TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 149, May 1970

BALANCED RINGS AND A PROBLEM OF THRALL(1)

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Abstract. Balanced ring is defined and related to Thrall's QF-1 rings. Several theorems are obtained which show that balanced rings enjoy strong homological and chain conditions. The structure of commutative balanced rings is determined. Also, the structure of commutative artinian QF-1 rings is gotten. This is a generalization of a theorem of Floyd.

Introduction. If M is a right R-module, then M is a natural left module over its endomorphism ring S. We call $T=\text{End}_S M$ the BiEndomorphism ring of M, and the elements of T are called BiEndomorphisms (notation BiEnd M). The mapping:

$$\eta\colon R \to \text{BiEnd } M_R, \qquad \eta\colon a \to a_d, \quad \text{where } (x)a_d = xa, \ \forall x \in M$$

is a ring homomorphism and $\ker \eta$ is the annihilator of M (notation $\ker \eta = \text{ann}_R M$). The elements of BiEnd M_R of the form a_d are called right multiplications of M. Every element of BiEnd M_R is a right multiplication iff the natural map $\eta\colon R \to \text{BiEnd } M$ is surjective, that is, a ring epimorphism. In this case, following Faith [2], we say M is balanced. If M is balanced, we have a complete description of BiEnd M_R, namely BiEnd $M_R \approx R/\text{ann}_R M$. Naturally, this is not always the case, as is well known. In this paper we study rings for which every right R-module is balanced, and call R right balanced in this case.

It appears that balanced rings have not been studied in this generality. Thrall [13] proposed the classification of finite dimensional algebras, called QF-1 algebras, having the property that every finitely generated faithful right R-module is balanced. The problem remains unsolved at the present, but there are results in special cases (Floyd [5], Fuller [7] and Morita [9]).

The point of departure of this paper, and the idea which led to our main results is the observation that first, the QF-1 hypothesis, when assumed for general rings and their quotients, actually implies chain conditions, and second that the determination of BiEnd M_R is what we want, for a general R-module M, not merely

Received by the editors September 8, 1969.

AMS Subject Classifications. Primary 1340, 1350.

Key Words and Phrases. Ring, quasi-Frobenius, perfect ring, BiEndomorphism, double centralizer.

(1) This paper is extracted from the author's doctoral dissertion written under the direction of Professor Carl Faith. The author would like to acknowledge Professor Faith's guidance and his warm and continuing friendship.

(2) This research was supported by the National Science Foundation during the author's tenure as an NSF trainee at Rutgers University.

the faithful ones. If we generalize Thrall's problem to an arbitrary ring R, and call R right QF-1 if every faithful right R-module is balanced, then our work aims at the classification of the rings R such that R/I is QF-1 for every factor ring of R. In this case, R is said to be balanced.

Some of our main results are:

16. THEOREM. *If R is balanced, then*:
1. rad R *is nil.*
2. *R is semiperfect.*

17. THEOREM. *If R is balanced and noetherian then R is artinian.*

20. THEOREM. *If R is balanced and commutative then R is artinian.*

21. THEOREM. *If R is a commutative artinian ring then the following three conditions are equivalent*:

A. *Every finitely generated faithful module is balanced.*
B. *Every faithful module is balanced.*
C. *R is QF (quasi-Frobenius).*

22. THEOREM. *If R is commutative then the following three statements are equivalent*:

A. *R is balanced.*
B. *R is a product of a finite number of local artinian principal ideal rings.*
C. *R/I is QF-*1 *for every ideal I.*

Theorem 21 is closely related to a theorem of Floyd who proved that a commutative finite dimensional algebra R is QF-1 (in the sense of Thrall) if and only if R is QF. Theorem 21 has also been independently proved by Fuller and Dickson, using different techniques. The author was required to use the hypothesis that $1/2 \in R$ until he learned of the result in its full generality from Fuller. He then saw that his own techniques also gave the complete theorem.

The artinian hypothesis is important in 21, since as Faith has observed, every finitely generated abelian group is balanced.

The notions of generator and QF ring are important in the study of QF-1 rings. This is true because every generator is balanced [3], and over a QF ring every faithful module is a generator [14], thus every QF ring is QF-1 [13].

