our treatment can in general yield impractical parameter values. When  $q = 3$  or  $4$ , the required BIBD again exists.



EXAMPLE 4.3. *The series  $tv = c_1c_2$ ,  $t > 1$ . This extends Example 4.2 in the same way that Examples 4.1.2—4.1.5 extended 4.1.1. We must again use Proposition 6. We illustrate with an extension for the  $v$  of Example 4.2.2 with  $t = 2$ :*

EXAMPLE 4.3.1. *The series  $v = 3(2q + 1)$ ,  $c_1 = 3$ ,  $c_2 = 2(2q + 1)$ . We now obtain, for  $(G_{21}, G_{22})$ , that  $r' = (3q + 1)J_2'$  and  $J_2' = 3J_2 - 1$ . Similarly, for  $(G'_{12}, G'_{22})$ , we obtain  $r' = (6q + 2)J_1'$  and  $J_1' = (2q + 1)J_1 - 2$ . We thus try to apply the technique of Proposition 6 to  $J_2'$  copies of  $(6q + 3, (2q + 1)(3q + 1), 3)$  and  $J_1'$  copies of  $(6q + 3, 3(3q + 1), 2(2q + 1))$ .*

For  $q > 1$  or  $J_1 > 1$ , we have  $J_1' \geq 3$  and thus the rows of  $G_{22}$  can be arbitrary (provided no variety occurs more than once per row and each variety occurs twice). Since  $J_2' \geq 2$ , we have at least two Steiner triples making up  $(G_{21}, G_{22})$ , so we can choose the first  $2q + 1$  columns of  $G_{22}$  from one Steiner triple so as to contain each variety once, and the last  $2q + 1$  columns from cyclic permutation of the rows of the first  $2q + 1$  columns.

There remains the case  $q = 1, J_1 = 1$ . The prescription of the previous paragraph then yields three rows of  $G_{22}$  which can be taken as blocks of the BIBD  $(9, 12, 6)$  (obtained as complements of the blocks of  $(9, 12, 3)$ ). We conclude:

*If the BIBD  $(6q + 3, 3(3q + 1), 2(2q + 1))$  exists, then there is a  $3[(2q + 1)J_1 - 2](3q + 1) \times (3J_2 - 1)(2q + 1)(3q + 1)$  GYD with  $v = 6q + 3$ , for all positive  $J_i$ .*

EXAMPLE 4.4. *Nonisomorphic GYD's.* These can of course occur when nonisomorphic BIBD's with the same parameter values exist and are used in our construction. More interesting is the existence of GYD's which are nonisomorphic because one is obtained from one of our patchwork methods and the other cannot be so obtained. For example, when  $v = 4$ , the  $6 \times 6$  GYD of [3], [6],

$$(4.6) \begin{matrix} 1 & 4 & 2 & 4 & 3 & 2 \\ 2 & 1 & 4 & 3 & 3 & 4 \\ 2 & 3 & 1 & 3 & 4 & 2 \\ 1 & 3 & 3 & 1 & 2 & 4 \\ 4 & 1 & 4 & 2 & 1 & 3 \\ 3 & 2 & 1 & 4 & 2 & 1 \end{matrix}$$

has no subarray of 4 rows and columns which constitute a LS(4), since any such  $4 \times 4$  array has at least one row or column with zero or two one's. Hence, this design is not isomorphic to (obtainable by row or column permutations or relabeling from) that of Example 4.1.1 with  $J_1 = J_2 = 1$ .

It appears that the methods of [8] are likely to yield designs of the form (4.6). Since all GYD's with the same  $v, k_1, k_2$  have the same covariance matrix for variety contrasts, optimality properties of GYD's cannot vary between two nonisomorphic designs. However, our preliminary investigations indicate that counterexamples to optimality, where they exist, may be more easily obtainable from slight modification of one of two such designs [6], [7], [8].

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