I. In this section we obtain some preliminary results. Rings are commutative, noncommutative, balanced, or QF-1, as indicated. All R-modules are right R-modules. We abbreviate $M^{\perp} = \mathrm{ann}_R\, M$, and if $m \in M$, $m^{\perp} = \{r \in R \mid mr = 0\}$.

1. LEMMA. *Let $R = \prod_{i \in I} R_i$, where the R_i are rings, and I is an infinite set. Let $S = \sum_{i \in I} \oplus R_i$ and let $S \subset M \subset R$, where M is a maximal right ideal of R. Then,* $\mathrm{Hom}_R(R/M, S) = 0$ *and* $\mathrm{Hom}_R(S, R/M) = 0$.

Proof. This is immediate from the fact that S is a two-sided ideal such that $S^2 = S$ and $(R/M)S = 0$.

The next lemma is one we use repeatedly in the text.

2. LEMMA. *If M is a faithful R-module, N arbitrary and not zero, and if $M \oplus N$ is balanced then either there is a nonzero $f\colon M \to N$ or a nonzero $f\colon N \to M$.*

Proof. If there are no such maps then $\operatorname{End}_R(M \oplus N) = \operatorname{End}_R M \times \operatorname{End}_R N$. This implies that the projection from $M \oplus N$ onto N is a BiEndomorphism, which must be given by right multiplication by an element $r \in R$. But then $Mr = 0$, a contradiction.

3. LEMMA. *Let $R = \prod_{i \in I} R_i$, where R is QF-1. Then I is finite.*

Proof. Let $S = \sum_{i \in I} \oplus R_i$. If $S \neq R$, then $S \subset M \subset R$ where M is a maximal right ideal of R. Clearly S is a faithful R-module. But, by 2 either $\operatorname{Hom}_R(R/M, S) \neq 0$, or $\operatorname{Hom}_R(S, R/M) \neq 0$. By Lemma 1 this is impossible, so $S = R$ and I is finite.

4. LEMMA. *Let $R = \prod_{i=1}^n R_i$. Then R is balanced (QF-1) iff each R_i is.*

Proof. Let M for instance be an R_1-module (faithful). We must show that every BiEndomorphism of M is given by right multiplication. Let $N = M \times \prod_{i=2}^n R_i$. Then N is an R-module (faithful). Let $P = \operatorname{End}_{R_1}(M, M)$. Then $\operatorname{End}_R(N, N) = P \times \prod_{i=2}^n R_i = S$. Let $t\colon {}_PM \to {}_PM$. Then t is extendable to a map $t\colon {}_SN \to {}_SN$ given by $(m, x_2, \ldots, x_n) \to (mt, 0, \ldots, 0)$. This is a BiEndomorphism of N, hence is given by right multiplication, so obviously t is.

Conversely, let M_R be an R-module and e_i, $i = 1, \ldots, n$ the central orthogonal idempotents which are the unit elements of R_i. Then $M = \sum_{i=1}^n \oplus Me_i$. If M is a faithful R-module, then Me_i is a faithful R_i-module, for if $(Me_i)e_is = 0$ then $(M)e_is = 0$ so $e_is = 0$. Now let t be a BiEndomorphism of M. Then $t\colon Me_i \to Me_i$, since t must preserve components, and is a BiEndomorphism of Me_i. By hypothesis there is an e_is_i such that $(xe_i)t = (xe_i)e_is_i$. Let $s = \sum_{i=1}^n e_is_i$. Then if $m \in M$,

$$(m)t = \left(\sum_{i=1}^n me_i\right)t = \sum_{i=1}^m (me_it) = \sum_{i=1}^n (me_i)e_is_i = \left(\sum_{i=1}^n me_i\right)\left(\sum_{i=1}^n e_is_i\right) = ms,$$

and the lemma is proved.

5. PROPOSITION. *If R is QF-1 and $J = \operatorname{rad} R = 0$, then R is semisimple.*

Proof. Let $V = \sum_{i \in I} \oplus V_i$ be a direct sum of the nonisomorphic simple right R-modules. Since $J = 0$, V is faithful. Clearly, $\operatorname{Hom}_R(V, V) = \prod_{i \in J} F_i$ where $F_i = \operatorname{Hom}_R(V_i, V_i)$ is a division ring. Then $R = \prod_{i \in I} L_i$ where the L_i are full linear rings. By Lemma 3, $R = \prod_{i=1}^n L_i$ for some integer n. We need only show that each V_i is finite dimensional.

By Lemma 4 each L_i is QF-1. Let $S_i = \operatorname{socle} L_i$, and suppose $L_i \neq S_i$. Let $S_i \subset M_i \subset L_i$ where M_i is a maximal right ideal of L_i. We note first that S_i is projective since each simple right ideal of L_i is generated by an idempotent and a direct sum of projective modules is projective. Now, L_i/M_i cannot be projective or else the map

$L_i \to L_i/M_i \to 0$ splits, and $L_i = A \oplus M_i$ where A is a simple right ideal of L_i not contained in M_i. But, S_i is faithful, so by 2 there is a nonzero map $L_i/M_i \to S_i$ or a map $S_i \to L_i/M_i$. In the first case L_i/M_i is a summand of a projective module since S_i is semisimple, and in the second case the kernel is a summand, and the complementary summand is isomorphic to L_i/M_i. So in either case L_i/M_i is projective, a contradiction that proves that V_i is finite dimensional and that R is therefore semisimple.

It will be revealed later in the study of balanced rings that local rings play a significant role. We now prove a result in this direction for QF-1 rings. But first we have a lemma.

6. LEMMA. *If R is a commutative ring, M a maximal ideal of R, and $E(R/M)$ the injective hull of R/M, then $E(R/M)^{\perp} = \{r \in R \mid r^{\perp} \not\subset M\}$.*

Proof. If $r \in E(R/M)^{\perp}$ then we must have $r^{\perp} \not\subset M$, for otherwise we have a map $f\colon rR \to R/M$ given by $f(r) = 1 + M$. But f may be extended to a map $\bar{f}\colon R \to E(R/M)$, and if $\bar{f}(1) = x$ we have:

$$1 + M = f(r) = \bar{f}(r) = \bar{f}(1) \cdot r = xr \neq 0.$$

On the other hand, if r is contained in the right hand side and $E(R/M) \cdot r \neq 0$, the simplcity of R/M and the definition of the injective hull implies that there is an $x \in E(R/M)$ such that $xr = 1 + M$, and this implies that $r^{\perp} \subset M$. The lemma is thus established.

7. THEOREM. *If R is a commutative ring such that there are no nonzero maps between injective hulls of distinct simple modules, then R is QF-1 if and only if R is a product of a finite number of local QF-1 rings.*

Proof. The sufficiency follows from Lemma 4.

Conversely, let $V = \sum_{i \in I} \oplus E(R/M_i)$ where the M_i are the complete set of maximal ideals. Since R is commutative, $R/M_i \approx R/M_j$ if and only if $i = j$. Now, V must be faithful, since $V^{\perp} = \bigcap_{i \in I} E(R/M_i)^{\perp} = \{r \in R \mid r^{\perp} \not\subset M_i, \forall i \in I\}$ by Lemma 6. But, this says that if $Vr = 0$, then $r^{\perp}$ is not contained in any maximal ideal, a clear impossibility unless $r = 0$. Thus, V is faithful.

We may now apply the same technique used in the proof of Theorem 5. Since V is faithful, our hypothesis gives that $R \approx \text{BiEnd } V$. But the fact that there are no nonzero maps between $E(R/M_i)$ and $E(R/M_j)$, $i \neq j$ gives us that $\text{BiEnd } V \approx \prod_{i \in I} K_i$ where $K_i = \text{BiEnd } E(R/M_i)$ (see the proof of Theorem 5).

But then, Lemma 3 gives us that $R \approx \prod_{i=1}^{n} K_i$, and counting simple modules on each side it is easy to see that each K_i must be local.

REMARK. The condition that there be no maps between injective hulls of nonisomorphic simple modules may be fairly general, and the reader may verify that it is true for Z-modules where Z is the ring of integers.

8. COROLLARY. *Let R be commutative. If R/J is regular, and idempotents lift*

modulo J (e.g., if R_R is injective [8] *or* [4]), *then R is QF-1 if and only if R is a product of a finite number of local QF-1 rings.*

Proof. We need only show that there are no maps between injective hulls of nonisomorphic simple modules. To this end, let M and N be distinct maximal ideals. Then there is an idempotent $e \in M-N$. For, if $x \in M-N$, then there is an $r \in R$ such that $x-xrx \in J$, and since xr is idempotent modulo J the hypothesis implies that there is an idempotent $e \in R$ such that $xr-e \in J$. Now, since $J \subset M$ and $x \in M$ we have $xr-e \in M$ and $xr \in M$ so $e \in M$. But $e \notin N$ or else we get $xr \in N$, or $xrx \in N$ or $x \in N$, a contradiction.

Next, let $E(R/M_i)$ be the injective hull of each simple R-module R/M_i where M_i is a maximal ideal of R. We claim that if $i \neq j$ there are no nonzero maps between $E(R/M_i)$ and $E(R/M_j)$. For, suppose f: $E(R/M_i) \to E(R/M_j)$ and f is not zero. Then, $R/M_j \subset \operatorname{Im} f$ so there is an $x \in E(R/M_i)$ such that $f(x)=1+M_j$. However, from the above there is an idempotent $e_i \in M_i - M_j$. Then $e_i^{\perp} \not\subset M_i$ so $E(R/M_i)e_i=0$ by 6, and $0=f(xe_i)=(1+M_j)e_i=e_i+M_j$. But this says that $e_i \in M_j$ contrary to the construction.

Now let R be commutative injective and QF-1. Then by the above, we have that $R=\prod_{i=1}^n R_i$, where each R_i is local injective and QF-1. It would be nice to know that in fact every faithful module is a generator. Rings which have this property are called PF rings. We have a proposition.

9. Proposition. *If R_R is commutative injective and $J=\operatorname{rad} R$ is finitely generated, then R is QF-1 if and only if R is PF.*

Proof. Utumi [14] has characterized PF rings. These are rings with the following properties: R_R is injective, R/J is semisimple, socle R_R is essential. (A right ideal is essential if and only if it has nonzero intersection with every nonzero right ideal.) It is known [8] that if R is injective then:

$$J = \{r \in R \mid r^{\perp} \text{ is essential}\}.$$

If R is PF then R is always QF-1. This is true because every faithful R-module is a generator and every generator is balanced (see introduction).

Suppose R is QF-1. Then by 7, $R=\prod_{i=1}^n R_i$ where each R_i is local. An examination of the three conditions which must be satisfied reveals that we may, without loss of generality, assume that R is local. Since R is assumed to be injective, and R/J is in fact simple, we must only show that socle R_R is essential. Since R is injective, $J=\{r \in R \mid r^{\perp}$ is essential$\}$. The intersection of any finite number of essential right ideals is never zero, and since J is finitely generated, J has nonzero annihilator. If $xJ=0$ then xR is simple, so R has nonzero socle. But the injectivity of R actually implies that xR is essential. For, let $r \in R$. Then there is a well defined map f: $rR \to xR$ where $f(r)=x$. Extend f to all of R and call the new map f also. Then if $a=f(1)$ we have $ra=x$, so xR is essential, and this proves the theorem.

It is interesting to note that in the above situation, if we consider the local ring case with maximal ideal M, we have that $M \neq M^2$ by Nakayama's Lemma. But even if M is not finitely generated we may still have a proper descending sequence $M \supset M^2, \ldots$, for if M contains no simple ideal, that is if R is not PF, then M is faithful. So there is a nonzero map $M \to R/M$ or $R/M \to M$ by 2, and if the latter is ruled out, then the former must hold. Thus, M has a maximal submodule, and $M \neq M^2$. By induction, we have $M^n \neq M^{n+1}$.

Before proceeding to the next chapter, we have one more word about QF-1 rings without chain conditions. The following proposition is somewhat reminiscent of the local ring case, where every projective module is free.

10. PROPOSITION. *If R is a QF-1 ring with zero socle, then every faithful projective module is a generator.*

Proof. Let P be a projective module. Then we have $R^{(I)} \approx P \oplus K$ for some index I and module K. In particular, this implies that P has zero socle. Now, by 2, if V is any simple R-module, there is either a nonzero map $f: V \to P$ or $g: P \to V$. Since the former is ruled out, the latter must hold and thus there is an epimorphism from P onto every simple module. As is well known [2], this implies that P is a generator.

We will have a little more to say in this direction in the next chapter (see Proposition 13).

II. In this section we begin the study of balanced rings, and QF-1 rings with chain conditions. We shall begin by determining certain BiEndomorphisms of direct sums of modules. We know that if A is an arbitrary R-module, and $r \in R$ then the map $\gamma_d: a \mapsto ar\ \forall a \in A$ is a BiEndomorphism of A. Suppose we have a direct sum of modules $A = \sum_{i \in I} \oplus A_i$, and a collection of elements $r_i \in R$, $i \in I$. Then one might want to know when the map t defined by $(a_1, a_2, \ldots, a_n, 0, \ldots)t = (a_1 r_1, \ldots, a_n r_n, 0, \ldots)$ is a BiEndomorphism. Note that this is a generalization of the usual situation. The lemma below is of central importance to our work.

By $T(M, N)$ we denote the trace of M in N. This is the submodule of N generated by images of elements of M under elements of $\text{Hom}_R(M, N)$. We have the following lemma:

11. LEMMA. *Let $\{A_i \mid i \in I\}$ be R-modules, $A = \sum_{i \in I} \oplus A_i$, and let $\{x_i \mid i \in I\}$ be elements of R. Then the map t defined by $(a_1, \ldots, a_n, 0, 0, \ldots)t = (a_1 x_1, \ldots, a_n x_n, 0, \ldots)$ is a BiEndomorphism if and only if $T(A_i, A_j)(x_i - x_j) = 0$ for every i and j.*

Proof. Necessity. Let $x \in T(A_i, A_j)$, say $x = \sum f_{ij}(a_i)$, $a_i \in A_i$, and $f_{ij} \in \text{Hom}(A_i, A_j)$. We must show that $f_{ij}(a_i)x_i = f_{ij}(a_i)x_j$. Let $f: A \to A$ be defined by requiring that f on A_i be contained in A_j and equal to f_{ij} and that f be zero elsewhere. Let (a_m) be that element of A which is a_m at the mth coordinate and zero elsewhere. Let $f(a_i) = a_j$. Then $[f(a_i)]t = (a_j)t = (a_j)x_j$ and $f[(a_i)t] = f((a_i)x_i) = (a_j)x_i$. So we have $T(A_i, A_j)(x_i - x_j) = 0$, since $[f(a_i)]t = f[(a_i)t]$.

Conversely, to show that t is a BiEndomorphism it is sufficient to check elements of the form (a_i), for if $(a)=(a_1, \ldots, a_n, 0, \ldots)$ then

$$[f(a)]t = \left[\sum_{i=1}^{n} f(a_i)\right]t = \sum_{i=1}^{n} [f(a_i)]t.$$

(The additivity of t is clear.) If $[f(a_i)]t=f[(a_i)t]$ then $[f(a)]t=f[(a)t]$.

So, let $f(a_i)=(b_1, \ldots, b_n, 0, \ldots)$ where $f \in \operatorname{End}_R A$. Then $b_j \in T(A_i, A_j)$ for $j=1, \ldots, n$. Then the hypothesis implies that $b_j(x_i - x_j)=0$. Thus we have

$$\begin{aligned}[f(a_i)]t &= [(b_1, \ldots, b_n, 0, \ldots)]t = (b_1x_1, \ldots, b_nx_n, 0, \ldots) = (b_1x_i, \ldots, b_nx_i, 0, \ldots) \\ &= [f(a_i)]x_i = f(0, \ldots, a_ix_i, 0, \ldots) = f[(a_i)t]\end{aligned}$$

which proves the lemma.

We now state a consequence to Lemma 11, and prove some results related to the previous section.

12. Lemma. *If $A \oplus B$ is balanced, then $T(A, B)^{\perp} \cap T(B, A)^{\perp} = A^{\perp} + B^{\perp}$.*

Proof. Clearly $A^{\perp}+B^{\perp} \subset T(A, B)^{\perp} \cap T(B, A)^{\perp}$. Let $w=1-x \in T(A, B)^{\perp} \cap T(B, A)^{\perp}$. Let $(a, b)t=(a, bx)$. By Lemma 11 this is a BiEndomorphism, so there is an $s \in R$ such that $(a, bx)=(as, bs)$, whence $1-s \in A^{\perp}$ and $s-x \in B^{\perp}$. So $w=1-x=1-s+s-x \in A^{\perp}+B^{\perp}$. This proves the lemma.

13. Proposition. *If R is QF-1 and I is an ideal of R such that $I^2=I$ and $I^{\perp} \cap I = {}^{\perp}I \cap I=0$, then I is a summand for R as a right R-module.*

Proof. Consider $I \oplus R/I$. This is a faithful R-module since $I^{\perp} \cap I=0$. Since $I^2=I$, $T(I, R/I)=0$, since $I \cap {}^{\perp}I=0$, $T(R/I, I)=0$. Now, $T(I, R/I)^{\perp} \cap T(R/I, I)^{\perp} = 0^{\perp} \cap 0^{\perp} = R$, so by 12, $R=I^{\perp} \oplus I$.

14. Proposition. *If R is a commutative QF-1 ring then every faithful projective module P is a generator.*

Proof. Let $T=T(P, R)$. Then as is well known, $T^2=T$ and $PT=P$ [2]. The faithfulness of P implies that $T^{\perp}=0$ and commutativity gives ${}^{\perp}T=0$, so the hypotheses of Lemma 13 are satisfied and we have $R=T \oplus T^{\perp}$, but $T^{\perp}=0$ so $R=T$.

Combining this with Proposition 9, we have that if R is QF-1 and is either commutative or has zero socle, every faithful projective module is a generator.

We now investigate the structure of balanced rings in more detail, and using the techniques to follow we characterize commutative artinian QF-1 rings.

Let R be any ring and $J=\operatorname{rad} R$. Then J is said to be right T-nilpotent if, given any sequence $\{a_i\}$ where $a_i \in J$, there is an integer n such that $a_n \cdot a_{n-1} \cdots a_1=0$. A module M is said to have a projective cover if there is a projective module P and an exact sequence $0 \to S \to P \to M \to 0$ where S is a superfluous submodule of P. The notion is dual to injective hull, but not every module has a projective cover. If every module in the category of modules over a ring R has a projective cover, then

R is said to be perfect, and if every finitely generated R-module has a projective cover, R is said to be semiperfect. Bass [1] has given necessary and sufficient internal conditions on a ring R that R be perfect or semiperfect. Some of these are:

A. R is perfect if and only if R/J is semisimple and J is right T-nilpotent.

B. R is semiperfect if and only if R/J is semisimple and idempotents lift modulo J.

Below, we show that commutative balanced rings are perfect, and noncommutative balanced rings are semiperfect. Bass has shown that if every right R-module contains a maximal submodule then J is right T-nilpotent. We first obtain a result similar to this where every module contains either a maximal or a minimal submodule.

To do this we consider sequences $\{a_i\}$, $i \in Z$, indexed by the integers, positive, negative and 0. We say that an ideal I is bi-T-nilpotent if, given any such sequence of elements from I there are integers n and m, $n \leqq 0 \leqq m$ such that $a_n \cdots a_m = 0$.

15. PROPOSITION. *If R has the property that every R-module contains a maximal or a minimal submodule then J is bi-T-nilpotent.*

Proof (Following Bass). Let F be the free R-module on countably many generators $\{x_i\}$, $i=1, 2, \ldots$, and let G be the submodule of F generated by $\{x_i - x_{i+1}a_{-i}\}$. Consider F/G. If $F/G=0$ then there is an integer $n<0$ such that $a_n \cdots a_{-1}=0$, and if G is a proper submodule of F then F/G has no maximal submodules. To see the latter assertion suppose $G \subset M \subset F$ where M is a maximal submodule of F. If $x_{i+1}a_{-i} \in M$ then $x_i \in M$. If $x_{i+1}a_{-i} \notin M$ then there is an $r \in R$ and an $m \in M$ such that $x_{i+1}a_{-i}r + m = x_{i+1}$. But then $m = x_{i+1}(1-a_{-i}r)$ and since $a_{-i} \in J$, $1-a_{-i}r$ is invertible so $x_{i+1} \in M$, whence $x_i \in M$, so $F=G$.

To see the first statement we may write $x_1 = \sum (x_i - x_{i+1}a_{-i})r_i$, and using the fact that the x_i are free generators of F we obtain $r_1=1$, $r_2=a_{-1}r_1$, and in general $r_{i+1}=a_{-i}r_i$. This implies that $r_{i+1}=a_{-i} \cdots a_{-1}$ and since there is a k such that $r_k=0$ we have $a_{-k+1} \cdots a_{-1}=0$.

Thus, to prove the proposition we are reduced to the case where $F/G \neq 0$ and F/G has no maximal submodules. This implies that no factor module of F/G has a maximal submodule and our hypothesis therefore implies that every factor module of F/G has nonzero socle. So, using the technique that Bass applied to the radical of the rings, we let $N_0=G$, N_1 be that module such that N_1/G is the socle of F/G and for each ordinal α, if α is a successor, define N_α to be such that $N_\alpha/N_{\alpha-1}$ is the socle of F/N_α and if α is a limit ordinal let $N_\alpha = \bigcup_{B<\alpha} N_B$. Now $F=\bigcup N_\alpha$, so for each $x \in F$ let $h(x)$ be the least ordinal α such that $x \in N_\alpha$. Then $h(x)$ is never a limit ordinal and if $h(x) \neq 0$, and $r \in J$, $h(xr)<h(x)$. We wish to show that there is an integer m such that $h(x_1a_0 \cdots a_m)=0$. But, this is clear, for if there is no such m we obtain a strictly decreasing sequence $h(x_1) > h(x_1a_0) \cdots$. This is an impossibility, so there is an m such that $x_1a_0 \cdots a_m \in G$. We may then write $x_1a_0 \cdots a_m = \sum (x_i - x_{i+1}a_{-i})r_i$. We then have $r_1 = a_0 \cdots a_m$, and $r_{i+1}=a_{-i} \cdots a_{-1}r_1$. So, picking $n=1$ where $r_{i+1}=0$ we have $a_n \cdots a_m = 0$, which is the result sought.

16. PROPOSITION. *Let A be a faithful module over a QF-1 ring R. Then every simple R-module V is either a quotient or a submodule of A.*

Proof. This is a direct consequence of Lemma 2.

17. THEOREM. *If R is balanced, then the following four statements are true of R:*
(1) rad *R is nil.*
(2) *If R is commutative then* rad *R is T-nilpotent.*
(3) *R is semiperfect.*
(4) *If R is commutative then R is perfect.*

Proof. Proposition 16 implies that every R-module has either a maximal or a minimal submodule. Thus, 15 applies and $J=\text{rad } R$ is bi-T-nilpotent. Considering cases:

(1) Let $c \in J$ and let $c=\{a_i\}$ as in 15, and the result follows.

(2) If R is commutative and $\{c_i\}$, $i=1, 2, \ldots$ is a sequence in J we let $a_0=c_1$, $a_1=c_3\cdots$ and $a_{-1}=c_2$, $a_{-2}=c_4\cdots$. Proposition 15 and commutativity then yield T-nilpotence.

(3) By 5, R/J is semisimple, and since idempotents lift modulo any nil ideal, R is semiperfect.

(4) J is T nilpotent, and R/J is semisimple; thus R is perfect.

18. COROLLARY. *If R is a balanced noerthian ring, then R is artinian.*

Proof. R/J is semisimple, and as is well known, in a noetherian ring every nil ideal is nilpotent. Since J^k/J^{k+1} is finitely generated, we obtain a composition series for R, so R is artinian.

We now strengthen our results on commutative balanced rings, and show that not only are they perfect, but in fact are artinian.

19. LEMMA. *If R is commutative, local, and QF-1, then socle R_R is either simple or zero.*

Proof. Let u and v generate distinct simple modules, so that the sum $uR+vR$ is direct. Let $A_1=uR$, $A_2=(u+v)R$, $A_3=vR$. We claim that $T(R/A_i, R/A_j)\subset J/A_j$, $i\neq j$. For example suppose $f(1+uR)=a+(u+v)R$, where a is a unit. Then we would have a $u\in(u+v)R$, or since a is invertible $u\in(u+v)R$ and thus $v\in(u+v)R$ so $uR=vR$ since $(u+v)R$ is simple. All cases are handled similarly.

We may now apply Lemma 11 with $x_1=u$, $x_2=u$, $x_3=0$. We have that $x_i-x_j\in\text{socle } R_R$, and $T(R/A_i, R/A_j)(x_i-x_j)=0$. Thus, the map t defined by $(a_1+A_1, a_2+A_2, a_3+A_3)t=(a_1u+A_1, a_2u+A_2, a_30+A_3)$ is a BiEndomorphism of a faithful module. Thus there is an $r\in R$ given by the hypothesis, and $s_1, s_2, s_3\in R$ such that (1) $r-u=us_1$, (2) $r-u=(u+v)s_2$, (3) $r=vs_3$. The first and the third equations give $vs_3-u=us$, and linear independence gives that $vs_3=0$. Using equations (2) and (3), we have $-u=(u+v)s_2$. But then, $vs_2=0$, so $us_2=0$ since $s_2\in J$, so $u=0$. This contradiction proves the lemma.

We now apply a lemma of Osofsky [12] to obtain the result we want. This lemma was generalized by Ornstein [11].

20. LEMMA (OSOFSKY). *If R is a perfect ring with J/J^2 finitely generated, then R is artinian.*

21. LEMMA. (See Theorem 22 to follow.) *If R is balanced and commutative, then R is artinian.*

Proof. By 17, R is perfect, and it is well known that R is a product of a finite number of local rings, each with T-nilpotent radical [1]. It is thus sufficient to show that each factor is artinian. But, since each of these rings R_i is balanced, R_i/J_i^2 is QF-1, where J_i is the radical of R_i. Proposition 19 then implies that J_i/J_i^2 is finitely generated, so Osofsky's lemma implies that each R_i is artinian.

At this point, we are able to prove a converse, and obtain a solution to a problem of Thrall for commutative rings. Recall that a ring R is quasi-Frobenius (QF) if R is right artinian and R_R is injective. A quasi-Frobenius ring R is QF-1, for R satisfies the three conditions of Utumi's theorem characterizing those rings for which every faithful module is a generator. Since every faithful module is a generator and since every generator is balanced, R is a QF-1 ring. Thrall's problem is to determine all finite dimensional QF-1 algebras, and he has shown that not every QF-1 algebra is QF, cf. [13]. His example is not commutative, but here we show that in the commutative case, one is able to drop the algebra structure, requiring only that the rings involved be artinian, and still prove that the two concepts are equivalent for a large class of commutative rings. Floyd [5] proved the same theorem for commutative finite dimensional algebras. We have:

22. THEOREM. *If R is a commutative artinian ring, then the following three statements are equivalent*:

A. *Every finitely generated faithful module is balanced.*

B. *Every faithful module is balanced.*

C. *R is QF.*

Proof. A $\Rightarrow$ C. Since R is commutative and artinian, we may write R as a product of a finite number of local rings. By 4 every factor has the property that every finitely generated faithful module is balanced, and since a finite product of QF rings is QF, it is sufficient to assume that R is local. If R is commutative, artinian, and local, then R is QF if and only if socle R_R is simple, cf. [2]. But this is a statement of Lemma 19. The fact that C $\Rightarrow$ B follows from the remarks previous to the theorem, and B $\Rightarrow$ A is trivial.

Using the same techniques as above we are able to characterize commutative balanced rings.

23. THEOREM. *If R is commutative then the following three conditions are equivalent*:

A. *R is balanced.*

B. *R is a product of a finite number of local artinian principal ideal rings.*

C. *R/I is QF for every ideal I.*

Proof. A $\Rightarrow$ B. Suppose R is balanced. Then, by 21, R is artinian. Write $R=\prod_{i=1}^{n} R_i$ where each R_i is local. Each R_i satisfies the conditions of 22 and in fact for every ideal $I_i \subset R_i$, R_i/I_i is QF-1. Thus, each R_i/I_i has simple socle, and this clearly means that R_i is uniserial, which in this case is equivalent to R_i being principal.

B $\Rightarrow$ C. The properties of 2 are clearly preserved under homomorphic images, and each factor is clearly QF.

C $\Rightarrow$ A. If R/I is QF for every ideal then R/I is QF-1, since every QF ring is QF-1, cf [13].

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